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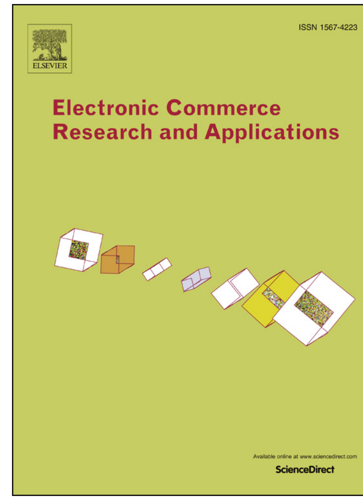

Combinatorial advertising internet auctions

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COMBINATORIAL ADVERTISING INTERNET AUCTIONS

The paper extends the Generalised Second Price (GSP) auction, adopted by search engines to allocate advertising slots, considering the possibility that players could use images to advertise their products, rather than text only ads. Since images occupy more than one slot in the template the extension consists in allowing players to bid for a bundle of slots, as well as for a single slot, where an image could be accommodated. For this reason the auctions we consider are combinatorial, though with constrained combinations. Indeed, to accommodate an image or a video slots will have to occupy consecutive positions. Unlike existing work, we assume that slots assignment is determined endogenously by the players, as the outcome of a Nash Equilibrium. This implies that the number of slots assigned to text only messages and to images is not predefined by the search engine. Based on the analysis, we argue that alternative ways to extend GSP can give rise to rather different slots allocation and revenues to the search engine.

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COMBINATORIAL ADVERTISING INTERNET AUCTIONS

1. Introduction

The success of Google position auctions, in selling slots of a template for advertising texts, has attracted much attention. As a result a large body of game theoretic analysis, with both complete (Edelman et. al. 2007; Edelman & Schwarz 2010a, Varian 2007, 2009; Easley & Steinberg, 2010; Varian & Harris, 2014) and incomplete information (Edelman & Schwarz 2010b, Gomes & Sweeney, 2014), has been developed to investigate their underlying properties.

Until now advertising texts have occupied only a single slot in a template. However, more recently bidders became interested in adding up to their texts videos and(or) images, which however occupy more than a single slot. Indeed, images or videos require a bundle of consecutive slots since otherwise they could not be accommodated in the template. Bidders' interest in more sophisticated advertisements is justified by the increase in the click rate, and related expected revenues, that more elaborated commercials may generate.

Proposals on how to extend the Google Generalized Second Price (GSP) position auction, to this new scenario, have been mostly based on the idea that the template composition of slots is pre-defined by the auctioneer (Deng et al 2014, Goel & Khani 2014, Bachrach et al, 2014, Agarwal & Mukhopadhyay 2016). That is, the search engine establishes whether to use a template only for images (plus text), for texts only or for both. In this last case however, it would fix a priori the number of slots dedicated to texts vs image bidding. As a consequence existing work, with few exceptions (Bapna et al, 2004; Bleier & Eisenbeiss, 2015; Chen & Stallaert, 2014,), so far concentrated on how the search engine could choose among alternative, feasible, templates and how prices for advertisements should be set by the engine. However, the optimal choice of a template composition in this case would put considerable informational burden on the search engine which, besides the click rates for single texts and images in alternative template positions, should also know enough about the bidders' values. Indeed, in a revenue maximizing perspective this information would be crucial to decide how many slots to occupy with text only messages and with images.

Because of such informational difficulty, in this paper we take a different approach and assume that the template composition of ads is not predefined by the search engine, but rather determined by the bidders according to their preferences. More explicitly, we suppose that players can freely choose whether to bid for a single slot, and so for a text only, or for a bundle of consecutive slots, hence for an image plus text. In so doing, based on the slots allocation rule chosen by the search engine, they would determine how many slots will be assigned to texts advertisements only and to images. As a result, the template composition of single texts and images would be left to bidders and emerge endogenously, as a Nash Equilibrium of the auction. To our knowledge, the paper is a first exploration of this new game theoretic scenario.

Since bundles of consecutive slots are needed to accommodate images, the auction would be combinatorial. However we do not allow players to bid freely for any possible single slot or bundle of slots, and for as many combinations as they wish. Indeed, as well as for GSP, in the paper players can only submit at most one bid: either for a single slot or for a single bundle but not for both.

More general settings where players can submit multiple offers could also be considered, but the framework we analyze is already rich and complex enough to provide interesting insights. Because our model extends the essential features of GSP to simplest combinations of slots, we shall refer to this framework as the Combinatorial Generalized Second Price auction (CGSP). However, in the paper we shall argue that the extension of GSP is far from being obvious, and may not be unique. For this reason we decided to introduce and compare two possible criteria, which we indicated as CGSPa and CGSPb. Indeed when some players bid for single slots while others for bundles there may be more than one criterion to compare and rank different bids, allocate slots and determine prices, in analogy with GSP. The challenging point in this case is the mixed composition of bids, for single slots and bundles of slots, whose comparison and ranking requires new perspectives.

The main goal of the paper is to introduce and discuss two such combinatorial models, which to us seem as *natural* extensions of GSP, and present a first exploration of the issue. As we shall see the reason why we believe they represent a natural way to generalize GSP, is because in our view they both embody its fundamental principle of "*paying as price the bid immediately below one's offer*". More specifically, they formalize two alternative, yet rather intuitive, views of capturing the same principle. To help illustrating their features and properties we shall also investigate several numerical examples, to convey some insights on the complexities that the selection and implementation of CGSPs may exhibit. The work is structured as follows: in Section 2 we introduce the combinatorial framework, in Section 3 we define GCSPa and GCSPb. Section 4 illustrates some numerical examples to show how the definitions apply while Section 5 presents some analytical results. Finally, Section 6 concludes the paper

2. The General Model

Let $a = 1,..,A$ be the set and the number of players, $s = 1,...,S$ the number of slots available for advertisements (ads), in a template, and $i = x,y$ the label of available ads type for which bidders can submit offers. For simplicity, in the paper an image can also mean a video. Conventionally $i = x$ will stand for a text only advertisement, with no image, while $i = y$ will stand for an image (plus text), which for simplicity we assume to be of the same size for all players. Each single slot can contain only a text message, while bundles of $1 < G \leq S$ consecutive slots can accommodate both an image plus text. The setting could be extended to include different types of images, requiring different numbers of consecutive slots. As said, because of the exploratory goal of the paper in what follows we confine the analysis to the simplest possible framework.

As well as for GSP, we assume that each bidder can at most make one bid, either for a single slot or for an image, but not for both. Therefore, if m is the number of images included in the template then $m = 0,1,..,M$, where $M = \min(A,\frac{S}{G})$ is the maximum number of images that could be inserted in the template. When m images are accommodated in the template then $r(m) = S - Gm = 0,..,S$ is the maximum number of slots, available to bidders as single slots.

Further suppose, as in Varian (2007), that single slots are ordered according to the expected number of clicks received, except that the model is now enriched by the presence of images that could occupy bundles of slots.

If c_{si} is the expected number of clicks for the generic slot $s = 1,..,S$, with ad $i = x,y$ and, with no loss of generality, assume

$$c_{1x} > c_{2x} > \dots > c_{Sx}$$

meaning that when single slots, with text only and no image, are compared then slot 1 obtains the highest expected number of clicks, slot 2 the second highest number of clicks and so on.

Likewise, we assume

$$c_{1y} > c_{2y} > \dots > c_{(S-G+1)y}$$

namely that an image starting from slot 1, and occupying slots from 1 to G, receives a higher number of clicks than an image starting from slot 2 and occupying positions from 2 to $G + 1$, and so on up to an image starting from slot $S - G + 1$ and covering slots from $S - G + 1$ to S.

Clearly, an image could be accommodated in the template only if at least G slots are available.

Since images increase the number of clicks it follows that $c_{sy} > c_{sx}$ for $s = 1,..,S - G + 1$

which implies $c_{sy} > c_{kx}$ with $s \leq k$ but not necessarily $c_{sy} > c_{kx}$, for $s > k$. That is, in principle the click rate may be larger for a single slot in a high position, than for an image occupying slots below that position (Asdemir et al., 2012).

Finally define as v_a the expected revenue, value, generated by a single click for the generic player a.

As previously said, like in GSP (Edelman et al 2007, Varian 2007) we assume that advertisers can submit at most one bid: in particular, b_{ai} is player a's bid for ad $i = x,y$.

If b_{ax}, for all a, are the submitted bids, that is everyone bids only for one slot, then this becomes the standard GSP ad auction, where players submit offers for single slots only. However now they have the possibility to bid alternatively for an image and, based on the rule adopted for awarding slots, define the composition of the search engine template.

If T stands for the template, then a convenient way to imagine T is as a vector of slots $T = (1,2,..,S)$ with slots from left to right obtaining a decreasing click rate. Let si, with $i = x,y$, stand for slot s assigned to ad i and suppose, for example, that $G = 2$ and that only one player bids for an image $i = y$. Assuming that all the slots are assigned by the search engine, the following could be the only possible configurations of the template

$$T = (1y,2y,3x,..,Sx);\ T = (1x,2y,3y,4x,..,Sx);\ldots;\ T = (1x,2x,3x,4x,..(S-1)y,Sy);$$

$$T = (1x,2x,3x,4x,..,Sx)$$

In the next chapter we introduce two combinational versions of GSP, CGSPa and CGSPb, and later conduct a related game theoretic analysis only under complete information.

3 Two Combinatorial Generalised Second Price auctions

As already mentioned, extension of the GSP to auctions where players bid for a combination of slots it is not straightforward. This is because the main principle underlying GSP, where each player pays the price *immediately below* his own bid, when offers for single and bundles of slots are coexisting, could reasonably be extended in more than one way.

In this chapter we present two such generalizations of GSP, CGSPa and CGSPb as an example to discuss the effect that alternative rules for slots award, and paid prices, may have on the template formation and the search engine revenue. We believe the two examples are somewhat *natural* extensions of GSP, though others could also be considered. Yet, we shall see that despite their apparent similarity they may induce very different outcomes.

As above, we assume bidders can submit at most one bid, either for a single slot or for an image but not for both. The auctioneer collects all the offers and if no player submits a price bid for an image then the CGSPs coincide with the standard GSP.

Suppose instead that if $n \leq A$ bidders submit a price offer then r of them bid for an image, while m for text only slots, so that $n = r + m$. Therefore, the total number of demanded slots is $Z = rG + m$. If $Z \leq S$ then all the requested slots would be assigned otherwise some bidder will be excluded, even if the number of bidders is lower than the number of available slots $n \leq S$.

Without losing generality bids for G consecutive slots, that is for the image, are such that

$$b_{(j)y} > b_{(j+1)y} \ with\ j = 1,..,r-1$$

and those for single slots such that

$$b_{(j)x} > b_{(j+1)x} \ with\ j = 1,..,m-1$$

where $b_{(j)i}$ stands for the $j-th$ largest bid submitted for $i = x,y$.

Moreover, to simplify notation and exposition, define $B(l) = \sum_{j=1}^{l} b_{(j)y}$ as the sum of the l highest bids for the image and $b(h) = \sum_{j=1}^{h} b_{(j)x}$ the sum of the h highest bids for single slots, with $l = 1,2,..$ and $h = 1,2...$, with $B(l) = B(r)$ for $l \geq r$ and $b(h) = b(m)$ for $h \geq r$. Finally, define $B(0) = 0 = b(0)$.

We begin with CGSPa

Definition (CGSPa) *(i) Consider slot s, with $s = 0,1,..,S-G$, such that s is the last slot assigned either to a bundle of slots or to a single slot. Interpret $s = 0$ as neither bundles, nor single slots, have been assigned up to that slot. Suppose that until slot s (included) Gk slots have been allocated to k images, with $k = 0,1,..,m-1$, and t as single slots, with $t = 0,1,..,r-1$, so that $Gk + t = s$.*

Then if $B(k+1) - B(k) > b(t+G) - b(t)$ *assign slots from the* $(s+1)th$ *to the* $(s+G)th$ *to the* $(k+1)th$ *best bidder for an image. In this case the best* $(k+1)th$ *bidder for an image pays as price*

$$max\,\{B(k+2) - B(k+1), b(t+G) - b(t)\}$$

(ii) if $B(k+1) - B(k) < b(t+G) - b(t)$ *assign the* $(s+1)th$ *slot to the* $(t+1)th$ *best bid for a single slot. In this case the best* $(t+1)th$ *bidder for a single slot will pay as price*

$$\begin{cases} b(t+2) - b(t+1) & if\ b(t+1+G) - b(t+1) > B(k+1) - B(k) \\ min\left[b(t+1) - b(t); \dfrac{2(B(k+1) - B(k))}{(G+1)}\right] & if\, b(t+1+G) - b(t+1) < B(k+1) - B(k) \end{cases}$$

If $b(t+1+G) - b(t+1) = B(k+1) - B(k)$ *randomly decide with probability* 0.5 *between the two prices.*

(iii) if $B(k+1) - B(k) = b(t+G) - b(t)$ *randomly decide with* 0.5 *probability whether the assignment, and related payment, is as in (i) or (ii).*

(iv) proceed accordingly until the last assigned slot s *is such that* $S - G < s < S$*. Then allocate the remaining slots to bids for text only advertisements, as in GSP.*

Some comments are in order. The main underlying principle of slots assignment is based on the separation between bids for single slots from package bids, according to the following intuition. First, bids for single slots and for bundles are separately ranked from larger to smaller. Then suppose $s = 0,1,..,S - G$ slots have been already allocated. Whether the next assignment should be to a single slot, or to a combination of G slots, depends on the sum of the remaining G largest bids for single slots $b(t+G) - b(t)$, as compared to the remaining largest bid for an image $B(k+1) - B(k)$. If $B(k+1) - B(k)$ is higher, then the next G slots will be assigned to that bidder. However, if $b(t+G)$

$-\, b(t)$ is higher then *only* slot $s+1$, will be assigned to the highest remaining bid for single slots. It is worth pointing out that, even though this would take place thanks to the *sum* of G bids for single slots, only the highest of them will be awarded a slot. The others will be part of the next, analogous, comparison with bids for images, in the continuation of the slots allocation procedure. As we shall see, this point represents a crucial difference with CGSPb.

Finally, if the next allocation is to an image it comes natural to generalize the GSP by setting a price to pay equal to the maximum between the package offer immediately below $B(k+2) - B(k+1)$ and $b(t+G) - b(t)$. However, if the next allocation is to a single slot then price determination is more involved. This is because the price depends on who obtains the following slot. If it is assigned to a text only ad then the price paid $b(t+2) - b(t+1)$ is the one offered by that bidder, immediately below $b(t+1) - b(t)$. If instead the $s+1$ slot is allocated to an image, then there is no such *obvious price* to refer to. The idea is to consider $\frac{2(B(k+1) - B(k))}{(G+1)}$ as a price estimate of the most valuable slot in the package, *implicitly* expressed by the offer for an image. This is obtained by first introducing weights $w_1 \geq w_2 \geq .. \geq w_G \geq 0$, such that $\sum_{g=1}^{G} w_g = 1$, associated to each of the G slots in the bundle, and then set the price to pay equal to the highest weighted package bid w_1 $(B(k+1) - B(k))$.

Indeed, since the bid for an image does not contain price expressions for single slots, we assume they take non-increasing values according to their position: the smaller their rank in the package the higher their value.

The simple criterion adopted both in GCSPa and CGSPb, for constructing such weights, is as follows. The best slot in the bundle will have a weight proportional to G, the second best slot a weight proportional to $(G-1)$ and so on up to the Gth slot which will have a weight proportional to 1. Since weights add up to one, each of the above values will be divided by their sum, that is by

$\frac{G(G+1)}{2}$, which implies that the best slot will have weight $\frac{2G}{G(G+1)} = \frac{2}{(G+1)}$, leading to $\frac{2(B(k+1)-B(k))}{(G+1)}$.

Finally, based on the GSP principle, that a bidder should never pay a sum higher than his own bid, the price paid by the relevant player will be

$$min\left[b(t+1)-b(t); \frac{2(B(k+1)-B(k))}{(G+1)}\right]$$

Though an extensive discussion of numerical examples illustrating the two CGSPs is postponed until next chapter, to gain some very early insights on how CGSPa works, below it could be useful to anticipate the relevant part of Table 1 of Chapter 4.

Table A			CGSPa		
Players	Bids for single slots	Bids for the image	Slots	Player	Price
1		B(1)=21	1y	1	18
2	b(1)=10		2y	1	
3	b(2)-b(1)=8		3x	2	8
4	b(3)-b(2)=3		4x	3	3
5	b(4)-b(3)=2		5x	4	2
6	b(5)-b(4)=1		6x	5	1
			7x	6	0

Applying the above definition, the first two slots are allocated to the bid for an image, as that bid $B(1) = 21$ is larger than the sum $b(2) = 10 + 18$ of the two highest bids for single slots. The rest of the remaining slots are assigned as singles according to the definition of CGSPa, which for such slots coincides with GSP.

We now introduce CGSPb, which requires a longer description.

Definition (CGSPb) *(i) Start with slot* $s = 0$ *Then if* $B(1) > b(G)$ *assign slots from the* $1st$ *to the* Gth *to the best bidder for an image, who will pay as price*

$$max\,\{B(2) - B(1), b(G)\}$$

(ii) if $B(1) < b(G)$ then assign slots from the $1st$ to the Gth, respectively, to the best bidder for a single slot up to the Gth best bidder for a single slot. In this case the best bidder for a single slot will pay as price the second highest bid, the second best bidder will pay the third highest bid and so on, until the best Gth bidder who will pay as price

$$\begin{cases} b(G+1) - b(G) & if\ b(G+1) - b(1) > B(1) \\ min\left[b(G) - b(G-1); \dfrac{2B(1)}{(G+1)}\right] & if\, b(G+1) - b(1) < B(1) \end{cases}$$

If $b(G+1) - b(1) = B(1)$ randomly decide with probability 0.5 between the two prices.

(iii) if $B(1) = b(G)$ randomly decide with 0.5 probability whether the assignment, and related payment, is as in (i) or (ii).

(iv) if (i) holds and $B(2) - B(1) > b(G)$ then assign slots from the $(G+1)th$ up to the $2Gth$ to the second best bidder for an image, who will pay as price

$$max\,\{B(3) - B(2), b(G)\}$$

(v) if (i) holds and $B(2) - B(1) < b(G)$ then assign slots from the $(G+1)th$ to the $2Gth$, respectively, to the best bidder for a single slot up to the Gth best bidder for a single slot. In this case the best bidder for a single slot will pay as price the second highest bid for a text only ad, the second best bidder will pay the third highest bid and so on, until the best Gth bidder who will pay as price

$$\begin{cases} b(G+1) - b(G) & if\ b(G+1) - b(1) > B(2) - B(1) \\ min\left[b(G) - b(G-1); \dfrac{2(B(2) - B(1))}{(G+1)}\right] & if\, b(G+1) - b(1) < B(2) - B(1) \end{cases}$$

If $b(G+1) - b(1) = B(2) - B(1)$ randomly decide with probability 0.5 between the two prices.

(vi) if (i) holds and $B(2) - B(1) = b(G)$ *randomly decide with* 0.5 *probability whether slots assignment, and related payment, is as in (iv) or (v).*

(vii) if (ii) holds and $B(1) > b(G+1) - b(1)$ *then assign slots from the* $(G+1)th$ *to the* $2Gth$ *to the best bidder for an image, who will pay as price*

$$max\,\{B(2) - B(1), b(G+1) - b(1)\}$$

(viii) if (ii) holds and $B(1) < b(G+1) - b(1)$ *then assign slot* $(G+1)th$ *to the* $(G+1)th$ *best bid for a single slot. In this case the* $(G+1)th$ *best bidder for a text only will pay as price*

$$\begin{cases} b(G+2) - b(G+1) & if\; b(G+2) - b(2) > B(1) \\ min\left[b(G+1) - b(G); \dfrac{2B(1)}{(G+1)}\right] & if\, b(G+2) - b(2) < B(1) \end{cases}$$

If $b(G+2) - b(2) = B(1)$ *randomly decide with probability* 0.5 *between the two prices.*

(ix) if (ii) holds and $B(1) = b(G+1) - b(1)$ *randomly decide with* 0.5 *probability whether the assignment, and related payment, is as in (vi) or (vii).*

(x) proceed accordingly until the last allocated slot s *is such that* $S - G < s < S$*. Then allocate the remaining slots to bids for text only advertisements, as in GSP.*

As previously anticipated, a first main difference between CGSPa and CGSPb is point (ii). Indeed in CGSPb if, initially, the sum of the bids for single slots $b(G)$ is higher than the bid for an image $B(1)$ then all those bids for single slots will occupy the next G slots, while in CGSPa only the best bid will do so. The second main difference is point (viii), which specifies that slots allocation to text only ads could be done also individually, as well as in blocks of G slots, however always comparing the sum of G bids for single slots with the bid for an image. If both extensions may as natural ways to generalise GSP, below we shall see how they could originate rather different equilibrium revenues to the search engine.

As well as for CGSPa, to immediately obtain some insights on how CGSPb works, below we anticipate from Chapter 4 the relevant part of Table 2.

Table B			CGSPb		
Players	Bids for single slots	Bids for the image	Slots	Player	Price
1		B(1)=17	1x	2	8
2	b(1)=10		2x	3	8
3	b(2)-b(1)=8		3y	1	5
4	b(3)-b(2)=3		4y	1	
5	b(4)-b(3)=2		5x	4	2
6	b(5)-b(4)=1		6x	5	1
			7x	6	0

Since $b(2) = 10 + 8 = 18 > 17 = B(1)$, both the best bids for single slots occupy the top two positions. We shall see later that this would not be the case with CGSPa.

Comparison between CGSPa and CGSPb will help discussing the implications of alternative ways to distribute slots between text only and image bids. In the next chapter we are going to apply the two CGSPs to see how allocation of positions and prices may change, despite such apparent similarity.

4 An appraisal of differences and similarities of the CGSPs based on numerical examples

In this chapter we investigate how the two CGSPs work when applied to numerical examples. The analysis will help establishing differences and similarities in the related outcomes. Start considering, for simplicity, only one bid for an image $r = 1$, and that with no loss of generality player 1 is the only bidder who submits an offer for $i = y$. Then in both CGSPa and CGSPb the auctioneer will assign, to player 1, consecutive slots from 1 to G if his price offer is higher than the sum of the price offers for single slots from the $1st$ to the Gth. That is if

$$b_{(1)y} = B(1) > b(G) \quad (1)$$

Defining $p_{(k)i}$ as the kth highest per-click price paid, for advertisement i, then when (1) is satisfied player 1 would pay a price $p_{(1)y}$ given by

$$p_{(1)y} = b(G)$$

The remaining slots, from the $(G+1)th$ onward, will be assigned in analogy with the standard GSP, according to the single slots ranking of bids $b_{(1)x} > \; b_{(2)x} > \ldots > b_{(m-1)x}$. In particular, $p_{(2)x} = \; b_{(2)x} = b(2) - b(1)$, $p_{(3)x} = \; b_{(3)x} = b(3) - b(2)$………

Notice that, since slots from 1 to G are allocated to player 1, the second highest price to be paid $p_{(2)x}$ refers to the single $(G+1)th$ slot, price $p_{(3)x}$ refers to the single $(G+2)th$ slot etc.

The numerical example in the following table where we assume that $A = 6$, an image occupies two slots, $G = 2$, the number of slots is $S > 6$, presents a benchmark situation to show how CGSPa and CGSPb may also produce the same outcome.

Table 1			Table 1a CGSPa			Table 1b CGSPb		
Players	Bids for single slots	Bids for the image	Slots	Player	Price	Slots	Player	Price

1		B(1)=21	1y	1	18	1y	1	18
2	b(1)=10		2y	1		2y	1	
3	b(2)-b(1)=8		3x	2	8	3x	2	8
4	b(3)-b(2)=3		4x	3	3	4x	3	3
5	b(4)-b(3)=2		5x	4	2	5x	4	2
6	b(5)-b(4)=1		6x	5	1	6x	5	1
			7x	6	0	7x	6	0

In this first example, where image $i = y$ occupies the most valuable slots in terms of click-through rate, the CGSPs entail the same assignment of positions and prices. For this reason, the example does not illustrate their difference; however, we shall see that things become different from the second example onward.

It is worth noticing also that as compared to GSP in CGSPs players have more alternative bidding options but, at the same time, because images occupy blocks of consecutive slots some specific changes for players may not be feasible.

To see this consider for example player 2 in Table 1a and 1b, and suppose he wants to occupy the first slot. Then, assuming the other players do not change their bids, he can do so by bidding for a single slot $b(1) > 13$ or bidding above 21 for the image. We see below that in the former case he would be assigned the first slot only, while in the latter case both the first and the second slots. However, for him to occupy precisely the second slot would be impossible when opponents do not vary their bids. Indeed, to do so he would need to change his own bid for a single slot in such a way that player 3 would occupy the first slot, but for that player 2 needs to lower his bid below 8, which leads to player 1 being assigned the first two slots.

Finally, if v_a is the value of an ad for bidder a, notice that the expected profit of player 2 is E $\Pi_2 = (v_2 - 8)c_{3x}$. Therefore, if 2 wants to occupy the first slot only, by bidding for instance 14, his expected profit would become $E\Pi_2 = (v_2 - \min(14, \frac{2}{3}21))c_{1x} = (v_2 - 14)c_{1x}$. Assuming both expected profits to be non-negative the change would be profitable if $(v_2 - 14)c_{1x} > (v_2 - 8)c_{3x}$, that is if the loss in the single click profit is more than compensated by the increase in the number of clicks, when moving from slot 3 to slot 1.

If (1) is reversed (ties are resolved by random drawings) then slot 1 will be assigned as single to the best offer, like in the GSP. However, through the example we now illustrate the CGSPs functioning and how they could entail different allocations. Indeed suppose

$$b(G) > B(1) > b(G+1) - b(1) \quad (2)$$

is satisfied.

Then in CGSPa slots from 2 to $G+1$ will be assigned to the image, while slots from the $(G+2)th$ onward will be allocated according to the single slots ranking $b_{(2)x} > \dots > b_{(m-1)x}$.

Therefore, based on the definition of CGSPa the price paid by the player occupying the first single slot will be

$$p_{(1)x} = Min\left(b_{(1)x}, \frac{2B(1)}{(G+1)}\right) \quad (3)$$

Notice that an even simpler, alternative, criterion to define the price to pay would be to set

$$p_{(1)x} = \frac{B(1)}{G}$$

that is to assume that each slot in the image has the same weight $w_g = \frac{1}{G}$, with $g = 1, \dots, G$. Unlike (3) this always guarantees $b_{(1)x} > p_{(1)x}$, but would disregard differences in the click rate along different slots. This is why in the paper we decided to adopt (3).

Consider now CGSPb, where we assume that as long as $b(G) > B(1)$, even if the right inequality in (2) is satisfied, all the top G bidders for single slots will occupy slots from 1 to G, and not the best bid for a single slot only as in CGSPa. As a consequence the image will occupy slots below the $(G+1)th$. As we shall see later, this difference in the assignment criterion would typically have meaningful implications on the search engine's revenue. The remaining positions will be allocated according to the bids ranking for single slots.

As an illustration, consider again the previous example, where all the bids are the same except that now $B(1) = 17$, a situation summarized below by Table 2.

Table 2			Table 2a CGSPa			Table 2b CGSPb		
Players	Bids for single slots	Bids for the image	Slots	Player	Price	Slots	Player	Price
1		B(1)=17	1x	2	10	1x	2	8
2	b(1)=10		2y	1	11	2x	3	8
3	b(2)-b(1)=8		3y	1		3y	1	5
4	b(3)-b(2)=3		4x	3	3	4y	1	
5	b(4)-b(3)=2		5x	4	2	5x	4	2
6	b(5)-b(4)=1		6x	5	1	6x	5	1
			7x	6	0	7x	6	0

As already mentioned in the previous chapter, in Table 2 it is $(2) = 18 > B(1) = 17 > b(3) - b(1) = 21 - 10 = 11$. Therefore, in CGSPa slot 1 will be allocated to player 2, while slots 2 and 3 to player 1. Since the price charged to player 2 is given by (3) then he has to pay $p_{(1)x} = Min(b(1), \frac{2}{3} 17) = 10$. This is because the price offered for the image is relatively high as compared to the best bids for single slots.

Moreover,

$$p_{(2)y} = b(G+1) - b(1) = b(3) - b(1) = 11$$

and $p_{(3)x} = b_{(2)x} = b(3) - b(2) = 3, p_{(4)x} = b_{(3)x} = b(4) - b(3) = 2$,...Summarizing, in this case the first slot would be allocated as single, then slots from 2 to $(G+1)$ to the image while the remaining slots to text only ads again.

Differently, in CGSPb the first two slots are both allocated to single bids, the image will occupy the next two slots and the remaining ones again assigned to single bids. Notice that player 2, occupying the top slot, will pay $p_{(1)x} = 8$ the second highest bid for single slots, as well as player 3 who occupies the second slot. Indeed $p_{(2)x} = min(8, \frac{2}{3}17) = 8$.

Likewise, continuing with the illustration of the CGSPs, if (1) and (2) are not satisfied but

$$b(G+1) - b(1) > B(1) > b(G+2) - b(2) \quad (4)$$

Is true then in CGSPa slots 1 and 2 would be allocated as single slots to the highest two bids, respectively $b_{(1)x}$ and $b_{(2)x}$, where $p_{(1)x} = b(2) - b(1)$. Moreover, in analogy with (3) the second highest price $p_{(2)x}$ will be given by

$$p_{(2)x} = Min\left(b_{(2)x}, \frac{2B(1)}{(G+1)}\right) (5)$$

with slots from 3 to $G+2$ assigned to player 1 for the image at the price

$$p_{(3)y} = b(G+3) - b(2)$$

and the remaining ones allocated as single slots at prices $p_{(4)x} = b(3) - b(2), p_{(5)x} = b(4) - b(3)$,...As an illustration of the CGSPs in this case consider again the previous example, modified as in Table 3, where all is like in Table 1 except for $B(1) = 9$ instead of $B(1) = 17$.

Table 3			Table 3a CGSPa			Table 3b CGSPb		
Players	Bids for single slots	Bids for the image	Slots	Player	Price	Slots	Player	Price
1		B(1)=9	1x	2	8	1x	2	8
2	b(1)=10		2x	3	6	2x	3	3
3	b(2)-b(1)=8		3y	1	5	3x	4	3
4	b(3)-b(2)=3		4y	1		4y	1	5
5	b(4)-b(3)=2		5x	4	2	5y	1	
6	b(5)-b(4)=1		6x	5	1	6x	5	1
			7x	6	0	7x	6	0

Therefore, now $b(2) = 18 > b(3) - b(1) = 11 > B(1) = 9 > b(4) - b(2) = 5$. Hence, according to CGSPa slots 1 and 2 are allocated as singles, 3 and 4 to the image, while 5 and 6 as singles. Based on (5), the price paid by player 3 who occupies the second single slot from the top, is given by

$$p_{(2)x} = Min\left(8, \frac{2}{3}9\right) = 6$$

Instead, according to CGSPb, the top three slots will be assigned as singles, the next two to the image and then singles again. Following a similar reasoning, all the remaining cases, with only one bid for the image, could be discussed to compose the optimal template for all possible bid profiles.

Extensions to more than one bid for images can be made along similar lines, according to the definitions of CGSPs

Finally, we observe that the examples discussed seem to suggest that CGSPa may tend to provide the auctioneer with a higher total revenue than CGSPb. Later, this point, will be taken up again .

5 Bidder's payoff and Nash Equilibrium with complete information

In this chapter we discuss some features of Nash Equilibria and of bidders' payoff.

As well as in GSP (Varian, 2007) the CGSPs formats can have multiple Nash Equilibria (NE) and, even with complete information, their full characterization would be difficult. Yet, because of its nature, in CGSPb it is possible to provide a general result on the NE structure, which we formulate below.

Proposition 1 *Consider a Nash Equilibrium with CGSPb. If at least two players bid for single slots then slots* 1 *and* 2 *can never be assigned to an image* $i = y$

Proof

Suppose, by contradiction, that in a CGSPb Nash Equilibrium slots from 1 to G are allocated to an image $i = y$, and that the first two slots assigned as single are $G + 1$ and $G + k$, with $k \geq 2$, respectively to player $r + 1$ and $r + k$ who bid $b_{(r+1)x} > b_{(r+k)x}$. Hence $B(1) > b_{(r+1)x} + b_{(r+k)x}$. However this means that player $r + 1$ is willing to pay as price $b_{(r+k)x}$, that is $v_{r+1} > b_{(r+k)x}$. Therefore, it would be optimal for him to raise his bid to $b'_{(r+1)x} > b_{(r+1)x}$, in such a way that $b'_{(r+1)x} + b_{(r+k)x} > B(1)$, since he could now increase his profit. Indeed, by doing so he would be assigned the first slot, augmenting his click through rate while still paying $b_{(r+k)x}$. Therefore the above allocation of slots cannot be part of a NE. It is easy to check that a similar reasoning applies if more than 2 bids are for single slots.

The above proposition has a simple, yet interesting, intuition. In CGSPb players' bids for single slots can cause significant externalities on other players, as one's slot assignment could vary meaningfully when opponents change their bids. Indeed, change of one's bid and position by more than one slot can also shift somebody else's position by many slots. Depending upon bids and values, externalities could be positive or negative.

Because full characterization of Nash Equilibria (NE) is not easy in this game, in the rest of this section we confine ourselves to gain some insights on Nash Equilibria by discussing numerical examples. A main intuition that we work with initially is that some NE in a CGSP could be found starting from NE in GSP. For instance, consider a GSP bids profile such as the one in Table 4

Table 4 Nash Equilibrium in GSP

Players	Bids for single slots	Slots allocation	Price Paid	Player value	Click rate single slot
1	b(1)=18	1x	10	20	7
2	b(2)-b(1)=10	2x	6	12	4
3	b(3)-b(2)=6	3x	2	8	2
4	b(4)-b(3)=2	4x	0	4	1

It can be checked that players' bids are a NE, since conditions for no unilateral deviation from those bids are met. With the above profile of bids the auctioneer would obtain as revenue $R = 10 * 7 + 6 * 4 + 2 * 2 = 98$. At the equilibrium $E\Pi_{1GSP} = (20 - 10)7 = 70$ is player 1's profit and notice the "meaningful" difference between players 1 and 2's value, which suggests that if the auctioneer would allow to bid for an image $i = y$, player 1 would probably do it and that it may be unprofitable for his opponents to try outbidding him.

Indeed, consider Table 5 below, which refers to a CGSPs version of the above GSP, with the image taking two consecutive slots $G = 2$.

Table 5 Nash Equilibrium with both CGSPa and CGSPb

Players	Bids for single	Bids for	Slots	Price	Player	Click rate single	Click rate

	slots	image	allocation	paid	value	slot	image
1	b(1)=18		1x	10	20	7	9
2	b(2)-b(1)=10		2x	6	12	4	5
3	b(3)-b(2)=6		3x	2	8	2	3
4	b(4)-b(3)=2		4x	0	4	1	2

It is easy to verify that, for both CGSPs, the bids of Table 6 are also a NE in Table 7, when players can choose between bidding for a text only or an image. The intuition is that although click rates increase with an image, they do not grow enough to make it profitable for any of the players to deviate and bid for a combination of slots. As a result the NE of the GSP is also a NE in CGSPa and CGSPb.

Suppose instead that now click rates for the image change as in Table 6, which coincides with Table 5 except for the click rate of the image starting from the first slot which is now 18, much larger than 9 as in Table 5

Table 6 Nash Equilibrium with CGSPa only

Players	Bids for single slots	Bids for image	Slots allocation	Price paid	Player value	Click rate single slot	Click rate image
1		B(1)=18	1y	16	20	7	18
2	b(1)=10		1y			4	5
3	b(2)-b(1)=6		2x	6	12	2	3
4	b(3)-b(2)=2		3x	2	8	1	2
			4x	0	4	1	2

The bids profile in Table 6 is a NE in CGSPa only since, consistently with Proposition 1, it is not a NE in CGSPb. Indeed, in CGSPb player 2 would have an incentive to raise his bid in such a way that $b(2) > B(1) = 18$ for him to obtain the first slot, pay a price of 6 and enjoy a profit equal to $(12 - 6)7 = 42 > (12 - 6)2 = 12$.

Notice that players bid the same price as in Table 5, but not for the same object. Now player 1 submits a price offer for the first two slots, because his profit would become equal to $E\Pi_{1CGSPa} = (20 - 16)18 = 72 > 70 = (20 - 10)7 = E\Pi_{1GSP}$, since a lower profit per single click is more than compensated by a higher number of clicks. Moreover, it is easy to verify that submitting the highest bid is player 1's best reply against the others' bids. For example, bids such that $10 > B(1) > 6$ would provide player 1 with a profit of $(20 - 8)5 = 60 < 72$. Likewise, if $6 > B(1) > 2$ then player 1's profit becomes $(20 - 2)3 = 54 < 72$ while if $B(1) < 2$ his profit would still be lower, and equal to 40. Similar considerations hold when checking that his opponents' bids are also best replies.

It is also interesting to point out that, as well as for GSP (Edelman et al 2007), truthful bidding may not be a NE in CGSPa. More specifically, if a bids profile is a NE in CGSPa it does not follow that a profile, where those bids are replaced with players' values, is still a NE. To see why consider Table 7, which includes the same type of bids as in Table 6, except that now submitted prices coincide with values.

Table 7 A truthful bidding profile which is not a Nash Equilibrium in CGSPa

Players	Bids for single slots	Bids for image	Slots allocation	Price paid	Player value	Click rate single slot	Click rate image
1		B(1)=20	1y	20	20	7	18
2	b(1)=12		1y			4	5
3	b(2)-b(1)=8		2x	8	12	2	3
4	b(3)-b(2)=4		3x	4	8	1	2
			4x	0	4	1	2

From the above Table it immediately follows that player 1 would obtain as expected profit $E\,\Pi_{1CGSPa} = (20 - 20)18 = 0$ and therefore he would be better off, for example, by still bidding his value however for a single slot rather than for an image.

In Table 8 we illustrate a NE according to CGSPb only. Still consistently with Proposition 1 single bids occupy the first two slots and the image the last two. To obtain an equilibrium player 2 bids above player 1 value while players 3 and 4 just about player 4's value. In the table the quantity $\varepsilon > 0$ is small enough.

Table 8 Nash Equilibrium with CGSPb only

Players	Bids for single slots	Bids for image	Slots allocation	Price paid	Player value	Click rate single slot	Click rate image
1		B(1)=18	2x	4+ε	12	7	18
2	b(1)=21		3x	4+ε	8	4	5
3	b(2)-b(1)=4+ε		1y	4	20	2	3
4	b(3)-b(2)=4		1y			1	2
			4x	0	4	1	2

The bids profile in Table 8 is not a NE in CGSPa, since player 2 would get the first slot and pay a price of $\frac{2(18)}{3} = 12$, obtaining zero profit, while bidding 12 he would be assigned the third slot, pay a price of 4+ε and make positive profits. Together, Tables 6 and 8 imply the following result

Proposition 2 *The sets of NE with CGSPa and CGSPb do not coincide.*

Finally, the above numerical examples may appear to suggest that, in terms of search engine revenue, for the auctioneer CGSPa is more rewarding than CGSPb. The following Table 9 shows that this is not the case.

Table 9			Table 9a CGSPa			Table 9b CGSPb		
Players	Bids for single slots	Bids for the image	Slots	Player	Price	Slots	Player	Price
1		B(1)=9	1x	2	10	1x	2	10
2	b(1)=12		2x	3	6	2x	3	5
3	b(2)-b(1)=10		3y	1	7	3x	4	5
4	b(3)-b(2)=5		4y	1		4y	1	7
5	b(4)-b(3)=2		5x	4	2	5y	1	
6	b(5)-b(4)=1		6x	5	1	6x	5	1
			7x	6	0	7x	6	0

We summarize the above considerations as follows

Proposition 3 *It is neither the case that for all profiles of bids the auctioneer's revenue under CGSPa is larger than under CGSPb, nor that under CGSPb is larger than under CGSPa.*

6 Conclusions

In the paper we introduced a class of advertising auctions extending GSP, adopted by Google, to allow sponsors to bid for images and texts only. As well as in GSP we considered players who could only submit a single bid, either for a text only or an image but not for both. Images occupy a set of consecutive slots and, unlike existing work, the allocation of slots in the template and its composition will be determined endogenously, as a Nash Equilibrium of the game, rather than being predefined exogenously by the search engine. Due to images occupying more than one consecutive slot, the auction is combinatorial, although constrained by the fact that package bidding can only be for consecutive slots.

When players can bid for packages there could be alternative ways to generalize the GSP, to allocate template slots based on the bids. In the paper we investigate two of them, providing an initial exploration of their features based on few general results and several numerical examples to gain some early insights. The main message is that if revenue maximization is the main goal of the search engine, then alternative extensions of GSP to when players bid for images can produce, possibly, large differences in the auctioneer's revenue. Therefore, the criterion used to extend GSP should be carefully chosen by the search engine.

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Combinatorial Advertising Internet Auctions

Highlights

- Two combinatorial extensions of the Generalised Second Price Auction are considered
- The extensions concern the possibility of bidding for consecutive advertising slots
- Combination of slots could host images and videos besides texts
- The auctioneer’s revenue is sensitive with respect to the extension