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The Ramsey Numbers of Paths Versus Kipases

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Abstract

For two given graphs G and H, the Ramsey number $R(G,H)$ is the smallest positive integer p such that for every graph F on p vertices the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we study the Ramsey numbers $R(P_n, \hat{K}_m)$, where P_n is a path on n vertices and $\hat{K}_m$ is the graph obtained from the join of K_1 and P_m. We determine the exact values of $R(P_n, \hat{K}_m)$ for the following values of n and m: $1 \le n \le 5$ and $m \ge 3$; $n \ge 6$ and (m is odd, $3 \le m \le 2n-1$) or (m is even, $4 \le m \le n+1$); $n = 6$ or 7 and $m = 2n-2$ or $m \ge 2n$; $n \ge 8$ and $m = 2n-2$ or $m = 2n$ or $(q \cdot n - 2q + 1 \le m \le q \cdot n - q + 2$ with $3 \le q \le n-5)$ or $m \ge (n-3)^2$; odd $n \ge 9$ and $(q \cdot n - 3q + 1 \le m \le q \cdot n - 2q$ with $3 \le q \le (n-3)/2)$ or $(q \cdot n - q - n + 4 \le m \le q \cdot n - 2q$ with $(n-1)/2 \le q \le n-4)$.

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1 Introduction

Throughout this paper, all graphs are finite and simple. Let G be such a graph. The graph $\overline{G}$ is the *complement* of G, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. A *kipas* $\hat{K}_m$ is the graph on $m+1$ vertices obtained from the join of K_1 and P_m. The vertex corresponding to K_1 is called the *hub* of the kipas. Given two graphs G and H, the *Ramsey number* $R(G,H)$ is defined as the smallest positive integer

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p such that every graph F on p vertices satisfies the following condition: F contains G as a subgraph or $\overline{F}$ contains H as a subgraph.

In 1967 Geréncser and Gyárfás [3] determined all Ramsey numbers for paths versus paths. After that, Ramsey numbers $R(P_n, H)$ for paths versus other graphs H have been investigated in several papers, for example in [5], [1], [6], [4], [2], [7] and [8]. We study Ramsey numbers for paths versus kipases.

2 Main results

We determine the Ramsey numbers $R(P_n, \hat{K}_m)$ for the following values of n and m: $1 \le n \le 5$ and $m \ge 3$; $n \ge 6$ and (m is odd, $3 \le m \le 2n-1$) or (m is even, $4 \le m \le n+1$); $n = 6$ or 7 and $m = 2n-2$ or $m \ge 2n$; $n \ge 8$ and $m = 2n-2$ or $m = 2n$ or $(q \cdot n - 2q + 1 \le m \le q \cdot n - q + 2$ with $3 \le q \le n-5)$ or $m \ge (n-3)^2$; odd $n \ge 9$ and $(q \cdot n - 3q + 1 \le m \le q \cdot n - 2q$ with $3 \le q \le (n-3)/2)$ or $(q \cdot n - q - n + 4 \le m \le q \cdot n - 2q$ with $(n-1)/2 \le q \le n-4)$.

Theorem 2.1

$$R(P_n, \hat{K}_m) = \begin{cases} 1 & \textit{for } n = 1 \textit{ and } m \ge 3 \\ m+1 & \textit{for either } (n = 2 \textit{ and } m \ge 3) \\ & \textit{or } (n = 3 \textit{ and even } m \ge 4) \\ m+2 & \textit{for } (n = 3 \textit{ and odd } m \ge 5) \\ 3n-2 & \textit{for either } (n = 3 \textit{ and } m = 3) \\ & \textit{or } (n \ge 4 \textit{ and } m \textit{ is odd}, 3 \le m \le 2n-1) \\ 2n-1 & \textit{for } n \ge 4 \textit{ and } m \textit{ is even}, 4 \le m \le n+1. \end{cases}$$

Theorem 2.1 can be obtained by indicating suitable graphs for providing sharp lower bounds, and using some result in [8] for getting the best upper bounds. We omit the details.

The next lemma plays a key role in our proofs of Lemma 2.3 and Lemma 2.5. The proof of this lemma has been given in [7].

Lemma 2.2 *Let $n \ge 4$ and G be a graph on at least n vertices containing no P_n. Let the paths $P^1, P^2, \ldots, P^k$ in G be chosen in the following way: $\bigcup_{j=1}^{k} V(P^j) = V(G)$, P^1 is a longest path in G, and, if $k > 1$, P^{i+1} is a longest path in $G - \bigcup_{j=1}^{i} V(P^j)$ for $1 \le i \le k-1$. Let z be an end vertex of P^k. Then:*

(i) $|V(P^1)| \ge |V(P^2)| \ge \ldots \ge |V(P^k)|$;

(ii) *If* $|V(P^k)| \geq \lfloor n/2 \rfloor$, *then* $|N(z)| \leq |V(P^k)| - 1$*;*

(iii) *If* $|V(P^k)| < \lfloor n/2 \rfloor$, *then* $|N(z)| \leq \lfloor n/2 \rfloor - 1$.

Lemma 2.3 *If* $n \geq 4$ *and* $m = 2n - 2$ *or* $m \geq 2n$, *then*

$$R(P_n, \hat{K}_m) \leq \begin{cases} m + n - 1 \text{ \textit{for} } m = 1 \textit{ mod}(n-1) \\ m + n - 2 \text{ \textit{for other values of} } m. \end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)| = \begin{cases} m + n - 1 \text{ for } m = 1 \textit{ mod}(n-1) \\ m + n - 2 \text{ for other values of } m. \end{cases} \tag{1}$$

Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 2.2. Because of (1), not all P^i can have $n-1$ vertices, so $|V(P^k)| \leq n-2$. By Lemma 2.2, $|N(z)| \leq n-3$. We will use the following result that has been proved in [1]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geq \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1 $|N(z)| \leq \lfloor n/2 \rfloor - 2$ or n is odd and $|N(z)| = \lfloor n/2 \rfloor - 1$. Since $|V(G) \setminus N[z]| \geq m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub.

Case 2 n is even and $|N(z)| = n/2 - 1$. Since $|V(G) \setminus N[z]| \geq (m+n-2) - n/2 = m + n/2 - 2$, we find that $\overline{G - N[z]}$ contains a C_{m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2 - 1$ vertices in $U = V(G) \setminus (V(C_{m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{n/2-1}$. If some vertex v_i $(i = 1, \ldots, m-1)$ is no neighbor of some vertex u_j $(j = 1, \ldots, n/2-1)$, w.l.o.g. assume $v_{m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains a $\hat{K}_m$ with hub z and its other vertices $v_1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, u_1$. Now let us assume each of the v_i is adjacent to all u_j in G. For every choice of a subset of $n/2$ vertices from $V(C_{m-1})$, there is a path on $n-1$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Since G contains no P_n, there are no edges $v_iv_j \in E(G)$ $(i, j \in \{1, \ldots, m-1\})$. This implies that $V(C_{m-1}) \cup \{z\}$ induces a K_m in $\overline{G}$. Since G contains no P_n, no v_i is adjacent to a vertex of $N(z)$. This implies that $\overline{G}$ contains a $K_{m+1} - e$ for some edge zw with $w \in N(z)$, and hence $\overline{G}$ contains a $\hat{K}_m$ with one of the v_i as a hub.

Case 3 Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geq \lfloor n/2 \rfloor$ for any end vertex w of a path on $|V(P^k)|$ vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $|V(P^k)| \geq \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k

we have that $|N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq |V(P^k)|/2$. By standard arguments in hamiltonian graph theory we obtain a cycle on $|V(P^k)|$ vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree in $\overline{G}$ at least

$$\begin{cases} m+1 & \text{if } |V(G)| = m+n-1 \\ m & \text{if } |V(G)| = m+n-2. \end{cases}$$

We now turn to P^{k-1} and consider one of its end vertices w. Since $|V(P^{k-1})| \geq |V(P^k)| \geq \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 2.2 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get a $\hat{K}_m$ in $\overline{G}$ as in Case 1 and 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geq \lfloor n/2 \rfloor \geq |V(P^{k-1})|/2$ for both end vertices w_1 and w_2 of P^{k-1}. By standard arguments in hamiltonian graph theory we obtain a cycle on $|V(P^{k-1})|$ vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree in $\overline{G}$ at least

$$\begin{cases} m & \text{if } |V(G)| = m+n-1 \\ m-1 & \text{if } |V(G)| = m+n-2. \end{cases} \tag{2}$$

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree in $\overline{G}$ at least as (2).

Now let $|V(P^k)| = \ell$ and $H = \overline{G} - V(P^k)$. If $V(G) = m+n-1$, then in the graph H all vertices have degree at least $m - \ell \geq m/2 + (n-1) - \ell \geq \frac{1}{2}(m+2n-2-\ell-(n-2)) = \frac{1}{2}(m+n-\ell) = \frac{1}{2}(|V(H)|+1)$. If $V(G) = m+n-2$, then in the graph H all vertices have degree at least $m-1-\ell \geq m/2 + (n-1) - 1 - \ell \geq \frac{1}{2}(m+2n-4-\ell-(n-2)) = \frac{1}{2}(m+n-2-\ell) = \frac{1}{2}|V(H)|$. Hence, there exists a Hamilton cycle in H. Since $|V(H)| \geq m$ and z is a neighbor of all vertices in H, it is clear that $\overline{G}$ contains a $\hat{K}_m$ with z as a hub. □

Corollary 2.4 *If* $(4 \leq n \leq 6$ *and* $m = 2n-2$ *or* $m \geq 2n)$ *or* $(n \geq 7$ *and* $m = 2n-2$ *or* $m = 2n$ *or* $m \geq (n-3)^2)$ *or* $(n \geq 8$ *and* $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ *for* $3 \leq q \leq n-5)$, *then*

$$R(P_n, \hat{K}_m) = \begin{cases} m+n-1 & \textit{for } m = 1 \textit{ mod}(n-1) \\ m+n-2 & \textit{for other values of } m. \end{cases}$$

Corollary 2.4 can be obtained by indicating suitable graphs for providing sharp lower bounds, and combining them with the upper bounds from Lemma

2.3. We omit the details.

Lemma 2.5 *If odd* $n \geq 7$ *and* $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ *with* $2 \leq q \leq n - 5$*, then* $R(P_n, \hat{K}_m) \leq m + n - 3$.

The proof of Lemma 2.5 is modeled along the lines of the proof of Lemma 2.3. We omit the details.

Corollary 2.6 *If* $(n = 7$ *and* $m = 15)$ *or* $($*odd* $n \geq 9$ *and* $(q \cdot n - 3q + 1 \leq m \leq q \cdot n - 2q$ *with* $3 \leq q \leq (n-3)/2)$ *or* $(q \cdot n - q - n + 4 \leq m \leq q \cdot n - 2q$ *with* $(n-1)/2 \leq q \leq n-4))$*, then* $R(P_n, \hat{K}_m) = m + n - 3$.

Proof. For $n = 7$ and $m = 15$, the graph $3K_6$ and for odd $n \geq 9$ and $m = q \cdot n - 2q - j$ with either $(3 \leq q \leq (n-3)/2$ and $0 \leq j \leq q-1)$ or $((n-1)/2 \leq q \leq n-5$ and $0 \leq j \leq n-q-4)$, the graph $(q-j-1)K_{n-2} \cup (j+2)K_{n-3}$ shows that $R(P_n, \hat{K}_m) > m + n - 4$. Using Lemma 2.5, we obtain that $R(P_n, \hat{K}_m) = m + n - 3$. □

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