On the structure of self-complementary graphs

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Abstract: A self-complementary graph is a graph isomorphic to its complement. An isomorphism between G and its complement, viewed as a permutation of V(G), is then called an antimorphism. A skew partition of G is a partition of V(G) into 4 sets A, B, C, D such that there is no edge between A, B and every possible edge between C, D. A symmetric partition of G is a partition of V(G) into 4 sets A, B, C, D such that there is no edge between A, D, no edge between B, C, every possible edge between A, B and every possible edge between C, D.

We give a new proof of a theorem of Gibbs saying that every selfcomplementary graph on 4k vertices has k disjoint paths on 4 vertices as induced subgraph. This new proof gives more structural information than the original one. We conjecture that every self-complementary graph on 4k vertices either has an induced cycle on 5 vertices, or a skew partition, or a symmetric partition. The new proof of Gibb's theorem yields a proof of the conjecture for the self-complementary graphs that have an antimorphism that is the product of a two circular permutations, one of them of length 4.

1 Introduction

In this paper graphs are simple, non-oriented, with no loop and finite. Several definitions that can be found in most handbooks (for instance [10] for graphs and [14] for algorithms) will not be given. We also refer the reader to a very complete survey on self-complementary graphs due to Farrugia [12].

If G is a graph, we denote the complement of G by \overline{G} . A graph is said to be *self-complementary* if G is isomorphic to its complement \overline{G} . We will often write "sc-graph" for "self-complementary graph". It is very easy to construct a lot of examples of sc-graphs: take any graph G, and consider 2 copies of G say G_1, G_2 , and 2 copies of \overline{G} say G_3, G_4 . Then join every vertex of G_1 to

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every vertex of G_3 , every vertex of G_3 to every vertex of G_4 and every vertex of G_4 to every vertex of G_2 . The graph obtained is self-complementary, and we call it the graph obtained from G by the P_4 -construction.



Figure 1: The P_4 -construction applied to G

An important problem in algorithmic graph theory is the *isomorphism* problem, known to be difficult and unsettled: is there a polynomial time algorithm that decides whether two graphs are isomorphic? If there is one, then we can easily decide in polynomial time if a graph is self-complementary, by just running the algorithm on G, G. Colbourn et al. [6] studied the converse and proved that the recognition of sc-graphs is *isomorphism-complete*. That is: if there exists a polynomial time algorithm that decides if a graph is self-complementary, then there exists a polynomial time algorithm for the isomorphism problem. The result of Colbourn et al. is not so surprising because of the P_4 -construction described above. Consider two graphs G, H. Consider G_1, G_2 , two copies of G and H_1, H_2 two copies of \overline{H} . Construct (like in the P_4 -construction) a new graph J: join every vertex of G_1 to every vertex of H_1 , every vertex of H_1 to every vertex of H_2 , and every vertex of H_2 to every vertex of G_2 . To decide if J is self-complementary, the obvious way is to decide if G, H are isomorphic. The result of Colbourn et al. says that there is no better way in general. So, it is to be feared that despite (or because of) formal equivalence, a study of the properties of sc-graphs will not help in solving the isomorphism problem.

However, the structure of sc-graphs is worth investigating for its own interest and because particular sc-graphs have sometime interesting properties, as being smallest counter-examples to several conjectures¹. It could also help for a general construction for every sc-graph, or at least for some substantial subclasses. Note that the P_4 -construction is not a good candidate: in a graph on at least 8 vertices obtained by the P_4 -construction, every vertex has degree at least 2, and on figure 3 page 14, there is an sc-graph with a vertex of degree one. Moreover, recognition algorithms for special classes of sc-graphs can be drastically easier than the isomorphism problem for the same class. For instance, it is easy to see that the only triangle-free sc-graphs are the isolated vertex, P_4 and C_5 , because by the Ramsey's famous Theorem, every

 $^{{}^{1}}C_{5}$ is the smallest non-perfect graph see [17], $L(K_{3,3} \setminus e)$ is the smallest perfect graph with no even pair and no even pair in its complement, see [11]. There are other examples.

graph on at least 6 vertices has a triangle or the complement of a triangle. Thus, recognizing triangle-free sc-graphs is trivial in constant time while the isomorphism problem for triangle-free graphs is difficult. It might be possible to recognize special non-trivial classes of sc-graphs in polynomial time. After reading this paper, the reader will maybe want to look for a general construction for C_5 -free sc-graphs, and why not for a recognition algorithm (he or she must be warned that most of the work is still to be done...).

In this paper, we aim at structural properties of sc-graphs, saying something like: every sc-graph either contains some prescribed induced subgraph or can be partitioned into sets of vertices with some prescribed adjacencies. There are really few such results. In his master's thesis that surveys more than 400 papers on sc-graphs, Farrugia [12] mentions only one theorem due to Gibbs:

Theorem 1.1 (Gibbs, [15]) An sc-graph on 4k vertices contains k disjoint induced P_4 's.

As pointed out by Farrugia, the theorem above has two major defaults in view of algorithmic applications. First, the problem of deciding whether the vertices of a graph can be partitioned into sets of 4 vertices, each of them inducing a P_4 , is NP-complete (proved by Kirkpatrick and Hell, [16]). Secondly, even if the partition into P_4 's of an sc-graph is obtained by any unexcepected mean, it will be of no use for recursion, since removing blindly one or some of the P_4 's may yield a graph that is no more self-complementary and that will have in general no forseeable properties.

We will investigate structural properties of sc-graphs that fix the first default: the structures that we will find (or conjecture) in sc-graphs will be detectable in polynomial time. Unfortunately, our results (and conjectures) will still have the second default: we will be able to break several sc-graphs into pieces with special adjacency properties, but without garanteeing any hereditary properties on these pieces.

We will first give a new proof of the theorem of Gibbs, that yields a slightly different result and gives more structural information (Section 2). This will allow us to prove a special case of a conjecture: every sc-graph on 4k vertices either contains a C_5 as an induced subgraph or can be broken in 4 pieces with special adjacencies properties (Section 3). Page 14, we show a picture of all the sc-graphs on 8 vertices.



Figure 2: The 3 sc-graphs on 4 or 5 vertices

2 A new proof of Gibb's theorem

If G is a graph, we denote by V(G) the vertex set of G, by E(G) the edge set of G. If $A \subset V(G)$, we denote by G[A] the subgraph of G induced by A. If v is a vertex of G, we denote by N(v) the set of the neighbours of v. We denote by $\overline{N}(v)$ the set of the non-neighbours of v. Note that $v \in \overline{N}(v)$. If $uv \in E(G)$, we say that u sees v, and if $uv \notin E(G)$, we say that u misses v.

By the definition, a graph G is self-complementary if and only if there exists a bijection τ from V(G) to V(G) such that for every pair $\{a, b\}$ of distinct vertices we have: $\{a, b\} \in E(G) \Leftrightarrow \{\tau(a), \tau(b)\} \notin E(G)$. Such a function τ is called an *antimorphism* of G.

Sachs [19] and Ringel [18] proved that any antimorphism is a product of circular permutations whose lengths are all multiples of 4, except possibly for one of length 1. Note that this implies a well known fact: the number of vertices of an sc-graph is equal to 0 or 1 modulo 4. Gibbs [15] also proved the following:

Theorem 2.1 (Gibbs [15]) If G is an sc-graph, then there exists an antimorphism τ of G such that every circular permutation of τ has length a power of 2.

It is convenient to denote by $(a_1a_2...a_k)$ the circular permutation of $\{a_1a_2...a_k\}$ that maps a_i to a_{i+1} , where the addition of the subscripts is taken modulo k. When a circular permutation has length 4k, we often denote it by $(a_1b_1c_1d_1a_2b_2c_2d_2...a_kb_kc_kd_k)$. Implicitly, the subscripts are then taken modulo k (for instance $a_{k+3} = a_3, d_0 = d_k, ...$).

We recall here a lemma used by Gibbs to prove Theorem 1.1. We give his proof with our notation.

Lemma 2.2 (Gibbs [15]) Let $k \ge 1$ be an integer and let G be an sc-graph with an antimorphism $\tau = (a_1b_1c_1d_1a_2b_2c_2d_2...a_kb_kc_kd_k)(...)\cdots(...)$. Then either:

- There exists $i \in \mathbb{N}$ such that $\{a_1, b_1, a_i, b_i\}$ induces a P_4 for which $(a_1b_1a_ib_i)$ is an antimorphism.
- There exists $i \in \mathbb{N}$ such that $\{a_1, b_1, c_i, d_i\}$ induces a P_4 for which $(a_1b_1c_id_i)$ is an antimorphism.

PROOF — Let us suppose without loss of generality that a_1 misses b_1 (if not, we may replay the same proof in \overline{G}). Applying τ^{-1} , we know that a_1 sees d_k . So there exists a smallest integer i > 1 such that: a_1 sees b_i or a_1 sees d_i .

If a_1 sees b_i then, $i \ge 2$. Applying $\tau^{4(i-1)}$ to a_1 and b_1 , we know that a_i misses b_i . By the definition of i, we know that a_1 misses d_{i-1} . Thus, applying τ , b_1 sees a_i . If a_1 sees a_i , then, applying τ , b_1 misses b_i and $\{a_1, b_1, a_i, b_i\}$ induces P_4 for which $(a_1b_1a_ib_i)$ is an antimorphism. In the same way, if a_1 misses a_i , then b_1 sees b_i and we reach the same conclusion.

If a_1 misses b_i , then by the definition of i, a_1 sees d_i . Applying $\tau^{4(i-1)+2}$ to a_1 and b_1 , we know that c_i misses d_i . Applying τ to a_1 and b_i , we know that b_1 sees c_i . If a_1 sees c_i , applying τ , b_1 misses d_i and $\{a_1, b_1, c_i, d_i\}$ induces a P_4 for which $(a_1b_1c_id_i)$ is an antimorphism. By the same way, if a_1 misses c_i then b_1 sees d_i and we reach the same conclusion.

We propose a new lemma of the same flavour that gives more structural information on sc-graphs. To state it, we need a definition. A symmetric partition in a graph G is a partition (A, B, C, D) of V(G) such that each of A, B, C, D is non-empty, there are no edges between A, D, no edges between B, C, every possible edges between A, B, and every possible edges between C, D.

Lemma 2.3 Let $k \ge 1$ be an integer and G be an sc-graph with an antimorphism $\tau = (a_1b_1c_1d_1a_2b_2c_2d_2\ldots a_kb_kc_kd_k)(\ldots)\cdots(\ldots)$. Put $A = \{a_1, \ldots, a_k\}, B = \{b_1, \ldots, b_k\}, C = \{c_1, \ldots, c_k\}, D = \{d_1, \ldots, d_k\}$. Then either:

- There exists $i, j \in \mathbb{N}$ such that $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 for which $(a_1b_ia_{1+j}b_{i+j})$ is an antimorphism.
- (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$.
- (B, C, D, A) is a symmetric partition of $G[A \cup B \cup C \cup D]$.

PROOF — If every vertex in A sees every vertex in B, then applying τ three times, we see that (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. Similarly, if every vertex in A misses every vertex in B then (B, C, D, A) is a symmetric partition of $G[A \cup B \cup C \cup D]$. Thus, we may assume that some vertex a_h in A has neighbours and non-neighbours in B, and applying $\tau^{4(h-1)}$, we see that a_1 has neighbours and non-neighbours in B.

Suppose first that a_1 has at least as many neighbours than non-neiboughs in B, more precisely: $|N(a_1) \cap B| \ge |\overline{N}(a_1) \cap B|$. Let i be such that $a_1b_i \notin E(G)$. There exists $j \not\equiv 0 \pmod{k}$ such that $a_1b_{i-j} \in E$ and $a_1b_{i+j} \in E$, for otherwise $|\overline{N}(a_1) \cap B| > k/2 \ge |N(a_1) \cap B|$, a contradiction. Note that we may have $b_{i-j} = b_{i+j}$ if $i - j \equiv i + j \pmod{k}$

We already know $a_1b_i \notin E$. Applying τ^{4j} to a_1b_i we know $a_{1+j}b_{i+j} \notin E$. We already know $a_1b_{i-j} \notin E$. Applying τ^{4j} to a_1b_{i-j} we know $a_{1+j}b_i \in E$. If a_1 sees a_{1+j} then applying $\tau^{1+4(i-1)}$, b_i misses b_{i+j} and $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 for which $(a_1b_ia_{1+j}b_{i+j})$ is an antimorphism. If a_1 misses a_{1+j} then applying $\tau^{1+4(i-1)}$, b_i sees b_{i+j} and we reach the same conclusion.

We are left with the case where $|N(a_1) \cap B| \leq |\overline{N}(a_1) \cap B|$. But then, the proof is similar up to a complementation of G.

Note that if (A, B, C, D) is a symmetric partition then for any $i, j, l \in \mathbb{N}$, the set $\{a_i, b_{i+j}, c_l, d_{l+j}\}$ induces a P_4 . Because by the definition of symmetric partitions, we have $a_i b_i \in E$, $b_i c_i \notin E$, $c_i d_i \in E$, $d_i a_i \notin E$, and applying τ , exactly one of $a_i c_i, b_i d_i$ is an edge. If (B, C, D, A) is a symmetric partition we reach the same conclusion. This remark allows us to follow the lines of Gibbs, and to prove again his theorem (Theorem 1.1) using Lemma 2.3 instead of Lemma 2.2. Let us do it for the sake of completeness.

Consider an sc-graph on 4k vertices and an antimorphism τ . By theorem 2.1 we may assume that every cycle of τ has length a power of 2. Let us consider a circular permutation $(a_1b_1c_1d_1a_2b_2c_2d_2\ldots a_kb_kc_kd_k)$ of τ . Put A = $\{a_1,\ldots a_k\}, B = \{b_1, b_2,\ldots, b_k\}, C = \{c_1, c_2,\ldots, c_k\}, D = \{d_1, d_2,\ldots, d_k\}.$ We claim that we may partition $A \cup B \cup C \cup D$ in sets of 4 vertices all of them inducing a P_4 , thus proving the theorem.

We have $\tau = (a_1b_1c_1d_1a_2b_2c_2d_2...a_kb_kc_kd_k)(...)\cdots(...)$. Apply Lemma 2.3. If one of (A, B, C, D), (B, C, D, A) is a symmetric partition, then for every $i \in \mathbb{N}$, $\{a_i, b_i, c_i, d_i\}$ induces a P_4 and we may easily partition $A \cup B \cup C \cup D$ into sets of 4 vertices all of them inducing a P_4 . So we are left with the case where there exists i such that $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 for which $(a_1b_ia_{1+j}b_{i+j})$ is an antimorphism. Let us put $k = 2^{\alpha}, \alpha \geq 1$. Note that for any l, $\{a_l, b_{l+i}, a_{l+j}, b_{l+i+j}\}$ induces a P_4 (apply $\tau^{4(l-1)}$) that we denote by P^l .

We claim that we can choose $l_1, l_2, \ldots, l_{k/2}$ such that $(P^{l_1}, P^{l_2}, \ldots, P^{l_{k/2}})$ partitions $A \cup B$. For that purpose it suffices to prove that for any integer $j \not\equiv 0 \pmod{2^{\alpha}}$, $\mathbb{Z}_{2^{\alpha}}$ can be partitioned into pairs of the form $\{l, l+j\}$. For $\alpha = 1$ this can be done, so let us prove it by induction. If j is even, then let us partition by the induction hypothesis $\mathbb{Z}_{2^{\alpha-1}}$ into pairs of the form $\{l, l+j/2\}$: $\mathbb{Z}_{2^{\alpha-1}} = \bigcup_i \{l_i, l_i+j/2\}$. Then we have $\mathbb{Z}_{2^{\alpha}} = \bigcup_i [\{2l_i, 2l_i+j\} \cup \{2l_i+1, 2l_i+1+j\}]$, so we manage to partition $\mathbb{Z}_{2^{\alpha}}$ as desired. If j is odd, then starting from 1 and adding j successively, we build a Hamiltonian cycle going through every element of its vertex set $\mathbb{Z}_{2^{\alpha}}$. Take then every second edge of this Hamiltonian cycle: here are the pairs that partition $\mathbb{Z}_{2^{\alpha}}$ as desired. So $A \cup B$ may be partitioned into P_4 's. Applying τ^2 , we see that $C \cup D$ can also be partitioned into P_4 's, hence $A \cup B \cup C \cup D$ can be partitioned into P_4 's.

3 A theorem and a conjecture

A skew partition in a graph G is a partition (A, B, C, D) of V(G) such that each of A, B, C, D is non-empty, there are no edges between A, B and every possible edges between C, D. We are now able to prove the following:

Theorem 3.1 Let G be an sc-graph with an antimorphism τ that is the product of two circular permutations, one of them of length 4. Then either:

- G contains a C_5 as an induced subgraph;
- G contains a skew partition;
- G contains a symmetric partition.

PROOF — Let us call a, b, c, d the four vertices of a cycle of τ . If the other cycle has length 1, then G is the C_5 or the bull which has a skew partition (see figure 2). So we may assume $\tau = (abcd)(a_1b_1c_1d_1a_2b_2c_2d_2\ldots a_kb_kc_kd_k)$ with $k \geq 1$. Put $A = \{a_1, \ldots, a_k\}, B = \{b_1, \ldots, b_k\}, C = \{c_1, \ldots, c_k\},\$ $D = \{d_1, \ldots, d_k\}$. Let us suppose up to a circular permutation of a, b, c, dthat the three edges of G[a, b, c, d] are ab, ac and cd. Note that if a sees a_1 , then a sees every vertex in A (apply $\tau^{4i}, i \in \mathbb{N}$). By the same way, if v is in $\{a, b, c, d\}$ and if H is one of A, B, C, D, then either v sees every vertex in H, or v misses every vertex in H. For every $v \in \{a, b, c, d\}$, put $N_v = N(v) \cap (A \cup B \cup C \cup D)$. We deal now with the $2^4 = 16$ following cases, according to N_a . Note that once N_a is known, N_b , N_c and N_d are also known, by applying τ three times. For some of the 16 cases, we apply Lemma 2.3 to the cycle $(a_1b_1c_1d_1a_2b_2c_2d_2\ldots a_kb_kc_kd_k)$ of τ . Then, up to a circular permutation of A, B, C, D, we may suppose that either (A, B, C, D)is a symmetric partition of $G[A \cup B \cup C \cup D]$, or there exist $i, j \in \mathbb{N}$ such that $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 with a_1a_{1+j} as central edge.

1. $N_a = A \cup B$.

Then $N_b = A \cup D$, $N_c = C \cup D$ and $N_d = B \cup C$. If $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 with a_1a_{1+j} as central edge, then $\{b, a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a C_5 . Else, (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup \{a\}, B \cup \{b\}, C \cup \{c\}, D \cup \{d\})$ is a symmetric partition of G.

2. $N_a = C \cup D$.

Then $N_b = B \cup C$, $N_c = A \cup B$ and $N_d = A \cup D$. If $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 with a_1a_{1+j} as central edge, then $\{b, a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a C_5 . Else, (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup \{c\}, B \cup \{d\}, C \cup \{a\}, D \cup \{b\})$ is a symmetric partition of G.

3. $N_a = A \cup C$.

Then $N_b = N_c = N_d = A \cup C$. Thus $(\{b, d\}, B \cup D, \{a, c\}, A \cup C)$ is a skew partition.

4. $N_a = B \cup D$.

Then $N_b = N_c = N_d = B \cup D$. Thus $(\{a, c\}, A \cup C, \{b, d\}, B \cup D)$ is a skew partition.

5. $N_a = A \cup D$.

Then $N_b = C \cup D$, $N_c = B \cup C$ and $N_d = A \cup B$. If $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 with a_1a_{1+j} as central edge, then $\{c, a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a C_5 . Else, (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup \{c\}, B \cup \{d\}, C \cup \{a\}, D \cup \{b\})$ is a symmetric partition of G.

6. $N_a = B \cup C$.

Then $N_b = A \cup B$, $N_c = A \cup D$ and $N_d = C \cup D$. If $\{a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a P_4 with a_1a_{1+j} as central edge, then $\{a, a_1, b_i, a_{1+j}, b_{i+j}\}$ induces a C_5 . Else, (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup \{a\}, B \cup \{b\}, C \cup \{c\}, D \cup \{d\})$ is a symmetric partition of G.

7. $N_a = \emptyset$.

Then a_1 sees b, misses c and sees d. Thus, $\{a_1, a, b, c, d\}$ induces C_5 .

8. $N_a = A \cup B \cup C \cup D$. Then $N_c = N_a = A \cup B \cup C \cup D$. Thus $(\{b\}, \{d\}, \{a, c\}, A \cup B \cup C \cup D)$ is a skew partition.

9. $N_a = A$.

Then $N_b = A \cup C \cup D$, $N_c = C$ and $N_d = A \cup B \cup C$. Thus $(\{a, c\}, B \cup D, \{b, d\}, A \cup C)$ is a skew partition.

10. $N_a = B \cup C \cup D$.

Then $N_b = B$, $N_c = A \cup B \cup D$ and $N_d = D$. Thus $(\{b, d\}, A \cup C, \{a, c\}, B \cup D)$ is a skew partition.

11. $N_a = B$.

Then $N_b = A \cup B \cup D$, $N_c = D$ and $N_d = B \cup C \cup D$. Thus $(\{a, c\}, A \cup C, \{b, d\}, B \cup D)$ is a skew partition.

12. $N_a = A \cup C \cup D$.

Then $N_b = C$, $N_c = A \cup B \cup C$ and $N_d = A$. Thus $(\{b, d\}, B \cup D, \{a, c\}, A \cup C)$ is a skew partition.

13. $N_a = C$.

Then $N_b = A \cup B \cup C$, $N_c = A$ and $N_d = A \cup C \cup D$. Thus $(\{a, c\}, B \cup D, \{b, d\}, A \cup C)$ is a skew partition.

14. $N_a = A \cup B \cup D$.

Then $N_b = D$, $N_c = B \cup C \cup D$ and $N_d = B$. Thus $(\{b, d\}, A \cup C, \{a, c\}, B \cup D)$ is a skew partition.

15. $N_a = D$.

Then $N_b = B \cup C \cup D$, $N_c = B$ and $N_d = A \cup B \cup D$. Thus $(\{a, c\}, A \cup C, \{b, d\}, B \cup D)$ is a skew partition.

16. $N_a = A \cup B \cup C$.

Then $N_b = A$, $N_c = A \cup C \cup D$ and $N_d = C$. Thus $(\{b, d\}, B \cup D, \{a, c\}, A \cup C)$ is a skew partition.

Note that as pointed out by Farrugia, a generalisation of Case 8 of the proof was implicitly known by Akiyama and Harary [1]. They proved that if an sc-graph G has at least an end-vertex, then G has exactly two end-vertices b, d and exactly two cut vertices a, c. They proved that $(\{b\}, \{d\}, \{a, c\}, V(G) \setminus \{a, b, c, d\})$ is then a skew partition of G.

Let us now discuss the motivivation and possible extensions of Theorem 3.1. Skew partitions were introduced by Chvátal for the study of perfect graphs [5], and play an important role in the proof of Strong Perfect Graph Conjecture by Chudnovsky, Robertson Seymour and Thomas [4]. Symmetric partitions may be seen as a very particular case of the 2-join defined by Cunningham and Cornuéjols, once again for the study of perfect graphs [7]. A 2-join in G is a partition (X_1, X_2) of V(G) so that there exist disjoint non-empty $A_i, B_i \subset X_i$, (i = 1, 2) satisfaying:

- 1. every vertex of A_1 is adjacent to every vertex of A_2 and every vertex of B_1 is adjacent to every vertex of B_2 ;
- 2. there are no other edges between X_1 and X_2 ;
- 3. for i = 1, 2, every component of $G[X_i]$ meets both A_i and B_i ;
- 4. for i = 1, 2, if $|A_i| = |B_i| = 1$, and if X_i induces a path of G joining the vertex of A_i and the vertex of B_i , then it has length at least 3.

The conditions 3, 4 are called the *technical requirements*. They are important for algorithms, and for applications to perfect graphs. If a graph G has a 2-join with the above notation, then (A_1, A_2, B_1, B_2) is a symmetric partition of $G[A_1 \cup A_2 \cup B_1 \cup B_2]$. In other words, if we forget the technical requirements, symmetric partitions may be seen as 2-joins such that $X_i \setminus (A_i \cup B_i) = \emptyset$, i = 1, 2.

A lot of work has been done recently on finding algorithms that decide if the vertices of a graph can be partitioned into several subsets with various restrictions on the adjacencies [2, 8, 13]). Symmetric partitions are detectable in linear time (see problem (31) in [8]). Skew partitions seem more difficult, but Figueiredo, Klein, Kohayakawa and Reed gave a polynomial time algorithm that decides whether a graph has or not a skew partition [9]. Note that detecting a C_5 in a graph can be done easily in $O(n^5)$. Thus, each of the outcome of Theorem 3.1 are testable in polynomial time. We conjecture that Theorem 3.1 holds for every sc-graph on 4k vertices:

Conjecture 3.2 Let G be an sc-graph on 4k vertices. Then either:

- G contains a C_5 as an induced subgraph;
- G contains a skew partition;
- G contains a symmetric partition.

This conjecture is motivated by several considerations. First, we are able to prove it in a quite general special case: Theorem 3.1. The proof shows how forbiding C_5 's can help for finding symmetric or skew partitions. Also, skew partitions and symmetric partitions arise naturally in P_4 -constructions of sc-graphs and in circular permutations of an antimorphism. Suppose $(a_1b_1c_1d_1 \dots a_kb_kc_kd_k)$ is such a permutation with our usual notation. If every vertex in A sees every vertex in B, then (A, B, C, D) is a symmetric partition of $G[A \cup B \cup C \cup D]$. If every vertex in A sees every vertex in C, then (B, D, C, A) is a skew partition of $G[A \cup B \cup C \cup D]$.

Secondly, the conjecture has an analogy with the theorem of Chudnovsky, Robertson, Seymour and Thomas for decomposing Berge Graphs. A hole in a graph is an induced cycle of length at least 4. A graph is *Berge* if in both G, \overline{G} , there is no hole of odd length. The *decomposition theorem* for Berge graphs is the following. Note that this theorem has been proved in two steps: first Chudnovsky, Robertson, Seymour and Thomas [4] proved a slightly weaker result, and then Chudnovsky [3] alone proved the form that we give:

Theorem 3.3 (Chudnovsky et al.[3, 4]) Let G be a Berge graph. Then either :

- One of G, \overline{G} is bipartite.
- One of G, \overline{G} is the line-graph of a bipartite graph.
- One of G, \overline{G} has a 2-join.
- G has a skew partition.

It would be nice to have a stronger theorem in the particular case of Berge sc-graphs. Conjecture 3.2 could be a candidate.

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