# On the structure of self-complementary graphs 

Nicolas Trotignon*

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#### Abstract

A self-complementary graph is a graph isomorphic to its complement. An isomorphism between $G$ and its complement, viewed as a permutation of $V(G)$, is then called an antimorphism. A skew partition of $G$ is a partition of $V(G)$ into 4 sets $A, B, C, D$ such that there is no edge between $A, B$ and every possible edge between $C, D$. A symmetric partition of $G$ is a partition of $V(G)$ into 4 sets $A, B, C, D$ such that there is no edge between $A, D$, no edge between $B, C$, every possible edge between $A, B$ and every possible edge between $C, D$.

We give a new proof of a theorem of Gibbs saying that every selfcomplementary graph on $4 k$ vertices has $k$ disjoint paths on 4 vertices as induced subgraph. This new proof gives more structural information than the original one. We conjecture that every self-complementary graph on $4 k$ vertices either has an induced cycle on 5 vertices, or a skew partition, or a symmetric partition. The new proof of Gibb's theorem yields a proof of the conjecture for the self-complementary graphs that have an antimorphism that is the product of a two circular permutations, one of them of length 4.


## 1 Introduction

In this paper graphs are simple, non-oriented, with no loop and finite. Several definitions that can be found in most handbooks (for instance 10 for graphs and [14] for algorithms) will not be given. We also refer the reader to a very complete survey on self-complementary graphs due to Farrugia 12.

If $G$ is a graph, we denote the complement of $G$ by $\bar{G}$. A graph is said to be self-complementary if $G$ is isomorphic to its complement $\bar{G}$. We will often write "sc-graph" for "self-complementary graph". It is very easy to construct a lot of examples of sc-graphs: take any graph $G$, and consider 2 copies of $G$ say $G_{1}, G_{2}$, and 2 copies of $\bar{G}$ say $G_{3}, G_{4}$. Then join every vertex of $G_{1}$ to

[^0]every vertex of $G_{3}$, every vertex of $G_{3}$ to every vertex of $G_{4}$ and every vertex of $G_{4}$ to every vertex of $G_{2}$. The graph obtained is self-complementary, and we call it the graph obtained from $G$ by the $P_{4}$-construction.


Figure 1: The $P_{4}$-construction applied to $G$
An important problem in algorithmic graph theory is the isomorphism problem, known to be difficult and unsettled: is there a polynomial time algorithm that decides whether two graphs are isomorphic ? If there is one, then we can easily decide in polynomial time if a graph is self-complementary, by just running the algorithm on $G, \bar{G}$. Colbourn et al. [6] studied the converse and proved that the recognition of sc-graphs is isomorphism-complete. That is: if there exists a polynomial time algorithm that decides if a graph is self-complementary, then there exists a polynomial time algorithm for the isomorphism problem. The result of Colbourn et al. is not so surprising because of the $P_{4}$-construction described above. Consider two graphs $G, H$. Consider $G_{1}, G_{2}$, two copies of $G$ and $H_{1}, H_{2}$ two copies of $\bar{H}$. Construct (like in the $P_{4}$-construction) a new graph $J$ : join every vertex of $G_{1}$ to every vertex of $H_{1}$, every vertex of $H_{1}$ to every vertex of $H_{2}$, and every vertex of $H_{2}$ to every vertex of $G_{2}$. To decide if $J$ is self-complementary, the obvious way is to decide if $G, H$ are isomorphic. The result of Colbourn et al. says that there is no better way in general. So, it is to be feared that despite (or because of) formal equivalence, a study of the properties of sc-graphs will not help in solving the isomorphism problem.

However, the structure of sc-graphs is worth investigating for its own interest and because particular sc-graphs have sometime interesting properties, as being smallest counter-examples to several conjectures ${ }^{1}$. It could also help for a general construction for every sc-graph, or at least for some substantial subclasses. Note that the $P_{4}$-construction is not a good candidate: in a graph on at least 8 vertices obtained by the $P_{4}$-construction, every vertex has degree at least 2 , and on figure 3 page 14, there is an sc-graph with a vertex of degree one. Moreover, recognition algorithms for special classes of sc-graphs can be drasticaly easier than the isomorphism problem for the same class. For instance, it is easy to see that the only triangle-free sc-graphs are the isolated vertex, $P_{4}$ and $C_{5}$, because by the Ramsey's famous Theorem, every

[^1]graph on at least 6 vertices has a triangle or the complement of a triangle. Thus, recognizing triangle-free sc-graphs is trivial in constant time while the isomorphism problem for triangle-free graphs is difficult. It might be possible to recognize special non-trivial classes of sc-graphs in polynomial time. After reading this paper, the reader will maybe want to look for a general construction for $C_{5}$-free sc-graphs, and why not for a recognition algorithm (he or she must be warned that most of the work is still to be done...).

In this paper, we aim at structural properties of sc-graphs, saying something like: every sc-graph either contains some prescribed induced subgraph or can be partitioned into sets of vertices with some prescribed adjacencies. There are really few such results. In his master's thesis that surveys more than 400 papers on sc-graphs, Farrugia [12] mentions only one theorem due to Gibbs:

Theorem 1.1 (Gibbs, [15]) An sc-graph on $4 k$ vertices contains $k$ disjoint induced $P_{4}$ 's.

As pointed out by Farrugia, the theorem above has two major defaults in view of algorithmic applications. First, the problem of deciding whether the vertices of a graph can be partitioned into sets of 4 vertices, each of them inducing a $P_{4}$, is NP-complete (proved by Kirkpatrick and Hell, [16]). Secondly, even if the partition into $P_{4}$ 's of an sc-graph is obtained by any unexcepected mean, it will be of no use for recursion, since removing blindly one or some of the $P_{4}$ 's may yield a graph that is no more self-complementary and that will have in general no forseeable properties.

We will investigate structural properties of sc-graphs that fix the first default: the structures that we will find (or conjecture) in sc-graphs will be detectable in polynomial time. Unfortunately, our results (and conjectures) will still have the second default: we will be able to break several sc-graphs into pieces with special adjacency properties, but without garanteeing any hereditary properties on these pieces.

We will first give a new proof of the theorem of Gibbs, that yields a slightly different result and gives more structural information (Section (2). This will allow us to prove a special case of a conjecture: every sc-graph on $4 k$ vertices either contains a $C_{5}$ as an induced subgraph or can be broken in 4 pieces with special adjacencies properties (Section (3). Page 14, we show a picture of all the sc-graphs on 8 vertices.

$P_{4}$

$C_{5}$


The bull

Figure 2: The 3 sc-graphs on 4 or 5 vertices

## 2 A new proof of Gibb's theorem

If $G$ is a graph, we denote by $V(G)$ the vertex set of $G$, by $E(G)$ the edge set of $G$. If $A \subset V(G)$, we denote by $G[A]$ the subgraph of $G$ induced by $A$. If $v$ is a vertex of $G$, we denote by $N(v)$ the set of the neighbours of $v$. We denote by $\bar{N}(v)$ the set of the non-neighbours of $v$. Note that $v \in \bar{N}(v)$. If $u v \in E(G)$, we say that $u$ sees $v$, and if $u v \notin E(G)$, we say that $u$ misses $v$.

By the definition, a graph $G$ is self-complementary if and only if there exists a bijection $\tau$ from $V(G)$ to $V(G)$ such that for every pair $\{a, b\}$ of distinct vertices we have: $\{a, b\} \in E(G) \Leftrightarrow\{\tau(a), \tau(b)\} \notin E(G)$. Such a function $\tau$ is called an antimorphism of $G$.

Sachs [19] and Ringel [18] proved that any antimorphism is a product of circular permutations whose lengths are all multiples of 4, except possibly for one of length 1 . Note that this implies a well known fact: the number of vertices of an sc-graph is equal to 0 or 1 modulo 4 . Gibbs [15] also proved the following:

Theorem 2.1 (Gibbs [15]) If $G$ is an sc-graph, then there exists an antimorphism $\tau$ of $G$ such that every circular permutation of $\tau$ has length a power of 2.

It is convenient to denote by $\left(a_{1} a_{2} \ldots a_{k}\right)$ the circular permutation of $\left\{a_{1} a_{2} \ldots a_{k}\right\}$ that maps $a_{i}$ to $a_{i+1}$, where the addition of the subscripts is taken modulo $k$. When a circular permutation has length $4 k$, we often denote it by $\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)$. Implicitly, the subscripts are then taken modulo $k$ (for instance $a_{k+3}=a_{3}, d_{0}=d_{k}, \ldots$ ).

We recall here a lemma used by Gibbs to prove Theorem 1.1. We give his proof with our notation.

Lemma 2.2 (Gibbs [15]) Let $k \geq 1$ be an integer and let $G$ be an sc-graph with an antimorphism $\tau=\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)(\ldots) \cdots(\ldots)$. Then either:

- There exists $i \in \mathbb{N}$ such that $\left\{a_{1}, b_{1}, a_{i}, b_{i}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{1} a_{i} b_{i}\right)$ is an antimorphism.
- There exists $i \in \mathbb{N}$ such that $\left\{a_{1}, b_{1}, c_{i}, d_{i}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{1} c_{i} d_{i}\right)$ is an antimorphism.

PROOF - Let us suppose without loss of generality that $a_{1}$ misses $b_{1}$ (if not, we may replay the same proof in $\bar{G})$. Applying $\tau^{-1}$, we know that $a_{1}$ sees $d_{k}$. So there exists a smallest integer $i>1$ such that: $a_{1}$ sees $b_{i}$ or $a_{1}$ sees $d_{i}$.

If $a_{1}$ sees $b_{i}$ then, $i \geq 2$. Applying $\tau^{4(i-1)}$ to $a_{1}$ and $b_{1}$, we know that $a_{i}$ misses $b_{i}$. By the definition of $i$, we know that $a_{1}$ misses $d_{i-1}$. Thus, applying $\tau, b_{1}$ sees $a_{i}$. If $a_{1}$ sees $a_{i}$, then, applying $\tau, b_{1}$ misses $b_{i}$ and $\left\{a_{1}, b_{1}, a_{i}, b_{i}\right\}$ induces $P_{4}$ for which $\left(a_{1} b_{1} a_{i} b_{i}\right)$ is an antimorphism. In the same way, if $a_{1}$ misses $a_{i}$, then $b_{1}$ sees $b_{i}$ and we reach the same conclusion.

If $a_{1}$ misses $b_{i}$, then by the definition of $i, a_{1}$ sees $d_{i}$. Applying $\tau^{4(i-1)+2}$ to $a_{1}$ and $b_{1}$, we know that $c_{i}$ misses $d_{i}$. Applying $\tau$ to $a_{1}$ and $b_{i}$, we know that $b_{1}$ sees $c_{i}$. If $a_{1}$ sees $c_{i}$, applying $\tau, b_{1}$ misses $d_{i}$ and $\left\{a_{1}, b_{1}, c_{i}, d_{i}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{1} c_{i} d_{i}\right)$ is an antimorphism. By the same way, if $a_{1}$ misses $c_{i}$ then $b_{1}$ sees $d_{i}$ and we reach the same conclusion.

We propose a new lemma of the same flavour that gives more structural information on sc-graphs. To state it, we need a definition. A symmetric partition in a graph $G$ is a partition $(A, B, C, D)$ of $V(G)$ such that each of $A, B, C, D$ is non-empty, there are no edges between $A, D$, no edges between $B, C$, every possible edges between $A, B$, and every possible edges between $C, D$.

Lemma 2.3 Let $k \geq 1$ be an integer and $G$ be an sc-graph with an antimorphism $\tau=\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)(\ldots) \cdots(\ldots)$.
Put $A=\left\{a_{1}, \ldots, a_{k}\right\}, B=\left\{b_{1}, \ldots, b_{k}\right\}, C=\left\{c_{1}, \ldots, c_{k}\right\}, D=\left\{d_{1}, \ldots, d_{k}\right\}$. Then either:

- There exists $i, j \in \mathbb{N}$ such that $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{i} a_{1+j} b_{i+j}\right)$ is an antimorphism.
- $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$.
- $(B, C, D, A)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$.

Proof - If every vertex in $A$ sees every vertex in $B$, then applying $\tau$ three times, we see that $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. Similarly, if every vertex in $A$ misses every vertex in $B$ then $(B, C, D, A)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. Thus, we may assume that
some vertex $a_{h}$ in $A$ has neighbours and non-neighbours in $B$, and applying $\tau^{4(h-1)}$, we see that $a_{1}$ has neighbours and non-neighbours in $B$.

Suppose first that $a_{1}$ has at least as many neighbours than non-neiboughs in $B$, more precisely: $\left|N\left(a_{1}\right) \cap B\right| \geq\left|\bar{N}\left(a_{1}\right) \cap B\right|$. Let $i$ be such that $a_{1} b_{i} \notin$ $E(G)$. There exists $j \not \equiv 0(\bmod k)$ such that $a_{1} b_{i-j} \in E$ and $a_{1} b_{i+j} \in E$, for otherwise $\left|\bar{N}\left(a_{1}\right) \cap B\right|>k / 2 \geq\left|N\left(a_{1}\right) \cap B\right|$, a contradiction. Note that we may have $b_{i-j}=b_{i+j}$ if $i-j \equiv i+j(\bmod k)$

We already know $a_{1} b_{i} \notin E$. Applying $\tau^{4 j}$ to $a_{1} b_{i}$ we know $a_{1+j} b_{i+j} \notin E$. We already know $a_{1} b_{i-j} \notin E$. Applying $\tau^{4 j}$ to $a_{1} b_{i-j}$ we know $a_{1+j} b_{i} \in E$. If $a_{1}$ sees $a_{1+j}$ then applying $\tau^{1+4(i-1)}, b_{i}$ misses $b_{i+j}$ and $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{i} a_{1+j} b_{i+j}\right)$ is an antimorphism. If $a_{1}$ misses $a_{1+j}$ then applying $\tau^{1+4(i-1)}, b_{i}$ sees $b_{i+j}$ and we reach the same conclusion.

We are left with the case where $\left|N\left(a_{1}\right) \cap B\right| \leq\left|\bar{N}\left(a_{1}\right) \cap B\right|$. But then, the proof is similar up to a complementation of $G$.

Note that if $(A, B, C, D)$ is a symmetric partition then for any $i, j, l \in \mathbb{N}$, the set $\left\{a_{i}, b_{i+j}, c_{l}, d_{l+j}\right\}$ induces a $P_{4}$. Because by the definition of symmetric partitions, we have $a_{i} b_{i} \in E, b_{i} c_{i} \notin E, c_{i} d_{i} \in E, d_{i} a_{i} \notin E$, and applying $\tau$, exactely one of $a_{i} c_{i}, b_{i} d_{i}$ is an edge. If $(B, C, D, A)$ is a symmetric partition we reach the same conclusion. This remark allows us to follow the lines of Gibbs, and to prove again his theorem (Theorem 1.1) using Lemma 2.3 instead of Lemma 2.2. Let us do it for the sake of completeness.

Consider an sc-graph on $4 k$ vertices and an antimorphism $\tau$. By theorem 2.1 we may assume that every cycle of $\tau$ has length a power of 2 . Let us consider a circular permutation $\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)$ of $\tau$. Put $A=$ $\left\{a_{1}, \ldots a_{k}\right\}, B=\left\{b_{1}, b_{2}, \ldots, b_{k}\right\}, C=\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}, D=\left\{d_{1}, d_{2}, \ldots, d_{k}\right\}$. We claim that we may partition $A \cup B \cup C \cup D$ in sets of 4 vertices all of them inducing a $P_{4}$, thus proving the theorem.

We have $\tau=\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)(\ldots) \cdots(\ldots)$. Apply Lemma 2.3, If one of $(A, B, C, D),(B, C, D, A)$ is a symmetric partition, then for every $i \in \mathbb{N},\left\{a_{i}, b_{i}, c_{i}, d_{i}\right\}$ induces a $P_{4}$ and we may easily partition $A \cup B \cup C \cup D$ into sets of 4 vertices all of them inducing a $P_{4}$. So we are left with the case where there exists $i$ such that $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ for which $\left(a_{1} b_{i} a_{1+j} b_{i+j}\right)$ is an antimorphism. Let us put $k=2^{\alpha}, \alpha \geq 1$. Note that for any $l,\left\{a_{l}, b_{l+i}, a_{l+j}, b_{l+i+j}\right\}$ induces a $P_{4}$ (apply $\left.\tau^{4(l-1)}\right)$ that we denote by $P^{l}$.

We claim that we can choose $l_{1}, l_{2}, \ldots, l_{k / 2}$ such that ( $P^{l_{1}}, P^{l_{2}}, \ldots, P^{l_{k / 2}}$ ) partitions $A \cup B$. For that purpose it suffices to prove that for any integer $j \not \equiv 0\left(\bmod 2^{\alpha}\right), \mathbb{Z}_{2^{\alpha}}$ can be partitioned into pairs of the form $\{l, l+j\}$. For $\alpha=1$ this can be done, so let us prove it by induction. If $j$ is even, then let us partition by the induction hypothesis $\mathbb{Z}_{2^{\alpha-1}}$ into pairs of the form
$\{l, l+j / 2\}: \mathbb{Z}_{2^{\alpha-1}}=\cup_{i}\left\{l_{i}, l_{i}+j / 2\right\}$. Then we have $\mathbb{Z}_{2^{\alpha}}=\cup_{i}\left[\left\{2 l_{i}, 2 l_{i}+j\right\} \cup\right.$ $\left.\left\{2 l_{i}+1,2 l_{i}+1+j\right\}\right]$, so we manage to partition $\mathbb{Z}_{2^{\alpha}}$ as desired. If $j$ is odd, then starting from 1 and adding $j$ successively, we build a Hamiltonian cycle going through every element of its vertex set $\mathbb{Z}_{2^{\alpha}}$. Take then every second edge of this Hamiltonian cycle: here are the pairs that partition $\mathbb{Z}_{2^{\alpha}}$ as desired. So $A \cup B$ may be partitioned into $P_{4}$ 's. Applying $\tau^{2}$, we see that $C \cup D$ can also be partitioned into $P_{4}$ 's, hence $A \cup B \cup C \cup D$ can be partitioned into $P_{4}$ 's.

## 3 A theorem and a conjecture

A skew partition in a graph $G$ is a partition $(A, B, C, D)$ of $V(G)$ such that each of $A, B, C, D$ is non-empty, there are no edges between $A, B$ and every possible edges between $C, D$. We are now able to prove the following:

Theorem 3.1 Let $G$ be an sc-graph with an antimorphism $\tau$ that is the product of two circular permutations, one of them of length 4. Then either:

- $G$ contains a $C_{5}$ as an induced subgraph;
- $G$ contains a skew partition;
- $G$ contains a symmetric partition.

Proof - Let us call $a, b, c, d$ the four vertices of a cycle of $\tau$. If the other cycle has length 1 , then $G$ is the $C_{5}$ or the bull which has a skew partition (see figure (2). So we may assume $\tau=(a b c d)\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)$ with $k \geq 1$. Put $A=\left\{a_{1}, \ldots, a_{k}\right\}, B=\left\{b_{1}, \ldots, b_{k}\right\}, C=\left\{c_{1}, \ldots, c_{k}\right\}$, $D=\left\{d_{1}, \ldots, d_{k}\right\}$. Let us suppose up to a circular permutation of $a, b, c, d$ that the three edges of $G[a, b, c, d]$ are $a b, a c$ and $c d$. Note that if $a$ sees $a_{1}$, then $a$ sees every vertex in $A$ (apply $\tau^{4 i}, i \in \mathbb{N}$ ). By the same way, if $v$ is in $\{a, b, c, d\}$ and if $H$ is one of $A, B, C, D$, then either $v$ sees every vertex in $H$, or $v$ misses every vertex in $H$. For every $v \in\{a, b, c, d\}$, put $N_{v}=N(v) \cap(A \cup B \cup C \cup D)$. We deal now with the $2^{4}=16$ following cases, according to $N_{a}$. Note that once $N_{a}$ is known, $N_{b}, N_{c}$ and $N_{d}$ are also known, by applying $\tau$ three times. For some of the 16 cases, we apply Lemma 2.3 to the cycle $\left(a_{1} b_{1} c_{1} d_{1} a_{2} b_{2} c_{2} d_{2} \ldots a_{k} b_{k} c_{k} d_{k}\right)$ of $\tau$. Then, up to a circular permutation of $A, B, C, D$, we may suppose that either $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$, or there exist $i, j \in \mathbb{N}$ such that $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ with $a_{1} a_{1+j}$ as central edge.

1. $N_{a}=A \cup B$.

Then $N_{b}=A \cup D, N_{c}=C \cup D$ and $N_{d}=B \cup C$. If $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ with $a_{1} a_{1+j}$ as central edge, then $\left\{b, a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $C_{5}$. Else, $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup\{a\}, B \cup\{b\}, C \cup\{c\}, D \cup\{d\})$ is a symmetric partition of $G$.
2. $N_{a}=C \cup D$.

Then $N_{b}=B \cup C, N_{c}=A \cup B$ and $N_{d}=A \cup D$. If $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ with $a_{1} a_{1+j}$ as central edge, then $\left\{b, a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $C_{5}$. Else, $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup\{c\}, B \cup\{d\}, C \cup\{a\}, D \cup\{b\})$ is a symmetric partition of $G$.
3. $N_{a}=A \cup C$.

Then $N_{b}=N_{c}=N_{d}=A \cup C$. Thus $(\{b, d\}, B \cup D,\{a, c\}, A \cup C)$ is a skew partition.
4. $N_{a}=B \cup D$.

Then $N_{b}=N_{c}=N_{d}=B \cup D$. Thus $(\{a, c\}, A \cup C,\{b, d\}, B \cup D)$ is a skew partition.
5. $N_{a}=A \cup D$.

Then $N_{b}=C \cup D, N_{c}=B \cup C$ and $N_{d}=A \cup B$. If $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ with $a_{1} a_{1+j}$ as central edge, then $\left\{c, a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $C_{5}$. Else, $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup\{c\}, B \cup\{d\}, C \cup\{a\}, D \cup\{b\})$ is a symmetric partition of $G$.
6. $N_{a}=B \cup C$.

Then $N_{b}=A \cup B, N_{c}=A \cup D$ and $N_{d}=C \cup D$. If $\left\{a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $P_{4}$ with $a_{1} a_{1+j}$ as central edge, then $\left\{a, a_{1}, b_{i}, a_{1+j}, b_{i+j}\right\}$ induces a $C_{5}$. Else, $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. We see that $(A \cup\{a\}, B \cup\{b\}, C \cup\{c\}, D \cup\{d\})$ is a symmetric partition of $G$.
7. $N_{a}=\emptyset$.

Then $a_{1}$ sees $b$, misses $c$ and sees $d$. Thus, $\left\{a_{1}, a, b, c, d\right\}$ induces $C_{5}$.
8. $N_{a}=A \cup B \cup C \cup D$. Then $N_{c}=N_{a}=A \cup B \cup C \cup D$. Thus ( $\{b\},\{d\},\{a, c\}, A \cup B \cup C \cup D$ ) is a skew partition.
9. $N_{a}=A$.

Then $N_{b}=A \cup C \cup D, N_{c}=C$ and $N_{d}=A \cup B \cup C$. Thus $(\{a, c\}, B \cup$ $D,\{b, d\}, A \cup C)$ is a skew partition.
10. $N_{a}=B \cup C \cup D$.

Then $N_{b}=B, N_{c}=A \cup B \cup D$ and $N_{d}=D$. Thus $(\{b, d\}, A \cup$ $C,\{a, c\}, B \cup D)$ is a skew partition.
11. $N_{a}=B$.

Then $N_{b}=A \cup B \cup D, N_{c}=D$ and $N_{d}=B \cup C \cup D$. Thus $(\{a, c\}, A \cup$ $C,\{b, d\}, B \cup D)$ is a skew partition.
12. $N_{a}=A \cup C \cup D$.

Then $N_{b}=C, N_{c}=A \cup B \cup C$ and $N_{d}=A$. Thus $(\{b, d\}, B \cup$ $D,\{a, c\}, A \cup C)$ is a skew partition.
13. $N_{a}=C$.

Then $N_{b}=A \cup B \cup C, N_{c}=A$ and $N_{d}=A \cup C \cup D$. Thus $(\{a, c\}, B \cup$ $D,\{b, d\}, A \cup C)$ is a skew partition.
14. $N_{a}=A \cup B \cup D$.

Then $N_{b}=D, N_{c}=B \cup C \cup D$ and $N_{d}=B$. Thus $(\{b, d\}, A \cup$ $C,\{a, c\}, B \cup D)$ is a skew partition.
15. $N_{a}=D$.

Then $N_{b}=B \cup C \cup D, N_{c}=B$ and $N_{d}=A \cup B \cup D$. Thus $(\{a, c\}, A \cup$ $C,\{b, d\}, B \cup D)$ is a skew partition.
16. $N_{a}=A \cup B \cup C$.

Then $N_{b}=A, N_{c}=A \cup C \cup D$ and $N_{d}=C$. Thus $(\{b, d\}, B \cup$ $D,\{a, c\}, A \cup C)$ is a skew partition.

Note that as pointed out by Farrugia, a generalisation of Case 8 of the proof was implicitly known by Akiyama and Harary [1]. They proved that if an sc-graph $G$ has at least an end-vertex, then $G$ has exactly two end-vertices $b, d$ and exactly two cut vertices $a, c$. They proved that $(\{b\},\{d\},\{a, c\}, V(G) \backslash\{a, b, c, d\})$ is then a skew partition of $G$.

Let us now discuss the motivivation and possible extensions of Theorem 3.1. Skew partitions were introduced by Chvátal for the study of perfect
graphs [5], and play an important role in the proof of Strong Perfect Graph Conjecture by Chudnovsky, Robertson Seymour and Thomas 4. Symmetric partitions may be seen as a very particular case of the 2-join defined by Cunningham and Cornuéjols, once again for the study of perfect graphs [7]. A 2-join in $G$ is a partition $\left(X_{1}, X_{2}\right)$ of $V(G)$ so that there exist disjoint non-empty $A_{i}, B_{i} \subset X_{i},(i=1,2)$ satisfaying:

1. every vertex of $A_{1}$ is adjacent to every vertex of $A_{2}$ and every vertex of $B_{1}$ is adjacent to every vertex of $B_{2}$;
2. there are no other edges between $X_{1}$ and $X_{2}$;
3. for $i=1,2$, every component of $G\left[X_{i}\right]$ meets both $A_{i}$ and $B_{i}$;
4. for $i=1,2$, if $\left|A_{i}\right|=\left|B_{i}\right|=1$, and if $X_{i}$ induces a path of $G$ joining the vertex of $A_{i}$ and the vertex of $B_{i}$, then it has length at least 3 .

The conditions 3, 4 are called the technical requirements. They are important for algorithms, and for applications to perfect graphs. If a graph $G$ has a 2 -join with the above notation, then $\left(A_{1}, A_{2}, B_{1}, B_{2}\right)$ is a symmetric partition of $G\left[A_{1} \cup A_{2} \cup B_{1} \cup B_{2}\right]$. In other words, if we forget the technical requirements, symmetric partitions may be seen as 2-joins such that $X_{i} \backslash\left(A_{i} \cup B_{i}\right)=\emptyset, i=1,2$.

A lot of work has been done recently on finding algorithms that decide if the vertices of a graph can be partitioned into several subsets with various restrictions on the adjacencies [2, 8, [13]). Symmetric partitions are detectable in linear time (see problem (31) in [8]). Skew partitions seem more difficult, but Figueiredo, Klein, Kohayakawa and Reed gave a polynomial time algorithm that decides whether a graph has or not a skew partition 9]. Note that detecting a $C_{5}$ in a graph can be done easily in $O\left(n^{5}\right)$. Thus, each of the outcome of Theorem 3.1 are testable in polynomial time. We conjecture that Theorem 3.1 holds for every sc-graph on $4 k$ vertices:

Conjecture 3.2 Let $G$ be an sc-graph on $4 k$ vertices. Then either:

- $G$ contains a $C_{5}$ as an induced subgraph;
- G contains a skew partition;
- $G$ contains a symmetric partition.

This conjecture is motivated by several considerations. First, we are able to prove it in a quite general special case: Theorem 3.1. The proof shows how forbiding $C_{5}$ 's can help for finding symmetric or skew partitions. Also, skew partitions and symmetric partitions arise naturally in $P_{4}$-constructions of sc-graphs and in circular permutations of an antimorphism. Suppose $\left(a_{1} b_{1} c_{1} d_{1} \ldots a_{k} b_{k} c_{k} d_{k}\right)$ is such a permutation with our usual notation. If every vertex in $A$ sees every vertex in $B$, then $(A, B, C, D)$ is a symmetric partition of $G[A \cup B \cup C \cup D]$. If every vertex in $A$ sees every vertex in $C$, then $(B, D, C, A)$ is a skew partition of $G[A \cup B \cup C \cup D]$.

Secondly, the conjecture has an analogy with the theorem of Chudnovsky, Robertson, Seymour and Thomas for decomposing Berge Graphs. A hole in a graph is an induced cycle of length at least 4. A graph is Berge if in both $G, \bar{G}$, there is no hole of odd length. The decomposition theorem for Berge graphs is the following. Note that this theorem has been proved in two steps: first Chudnovsky, Robertson, Seymour and Thomas [4] proved a slightly weaker result, and then Chudnovsky [3] alone proved the form that we give:

Theorem 3.3 (Chudnovsky et al.[3, 4]) Let $G$ be a Berge graph. Then either :

- One of $G, \bar{G}$ is bipartite.
- One of $G, \bar{G}$ is the line-graph of a bipartite graph.
- One of $G, \bar{G}$ has a 2-join.
- G has a skew partition.

It would be nice to have a stronger theorem in the particular case of Berge sc-graphs. Conjecture 3.2 could be a candidate.

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Figure 3: The 10 sc-graphs on 8 vertices



[^0]:    *Laboratoire Leibniz, IMAG, 46 av Félix Viallet, 38041 cedex, Grenoble, France. nicolas.trotignon@imag.fr

[^1]:    ${ }^{1} C_{5}$ is the smallest non-perfect graph see [17, $L\left(K_{3,3} \backslash e\right)$ is the smallest perfect graph with no even pair and no even pair in its complement, see [11. There are other examples.

