Additive number theory and the ring of quantum integers^{*}

Melvyn B. Nathanson[†] Department of Mathematics Lehman College (CUNY) Bronx, New York 10468 Email: nathansn@alpha.lehman.cuny.edu

November 1, 2018

Abstract

Let m and n be positive integers. For the quantum integer $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ there is a natural polynomial addition such that $[m]_q \oplus_q [n]_q = [m+n]_q$ and a natural polynomial multiplication such that $[m]_q \otimes_q [n]_q = [mn]_q$. These definitions lead to the construction of the ring of quantum integers and the field of quantum rational numbers. It is also shown that addition and multiplication of quantum integers are equivalent to elementary decompositions of intervals of integers in additive number theory.

1 Addition and multiplication

Let \mathbf{N}, \mathbf{Z} and \mathbf{Q} be the sets of positive integers, integers, and rational numbers, respectively. We define the function

$$[x]_q = \frac{1-q^x}{1-q}$$

of two variables x and q. This is called the quantum number $[x]_q$. Then

$$[0]_q = 0$$

and for every positive integer n we have

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1},$$

 $^{^{*}2000}$ Mathematics Subject Classification: Primary 30B12, 81R50. Secondary 11B13. Key words and phrases. Quantum integers, quantum polynomial, polynomial functional equations, additive bases

[†]This work was supported in part by grants from the NSA Mathematical Sciences Program and the PSC-CUNY Research Award Program.

which is the usual quantum integer n. The negative quantum integers are

$$[-n]_q = \frac{1-q^{-n}}{1-q} = -\frac{1-q^n}{q^n(1-q)}$$
$$= -\frac{1}{q^n}[n]_q = -q^{-1}[n]_{q^{-1}}$$
$$= -\left(\frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^n}\right).$$

Define quantum addition \oplus_q as follows:

$$[x]_q \oplus_q [y]_q = [x]_q + q^x [y]_q.$$

Then

$$\begin{split} [x]_q \oplus_q [y]_q &= [x]_q + q^x [y]_q \\ &= \frac{1 - q^x}{1 - q} + q^x \frac{1 - q^y}{1 - q} \\ &= \frac{1 - q^{x+y}}{1 - q} \\ &= [x + y]_q. \end{split}$$

Define quantum multiplication \otimes_q as follows:

$$[x]_q \otimes_q [y]_q = [x]_q [y]_{q^x}$$

Then

$$\begin{split} [x]_q \otimes_q [y]_q &= [x]_q [y]_{q^x} \\ &= \frac{1-q^x}{1-q} \frac{1-q^{xy}}{1-q^x} \\ &= \frac{1-q^{xy}}{1-q} \\ &= [xy]_q. \end{split}$$

The identities

$$[x]_q \oplus_q [y]_q = [x+y]_q \quad \text{and} \quad [x]_q \otimes_q [y]_q = [xy]_q \quad (1)$$

immediately imply that the set

$$[\mathbf{Z}]_q = \{[n]_q : n \in \mathbf{Z}\}$$

is a commutative ring with the operations of quantum addition \oplus_q and quantum multiplication \otimes_q . The ring $[\mathbf{Z}]_q$ is called the *ring of quantum integers*. The map $n \mapsto [n]_q$ from \mathbf{Z} to $[\mathbf{Z}]_q$ is a ring isomorphism.

For any rational number m/n, the quantum rational number $[m/n]_q$ is

$$[m/n]_q = \frac{1 - q^{m/n}}{1 - q} = \frac{1 - (q^{1/n})^m}{1 - (q^{1/n})^n}$$
$$= \frac{\frac{1 - (q^{1/n})^m}{1 - (q^{1/n})^m}}{\frac{1 - (q^{1/n})^n}{1 - q^{1/n}}} = \frac{[m]_{q^{1/n}}}{[n]_{q^{1/n}}}.$$

Identities (1) imply that addition and multiplication of quantum rational numbers are well-defined. We call

$$[\mathbf{Q}]_q = \{[m/n]_q : m/n \in \mathbf{Q}\}$$

the field of quantum rational numbers.

If we consider $[x]_q$ as a function of real variables x and q, then

$$\lim_{q \to 1} [x]_q = x$$

for every real number x.

We can generalize the results in this section as follows:

Theorem 1 Consider the function

$$[x]_q = \frac{1 - q^x}{1 - q}$$

in the variables x and q. For any ring R, not necessarily commutative, the set

$$[R]_q = \{ [x]_q : x \in R \}$$

is a ring with addition defined by

$$[x]_q \oplus_q [y]_q = [x]_q + q^x [y]_q.$$

and multiplication by

$$[x]_q \otimes_q [y]_q = [x]_q [y]_{q^x}$$

The map from R to $[R]_q$ defined by $x \mapsto [x]_q$ is a ring isomorphism.

Proof. This is true for an arbitrary ring R because the two identities in (1) are formal.

2 Uniqueness of quantum arithmetic

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials in the variable q that satisfies the addition and multiplication rules for quantum integers, that is, \mathcal{F} satisfies the *additive functional equation*

$$f_{m+n}(q) = f_m(q) + q^m f_n(q)$$
(2)

and the multiplicative functional equation

$$f_{mn}(q) = f_m(q)f_n(q^m) \tag{3}$$

for all positive integers m and n. Nathanson [1] showed that there is a rich variety of sequences of polynomials that satisfy the multiplicative functional equation (3). There is not yet a complete classification of solutions of (3), but there is a simple description of all solutions of the additive functional equation (2).

Theorem 2 Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of functions that satisfies the additive functional equation (2). Let $h(q) = f_1(q)$. Then

$$f_n(q) = h(q)[n]_q \quad \text{for all } n \in \mathbf{N}.$$
(4)

Conversely, for any function h(q) the sequence of functions $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ defined by (4) is a solution of (2). In particular, if h(q) is a polynomial in q, then $h(q)[n]_q$ is a polynomial in q for all positive integers n, and all polynomial solutions of (2) are of this form.

Proof. Suppose that $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ is a solution of the additive functional equation (2). Define $h(q) = f_1(q)$. Since $[1]_q = 1$ we have

$$f_1(q) = h(q)[1]_q.$$

Let $n \ge 2$ and suppose that $f_{n-1}(q) = h(q)[n-1]_q$. From (2) we have

$$f_n(q) = f_1(q) + q f_{n-1}(q)$$

= $h(q)[1]_q + q h(q)[n-1]_q$
= $h(q)([1]_q + q[n-1]_q)$
= $h(q)[n]_q.$

It follows by induction that $f_n(q) = h(q)[n]_q$ for all $n \in \mathbb{N}$. Conversely, multiplying (2) by h(q), we obtain

$$h(q)[m+n]_q = h(q)[m]_q + q^m h(q)[n]_q,$$

and so the sequence $\{h(q)[n]_q\}_{n=1}^{\infty}$ is a solution of the additive functional equation (2) for any function h(q). This completes the proof.

We can now prove that the sequence of quantum integers is the only nontrivial solution of the additive and multiplicative functional equations (2) and (3).

Theorem 3 Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of functions that satisfies both functional equations (2) and (3). Then either $f_n(q) = 0$ for all $n \in \mathbf{N}$ or $f_n(q) = [n]_q$ for all $n \in \mathbf{N}$.

Proof. The multiplicative functional equation implies that $f_1(q) = f_1(q)^2$, and so either $f_1(q) = 0$ or $f_1(q) = 1$. Since $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ also satisfies the additive functional equation, it follows from Theorem 2 that either $f_n(q) = 0$ for all n or $f_n(q) = [n]_q$ for all n. This completes the proof.

3 Additive number theory

In this section we show that the addition and multiplication rules for quantum integers correspond to elementary decompositions of finite sets of integers in additive number theory.

Let A and B be sets of integers, and let m be an integer. We define the *dilation*

$$m * A = \{ma : a \in A\},\$$

the $\ translation$

$$m+A=\{m+a:a\in A\},$$

and the sumset

 $A+B=\{a+b:a\in A,b\in B\}.$

We write

$$A \oplus B = C$$

if A + B = C and every integer in C can be written uniquely in the form a + b for some $a \in A$ and $b \in B$.

In additive number theory we consider partitions of a set of integers into a disjoint union of subsets, and decompositions of a set of integers into a sum of sets of integers. Denote by [n] the set of the first n nonnegative integers, that is,

$$[n] = \{0, 1, 2, \dots, n-1\}.$$

We have the partition

$$[m+n] = [m] \cup (m+[n]), \quad \text{where } [m] \cap (m+[n]) = \emptyset, \quad (5)$$

and the direct sum decomposition

$$[mn] = [m] \oplus m * [n]. \tag{6}$$

If m_1, \ldots, m_r are positive integers, then, by induction, we have the partition

$$[m_1 + m_2 + \dots + m_r] = \bigcup_{j=1}^r \left(\sum_{i=1}^{j-1} m_i + [m_j] \right)$$
(7)

into pairwise disjoint sets, and the direct sum decomposition

$$[m_1 m_2 \cdots m_r] = \bigoplus_{j=1}^r \left(\prod_{i=1}^{j-1} m_i * [m_j] \right).$$
 (8)

To each finite set A of integers we associate the Laurent polynomial

$$F_A(q) = \sum_{a \in A} q^a.$$

This is called the *generating function* for A. From the definitions of dilation, translation, and sumset, we have the generating function identities

$$F_{m*A}(q) = F_A(q^m),$$

$$F_{m+A}(q) = q^m F_A(q),$$

and

$$F_{A\oplus B}(q) = F_A(q)F_B(q).$$

If $A \cap B = \emptyset$, then

$$F_{A\cup B}(q) = F_A(q) + F_B(q).$$

The generating function for the set [n] is the quantum integer $[n]_q$, since

$$F_{[n]}(q) = 1 + q + \dots + q^{n-1} = [n]_q$$

Rewriting the partition identity (5) in terms of generating functions, we obtain

$$[m+n]_q = F_{[m+n]}(q)$$

= $F_{[m]\cup(m+[n])}(q)$
= $F_{[m]}(q) + F_{m+[n]}(q)$
= $F_{[m]}(q) + q^m F_{[n]}(q)$
= $[m]_q + q^m [n]_q.$

The sumset decomposition (6) of the interval [mn] gives

$$\begin{split} [mn]_q &= F_{[mn]}(q) \\ &= F_{[m]\oplus m*[n]}(q) \\ &= F_{[m]}(q)F_{m*[n]}(q) \\ &= F_{[m]}(q)F_{[n]}(q^m) \\ &= [m]_q [n]_{q^m}. \end{split}$$

Similarly, the additive number theoretic identities (7) and (8) yield the quantum integer identities

$$[m_1 + m_2 + \dots + m_r]_q = \sum_{j=1}^r q^{\sum_{i=1}^{j-1} m_i} [m_j]_q$$

and

$$[m_1 m_2 \cdots m_r]_q = \prod_{j=1}^r [m_j]_{q \prod_{i=1}^{j-1} m_i}.$$

In this way we see that the addition and multiplication rules for quantum integers are equivalent to elementary statements in additive number theory.

References

[1] M. B. Nathanson, A functional equation arising from multiplication of quantum integers, www.arXiv.org: math.NT/0203217.