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# The number of tree stars is $O^*(1.357^k)$

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**Abstract**

Every rectilinear Steiner tree problem admits an optimal tree $T^*$ which is composed of *tree stars*. Moreover, the currently fastest algorithms for the rectilinear Steiner tree problem proceed by composing an optimum tree $T^*$ from tree star components in the cheapest way. The efficiency of such algorithms depends heavily on the number of tree stars (candidate components). Fößmeier and Kaufmann [4] showed that any problem instance with $k$ terminals has a number of tree stars in between $1.32^k$ and $1.38^k$ (modulo polynomial factors) in the worst case. We determine the exact bound of $O^*(\alpha^k)$ where $\alpha \approx 1.357$ and mention some consequences of this result.

*Keywords:* Rectilinear Steiner tree, Terminal points, Tree star, Components

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Given a weighted graph $(V, E)$ on $n = |V|$ nodes, non-negative edge weight $c : E \to \mathbb{R}_+$, $Y \subseteq V$ of $k$ *terminal nodes* (or *terminals*, for short), the *Steiner tree problem* asks for exhibiting a shortest (*i.e.*, min cost) subtree $T^* = T^*(Y)$ of $(V, E)$ spanning all terminals.

The most well-known algorithm for solving Steiner tree problem is the so-called Dreyfus-Wagner algorithm [1], a certain dynamic programming approach that computes an optimum tree $T^*$ in time $O^*(3^k)$. Here and in what follows, we use the $O^*$-notation to indicate that polynomial factors are suppressed. The currently fastest algorithm, due to [5] resolves the problem in $O^*((2 + \epsilon)^k)$ for any $\epsilon > 0$. We admit, however, that the result is purely theoretical and the algorithm is not expected to be of any use in practice.

The most interesting problems in practice are so-called rectilinear problems, where the terminal set is a finite set $Y \subseteq \mathbb{R}^2$ and the underlying graph is the Hanan grid gererated by $Y$.

In general, every leaf of an optimal Steiner tree $T^*$ is necessarily a terminal. But $T^*$ may also contain some terminals in its interior. These *interior terminals* define a partition of $T^*$ into *components*. In the rectilinear case, a well known result of Hwang [2] states the existence of an optimum Steiner tree $T^* = T^*(Y)$ with each component of special form (*Hwang topology*).

We call a Steiner tree with Hwang topology a *Hwang tree*. Any set $X \subseteq Y$ which is the terminal set of a Hwang tree is called a *Hwang set*. For a Hwang set $X \subseteq Y$, we let $H(X)$ denote the shortest Hwang tree connecting $X$. So Hwang trees are candidates for $T^*$-components. Ganley and Cohoon [3] present a straightforward dynamic program for computing an optimal Steiner tree $T^*$:

- Compute $H(X_0)$ for all Hwang sets $X_0 \subseteq Y$.
- Compute recursively for all $X \subseteq Y$

$$T^*(X) \;=\; \min_{X = X_1 \bowtie X_0} T^*(X_1) \;\cup\; H(X_0),$$

where

$$X = X_1 \bowtie X_0 \;\;\Leftrightarrow\;\; X = X_1 \cup X_0, \text{ and } |X_1 \cap X_0| = 1.$$

In [3], it is shown that there are at most $1.62^k$ Hwang sets $X_0 \subseteq Y$. More generally, every $X \subseteq Y$ of size $i \leq k$ has at most $1.62^i$ Hwang subset $X_0 \subseteq X$. So the above dynamic program has a running time of order

$$O^*(\sum_{i=1}^{k} \binom{k}{i} 1.62^i) = O^*(2.62^k).$$

Fößmeier and Kaufmann [4] further restrict the set of candidates for $T^*$-components by showing that each $T^*$-components can be assumed to be a so-called *tree star*. They showed that, in the worst case, the number of tree stars is in between $1.32^k$ and $1.38^k$, yielding an improvement of the running time in the above dynamic program to $O^*(1.38^k)$. We present a somewhat simper approach, yielding a tight bound of $O^*(\alpha^k)$ for $\alpha \approx 1.357$.

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