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Contractible Subgraphs, Thomassen's Conjecture and the Dominating Cycle Conjecture for Snarks

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Abstract

We show that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is hamiltonian), by Thomassen (every 4-connected line graph is hamiltonian) and by Fleischner (every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle), which are known to be equivalent, are equivalent with the statement that every snark (i.e. a cyclically 4-edge-connected cubic graph of girth at least five that is not 3-edge-colorable) has a dominating cycle.

We use a refinement of the contractibility technique which was introduced by Ryjáček and Schelp in 2003 as a common generalization and strengthening of the reduction techniques by Catlin and Veldman and of the closure concept introduced by Ryjáček in 1997.

Keywords: dominating cycle, contractible graph, cubic graph, snark, line graph, hamiltonian graph

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1 Introduction

In this paper we consider finite undirected graphs. All graphs we consider are loopless, however we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation and for concepts and notation not defined here we refer the reader to [2]. If F, G are graphs then $G - F$ denotes the graph $G - V(F)$ and by an a, b*-path* we mean a path with end vertices a, b. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$.

In 1984, Matthews and Sumner [9] posed the following conjecture.

Conjecture 1.1 [9] *Every 4-connected claw-free graph is hamiltonian.*

Since every line graph is claw-free (see [1]), the following conjecture by Thomassen is a special case of Conjecture 1.1.

Conjecture 1.2 [13] *Every 4-connected line graph is hamiltonian.*

A closed trail T in a graph G is said to be *dominating*, if every edge of G has at least one vertex on T, i.e., the graph $G-T$ is edgeless (a closed trail is defined as usual, except that we allow a single vertex to be such a trail). The following result by Harary and Nash-Williams [6] shows the relation between the existence of a dominating closed trail (abbreviated DCT) in a graph and hamiltonicity of its line graph.

Theorem 1.3 [6] *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G contains a DCT.*

Let k be an integer and let G be a graph with $|E(G)| > k$. The graph G is said to be *essentially k-edge-connected* if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge). If G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain a cycle, G is said to be *cyclically k-edge-connected*.

It is well-known that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected. Thus, the following statement is an equivalent formulation of Conjecture 1.2.

Conjecture 1.4 *Every essentially 4-edge-connected graph contains a DCT.*

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Specifically, if G is *cubic* (i.e. regular of degree 3), then a DCT becomes a dominating cycle (abbreviated DC). It is easy to observe that every essentially 4-edge-connected cubic graph must be triangle-free, with a single exception of the graph K_4. To avoid this exceptional case, we will always consider only essentially 4-edge-connected cubic graphs on at least 5 vertices.

Since a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected (see [5], Corollary 1), the following statement, known as the Dominating Cycle Conjecture, is a special case of Conjecture 1.4.

Conjecture 1.5 *Every cyclically 4-edge-connected cubic graph has a DC.*

Restricting to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Fleischner [4].

Conjecture 1.6 [4] *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

In [11], a closure technique was used to prove that Conjectures 1.1 and 1.2 are equivalent. Fleischner and Jackson [5] showed that Conjectures 1.2, 1.4 and 1.5 are equivalent. Finally, Kochol [7] established the equivalence of these conjectures with Conjecture 1.6. Thus, we have the following result.

Theorem 1.7 **[5], [7], [11]** *Conjectures 1.1, 1.2, 1.4, 1.5 and 1.6 are equivalent.*

Note that recently Kužel and Xiong [8] showed the equivalence of these conjectures with the statement that every 4-connected line graph is hamiltonian-connected.

2 Main result

A cyclically 4-edge-connected cubic graph G of girth $g(G) \geq 5$ that is not 3-edge-colorable is called a *snark*. Snarks have turned out to be an important class of graphs for example in the context of nowhere zero flows. For more information about snarks see the paper [10]. Restricting our considerations to snarks, we obtain the following special case of Conjecture 1.6.

Conjecture 2.1 *Every snark has a DC.*

The following theorem, which is the main result of this paper, shows that Conjecture 2.1 is equivalent with the previous ones.

Theorem 2.2 *Conjecture 2.1 is equivalent with Conjectures 1.1, 1.2, 1.4, 1.5 and 1.6.*

As already noted, every cyclically 4-edge-connected cubic graph other than K_4 must be triangle-free. Thus, the difference between Conjectures E and F consists in restricting to graphs which do not contain a 4-cycle. For the proof of the equivalence of these conjectures we first develop a refinement of the technique of contractible subgraphs that was developed in [12] as a common generalization of the closure concept [11] and Catlin's collapsibility technique [3], and then a technique that allows to handle the (non)existence of a DC while replacing a subgraph of a graph by another one.

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