# Detecting induced subgraphs

Benjamin Lévêque\*, David Y. Lin<sup>†</sup>, Frédéric Maffray\*, Nicolas Trotignon<sup>‡</sup>

October 9, 2007 Revised June 2, 2008

#### Abstract

An s-graph is a graph with two kinds of edges: subdivisible edges and real edges. A realisation of an s-graph B is any graph obtained by subdividing subdivisible edges of B into paths of arbitrary length (at least one). Given an s-graph B, we study the decision problem  $\Pi_B$  whose instance is a graph G and question is "Does G contain a realisation of B as an induced subgraph?". For several B's, the complexity of  $\Pi_B$  is known and here we give the complexity for several more.

Our NP-completeness proofs for  $\Pi_B$ 's rely on the NP-completeness proof of the following problem. Let  $\mathcal{S}$  be a set of graphs and d be an integer. Let  $\Gamma^d_{\mathcal{S}}$  be the problem whose instance is (G, x, y) where G is a graph whose maximum degree is at most d, with no induced subgraph in  $\mathcal{S}$  and  $x, y \in V(G)$  are two non-adjacent vertices of degree 2. The question is "Does G contain an induced cycle passing through x, y?". Among several results, we prove that  $\Gamma^3_{\emptyset}$  is NP-complete. We give a simple criterion on a connected graph H to decide whether  $\Gamma^{+\infty}_{\{H\}}$  is polynomial or NP-complete. The polynomial cases rely on the algorithm three-in-a-tree, due to Chudnovsky and Seymour.

AMS Mathematics Subject Classification: 05C85, 68R10, 68W05, 90C35 Key words: detecting, induced, subgraphs

# 1 Introduction

In this paper graphs are simple and finite. A subdivisible graph (s-graph for short) is a triple B = (V, D, F) such that  $(V, D \cup F)$  is a graph and  $D \cap F = \emptyset$ . The edges in D are said to be real edges of B while the edges in F are said to be subdivisible edges of F. A realisation of F is a graph obtained from F by

<sup>\*</sup>CNRS, Laboratoire G-SCOP, 46 Avenue Félix Viallet, 38031 Grenoble Cedex, France. (benjamin.leveque@g-scop.inpg.fr, Frederic.Maffray@g-scop.inpg.fr)

<sup>&</sup>lt;sup>†</sup>Princeton University, Princeton, NJ, 08544. dylin@princeton.edu

<sup>&</sup>lt;sup>‡</sup>CNRS, Université Paris 7, Paris Diderot, LIAFA, Case 7014, 75205 Paris Cedex 13, France. E-mail: nicolas.trotignon@liafa.jussieu.fr.

This work has been partially supported by ADONET network, a Marie Curie training network of the European Community.

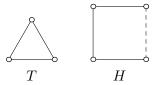


Figure 1: s-graphs yielding trivially polynomial problems

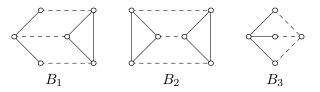


Figure 2: Pyramids, prisms and thetas

subdividing edges of F into paths of arbitrary length (at least one). The problem  $\Pi_B$  is the decision problem whose input is a graph G and whose question is "Does G contain a realisation of B as an induced subgraph?". On figures, we depict real edges of an s-graph with straight lines, and subdivisible edges with dashed lines.

Several interesting instance of  $\Pi_B$  are studied in the literature. For some of them, the existence of a polynomial time algorithm is trivial, but efforts are devoted toward optimized algorithms. For example, Alon, Yuster and Zwick [2] solve  $\Pi_T$  in time  $O(m^{1.41})$  (instead of the obvious  $O(n^3)$  algorithm), where T is the s-graph depicted on Figure 1. This problem is known as triangle detection. Rose, Tarjan and Lueker [10] solve  $\Pi_H$  in time O(n+m) where H is the s-graph depicted on Figure 1.

For some  $\Pi_B$ 's, the existence of a polynomial time algorithm is non-trivial. A pyramid (resp. prism, theta) is any realisation of the s-graph  $B_1$  (resp.  $B_2$ ,  $B_3$ ) depicted on Figure 2. Chudnovsky and Seymour [5] gave an  $O(n^9)$ -time algorithm for  $\Pi_{B_1}$  (or equivalently, for detecting a pyramid). As far as we know, that is the first example of a solution to a  $\Pi_B$  whose complexity is non-trivial to settle. In contrast, Maffray and Trotignon [8] proved that  $\Pi_{B_2}$  (or detecting a prism) is NP-complete. Chudnovsky and Seymour [4] gave an  $O(n^{11})$ -time algorithm for  $P_{B_3}$  (or detecting a theta). Their algorithm relies on the solution of a problem called "three-in-a-tree", that we will define precisely and use in Section 2. The three-in-tree algorithm is quite general since it can be used to solve a lot of  $\Pi_B$  problems, including the detection of pyramids.

These facts are a motivation for a systematic study of  $\Pi_B$ . A further motivation is that very similar s-graphs can lead to a drastically different complexity. The following example may be more striking than pyramid/prism/theta:  $\Pi_{B_4}, \Pi_{B_6}$  are polynomial and  $\Pi_{B_5}, \Pi_{B_7}$  are NP-complete, where  $B_4, \ldots, B_7$  are the s-graphs depicted on Figure 3. This will be proved in section 3.1.

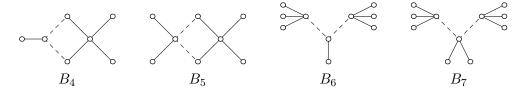


Figure 3: Some s-graphs with pending edges



Figure 4:  $I_1$ 

#### Notation and remarks

By  $C_k$   $(k \ge 3)$  we denote the cycle on k vertices, by  $K_l$   $(l \ge 1)$  the clique on l vertices. A *hole* in a graph is an induced cycle on at least four vertices. We denote by  $I_l$   $(l \ge 1)$  the tree on l+5 vertices obtained by taking a path of length l with ends a, b, and adding four vertices, two of them adjacent to a, the other two to b; see Figure 4. When a graph G contains a graph isomorphic to H as an induced subgraph, we will often say "G contains an H".

Let (V,D,F) be an s-graph. Suppose that  $(V,D\cup F)$  has a vertex of degree one incident to an edge e. Then  $\Pi_{(V,D\cup\{e\},F\setminus\{e\})}$  and  $\Pi_{(V,D\setminus\{e\},F\cup\{e\})}$  have the same complexity, because a graph G contains a realisation of  $(V,D\cup\{e\},F\setminus\{e\})$  if and only if it contains a realisation of  $(V,D\setminus\{e\},F\cup\{e\})$ . For the same reason, if  $(V,D\cup F)$  has a vertex of degree two incident to the edges  $e\neq f$  then  $\Pi_{(V,D\setminus\{e\}\cup\{f\},F\setminus\{f\}\cup\{e\})}$ ,  $\Pi_{(V,D\setminus\{f\}\cup\{e\},F\setminus\{e\}\cup\{f\})}$  and  $\Pi_{(V,D\setminus\{e,f\},F\cup\{e,f\})}$  have the same complexity. If  $|F|\leq 1$  then  $\Pi_{(V,D,F)}$  is clearly polynomial. Thus in the rest of the paper, we will consider only s-graphs (V,D,F) such that:

- $|F| \ge 2$ ;
- no vertex of degree one is incident to an edge of F;
- every induced path of  $(V, D \cup F)$  with all interior vertices of degree 2 and whose ends have degree  $\neq 2$  has at most one edge in F. Moreover, this edge is incident to an end of the path;
- every induced cycle with at most one vertex v of degree at least 3 in  $(V, D \cup F)$  has at most one edge in F and this edge is incident to v if v exists (if it does not then the cycle is a component of  $(V, D \cup F)$ ).

# 2 Detection of holes with prescribed vertices

Let  $\Delta(G)$  be the maximum degree of G. Let  $\mathcal{S}$  be a set of graphs and d be an integer. Let  $\Gamma^d_{\mathcal{S}}$  be the problem whose instance is (G, x, y) where G is a graph such that  $\Delta(G) \leq d$ , with no induced subgraph in  $\mathcal{S}$  and  $x, y \in V(G)$ 

are two non-adjacent vertices of degree 2. The question is "Does G contain a hole passing through x, y?". For simplicity, we write  $\Gamma_{\mathcal{S}}$  instead of  $\Gamma_{\mathcal{S}}^{+\infty}$  (so, the graph in the instance of  $\Gamma_{\mathcal{S}}$  has unbounded degree). Also we write  $\Gamma^d$  instead of  $\Gamma^d$  (so the graph in the instance of  $\Gamma^d$  has no restriction on its induced subgraphs). Bienstock [3] proved that  $\Gamma = \Gamma_{\emptyset}$  is NP-complete. For  $\mathcal{S} = \{K_3\}$  and  $\mathcal{S} = \{K_{1,4}\}$ ,  $\Gamma_{\mathcal{S}}$  can be shown to be NP-complete, and a consequence is the NP-completeness of several problems of interest: see [8] and [9].

In this section, we try to settle  $\Gamma^d_{\mathcal{S}}$  for as many  $\mathcal{S}$ 's and d's as we can. In particular, we give the complexity of  $\Gamma_{\mathcal{S}}$  when  $\mathcal{S}$  contains only one connected graph and of  $\Gamma^d$  for all d. We also settle  $\Gamma^d_{\mathcal{S}}$  for some cases when  $\mathcal{S}$  is a set of cycles. The polynomial cases are either trivial, or are a direct consequence of an algorithm of Chudnovsky and Seymour. The NP-complete cases follow from several extensions of Bienstock's construction.

#### 2.1 Polynomial cases

Chudnovsky and Seymour [4] proved that the problem whose instance is a graph G together with three vertices a, b, c and whose question is "Does G contain a tree passing through a, b, c as an induced subgraph?" can be solved in time  $O(n^4)$ . We call this algorithm "three-in-a-tree". Three-in-a-tree can be used directly to solve  $\Gamma_S$  for several S's. Let us call subdivided claw any tree with one vertex u of degree 3, three vertices  $v_1, v_2, v_3$  of degree 1 and all the other vertices of degree 2.

**Theorem 2.1** Let H be a graph on k vertices that is either a path or a subdivided claw. There is an  $O(n^k)$ -time algorithm for  $\Gamma_{\{H\}}$ .

PROOF — Here is an algorithm for  $\Gamma_{\{H\}}$ . Let (G, x, y) be an instance of  $\Gamma_H$ . If H is a path on k vertices then every hole in G is on at most k vertices. Hence, by a brute-force search on every k-tuple, we will find a hole through x, y if there is any. Now we suppose that H is a subdivided claw. So  $k \geq 4$ . For convenience, we put  $x_1 = x$ ,  $y_1 = y$ . Let  $x_0, x_2$  (resp.  $y_0, y_2$ ) be the two neighbors of  $x_1$  (resp.  $y_1$ ).

First check whether there is in G a hole C through  $x_1, y_1$  such that the distance between  $x_1$  and  $y_1$  in C is at most k-2. If k=4 or k=5 then  $\{x_0, x_1, x_2, y_0, y_1, y_2\}$  either induces a hole (that we output) or a path P that is contained in every hole through x, y. In this last case, the existence of a hole through x, y can be decided in linear time by deleting the interior of P, deleting the neighbors in  $G \setminus P$  of the interior vertices of P and by checking the connectivity of the resulting graph. Now suppose  $k \geq 6$ . For every l-tuple  $(x_3, \ldots, x_{l+2})$  of vertices of G, with  $l \leq k-5$ , test whether  $P = x_0 - x_1 - \cdots - x_{l+2} - y_2 - y_1 - y_0$  is an induced path, and if so delete the interior vertices of P and their neighbors except  $x_0, y_0$ , and look for a shortest path from  $x_0$  to  $y_0$ . This will find the desired hole if there is one, after possibly swapping  $x_0, x_2$  and doing the work again. This takes time  $O(n^{k-3})$ .

Now we may assume that in every hole through  $x_1, y_1$ , the distance between  $x_1, y_1$  is at least k-1.

Let  $k_i$  be the length of the unique path of H from u to  $v_i$ , i=1,2,3. Note that  $k=k_1+k_2+k_3+1$ . Let us check every (k-4)-tuple z=1  $(x_3, \ldots, x_{k_1+1}, y_3, \ldots, y_{k_2+k_3})$  of vertices of G. For such a (k-4)-tuple, test whether  $x_0 - x_1 - \cdots - x_{k_1+1}$  and  $P = y_0 - y_1 - \cdots - y_{k_2+k_3}$  are induced paths of G with no edge between them except possibly  $x_{k_1+1}y_{k_2+k_3}$ . If not, go to the next (k-4)-tuple, but if yes, delete the interior vertices of P and their neighbors except  $y_0, y_{k_2+k_3}$ . Also delete the neighbors of  $x_2, \ldots, x_{k_1}$ , except  $x_1, x_2, \ldots, x_{k_1}, x_{k_1+1}$ . Call  $G_z$  the resulting graph and run three-in-a-tree in  $G_z$  for the vertices  $x_1, y_{k_2+k_3}, y_0$ . We claim that the answer to three-in-a-tree is YES for some (k-4)-tuple if and only if G contains a hole through  $x_1, y_1$  (after possibly swapping  $x_0, x_2$  and doing the work again).

To prove this, first assume that G contains a hole C through  $x_1, y_1$  then up to a symmetry this hole visits  $x_0, x_1, x_2, y_2, y_1, y_0$  in this order. Let us name  $x_3, \ldots, x_{k_1+1}$  the vertices of C that follow after  $x_1, x_2$  (in this order), and let us name  $y_3, \ldots, y_{k_2+k_3}$  those that follow after  $y_1, y_2$  (in reverse order). Note that all these vertices exist and are pairwise distinct since in every hole through  $x_1, y_1$  the distance between  $x_1, y_1$  is at least k-1. So the path from  $y_0$  to  $y_{k_2+k_3}$  in  $C \setminus y_1$  is a tree of  $G_z$  passing through  $x_1, y_{k_2+k_3}, y_0$ , where z is the (k-4)-tuple  $(x_3, \ldots, x_{k_1+1}, y_3, \ldots, y_{k_2+k_3})$ .

Conversely, suppose that  $G_z$  contains a tree T passing through  $x_1, y_{k_2+k_3}, y_0$ , for some (k-4)-tuple z. We suppose that T is vertex-inclusion-wise minimal. If T is a path visiting  $y_0, x_1, y_{k_2+k_3}$  in this order, then we obtain the desired hole of G by adding  $y_1, y_2, \ldots, y_{k_2+k_3-1}$  to T. If T is a path visiting  $x_1, y_0, y_{k_2+k_3}$  in this order, then we denote by  $y_{k_2+k_3+1}$  the neighbor of  $y_{k_2+k_3}$  along T. Note that T contains either  $x_0$  or  $x_2$ . If T contains  $x_0$ , then there are three paths in G:  $y_0-T-x_0-x_1-\cdots-x_{k_1}, y_0-T-y_{k_2+k_3+1}-\cdots-y_{k_3+2}$  and  $y_0-y_1-\cdots-y_{k_3}$ . These three paths form a subdivided claw centered at  $y_0$  that is long enough to contain an induced subgraph isomorphic to H, a contradiction. If T contains  $x_2$  then the proof works similarly with  $y_0-T-x_{k_1+1}-x_{k_1}-\cdots-x_1$  instead of  $y_0-T-x_0-x_1-\cdots-x_{k_1}$ . If T is a path visiting  $x_1, y_{k_2+k_3}, y_0$  in this order, the proof is similar, except that we find a subdivided claw centered at  $y_{k_2+k_3}$ . If T is not a path, then it is a subdivided claw centered at a vertex  $y_0$  of  $y_0$ . We obtain again an induced subgraph of  $y_0$  isomorphic to  $y_0$  by adding to  $y_0$  sufficiently many vertices of  $y_0, \dots, y_{k_1+1}, y_0, \dots, y_{k_2+k_3}$ .

# 2.2 NP-complete cases (unbounded degree)

Many NP-completeness results can be proved by adapting Bienstock's construction. We give here several polynomial reductions from the problem 3-SATISFIABILITY of Boolean functions. These results are given in a framework that involves a few parameters, so that our result can possibly be used for different problems of the same type. Recall that a Boolean function with n variables is a mapping f from  $\{0,1\}^n$  to  $\{0,1\}$ . A Boolean vector  $\xi \in \{0,1\}^n$  is a truth assignment satisfying f if  $f(\xi) = 1$ . For any Boolean variable z on  $\{0,1\}$ , we write  $\overline{z} := 1 - z$ , and each of  $z,\overline{z}$  is called a literal. An instance of 3-SATISFIABILITY is a Boolean function f given as a product of clauses, each clause being the Boolean sum  $\vee$  of three literals; the question is whether f is satisfied by a truth assignment. The NP-completeness of 3-SATISFIABILITY is a fundamental result in complexity theory, see [6].

Let f be an instance of 3-Satisfiability, consisting of m clauses  $C_1, \ldots, C_m$  on n variables  $z_1, \ldots, z_n$ . For every integer  $k \geq 3$  and parameters  $\alpha \in \{1, 2\}$ ,  $\beta \in \{0, 1\}$ ,  $\gamma \in \{0, 1\}$ ,  $\delta \in \{0, 1, 2, 3\}$ ,  $\varepsilon \in \{0, 1\}$ ,  $\zeta \in \{0, 1\}$  such that if  $\alpha = 2$  then  $\varepsilon = \beta = \gamma$ , let us build a graph  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  with two specified vertices x, y of degree 2. There will be a hole containing x and y in  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  if and only if there exists a truth assignment satisfying f. In  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  (we will sometimes write  $G_f$  for short), there will be two kinds of edges: blue and red. The reason for this distinction will appear later. Let us now describe  $G_f$ .

# Pieces of $G_f$ arising from variables

For each variable  $z_i$   $(i=1,\ldots,n)$ , prepare a graph  $G(z_i)$  with 4k vertices  $a_{i,r}, b_{i,r}, a'_{i,r}, b'_{i,r}, r \in \{1,\ldots,k\}$  and 4(m+2)2k vertices  $t_{i,2pk+r}, f_{i,2pk+r}, t'_{i,2pk+r}, f'_{i,2pk+r}, p \in \{0,\ldots,m+1\}, r \in \{0,\ldots,2k-1\}$ . Add blue edges so that the four sets  $\{a_{i,1},\ldots a_{i,k}, t_{i,0},\ldots, t_{i,2k(m+2)-1}, b_{i,1},\ldots,b_{i,k}\}, \{a_{i,1},\ldots a_{i,k}, f_{i,0},\ldots,f_{i,2k(m+2)-1},b_{i,1},\ldots,b'_{i,k}\}, \{a'_{i,1},\ldots a'_{i,k}, f'_{i,0},\ldots,f'_{i,2k(m+2)-1},b'_{i,1},\ldots,b'_{i,k}\}$  all induce paths (and the vertices appear in this order along these paths). See Figure 5.

Add red edges according to the value of  $\alpha$ ,  $\beta$ ,  $\gamma$ , as follows:

- If  $\alpha = 1$  then, for every p = 1, ..., m + 1, add all edges between  $\{t_{i,2kp}, t_{i,2kp+\beta}\}$  and  $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$ , between  $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$  and  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$ , between  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$  and  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$ , between  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$  and  $\{t_{i,2kp}, t_{i,2kp+\beta}\}$ .
- If  $\alpha=2$  then, for every  $p=1,\ldots,m$ , add all edges between  $\{t_{i,2kp+k-1},t_{i,2kp+k-1+\beta}\}$  and  $\{f_{i,2kp+k-1},f_{i,2kp+k-1+\gamma}\}$ ; for every  $p=1,\ldots,m+1$ , add all edges between  $\{f_{i,2kp+k-1},f_{i,2kp+k-1+\gamma}\}$  and  $\{t'_{i,2kp},t'_{i,2kp+\beta}\}$ , between  $\{t'_{i,2kp},t'_{i,2kp+\beta}\}$  and  $\{f'_{i,2kp},f'_{i,2kp+\gamma}\}$ , between  $\{f'_{i,2kp},f'_{i,2kp+\gamma}\}$  and  $\{t_{i,2k(p-1)+k-1},t_{i,2k(p-1)+k-1+\beta}\}$ . See Figures 6, 7.

#### Pieces of $G_f$ arising from clauses

For each clause  $C_j$   $(j=1,\ldots,m)$ , with  $C_j=y_j^1\vee y_j^2\vee y_j^3$ , where each  $y_j^q$  (q=1,2,3) is a literal from  $\{z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n\}$ , prepare a graph  $G(C_j)$  with 2k vertices  $c_{j,p},\,d_{j,p},\,p\in\{1,\ldots,k\}$  and 6k vertices  $u_{j,p}^q,\,q\in\{1,2,3\},\,p\in\{1,\ldots,2k\}$ . Add blue edges so that the three sets  $\{c_{j,1},\ldots,c_{j,k},u_{j,1}^q,\ldots,u_{j,2k}^q,d_{j,1},\ldots d_{j,k}\},\,q\in\{1,2,3\}$  all induce paths (and the vertices appear in this order along these paths).

Add red edges according to the value of  $\delta$ :

- If  $\delta = 0$ , add no edge.
- If  $\delta = 1$ , add  $u_{j,1}^1 u_{j,1}^2$ ,  $u_{j,2k}^1 u_{j,2k}^2$ .
- If  $\delta=2$ , add  $u_{j,1}^1u_{j,1}^2$ ,  $u_{j,2k}^1u_{j,2k}^2$ ,  $u_{j,1}^1u_{j,1}^3$ ,  $u_{j,2k}^1u_{j,2k}^3$ .

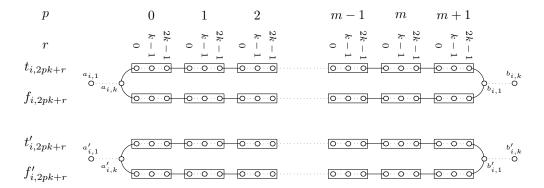


Figure 5: The graph  $G(z_i)$  (only blue edges are depicted)

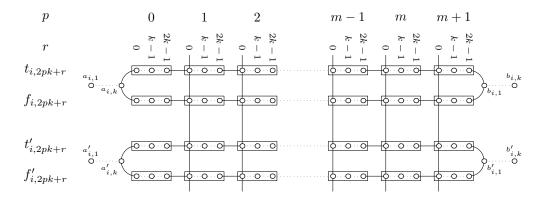


Figure 6: The graph  $G(z_i)$  when  $\alpha = 1, \beta = 0, \gamma = 0$ 

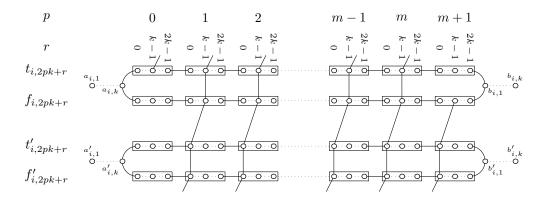


Figure 7: The graph  $G(z_i)$  when  $\alpha=2,\,\beta=0,\,\gamma=0$ 

• If  $\delta=3$ , add  $u_{j,1}^1u_{j,1}^2$ ,  $u_{j,2k}^1u_{j,2k}^2$ ,  $u_{j,1}^1u_{j,1}^3$ ,  $u_{j,2k}^1u_{j,2k}^3$ ,  $u_{j,1}^2u_{j,1}^3$ ,  $u_{j,2k}^2u_{j,2k}^3$ . See Figure 8.

#### Gluing the pieces of $G_f$

The graph  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  is obtained from the disjoint union of the  $G(z_i)$ 's and the  $G(C_j)$ 's as follows. For  $i = 1, \ldots, n-1$ , add blue edges  $b_{i,k}a_{i+1,1}$  and  $b'_{i,k}a'_{i+1,1}$ . Add a blue edge  $b'_{n,k}c_{1,1}$ . For  $j = 1, \ldots, m-1$ , add a blue edge  $d_{j,k}c_{j+1,1}$ . Introduce the two special vertices x, y and add blue edges  $xa_{1,1}, xa'_{1,1}$  and  $yd_{m,k}, yb_{n,k}$ . See Figure 9.

Add red edges according to  $f, \varepsilon, \zeta$ . For q=1,2,3, if  $y_j^q=z_i$ , then add all possible edges between  $\{f_{i,2kj+k-1}, f_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$  and between  $\{f'_{i,2kj+k-1}, f'_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$ ; if  $y_j^q=\overline{z}_i$  then add all possible edges between  $\{t_{i,2kj+k-1}, t_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$  and between  $\{t'_{i,2kj+k-1}, t'_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$ . See Figure 10.

Clearly the size of  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  is polynomial (actually quadratic) in the size n + m of f, and x, y are non-adjacent and both have degree two.

**Lemma 2.2** f is satisfied by a truth assignment if and only if  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  contains a hole passing through x, y.

PROOF — Recall that if  $\alpha=2$  then  $\varepsilon=\beta=\gamma$ . We will prove the lemma for  $\beta=0,\ \gamma=0,\ \varepsilon=0,\ \zeta=0$  because the proof is essentially the same for the other possible values.

Suppose that f is satisfied by a truth assignment  $\xi \in \{0,1\}^n$ . We can build a hole in G by selecting vertices as follows. Select x,y. For  $i=1,\ldots,n$ , select  $a_{i,p},b_{i,p},a'_{i,p},b'_{i,p}$  for all  $p \in \{1,\ldots,k\}$ . For  $j=1,\ldots,m$ , select  $c_{j,p},d_{j,p}$  for all  $p \in \{1,\ldots,k\}$ . If  $\xi_i=1$  select  $t_{i,p},t'_{i,p}$  for all  $p \in \{0,\ldots,2k(m+2)-1\}$ . If  $\xi_i=0$  select  $f_{i,p},f'_{i,p}$  for all  $p \in \{0,\ldots,2k(m+2)-1\}$ . For  $j=1,\ldots,m$ , since  $\xi$  is a truth assignment satisfying f, at least one of the three literals of  $C_j$  is equal to 1, say  $y_j^q=1$  for some  $q \in \{1,2,3\}$ . Then select  $u_{j,p}^q$  for all  $p \in \{1,\ldots,2k\}$ . Now it is a routine matter to check that the selected vertices induce a cycle Z that contains x,y, and that Z is chordless, so it is a hole. The main point is that there is no chord in Z between some subgraph  $G(C_j)$  and some subgraph  $G(z_i)$ , for that would be either an edge  $t_{i,p}u_{j,r}^q$  with  $y_j^q=z_i$  and  $\xi_i=1$ , or, symmetrically, an edge  $f_{i,p}u_{j,r}^q$  with  $y_j^q=\overline{z}_i$  and  $\xi_i=0$ , and in either case this would contradict the way the vertices of Z were selected.

Conversely, suppose that  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  admits a hole Z that contains x, y.

(1) For  $i=1,\ldots,n$ , Z contains at least 4k+4k(m+2) vertices of  $G(z_i)$ : 4k of these are  $a_{i,p},a'_{i,p},b_{i,p},b'_{i,p}$  where  $p \in \{1,\ldots,k\}$ , and the others are either the  $t_{i,p},t'_{i,p}$ 's or the  $f_{i,p},f'_{i,p}$ 's where  $p \in \{0,\ldots,2k(m+2)-1\}$ .

Let us first deal with the case i=1. Since  $x \in Z$  has degree 2, Z contains  $a_{1,1},\ldots,a_{1,k}$  and  $a'_{1,1},\ldots,a'_{1,k}$ . Hence exactly one of  $t_{1,0},f_{1,0}$  is in Z. Likewise exactly one of  $t'_{1,0},f'_{1,0}$  is in Z. If  $t_{1,0},f'_{1,0}$  are both in Z then there is a contradiction: indeed, if  $\alpha=1$  then,  $t_{1,0},\ldots,t_{1,2k}$  and  $f'_{1,0},\ldots,f'_{1,2k}$  must all be in Z, and since  $t_{1,2k}$  sees  $f'_{1,2k},Z$  cannot go through y; and if  $\alpha=2$  the proof

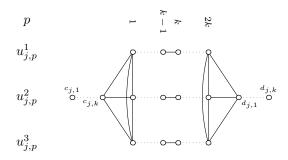


Figure 8: The graph  $G(C_j)$  when  $\delta=3$ 



Figure 9: The whole graph  $G_f$ 

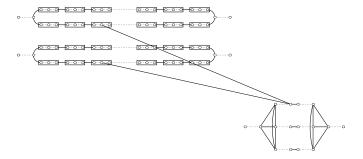


Figure 10: Red edges between  $G(z_i)$  and  $G(C_j)$  when  $\varepsilon=\zeta=0$ 

is similar. Similarly,  $t'_{1,0}, f_{1,0}$  cannot both be in Z. So, there exists a largest integer  $p \leq 2k(m+2)-1$  such that either  $t_{1,0},\ldots,t_{1,p}$  and  $t'_{1,0},\ldots,t'_{1,p}$  are all

in Z or  $f_{1,0}, \ldots, f_{1,p}$  and  $f'_{1,0}, \ldots, f'_{1,p}$  are all in Z. We claim that p = 2k(m+2) - 1. For otherwise, some vertex w in  $\{t_{1,p}, t'_{1,p}, t$  $f_{1,p}, f'_{1,p}$  is incident to a red edge e of Z. If  $\alpha = 1$  then, up to a symmetry, we assume that  $t_{1,0},\ldots,t_{1,p}$  and  $t'_{1,0},\ldots,t'_{1,p}$  are all in Z. Let w' be the vertex of e that is not w. Then w' (which is either an  $f_{1,\cdot}$ , an  $f'_{1,\cdot}$  or a  $u'_{i,\cdot}$ ) is a neighbor of both  $t_{1,p}$ ,  $t'_{1,p}$ . Hence, Z cannot go through y, a contradiction. This proves our claim when  $\alpha = 1$ . If  $\alpha = 2$ , we distinguish between the following six cases. Case 1: p = k - 1. Then  $e = t_{1,k-1} f'_{1,2k}$ . Clearly  $t_{1,0}, \ldots, t_{1,k-1}$  must all be in Z. If  $t'_{1,0}, \ldots, t'_{1,2k}$  are in Z, there is a contradiction because of  $t'_{1,2k}f'_{1,2k}$ , and if  $f'_{1,0}, \ldots, f'_{1,2k}$  are in Z, there is a contradiction because of e.

Case 2: p = 2kl where  $1 \le l \le m+1$  and  $w = t'_{1,2kl}$ . Then e is  $t'_{1,2kl}f_{1,2kl+k-1}$ or  $t'_{1,2kl}f'_{1,2kl}$ . In either case,  $t_{1,2kl},\ldots,t_{1,2kl+k-1}$  are all in Z, and there is a contradiction because of the red edge  $f_{1,2kl+k-1}t_{1,2kl+k-1}$  or  $t_{1,2(l-1)k+k-1}f'_{1,2kl}$ , or when l = m + 1 because of  $b_{1,1}$ .

Case 3: p = 2kl where  $1 \le l \le m+1$  and  $w = f'_{1,2kl}$ . e is  $f'_{1,2kl}t_{1,2(l-1)k+k-1}$  or  $t'_{1,2kl}f'_{1,2kl}$ . In either case,  $f_{1,2kl},\ldots,f_{1,2kl+k-1}$  are all in Z, and there is a contradiction because of the red edge  $t_{1,2(l-1)k+k-1}f_{1,2(l-1)k+k-1}$  or  $t'_{1,2kl}f_{1,2kl+k-1}$ , or when l=1 because of  $a_{1,k}$ . Case 4: p = 2kl + k - 1 where  $1 \le l \le m$  and  $w = t_{1,2kl+k-1}$ . Then e is  $t_{1,2kl+k-1}f_{1,2kl+k-1}, t_{1,2kl+k-1}f'_{1,2(l+1)k}, \text{ or } t_{1,2kl+k-1}u^q_{j,k} \text{ for some } j,q.$  In the last case, there is a contradiction since  $t'_{1,2kl+k-1} \in Z$  also sees  $u^q_{j,k}$ . For the same reason,  $t'_{1,2kl+k-1}u^q_{j,k}$  is not an edge of Z and  $t'_{1,2kl+k-1},\ldots,t'_{1,2(l+1)k}$  are all in Z. So there is a contradiction because of the red edge  $t'_{1,2kl}f_{1,2kl+k-1}$  or

 $t'_{1,2(l+1)k}f'_{1,2(l+1)k}$ . Case 5: p=2kl+k-1 where  $2 \leq l \leq m$  and  $w=f_{1,2kl+k-1}$ . Then e is either  $f_{1,2kl+k-1}t_{1,2kl+k-1}$  or  $f_{1,2kl+k-1}t'_{1,2kl}$  or  $f_{1,2kl+k-1}u^q_{j,k}$  for some j,q. In the last case, there is a contradiction since  $f'_{1,2kl+k-1} \in Z$  also sees  $u^q_{j,k}$ . For the same reason,  $f'_{1,2kl+k-1}u^q_{j,k}$  is not an edge of Z and  $f'_{1,2kl+k-1}, \dots, f'_{1,2(l+1)k}$ are all in Z. So there is a contradiction because of the red edge  $t'_{1,2kl}f'_{1,2kl}$  or  $t_{1,2kl+k-1}f'_{1,2(l+1)}$ .

Case 6: p = 2k(m+1) + k - 1 and  $w = f_{1,2k(m+1)+k-1}$ . Then there is a con-

tradiction because of the red edge  $t'_{1,2k(m+1)}f'_{1,2k(m+1)}$ . This proves our claim. Since p=2k(m+2)-1,  $b_{1,1}$  is in Z. We claim that  $b_{1,2}$  is in Z. For otherwise, the two neighbors of  $b_{1,1}$  in Z are  $t_{1,2k(m+2)-1}$  and  $f_{1,2k(m+2)-1}$ . This is a contradiction because of the red edges  $t_{1,2km+k-1}f'_{1,2k(m+1)}$ ,  $t'_{1,2k(m+1)}f_{1,2k(m+1)+k-1}$  (if  $\alpha=2$ ) or  $t_{1,2k(m+1)}f'_{1,2k(m+1)}, t'_{1,2k(m+1)}f_{1,2k(m+1)}$ (if  $\alpha = 1$ ). Similarly,  $b'_{1,1}, b'_{1,2}$  are in Z. So  $b_{1,1}, \ldots, b_{1,k}$  and  $b'_{1,1}, \ldots, b'_{1,k}$  are all in Z.

This proves (1) for i = 1. The proof for i = 2, ..., n is essentially the same as for i = 1. This proves (1).

(2) For  $j = 1, \ldots, m, Z$  contains  $c_{j,1}, \ldots, c_{j,k}, d_{j,1}, \ldots, d_{j,k}$  and exactly one of  $\{u_{j,1}^1, \ldots, u_{j,2k}^1\}, \{u_{j,1}^2, \ldots, u_{j,2k}^2\}, \{u_{j,1}^3, \ldots, u_{j,2k}^3\}.$ 

Let us first deal with the case j = 1. By (1),  $b'_{n,k}$  is in Z and so  $c_{1,1}, \ldots, c_{1,k}$ are all in Z. Consequently exactly one of  $u_{1,1}^1, u_{1,1}^2, u_{1,1}^3$  is in Z, say  $u_{1,1}^1$  up to a symmetry. Note that the neighbour of  $u_1^1$  in  $Z \setminus c_{1,k}$  cannot be a vertex among  $u_{1,1}^2, u_{1,1}^3$  for this would imply that Z contains a triangle. Hence  $u_{1,2}^1, \ldots, u_{1,k}^1$  are all in Z. The neighbour of  $u_{1,k}^1$  in  $Z \setminus u_{1,k-1}^1$  cannot be in some  $G(z_i)$   $(1 \le i \le n)$ . Else, up to a symmetry we assume that this neighbor is  $t_{1,p}$ ,  $p \in \{0, \ldots, 2k(m+2)-1\}$ . If  $t_{1,p} \in Z$ , there is a contradiction because then  $t_{1,p}'$  is also in Z by (1) and  $t_{1,p}'$  would be a third neighbour of  $u_{1,k}^1$  in Z. If  $t_{1,p} \notin Z$ , there is a contradiction because then the neighbor of  $t_{1,p}$  in  $Z \setminus u_{1,k}^1$  must be  $t_{1,p+1}$  (or symmetrically  $t_{1,p-1}$ ) for otherwise Z contains a triangle. So,  $t_{1,p+1}, t_{1,p+2}, \ldots$  must be in Z, till reaching a vertex having a neighbor  $f_{1,p'}$  or  $f_{1,p'}'$  in Z (whatever  $\alpha$ ). Thus the neighbour of  $u_{1,k}^1$  in  $Z \setminus u_{1,k-1}^1$  is  $u_{1,k+1}^1$ . Similarly, we prove that  $u_{1,k+2}, \ldots, u_{1,2k}$  are in Z, that  $d_{1,1}, \ldots, d_{1,k}$  are in Z, and so the claim holds for j=1. The proof of the claim for  $j=2,\ldots,m$  is essentially the same. This proves (2).

Together with x,y, the vertices of Z found in (1) and (2) actually induce a cycle. So, since Z is a hole, they are the members of Z and we can replace "at least" by "exactly" in (1). We can now make a Boolean vector  $\xi$  as follows. For  $i=1,\ldots,n$ , if Z contains  $t_{i,0},t'_{i,0}$  set  $\xi_i=1$ ; if Z contains  $f_{i,0},f'_{i,0}$  set  $\xi_i=0$ . By (1) this is consistent. Consider any clause  $C_j$   $(1 \le j \le m)$ . By (2) and up to symmetry we may assume that  $u^1_{j,k}$  is in Z. If  $y^1_j=z_i$  for some  $i\in\{1,\ldots,n\}$ , then the construction of G implies that  $f_{i,2kj+k-1},f'_{i,2j+k-1}$  are not in Z, so  $t_{i,2kj+k-1},t'_{i,2kj+k-1}$  are in Z, so  $\xi_i=1$ , so clause  $C_j$  is satisfied by  $x_i$ . If  $y^1_j=\overline{z}_i$  for some  $i\in\{1,\ldots,n\}$ , then the construction of  $G_f$  implies that  $t_{i,2kj+k-1},t'_{i,2kj+k-1}$  are not in Z, so  $f_{i,2kj+k-1},f'_{i,2kj+k-1}$  are in Z, so  $\xi_i=0$ , so clause  $C_j$  is satisfied by  $\overline{z}_i$ . Thus  $\xi$  is a truth assignment satisfying f.

**Theorem 2.3** Let  $k \geq 5$  be an integer. Then  $\Gamma_{\{C_3,...,C_k,K_{1,6}\}}$  and  $\Gamma_{\{I_1,...,I_k,C_5,...,C_k,K_{1,4}\}}$  are NP-complete.

PROOF — It is a routine matter to check that the graph  $G_f(k, 2, 0, 0, 0, 0, 0)$  contains no  $C_l$  ( $3 \le l \le k$ ) and no  $K_{1,6}$  (in fact it has no vertex of degree at least 6). So Lemma 2.2 implies that  $\Gamma_{\{C_3,\ldots,C_k,K_{1,6}\}}$  is NP-complete.

It is a routine matter to check that the graph  $G_f(k, 1, 1, 1, 3, 1, 1)$  contains no  $K_{1,4}$ , no  $C_l$   $(5 \le l \le k)$  and no  $I_{l'}$   $(1 \le l' \le k)$ . So Lemma 2.2 implies that  $\Gamma_{\{K_{1,4},C_5,...,C_k,I_1,...,I_k\}}$  is NP-complete.

# 2.3 Complexity of $\Gamma_{\{H\}}$ when H is a connected graph

**Theorem 2.4** Let H be a connected graph. Then one of the following holds:

- H is a path or a subdivided claw and  $\Gamma_{\{H\}}$  is polynomial.
- H contains one of  $K_{1,4}$ ,  $I_k$  for some  $k \ge 1$ , or  $C_l$  for some  $l \ge 3$  as an induced subgraph and  $\Gamma_{\{H\}}$  is NP-complete.

PROOF — If H contains one of  $K_{1,4}$ ,  $I_k$  for some  $k \geq 1$ , or  $C_l$  for some  $l \geq 3$  as an induced subgraph then  $\Gamma_{\{H\}}$  is NP-complete by Theorem 2.3. Else, H is a tree since it contains no  $C_l$ ,  $l \geq 3$ . If H has no vertex of degree at least 3, then H is a path and  $\Gamma_{\{H\}}$  is polynomial by Theorem 2.1. If H has a single

vertex of degree at least 3, then this vertex has degree 3 because H contains no  $K_{1,4}$ . So, H is a subdivided claw and  $\Gamma_{\{H\}}$  is polynomial by Theorem 2.1. If H has at least two vertices of degree at least 3 then H contains an  $I_l$ , where l is the minimum length of a path of H joining two such vertices. This is a contradiction.

Interestingly, the following analogous result for finding maximum stable sets in H-free graphs was proved by Alekseev:

**Theorem 2.5 (Alekseev, [1])** Let H be a connected graph that is not a path nor a subdivided claw. Then the problem of finding a maximum stable set in H-free graphs is NP-hard.

But the complexity of the maximum stable set problem is not known in general for H-free graphs when H is a path or a subdivided claw. See [7] for a survey.

# 2.4 NP-complete cases (bounded degree)

Here, we will show that  $\Gamma^d$  is NP-complete when  $d \geq 3$  and polynomial when d = 2. If S is any finite list of cycles  $C_{k_1}, C_{k_2}, \ldots, C_{k_m}$ , then we will also show that  $\Gamma^3_S$  is NP-complete as long as  $C_6 \notin S$ .

Let f be an instance of 3-SATISFIABILITY, consisting of m clauses  $C_1, \ldots, C_m$  on n variables  $z_1, \ldots, z_n$ . For each clause  $C_j$   $(j = 1, \ldots, m)$ , with  $C_j = y_{3j-2} \lor y_{3j-1} \lor y_{3j}$ , then  $y_i$   $(i = 1, \ldots, 3m)$  is a literal from  $\{z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}\}$ .

Let us build a graph  $G_f$  with two specified vertices x and y of degree 2 such that  $\Delta(G_f) = 3$ . There will be a hole containing x and y in  $G_f$  if and only if there exists a truth assignment satisfying f.

For each literal  $y_j$  (j = 1, ..., 3m), prepare a graph  $G(y_j)$  on 20 vertices  $\alpha$ ,  $\alpha'$ ,  $\alpha^{1+}$ , ...,  $\alpha^{4+}$ ,  $\alpha^{1-}$ , ...,  $\alpha^{4-}$ ,  $\beta$ ,  $\beta'$ ,  $\beta^{1+}$ , ...,  $\beta^{4+}$ ,  $\beta^{1-}$ , ...,  $\beta^{4-}$ . (We drop the subscript j in the labels of the vertices for clarity).

For i=1,2,3 add the edges  $\alpha^{i+}\alpha^{(i+1)+}$ ,  $\beta^{i+}\beta^{(i+1)+}$ ,  $\alpha^{i-}\alpha^{(i+1)-}$ ,  $\beta^{i-}\beta^{(i+1)-}$ . Also add the edges  $\alpha^{1+}\beta^{1-}$ ,  $\alpha^{1-}\beta^{1+}$ ,  $\alpha^{4+}\beta^{4-}$ ,  $\alpha^{4-}\beta^{4+}$ ,  $\alpha\alpha^{1+}$ ,  $\alpha\alpha^{1-}$ ,  $\alpha^{4+}\alpha'$ ,  $\alpha^{4-}\alpha'$ ,  $\beta\beta^{1+}$ ,  $\beta\beta^{1-}$ ,  $\beta^{4+}\beta'$ ,  $\beta^{4-}\beta'$ . See Figure 11.

For each clause  $C_j$   $(j=1,\ldots,m)$ , prepare a graph  $G(C_j)$  with 10 vertices  $c^{1+}$ ,  $c^{2+}$ ,  $c^{3+}$ ,  $c^{1-}$ ,  $c^{2-}$ ,  $c^{3-}$ ,  $c^{0+}$ ,  $c^{12+}$ ,  $c^{0-}$ ,  $c^{12-}$ . (We drop the subscript j in the labels of the vertices for clarity).

Add the edges  $c^{12+}c^{1+}$ ,  $c^{12+}c^{2+}$ ,  $c^{12-}c^{1-}$ ,  $c^{12-}c^{2-}$ ,  $c^{0+}c^{12+}$ ,  $c^{0+}c^{3+}$ ,  $c^{0-}c^{12-}$ ,  $c^{0-}c^{3-}$ . See Figure 12.

For each variable  $z_i$  (i = 1, ..., n), prepare a graph  $G(z_i)$  with  $2z_i^- + 2z_i^+$  vertices, where  $z_i^-$  is the number of times  $\overline{z}_i$  appears in clauses  $C_1, ..., C_m$  and  $z_i^+$  is the number of times  $z_i$  appears in clauses  $C_1, ..., C_m$ .

Let  $G(z_i)$  consist of two internally disjoint paths  $P_i^+$  and  $P_i^-$  with common endpoints  $d_i^+$  and  $d_i^-$  and lengths  $1+2z_i^-$  and  $1+2z_i^+$  respectively. Label the vertices of  $P_i^+$  as  $d_i^+$ ,  $p_{i,1}^+$ , ...,  $p_{i,2f_i}^+$ ,  $d_i^-$  and label the vertices of  $P_i^-$  as  $d_i^+$ ,  $p_{i,1}^-$ , ...,  $p_{i,2g_i}^-$ ,  $d_i^-$ . See Figure 13.

The final graph  $G_f$  (see Figure 14) will be constructed from the disjoint union of all the graphs  $G(y_i)$ ,  $G(C_i)$ , and  $G(z_i)$  with the following modifications:

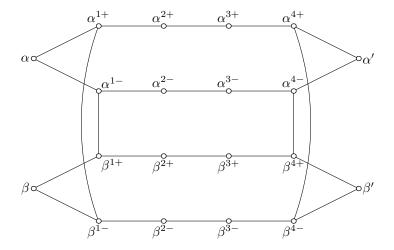


Figure 11: The graph  $G(y_j)$ 

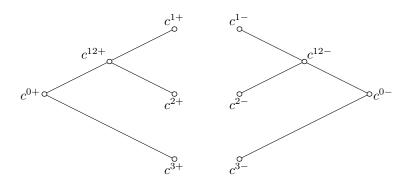


Figure 12: The graph  $G(C_j)$ 

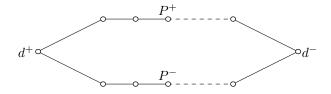


Figure 13: The graph  $G(z_i)$ 

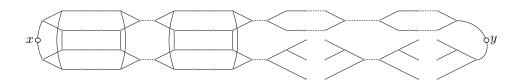


Figure 14: The final graph  $G_f$ 

- For j = 1, ..., 3m 1, add the edges  $\alpha'_j \alpha_{j+1}$  and  $\beta'_j \beta_{j+1}$ .
- For j = 1, ..., m 1, add the edge  $c_i^{0-} c_{i+1}^{0+}$ .
- For i = 1, ..., n 1, add the edge  $d_i^- d_{i+1}^+$ .
- For  $i=1,\ldots,n,$  let  $y_{n_1},\ldots,y_{n_{z_i^-}}$  be the occurrences of  $\overline{z}_i$  over all literals. For  $j=1,\ldots,z_i^-,$  delete the edge  $p_{i,2j-1}^+p_{i,2j}^+$  and add the four edges  $p_{i,2j-1}^+\alpha_{n_j}^{2+},\,p_{i,2j-1}^+\beta_{n_j}^{2+},\,p_{i,2j}^+\alpha_{n_j}^{3+},\,p_{i,2j}^+\beta_{n_j}^{3+}.$
- For i = 1, ..., n, let  $y_{n_1}, ..., y_{n_{z_i^+}}$  be the occurrences of  $z_i$  over all literals. For  $j = 1, 2, ..., z_i^+$ , delete the edge  $p_{i,2j-1}^- p_{i,2j}^-$  and add the four edges  $p_{i,2j-1}^- \alpha_{n_i}^{2+}, p_{i,2j-1}^- \beta_{n_i}^{2+}, p_{i,2j}^- \alpha_{n_i}^{3+}, p_{i,2j}^- \beta_{n_i}^{3+}$ .
- For i = 1, ..., m and j = 1, 2, 3, add the edges  $\alpha_{3(i-1)+j}^{2-} c_i^{j+}, \alpha_{3(i-1)+j}^{3-} c_i^{j-}, \beta_{3(i-1)+j}^{2-} c_i^{j+}, \beta_{3(i-1)+j}^{3-} c_i^{j-}$ .
- Add the edges  $\alpha'_{3m}d_1^+$  and  $\beta'_{3m}c_1^{0+}$
- Add the vertex x and add the edges  $x\alpha_1$  and  $x\beta_1$ .
- Add the vertex y and add the edges  $yc_m^{0-}$  and  $yd_n^{-}$

It is easy to verify that  $\Delta(G_f) = 3$ , that the size of  $G_f$  is polynomial (actually linear) in the size n + m of f, and that x, y are non-adjacent and both have degree two.

**Lemma 2.6** f is satisfied by a truth assignment if and only if  $G_f$  contains a hole passing through x and y.

PROOF — First assume that f is satisfied by a truth assignment  $\xi \in \{0,1\}^n$ . We will pick a set of vertices that induce a hole containing x and y.

- 1. Pick vertices x and y.
- 2. For i = 1, ..., 3m, pick the vertices  $\alpha_i, \alpha'_i, \beta_i, \beta'_i$ .
- 3. For  $i=1,\ldots,3m$ , if  $y_i$  is satisfied by  $\xi$ , then pick the vertices  $\alpha_i^{1+}, \alpha_i^{2+}, \alpha_i^{3+}, \alpha_i^{4+}, \beta_i^{1+}, \beta_i^{2+}, \beta_i^{3+}$ , and  $\beta_i^{4+}$ . Otherwise, pick the vertices  $\alpha_i^{1-}, \alpha_i^{2-}, \alpha_i^{3-}, \alpha_i^{4-}, \beta_i^{1-}, \beta_i^{2-}, \beta_i^{3-}$ , and  $\beta_i^{4-}$ .
- 4. For  $i=1,\ldots,n$ , if  $\xi_i=1$ , then pick all the vertices of the path  $P_i^+$  and all the neighbors of the vertices in  $P_i^+$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any k.
- 5. For  $i=1,\ldots,n$ , if  $\xi_i=0$ , then pick all the vertices of the path  $P_i^-$  and all the neighbors of the vertices in  $P_i^-$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any k.
- 6. For  $i=1,\ldots,m$ , pick the vertices  $c_i^{0+}$  and  $c_i^{0-}$ . Choose any  $j\in\{3i-2,3i-1,3i\}$  such that  $\xi$  satisfies  $y_j$ . Pick vertices  $\alpha_j^{2-}$ , and  $\alpha_j^{3-}$ . If j=3i-2, then pick the vertices  $c_i^{12+}$ ,  $c_i^{1+}$ ,  $c_i^{1-}$ ,  $c_i^{12-}$ . If j=3i-1, then pick the vertices  $c_i^{12+}$ ,  $c_i^{2-}$ ,  $c_i^{12-}$ . If j=3i, then pick the vertices  $c_i^{3+}$  and  $c_i^{3-}$ .

It suffices to show that the chosen vertices induce a hole containing x and y. The only potential problem is that for some k, one of the vertices  $\alpha_k^{2+}$ ,  $\alpha_k^{3+}$ ,  $\alpha_k^{2-}$ , or  $\alpha_k^{3-}$  was chosen more than once. If  $\alpha_k^{2+}$  and  $\alpha_k^{3+}$  were picked in Step 3,

then  $y_k$  is satisfied by  $\xi$ . Therefore,  $\alpha_k^{2+}$  and  $\alpha_k^{3+}$  were not chosen in Step 4 or Step 5. Similarly, if  $\alpha_k^{2-}$  and  $\alpha_k^{3-}$  were picked in Step 6, then  $y_k$  is satisfied by  $\xi$  and  $\alpha_k^{2-}$  and  $\alpha_k^{3-}$  were not picked in Step 3. Thus, the chosen vertices induce a hole in G containing vertices x and y.

Now assume  $G_f$  contains a hole H passing through x and y. The hole Hmust contain  $\alpha_1$  and  $\beta_1$  since they are the only two neighbors of x. Next, either

both  $\alpha_1^{1+}$  and  $\beta_1^{1+}$  are in H, or both  $\alpha_1^{1-}$  and  $\beta_1^{1-}$  are in H.

Without loss of generality, let  $\alpha_1^{1+}$  and  $\beta_1^{1+}$  be in H (the same reasoning that follows will hold true for the other case). Since  $\beta_1^{1-}$  and  $\alpha_1^{1-}$  are both neighbors of two members in H, they cannot be in H. Thus,  $\alpha_1^{2+}$  and  $\beta_1^{2+}$ must be in H. Since  $\alpha_1^{2+}$  and  $\beta_1^{2+}$  have the same neighbor outside  $G(y_1)$ , it follows that H must contain  $\alpha_1^{3+}$  and  $\beta_1^{3+}$ . Also, H must contain  $\alpha_1^{4+}$  and  $\beta_1^{4+}$ . Suppose that  $\alpha_1^{4-}$  and  $\beta_1^{4-}$  are in H. Because  $\alpha_1^{i-}$  has the same neighbor as  $\beta_1^{i-1}$  outside  $G(y_1)$  for i=2,3, it follows that H must contain  $\alpha_1^{3-}$ ,  $\alpha_1^{2-}$ , and  $\alpha_1^{1-}$ . But then H is not a hole containing b, a contradiction. Therefore,  $\alpha_1^{4-}$ and  $\beta_1^{4-}$  cannot both be in H, so H must contain  $\alpha_1'$ ,  $\beta_1'$ ,  $\alpha_2$ , and  $\beta_2$ .

By induction, we see for  $i=1,2,\ldots,3m$  that H must contain  $\alpha_i,\alpha_i',\beta_i,\beta_i'$ . Also, for each i, either H contains  $\alpha_i^{1+},\alpha_i^{2+},\alpha_i^{3+},\alpha_i^{4+},\beta_i^{1+},\beta_i^{2+},\beta_i^{3+},\beta_i^{4+}$  or H contains  $\alpha_i^{1-},\alpha_i^{2-},\alpha_i^{3-},\alpha_i^{4-},\beta_i^{1-},\beta_i^{2-},\beta_i^{3-},\beta_i^{4-}$ . As a result, H must also contain  $d_1^+$  and  $d_1^{0+}$ . By symmetry, we may assume H contains  $d_1^+$  and  $d_1^+$  for some  $d_1^+$ . Since  $d_1^+$  is adjacent to two vertices in  $d_1^+$  and  $d_1^$ 

in H, H must contain  $\alpha_k^{3+}$ . Similarly, H cannot contain  $\alpha_k^{4+}$ , so H contains  $p_{1,2}^+$  and  $p_{1,3}^+$ . By induction, we see that H contains  $p_{1,i}^+$  for  $i=1,2,\ldots,z_i^+$ and  $d_1^-$ . If H contains  $p_{1,z_i^-}^-$ , then H must contain  $p_{1,i}^-$  for  $i=z_i^-,\ldots,1,$  a contradiction. Thus, H must contain  $d_2^+$ . By induction, for  $i=1,2,\ldots,n$ , we see that H contains all the vertices of the path  $P_i^+$  or  $P_i^-$  and by symmetry, we may assume H contains all the neighbors of the vertices in  $P_i^+$  or  $P_i^-$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any k.

Similarly, for i = 1, 2, ..., m, it follows that H must contain  $c_i^{0+}$  and  $c_i^{0-}$ . Also, H contains one of the following:

- $c_i^{12+}, c_i^{1+}, c_i^{1-}, c_i^{12-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$  is adjacent to  $c_i^{1+}$ ).
- $c_i^{12+}, c_i^{2+}, c_i^{2-}, c_i^{12-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$
- $c_i^{3+}$  and  $c_i^{3-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$  is adjacent to  $c_i^{3+}$ ).

We can recover the satisfying assignment  $\xi$  as follows. For i = 1, 2, ..., n, set  $\xi_i = 1$  if the vertices of  $P_i^+$  are in H and set  $\xi_i = 0$  if the vertices of  $P_i^$ are in H. By construction, it is easy to verify that at least one literal in every clause is satisfied, so  $\xi$  is indeed a satisfying assignment.

Note that the graph  $G_f$  used above contains several  $C_6$ 's that we could not eliminate, induced for instance by  $\alpha, \alpha^{1+}, \beta^{1-}, \beta, \beta^{1+}, \alpha^{1-}$ .

**Theorem 2.7** The following statements hold:

- For any  $d \in \mathbb{Z}$  with  $d \geq 2$ , the problem  $\Gamma^d$  is NP-complete when  $d \geq 3$  and polynomial when d = 2.
- If  $\mathcal{H}$  is any finite list of cycles  $C_{k_1}, C_{k_2}, \ldots, C_{k_m}$  such that  $C_6 \notin \mathcal{H}$ , then  $\Gamma^3_{\mathcal{H}}$  is NP-complete.

PROOF — In the above reduction,  $\Delta(G_f) = 3$  so  $\Gamma^d$  is NP-complete for  $d \geq 3$ . When d = 2, there is a simple O(n) algorithm. Any hole containing x and y must be a component of G so pick the vertex x and consider the component C of G that contains x. It takes O(n) time to verify whether C is a hole containing x and y or not.

To show the second statement, let K be the length of the longest cycle in  $\mathcal{H}$ . In the above reduction, do the following modifications.

- For i=1,2,3 and  $j=1,2,\ldots,3m$ , replace the edges  $\alpha_j^{i+}\alpha_j^{(i+1)+}$ ,  $\alpha_j^{i-}\alpha_j^{(i+1)-}$ ,  $\beta_j^{i+}\beta_j^{(i+1)+}$ , and  $\beta_j^{i-}\beta_j^{(i+1)-}$  by paths of length K.
- For j = 1, 2, ..., 3m 1, replace the edges  $\alpha'_j \alpha_{j+1}$  and  $\beta'_j \beta_{j+1}$  by paths of length K.
- Replace the edges  $x\alpha_1$  and  $x\beta_1$  by paths of length K.

This new reduction is polynomial in n, m and contains no graph of the list  $\mathcal{H}$ . The proof of Lemma 2.6 still holds for this new reduction therefore  $\Gamma^3_{\mathcal{H}}$  is NP-complete.

# 3 $\Pi_B$ for some special s-graphs

### 3.1 Holes with pending edges and trees

Here, we study  $\Pi_{B_4}, \ldots, \Pi_{B_7}$  where  $B_4, \ldots, B_7$  are the s-graphs depicted on Figure 3. Our motivation is simply to give a striking example and to point out that surprisingly, pending edges of s-graphs matter and that even an s-graph with no cycle can lead to NP-complete problems.

**Theorem 3.1** There is an  $O(n^{13})$ -time algorithm for  $\Pi_{B_4}$  but  $\Pi_{B_5}$  is NP-complete.

PROOF — A realisation of  $B_4$  has exactly one vertex of degree 3 and one vertex of degree 4. Let us say that the realisation H is short if the distance between these two vertices in H is at most 3. Detecting short realisations of  $B_4$  can be done in time  $n^9$  as follows: for every 6-tuple  $F = (a, b, x_1, x_2, x_3, x_4)$  such that G[F] has edge-set  $\{x_1a, ax_2, x_2b, bx_3, bx_4\}$  and for every 7-tuple  $F = (a, b, x_1, x_2, x_3, x_4, x_5)$  such that G[F] has edge-set  $\{x_1a, ax_2, x_2x_3, x_3b, bx_4, bx_5\}$ , delete  $x_1, \ldots, x_5$  and their neighbors except a, b. In the resulting graph, check whether a and b are in the same component. The answer is YES for at least one 7-or-6-tuple if and only if G contains at least one short realisation of  $B_4$ .

Here is an algorithm for  $\Pi_{B_4}$ , assuming that the entry graph G has no short realisation of  $B_4$ . For every 9-tuple  $F=(a,b,c,x_1,\ldots,x_6)$  such that G[F] has edge-set  $\{x_1a,bx_2,x_2x_3,x_3x_4,cx_5,x_5x_3,x_3x_6\}$  delete  $x_1,\ldots,x_6$  and their neighbors except a,b,c. In the resulting graph, run three-in-a-tree for a,b,c. It is easily checked that the answer is YES for some 9-tuple if and only if G contains a realisation of  $B_4$ .

Let us prove that  $\Pi_{B_5}$  is NP-complete by a reduction of  $\Gamma^3$  to  $\Pi_{B_5}$ . Since by Theorem 2.7,  $\Gamma^3$  is NP-complete, this will complete the proof. Let (G, x, y) be an instance of  $\Gamma^3$ . Prepare a new graph G': add four vertices x', x'', y', y'' to G and add four edges xx', xx'', yy', yy''. Since  $\Delta(G) \leq 3$ , it is easily seen that G contains a hole passing through x, y if and only if G' contains a realisation of  $B_5$ .

The proof of the theorem below is omitted since it is similar to the proof of Theorem 3.1.

**Theorem 3.2** There is an  $O(n^{14})$ -time algorithm for  $\Pi_{B_6}$  but  $\Pi_{B_7}$  is NP-complete.

#### 3.2 Induced subdivisions of $K_5$

Here, we study the problem of deciding whether a graph contains an induced subdivision of  $K_5$ . More precisely, we put :  $sK_5 = (\{a, b, c, d, e\}, \emptyset, \binom{\{a, b, c, d, e\}}{2})$ .

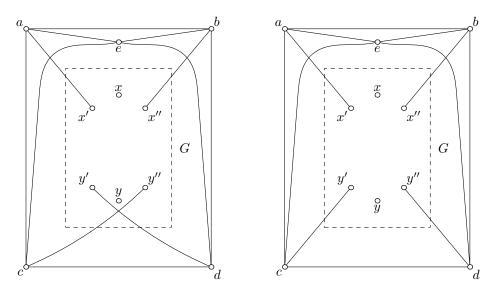


Figure 15: Graphs G' and G''

**Theorem 3.3**  $\Pi_{sK_5}$  is NP-complete.

PROOF — We consider an instance (G, x, y) of  $\Gamma^3$ . Let us denote by x', x'' the two neighbors of x and by y', y'' the two neighbors of y.

Let us build a graph G' by adding five vertices a, b, c, d, e. We add the edges ab, bd, dc, ca, ea, eb, ec, ed, ax', bx'', cy'', dy'. We delete the edges xx', xx'', yy', yy''. We define a very similar graph G'', the only change being that we do not add edges cy'', dy' but edges cy', dy'' instead. See Figure 15.

Now in G' (and similarly G'') every vertex has degree at most 3, except for a, b, c, d, e. We claim that G contains a hole going through x and y if and only if at least one of G', G'' contains an induced subdivision of  $K_5$ . Indeed, if G contains a hole passing through x, x', y', y, y'', x'' in that order then G' obviously contains an induced subdivision of  $K_5$ , and if the hole passes in order through x, x', y'', y, y', x'' then G'' contains such a subgraph. Conversely, if G' (or symmetrically G'') contains an induced subdivision of  $K_5$  then a, b, c, d, e must be the vertices of the underlying  $K_5$ , because they are the only vertices with degree at least 4. Hence there is a path from x' to y' in  $G \setminus \{x, y\}$  and a path from x'' to y'' in  $G \setminus \{x, y\}$ , and consequently a hole going through x, y in G.

# 3.3 $\Pi_B$ for small B's

Here, we survey the complexity  $\Pi_B$  when B has at most four vertices. By the remarks in the introduction, if  $|V| \leq 3$  then  $\Pi_{(V,D,F)}$  is polynomial. Up to symmetries, we are left with twelve s-graphs on four vertices as shown below.

For the following two s-graphs, there is a polynomial algorithm using three-in-a-tree. The two algorithms are essentially similar to those for thetas and pyramids (see Figure 2). See [4] for details.



The next two s-graphs yield an NP-complete problem:

For the next seven graphs on four vertices, we could not get an answer:

For the last graph represented below, it was proved recently by Trotignon and Vušković [11] that the problem can be solved in time O(nm), using a method based on decompositions.



In conclusion we would like to point out that, except for the problem solved in [11], every detection problem associated with an s-graph for which a polynomial time algorithm is known can be solved either by using three-in-a-tree or by some easy brute-force enumeration.

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