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On components of 2-factors in claw-free graphs

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Abstract

For a non-hamiltonian claw-free graph G with order n and minimum degree δ we show the following. If $\delta = 4$, then G has a 2-factor with at most $(5n - 14)/18$ components, unless G belongs to a finite class of exceptional graphs. If $\delta \geq 5$, then G has a 2-factor with at most $(n-3)/(\delta-1)$ components. These bounds are sharp in the sense that we can replace nor $5/18$ by a smaller quotient nor $\delta - 1$ by δ.

Keywords: claw-free graph, 2-factor, minimum degree, edge degree

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1 Introduction

Let $G = (V_G, E_G)$ be a finite and simple graph of order $|G| = |V_G| = n$ and of size $e(G) = |E(G)|$. We denote the minimum (vertex) degree of G by $\delta(G)$. The neighbor set of a vertex x in G is denoted by $N_G(x)$, and its cardinality by $d_G(x)$. If no confusion is possible we omit the subscripts. A *2-factor* of a graph G is a spanning 2-regular subgraph of G.

In this paper we study *claw-free* graphs, i.e., graphs that do not contain an induced four-vertex star $K_{1,3}$. We try to obtain a sharp upper bound on the minimum number of components of a 2-factor in a claw-free graph. Our research is motivated by the fact that any hamiltonian cycle is a 2-factor of only one component, and hence the smallest number of components in a 2-factor can be seen as a measure for how close a graph is to being hamiltonian.

Results of both Egawa & Ota [2] and Choudum & Paulraj [1] imply that every claw-free graph with $\delta \geq 4$ contains a 2-factor. We observe that every 4-connected claw-free graph has minimum degree at least four, and hence has a 2-factor. A 2-connected claw-free graph already has a 2-factor if $\delta = 3$ [8]. However, in general a claw-free graph with $\delta \leq 3$ does not have to contain a 2-factor. If it does, we can not do better than the trivial bound $n/3$. Hence, it is natural to consider claw-free graphs with $\delta \geq 4$.

Faudree et al. [3] showed that every claw-free graph with $\delta \geq 4$ has a 2-factor with at most $6n/(\delta+2) - 1$ components. Gould & Jacobson [5] proved that every claw-free graph G with $\delta \geq (4n)^{\frac{2}{3}}$ has a 2-factor with at most n/δ components. Our two main results provide answers to two open questions posed in [8]. They improve the previously mentioned results.

Theorem 1.1 *A non-hamiltonian claw-free graph on n vertices with $\delta \geq 5$ has a 2-factor with at most $(n-3)/(\delta-1)$ components.*

This result is tight in the following sense. Let $f_2(G)$ denote the minimum number of components in a 2-factor of G. Then in [8] an infinite family $\{G_i\}$ of claw-free graphs with $\delta(G_i) \geq 4$ is given such that $f_2(G_i) > |G_i|/\delta(G_i)$. For $\delta = 4$ we are able to prove a stronger bound.

Theorem 1.2 *A claw-free graph G on n vertices with $\delta = 4$ has a 2-factor with at most $(5n-14)/18$ components, unless G belongs to a finite class of exceptional graphs.*

The bound in Theorem 1.2 is tight in the following sense. There exists an infinite family $\{H_i\}$ of claw-free graphs with $\delta(H_i) = 4$ such that

$\lim\limits_{|H_i|\to\infty} \dfrac{f_2(H_i)}{|H_i|} = \dfrac{5}{18}$. This family can be found in [8] as well.

2 Proof sketch of Theorem 1.1

We will sketch the proof of Theorem 1.1 in four different parts. The proof of Theorem 1.2 uses different counting arguments but follows the same line.

Step 1: restrict to the line graph of a triangle-free graph
We show that we can restrict ourselves to a subclass of claw-free graphs, namely the class of line graphs of triangle-free graphs. For this purpose we use the *closure* concept as defined in [6].

The closure of a graph is defined as follows. Let G be a claw-free graph. If the subgraph induced by the neighbor set $N(x)$ of some vertex x in G is connected, we add edges joining all pairs of nonadjacent vertices in $N(x)$. This operation is called *local completion of G at x*. The *closure* $cl(G)$ of G is a graph obtained by recursively repeating the local completion operation, as long as this is possible. Ryjáček [6] showed that the closure of G is uniquely determined. Ryjáček, Saito & Schelp [7] showed the following result.

Theorem 2.1 ([7]) *A claw-free graph G has a 2-factor with at most k components if and only if $cl(G)$ has a 2-factor with at most k components.*

For a graph G, we call a graph H with $L(H) = cl(G)$ the *preimage graph* of G. Ryjáček [6] showed that every claw-free graph G has a (triangle-free) preimage graph.

Theorem 2.2 ([6]) *If G is a claw-free graph, then there is a triangle-free graph H such that $L(H) = cl(G)$.*

By Theorem 2.1 and Theorem 2.2, we deduce that for a claw-free graph G, $f_2(G) = f_2(cl(G)) = f_2(L(H))$, where H is the preimage graph of G. This implies that we can restrict ourselves to line graphs of triangle-free graphs.

Step 2: translate the problem into finding a dominating system
An *even* graph is a graph in which every vertex has even degree at least two. A connected even graph is called a *circuit*. Let H be a graph that contains a set $\mathcal{S}$ of stars with at least three edges and circuits, all (stars and circuits) mutually edge-disjoint. We call $\mathcal{S}$ a *system that dominates H* or simply a *dominating system* if for every edge e of H the following holds:

- e is contained in one of the stars of $\mathcal{S}$, or

- e is contained in one of the circuits of $\mathcal{S}$, or
- e shares an end vertex with an edge of at least one of the circuits in $\mathcal{S}$.

Gould & Hynds [4] proved the following result.

Theorem 2.3 ([4]) *The line graph $L(H)$ of a graph H has a 2-factor with k components if and only if H has a dominating system with k elements.*

Combining Theorem 2.1 and Theorem 2.2 with Theorem 2.3 yields the following result.

Theorem 2.4 *A claw-free graph G has a 2-factor with at most k components if and only if the (triangle-free) preimage graph of $cl(G)$ has a dominating system with at most k elements.*

The *edge degree* of an edge xy in a graph H is defined as $d_H(x)+d_H(y)-2$. We denote the minimum edge degree of H by $\delta_e(H)$. Due to Theorem 2.4 we have proven Theorem 1.1 if the following theorem holds.

Theorem 2.5 *A triangle-free graph H with $\delta_e \geq 5$ has a dominating system with at most* $\max\{1, (e-3)/(\delta_e - 1)\}$ *elements.*

Step 3: prove Theorem 2.5 for trees

We use a proof by contradiction and choose a tree H to be a counterexample of the theorem with minimum number of edges.

Step 4: prove Theorem 2.5 for any triangle-free graph

Let H be a triangle-free graph that is not a tree. A *maximum* even subgraph of H is an even subgraph of H with maximum number of edges. Let X be a maximum even subgraph of H with minimum number of components. The proof idea is to "break" the circuits in X by removing a number of edges, such that we obtain a new graph F that is a forest. Then we can apply the result obtained in Step 3 to each component of F. We then translate the dominating system of F into one of H, and counting arguments complete the proof. There are two problems to take care of. Firstly, we have to show that F is indeed a forest. Obviously $H - E(X)$ is a forest. For our proof however, we need a stronger statement.

Lemma 2.6 *Let X be a maximum even subgraph of a graph H such that the number of components in X is as small as possible. Let $\mathcal{C}$ be the set of all components in X and $\mathcal{C}_4 \subset \mathcal{C}$ the set of all components of order 4. For each $C \in \mathcal{C} \backslash \mathcal{C}_4$, let e_C be an edge in C, and for each $C \in \mathcal{C}_4$, let e_C, e'_C be two* *independent edges in C. Then*

$$(H - E(X)) \cup \{e_C \mid C \in \mathcal{C}\backslash\mathcal{C}_4\} \cup \{e_C, e'_C \mid C \in \mathcal{C}_4\}$$

is a forest or there is a component $C \in \mathcal{C}_4$ that induces a K_4 in H.

Secondly, we may only use the result of Step 3 if $\delta_e(F) \geq 4$. We guarantee this by adding a sufficient number of pendant edges to F. This will create a number of extra stars that will be in any dominating system of F. We show that the number of extra stars compensate for the number of extra edges.

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