List Colouring Squares of Planar Graphs

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Abstract

In 1977, Wegner conjectured that the chromatic number of the square of every planar graph with maximum degree $\Delta \geq 8$ is at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. We show that it is at most $\frac{3}{2}\Delta(1+o(1))$ (where the $o(1)$ is as $\Delta \to +\infty$), and indeed that this is true for the list chromatic number and for more general classes of graphs.

1 Introduction

Most of the terminology and notation we use in this paper is standard and can be found in any text book on graph theory (such as [6] or [9]). All our graphs and multigraphs will be finite. A *multigraph* can have multiple edges; a *graph* is supposed to be simple. We will not allow loops.

The *degree* of a vertex is the number of edges incident with that vertex. We require all colourings, whether we are discussing vertex, edge or list colouring, to be *proper*: neighbouring objects must receive different colours. We also always assume that colours are integers, which allows us to talk about the "distance" $|\gamma_1 - \gamma_2|$ between two colours γ_1, γ_2.

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Given a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours required so that we can properly colour its vertices using those colours. If we colour the edges of G, we get the *chromatic index*, denoted $\chi'(G)$.

Given a list $L(v)$ of colours for each vertex v of G, we say a colouring is *acceptable* (with respect to the lists) if it is proper and every vertex gets assigned a colour from its own private list. The *list chromatic number* or *choice number* $\mathrm{ch}(G)$ is the minimum value k such that, if we give each vertex of G a list of size k, then there is an acceptable colouring. The *list chromatic index* is defined analogously for edges. See [46] for a survey of research on list colouring of graphs. Note that the list $L(v)$ is really just a set, but as is standard we refer to it as a list.

1.1 Colouring the Square of a Graph

Given a graph G, the *square of* G, denoted G^2, is the graph with the same vertex set as G and with an edge between each pair of distinct vertices that have distance at most two in G. If G has maximum degree Δ, then a vertex colouring of its square will need at least $\Delta+1$ colours; the greedy algorithm shows it is always possible with Δ^2+1 colours. Diameter two cages such as the 5-cycle, the Petersen graph and the Hoffman-Singleton graph (see [6, page 84]) show that there exist graphs that in fact require Δ^2+1 colours, for $\Delta = 2, 3, 7$, and possibly one for $\Delta = 57$.

We are particularly interested in planar graphs. The celebrated Four Colour Theorem by Appel and Haken [3, 4, 5] states that $\chi(G) \leq 4$ for planar graphs G. Regarding the chromatic number of the square of a planar graph, Wegner [44] posed the following conjecture (see also the book of Jensen and Toft [17, Section 2.18]), suggesting that for planar graphs far less than Δ^2+1 colours suffice.

Conjecture 1.1 (Wegner [44])
For a planar graph G with maximum degree Δ,

$$\chi(G^2) \;\leq\; \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also gave examples showing that these bounds would be tight. For even $\Delta \geq 8$, these examples are sketched in Figure 1. The graph G_k consists of three vertices x, y and z together with $3k-1$ additional vertices with degree two, such that z has k common neighbours with x and k common neighbours with y, and x and y are adjacent and have $k-1$ common neighbours. This graph has maximum degree $2k$ and yet all the vertices except z are adjacent in its square. Hence to colour these $3k+1$ vertices, we need at least $3k+1 = \frac{3}{2}\Delta + 1$ colours.

Kostochka and Woodall [28] conjectured that for every square of a graph the list chromatic number equals the chromatic number. This conjecture was first disproved by Kim and Park [23]. Since then more counterexamples have been found [22, 24, 26]. All these counterexamples are not planar, which gives us hope that Kostochka and Woodall's original conjecture is true for planar graphs.

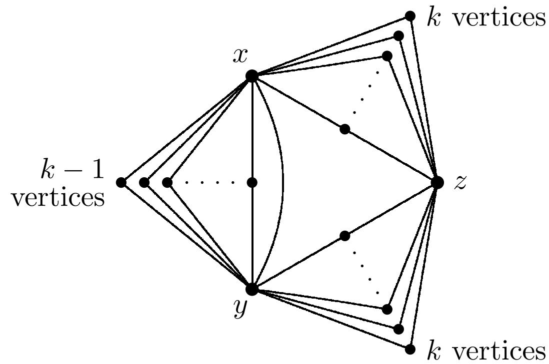

Figure 1: The planar graph G_k.

Conjecture 1.2
For a planar graph G with maximum degree Δ,

$$\mathrm{ch}(G^2) \;\le\; \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7, \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1, & \text{if } \Delta \ge 8. \end{cases}$$

Wegner also showed that if G is a planar graph with $\Delta = 3$, then G^2 can be 8-coloured. Thomassen [43] established Wegner's conjecture for $\Delta = 3$ using an involved structural result on subcubic (i.e. with $\Delta \le 3$) graphs; while Hartke *et al.* [13] proved the same using the discharging method and a serious amount of computer time. Cranston and Kim [8] showed that the square of every connected graph (not necessarily planar) which is subcubic is 8-choosable, except for the Petersen graph. However, the 7-choosability of the squares of subcubic planar graphs is still open.

The first upper bound on $\chi(G^2)$ for planar graphs that is linear in Δ, namely $\chi(G^2) \le 8\Delta - 22$, was implicit in the work of Jonas [18]. (The results in [18] deal with $L(2,1)$-labellings, see below, but the proofs are easily seen to be applicable to colouring the square of a graph as well.) This bound was later improved by Wong [45] to $\chi(G^2) \le 3\Delta + 5$, and then by Van den Heuvel and McGuinness [15] to $\chi(G^2) \le 2\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \le \left\lceil \frac{9}{5}\Delta \right\rceil + 1$ for $\Delta \ge 750$ by Agnarsson and Halldórsson [1], and the same bound for $\Delta \ge 47$ by Borodin *et al.* [7]. Finally, the asymptotically best known upper bound so far has been obtained by Molloy and Salavatipour [38] as a special case of Theorem 1.6 below.

Theorem 1.3 (Molloy and Salavatipour [38])
For a planar graph G with maximum degree Δ,

$$\chi(G^2) \;\le\; \tfrac{5}{3}\Delta + 78.$$

As mentioned in [38], the constant 78 can be reduced for sufficiently large Δ; the paper improves it to 24 when $\Delta \geq 241$.

In this paper we prove the following theorem.

Theorem 1.4
The square of every planar graph G with maximum degree Δ has list chromatic number at most $(1+o(1))\frac{3}{2}\Delta$. Moreover, given lists of this size, there is an acceptable colouring in which the colours on every pair of adjacent vertices of G differ by at least $\Delta^{1/4}$.

A more precise statement is as follows. For each $\epsilon > 0$, there is a Δ_ϵ such that for every $\Delta \geq \Delta_\epsilon$ we have: for every planar graph G with maximum degree at most Δ, and for all vertex lists each of size at least $\left(\frac{3}{2}+\epsilon\right)\Delta$, there is an acceptable colouring of G, with the further property that the colours on every pair of adjacent vertices of G differ by at least $\Delta^{1/4}$.

The $o(1)$ term in the theorem is as $\Delta \longrightarrow +\infty$. The first order term $\frac{3}{2}\Delta$ in Theorem 1.5 is best possible, as the examples in Figure 1 show. On the other hand, the term $\Delta^{1/4}$ is probably far from best possible; it was chosen to keep the proof simple. The main point, to our minds, is that this parameter tends to infinity as $\Delta \longrightarrow +\infty$.

In [2], the first part of Theorem 1.4 is extended to graphs G embeddable in any fixed surface[1]. That paper also considers the more general framework of Σ-colourings, where for each vertex v a subset $\Sigma(v) \subseteq N_G(v)$ of the neighbourhood of v is given, and two vertices u, w only need to receive a different colour if $uw \in E(G)$ or $u, w \in \Sigma(v)$ for some v. This concept unifies ordinary colourings (taking $\Sigma(v) = \varnothing$ for all v) and colourings of the square G^2 (taking $\Sigma(v) = N_G(v)$ for all v). It also includes so-called *cyclic colourings* of graphs that are embedded in a surface, where vertices that share a face must be coloured differently.

Here we extend Theorem 1.4 to every *nice family of graphs*, which are those minor-closed families of graphs such that there is some k for which the complete bipartite graph $K_{3,k}$ is not in the family.

Theorem 1.5
Let $\mathcal{F}$ be a nice family of graphs. The square of every graph G in $\mathcal{F}$ with maximum degree Δ has list chromatic number at most $\left(\frac{3}{2}+o(1)\right)\Delta$. Moreover, given lists of this size, there is a proper colouring in which the colours on every pair of adjacent vertices of G differ by at least $\Delta^{1/4}$.

Kuratowski's theorem tells us that planar graphs form a nice family. So do graphs which are embeddable in a fixed surface. For, by Euler's formula, if a bipartite graph with n vertices and e edges embeds in a surface Σ of Euler genus g, then $e \leq 2(n+g-2)$; and so $K_{3,k}$ does not embed in Σ if $k > 2g+2$.

Note that $K_{3,3}$ has K_4 as a minor, and so K_4-minor-free graphs (that is, series-parallel graphs) form a nice class. Lih, Wang and Zhu [33] showed that the square of a K_4-minor-free

[1]We note that [2] was written after the results in this paper were obtained, due to the lengthy amount of time this paper has spent in the revision process (which is the fault of the authors), and combines the techniques developed in this paper with other arguments

graph with maximum degree Δ has chromatic number at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ if $\Delta \geq 4$ and $\Delta + 3$ if $\Delta = 2, 3$. The same bounds, but then for the list chromatic number of the square of a K_4-minor-free graph, were proved by Hetherington and Woodall [16].

1.2 $L(p, q)$-Labellings of Graphs

Vertex colourings of squares of graphs can be considered a special case of a more general concept: $L(p, q)$-labellings of graphs. This topic takes some of its inspiration from so-called channel assignment problems in radio or cellular phone networks, see for example [32]. The basic channel assignment problem is the following: we need to assign radio frequency channels to transmitters (each gets one channel which corresponds to an integer). In order to avoid interference, if two transmitters are very close, then the separation of the channels assigned to them has to be large enough. Moreover, if two transmitters are close but not very close, then they must also receive channels that are sufficiently far apart.

An idealised version of such a problem may be modelled by $L(p, q)$-labellings of a graph G, where p and q are non-negative integers. The vertices of this graph correspond to the transmitters and two vertices are linked by an edge if they are very close. Two vertices are then considered close if they are at distance two in the graph. Let $\text{dist}(u, v)$ denote the distance between the two vertices u and v. An $L(p, q)$-*labelling of* G is an integer assignment f to the vertex set $V(G)$ such that:

- $|f(u) - f(v)| \geq p$ if $\text{dist}(u, v) = 1$, and
- $|f(u) - f(v)| \geq q$ if $\text{dist}(u, v) = 2$.

It is natural to assume that $p \geq q$, and we do so throughout.

The *span* of f is the difference between the largest and the smallest labels of f plus one. The $\lambda_{p,q}$-*number* of G, denoted by $\lambda_{p,q}(G)$, is the minimum span over all $L(p, q)$-labellings of G.

The problem of determining $\lambda_{p,q}(G)$ has been studied for some specific classes of graphs (see the survey of Yeh [47]). Generalisations of $L(p, q)$-labellings have also been studied in which a minimum gap of p_i is required for channels assigned to vertices at distance i, for several values $i = 1, 2, \ldots$ (see for example [29] or [34]).

Moreover, very often, because of technical reasons or dynamicity, the set of channels available varies from transmitter to transmitter. Therefore one has to consider the list version of $L(p, q)$-labellings. A k-*list assignment* L of a graph is a function which assigns to each vertex v of the graph a list $L(v)$ of k prescribed integers. Given a graph G, the *list* $\lambda_{p,q}$-*number*, denoted $\lambda^l_{p,q}(G)$, is the smallest integer k such that, for every k-list assignment L of G, there exists an $L(p, q)$-labelling f such that $f(v) \in L(v)$ for every vertex v. Surprisingly, list $L(p, q)$-labellings have received very little attention and appear only quite recently in the literature [25]. However, some of the proofs for $L(p, q)$-labellings also work for list $L(p, q)$-labellings.

Note that $L(1, 0)$-labellings of G correspond to ordinary vertex colourings of G and $L(1, 1)$-labellings of G to vertex colourings of the square of G: thus $\lambda_{1,0}(G) = \chi(G)$, $\lambda^l_{1,0}(G) = \text{ch}(G)$, $\lambda_{1,1}(G) = \chi(G^2)$, and $\lambda^l_{1,1}(G) = \text{ch}(G^2)$.

It is well known that for a graph G with clique number ω (the size of a maximum clique in G) and maximum degree Δ we have $\omega \le \chi(G) \le \mathrm{ch}(G) \le \Delta + 1$. Similar easy inequalities may be obtained for $L(p,q)$-labellings:

$$q\,\omega(G^2) - q + 1 \;\le\; \lambda_{p,q}(G) \;\le\; d\lambda^l_{p,q}(G) \;\le\; p\,\Delta(G^2) + 1.$$

(Recall that we assume throughout that $p \ge q$.) As $\omega(G^2) \ge \Delta(G) + 1$, the previous inequality gives $\lambda_{p,q}(G) \ge q\Delta + 1$. However, a straightforward argument shows that in fact we must have $\lambda_{p,q}(G) \ge q\Delta + p - q + 1$. In the same way, $\Delta(G^2) \le (\Delta(G))^2$ so $\lambda^l_{p,q}(G) \le p(\Delta(G))^2 + 1$. The "many-passes" greedy algorithm (see [36]) gives the alternative bound

$$\lambda^l_{p,q}(G) \;\le\; q\Delta(G)(\Delta(G) - 1) + p\Delta(G) + 1 \;=\; q(\Delta(G))^2 + (p-q)\Delta(G) + 1.$$

Because for many large-scale networks the transmitters are laid out on the surface of the earth, $L(p,q)$-labellings of planar graphs are of particular interest. There are planar graphs for which $\lambda_{p,q} \ge \frac{3}{2}q\Delta + c(p,q)$, where $c(p,q)$ is a constant depending on p and q. We already saw some of those examples in Figure 1. The graph G_k has maximum degree $2k$ and yet its square contains a clique with $3k+1$ vertices (all the vertices except z). Labelling the vertices in the clique already requires a span of at least $q \cdot 3k + 1 = \frac{3}{2}q\Delta + 1$.

A first upper bound on $\lambda_{p,q}(G)$, for planar graphs G and positive integers $p \ge q$ was proved by Van den Heuvel and McGuinness [15]: $\lambda_{p,q}(G) \le 2(2q-1)\Delta + 10p + 38q - 24$. Molloy and Salavatipour [38] improved this bound by showing the following.

Theorem 1.6 (Molloy and Salavatipour [38])
For positive integers $p \ge q$, and a planar graph G with maximum degree Δ,

$$\lambda_{p,q}(G) \;\le\; q\left\lceil \tfrac{5}{3}\Delta \right\rceil + 18p + 77q - 18.$$

Moreover, they described an $O(n^2)$ time algorithm for finding an $L(p,q)$-labelling with span at most the bound in their theorem.

As a corollary to our main result Theorem 1.5 we get that, for any fixed p and every nice family $\mathcal{F}$ of graphs, we have $\lambda^l_{p,1}(G) \le (1+o(1))\frac{3}{2}\Delta(G)$ for $G \in \mathcal{F}$. Taking an $L(\lceil p/k \rceil, \lceil q/k \rceil)$-labelling and multiplying each label by k, for some positive integer k, we obtain an $L(p,q)$-labelling. This gives the following corollary.

Corollary 1.7
Let $\mathcal{F}$ be a nice family of graphs and let $p \ge q$ be positive integers. Then for graphs G in $\mathcal{F}$ we have $\lambda_{p,q}(G) \le (1+o(1))\frac{3}{2}q\Delta(G)$.

Note that the examples discussed earlier show that for each positive integer q the factor $\frac{3}{2}q$ is optimal.

2 Nice Families of Graphs

Recall that we call a family $\mathcal{F}$ of graphs *nice* if (a) it is closed under taking minors and (b) there is some k for which $K_{3,k}$ is not in the family. In this section we prove a number of properties of nice families, eventually showing that we obtain an equivalent definition if we replace condition (b) by the following condition:

(c) there is a constant $\beta_{\mathcal{F}}$ such that for any graph $G \in \mathcal{F}$ and any vertex set $B \subseteq V(G)$, if we let A be the set of vertices in $V(G) \backslash B$ which have at least three neighbours in B, then the number of edges between A and B is at most $\beta_{\mathcal{F}}|B|$.

To prove this equivalence, we need the following result.

Theorem 2.1 (Mader [35])
For any graph H, there is a constant C_H such that every H-minor free graph has average degree at most C_H.

In the proof of Theorem 2.1, Mader showed that $C_H \leq c|V(H)| \log |V(H)|$, for some constant c. This upper bound was later lowered independently by Kostochka [27] and Thomason [42] to $C_H \leq c'|V(H)|\sqrt{\log |V(H)|}$, for some constant c'.

Corollary 2.2
Any H-minor-free graph with n vertices has at most $\binom{\lfloor C_H \rfloor}{2} \cdot n$ triangles.

Proof We prove the result by induction on n, the result holding trivially if $n \leq 2$. Let G be an H-minor-free graph with n vertices. By Theorem 2.1, its average degree is at most C_H. So G has a vertex v with degree at most $\lfloor C_H \rfloor$. The vertex v is in at most $\binom{\lfloor C_H \rfloor}{2}$ triangles. By induction, $G - v$ has at most $\binom{\lfloor C_H \rfloor}{2} \cdot (n-1)$ triangles. Hence G has at most $\binom{\lfloor C_H \rfloor}{2} \cdot n$ triangles. $\square$

For an extension of this result see Lemma 2.1 of Norine *et al.* [39].

Theorem 2.3
A class $\mathcal{F}$ of graphs is nice if and only if it is minor-closed and satisfies condition (c).

Proof First suppose that (c) holds for $\mathcal{F}$. By taking B the set of three vertices in $K_{3,k}$ from one part of the bipartition, and A the remaining k vertices, we see that $K_{3,k}$ cannot be in $\mathcal{F}$ for $k > \beta_{\mathcal{F}}$. It follows that every graph in $\mathcal{F}$ is $K_{3,k}$-minor-free if $k > \beta_{\mathcal{F}}$.

Next suppose that $\mathcal{F}$ is a minor-closed family not containing $K_{3,k}$ for some k. We want to prove that (c) holds for $\mathcal{F}$. Note that by Theorem 2.1, the average degree of a $K_{3,k}$-minor-free graph is bounded by some constant C_k.

Let $G \in \mathcal{F}$, let B be a set of vertices of G, and let A be the set of vertices in $V \backslash B$ having at least three neighbours in B. Construct a graph H with vertex set B as follows: For each vertex of A, one after another, if two of its neighbours in B are not yet adjacent in H, choose a pair of those non-adjacent neighbours and add an edge between them.

Let $A' \subseteq A$ be the set of vertices for which an edge has been added to H, and set $A'' = A \backslash A'$. Then H is $K_{3,k}$-minor-free because G was, and hence $|A'| = |E(H)| \le \frac{1}{2} C_k |B|$. Now for every vertex $a \in A''$, the neighbours of a in B form a clique in H (otherwise we would have used a to link two of its neighbours in B). Moreover, k vertices of A'' may not be complete to (that is, adjacent to each vertex of) the same triangle of H, since otherwise G would contain a $K_{3,k}$-minor. Hence $|A''|$ is at most $k-1$ times the number of triangles in H, which is at most $\binom{\lfloor C_H \rfloor}{2}|B|$ by Corollary 2.2. We find that $|A''| \le (k-1)\binom{\lfloor C_H \rfloor}{2}|B|$, and hence $|A| = |A'| + |A''| \le \left(\frac{1}{2}C_k + (k-1)\binom{\lfloor C_H \rfloor}{2}\right)|B|$.

Since the subgraph of G induced on $A \cup B$ is $K_{3,k}$-minor-free, there are at most $\frac{1}{2}C_k(|A|+|B|)$ edges between A and B; that is, at most $\frac{1}{2}C_k\left(\frac{1}{2}C_k + (k-1)\binom{\lfloor C_H \rfloor}{2} + 1\right)|B|$. So we are done with $\beta_{\mathcal{F}} = \frac{1}{2}C_k\left(\frac{1}{2}C_k + (k-1)\binom{\lfloor C_H \rfloor}{2} + 1\right)$. □

3 Overview of the proof of Theorem 1.5

To prove Theorem 1.5, for a fixed nice family $\mathcal{F}$, we need to show that for every $\epsilon > 0$ there is a Δ_ϵ such that for every $\Delta \ge \Delta_\epsilon$ we have: for every graph $G \in \mathcal{F}$ with maximum degree at most Δ, given lists of size

$$\ell^* = \ell^*(\Delta, \epsilon) = \left\lfloor \left(\tfrac{3}{2} + \epsilon\right)\Delta \right\rfloor$$

for each vertex v of G, we can find the desired colouring.

Given a graph G with vertex set V, and $R \subseteq V$, we write $G - R$ for the graph obtained from G by deleting the vertices in R (and any incident edges), and write $G - v$ for $G - \{v\}$. Similarly, we may write $V \backslash v$ for $V \backslash \{v\}$.

We proceed by induction on the number of vertices of G. Our proof is a recursive algorithm. In each iteration, we split off a set R of vertices of the graph which are easy to handle, recursively colour $G^2 - R$ (which we can do by the induction hypothesis), and then extend this colouring to the vertices of R. In extending the colouring, we must ensure that no vertex v in R receives a colour which is either used on a vertex in $V \backslash R$ which is adjacent to v in G^2 or is too close to a colour on a vertex in $V \backslash R$ which is adjacent to v in G. Thus, we modify the list $L(v)$ of colours available for v by deleting those which are forbidden because of such neighbours.

We note that $(G - R)^2$ need not be equal to $G^2 - R$, as there may be non-adjacent vertices of $G - R$ with a common neighbour in R but no common neighbour in $G - R$. When choosing R, we need to ensure that we can construct a graph G_1 in $\mathcal{F}$ on $V \backslash R$ such that $G^2 - R$ is a subgraph of G_1^2. We also need to ensure that the connections between R and $V \backslash R$ are limited, so that the modified lists used when list colouring the induced subgraph $G^2[R]$ are still reasonably large. Finally, we will want $G^2[R]$ to have a simple structure so that we can prove that we can list colour it as desired.

We begin with a simple example of such a set R. We say a vertex v of G is *removable* if it has at most $\Delta^{1/4}$ neighbours in G and at most two neighbours in G which have degree at least $\Delta^{1/4}$. We note that if v is a (removable) vertex with at most one neighbour, then $(G-v)^2$ is $G^2[V\backslash v]$, while if v has exactly two neighbours x and y, then forming G_1 from $G-v$ by adding an edge between x and y if they are not already adjacent, we have that G_1 is in $\mathcal{F}$ and G^2-v is a subgraph of G_1^2. On the other hand, if v is a removable vertex with at least three neighbours, then it must have a neighbour w with degree at most $\Delta^{1/4}$. In this case, the graph G_2 obtained from G–v by adding an edge from w to every other neighbour of v in G is a graph with maximum degree at most Δ such that G^2-v is a subgraph of G_2^2. Furthermore, $G_2 \in \mathcal{F}$ as it is obtained from G by contracting the edge wv.

Thus, for any removable vertex v, we can recursively list colour G^2-v using our algorithm. If, in addition, v has at most $\ell^*-1-2\Delta^{1/2}$ neighbours in G^2, then there will be a colour in $L(v)$ which appears on no vertex adjacent to v in G^2 and is not within $\Delta^{1/4}$ of any colour assigned to a neighbour of v in G. To complete the colouring we give v any such colour.

The above remarks show that no minimal counterexample to our theorem can contain a removable vertex of low degree in G^2. We are about to describe another, more complicated, reduction we will use. It relies on the following easy result.

Lemma 3.1
If R is a set of removable vertices of G, then there is a graph $G_1 \in \mathcal{F}$ with vertex set $V\setminus R$ and maximum degree at most Δ such that G^2-R is a subgraph of G_1^2.

Proof For each $v \in R$ with at least three neighbours in $V\setminus R$, choose one of these neighbours with degree less than $\Delta^{1/4}$ onto which we will contract v. Add an edge between the two neighbours of any vertex in R with exactly two neighbours in $V\setminus R$ (if they are not already adjacent). The degree of a vertex x in the resultant graph G_1 is at most $\max\{\Delta^{1/2}, d_G(x)\}$. $\square$

For any multigraph H, we let H^* be the graph obtained from H by subdividing each edge exactly once. For each edge e of H, we let e^* be the vertex of H^* which we placed in the middle of e and we let E^* be the set of all such vertices. We call this set of vertices corresponding to the edges of H the *core of* H^*.

A *removable copy of* H^* is a subgraph of G isomorphic to H^* such that the vertices of G corresponding to the vertices of the core of H^* are removable, and each vertex of H^* corresponding to a vertex of H (i.e. not in the core) has degree at least $\Delta^{1/4}$ (in G). It follows that if v^*, w^*, e^* are vertices of G corresponding to the vertices v, w and edge $e = vw$ of H, then e^* and all its neighbours other than v^* and w^* have degree at most $\Delta^{1/4}$.

Note that the subgraph J of G^2 induced by the core of some copy of H^* in G contains a subgraph isomorphic to $L(H)$, the line graph of H. So the list chromatic number of J is at least the list chromatic number of $L(H)$. If the copy of H^* is removable, then removing the edges of this copy of $L(H)$ from J yields a graph J' which has as vertex set the core of H^*, and has maximum degree at most $\Delta^{1/2}$. (To see this, note that if v^*, w^*, e^* are as above, and we denote by $\tilde{N}(e^*)$ the set of neighbours in G of e^* other than v^* and w^*, then $|\tilde{N}(e^*)| \leq \Delta^{1/4}$, each vertex in $\tilde{N}(e^*)$ has degree at most $\Delta^{1/4}$ in G, and each neighbour in J' of e^* is in $\tilde{N}(e^*)$ or is a neighbour in G of a vertex in $\tilde{N}(e^*)$.) Thus, the key to list colouring J will be to list colour $L(H)$. Fortunately, list colouring line graphs is much easier than list colouring arbitrary graphs (see e.g. [19, 21, 37]). In particular, using a sophisticated argument due to Kahn [19], we can prove the following lemma which specifies certain sets of removable vertices which we can use to perform reductions.

Given a multigraph H and sets U and W of vertices in H, we let $e_H(U, W)$ denote the number of edges between U and W, with any edge between two vertices in $U \cap W$ counting twice. If the graph H is clear from the context, we may write just $e(U, W)$.

Lemma 3.2
For any $\epsilon > 0$, there exists Δ_ϵ such that the following holds for every graph G with $\Delta = \Delta(G) \geq \Delta_\epsilon$. Suppose R is the core of a removable copy of H^* in G, for some multigraph H, such that for any set X of vertices of H and corresponding set X^* of vertices of the copy of H^*,

$$\sum_{x \in X^*} d_{G-R}(x) \ \leq \ e_H(X, V(H) \backslash X) + \tfrac{1}{30}\epsilon|X|\Delta.$$

Then, given lists of size ℓ^* for every vertex, any acceptable colouring on $G^2 - R$ can be extended to an acceptable colouring of G^2.

The following lemma shows that we will indeed be able to find a removable set of vertices which we can use to perform a reduction.

Lemma 3.3
For any $\epsilon > 0$, there exists Δ_ϵ such that every graph $G \in \mathcal{F}$ with maximum degree at most $\Delta \geq \Delta_\epsilon$ contains at least one of the following:

(a) *a removable vertex v which has degree less than $\frac{3}{2}\Delta + \Delta^{1/2}$ in G^2, or*

(b) *a removable copy of H^* with core R, for some multigraph H which contains an edge and is such that for any set X of vertices of H and corresponding set X^* of vertices of H,*

$$\sum_{x \in X^*} d_{G-R}(x) \ \leq \ e_H(X, V(H) \backslash X) + |X|\, \Delta^{9/10}.$$

Combining Lemmas 3.1, 3.2, and 3.3 and our observations on removing a removable vertex, yields Theorem 1.5 (with $o(1)$ replaced by ϵ), provided that we choose Δ large enough so that $3\Delta^{1/2} + 2 \leq \epsilon\Delta$ (since then $\frac{3}{2}\Delta + \Delta^{1/2} < \ell^* - 1 - 2\Delta^{1/2}$) and $\Delta^{9/10} \leq \frac{1}{30}\epsilon\Delta$.

Thus, we need only prove the last two of these lemmas. The proof of Lemma 3.3 is given in the next section. The proof of Lemma 3.2 is much more complicated and forms the bulk of the paper. We follow the approach developed by Kahn [19] for his proof that the list chromatic index of a multigraph is asymptotically equal to its fractional chromatic number. We need to modify the proof so it can handle our situation in which we have a graph which is slightly more than a line graph and in which we have lists with fewer colours than Kahn permitted. We defer any further discussion to Section 5.

4 Proof of Lemma 3.3: Finding a Reduction

In this section we prove Lemma 3.3. Throughout the section we assume that $\mathcal{F}$ is a nice family of graphs. Since there is a k such that no graph in $\mathcal{F}$ contains $K_{3,k}$ as a minor, Theorem 2.1 implies every graph in $\mathcal{F}$ has average degree at most $C_{\mathcal{F}}$ for some constant $C_{\mathcal{F}}$.

Let G be a graph in $\mathcal{F}$ with vertex set V and maximum degree at most Δ, and let $n = |V|$. We let B be the set of vertices of degree exceeding $\Delta^{1/4}$. Since the average degree of G is at most $C_{\mathcal{F}}$, we have $|B| < \dfrac{C_{\mathcal{F}}\, n}{\Delta^{1/4}}$. Hence, another application of Theorem 2.3 implies that G contains a set R_0 of at least $n - O\Big(\dfrac{n}{\Delta^{1/4}}\Big)$ removable vertices. We note that if a vertex in R_0 is adjacent to a vertex in B with degree less than $\frac{1}{2}\Delta$, or is adjacent to at most one vertex in B, then its total degree in the square G^2 is less than $\frac{3}{2}\Delta + \Delta^{1/2}$ and conclusion (a) of Lemma 3.3 holds. So, we can assume this is not the case.

We let V_0 be the set of vertices of G which have degree at least $\frac{1}{2}\Delta$. Note that $V_0 \subseteq B \subseteq V \backslash R_0$. Since every vertex in R_0 has exactly two neighbours in V_0, the sum of the degrees of the vertices in V_0 is at least $2|R_0|$. This gives $|V_0| \geq \dfrac{2|R_0|}{\Delta} \geq \dfrac{2n}{\Delta} - O\Big(\dfrac{n}{\Delta^{5/4}}\Big)$.

We let S_0 be the set of vertices in V_0 which are adjacent to more than $\Delta^{7/8}$ vertices of $V \backslash R_0$. Since every subgraph of G has average degree at most $C_{\mathcal{F}}$, the total number of edges within $V \backslash R_0$ is $O\Big(\dfrac{n}{\Delta^{1/4}}\Big)$. This implies that $|S_0| = O\Big(\dfrac{n}{\Delta^{9/8}}\Big)$. We set $V_1 = V_0 \backslash S_0$ and note that $|V_1| \geq \dfrac{2n}{\Delta} - O\Big(\dfrac{n}{\Delta^{9/8}}\Big)$. We can conclude that

$$|V_1| \;\geq\; \frac{n}{\Delta} \quad \text{for large enough } \Delta. \tag{1}$$

We let R_1 be the set of vertices in R_0 adjacent to (exactly) two vertices in V_1. So every vertex in $R_0 \backslash R_1$ has one or two neighbours in S_0. By our bound on the size of S_0, this means $|R_0 \backslash R_1| \leq |S_0|\,\Delta = O\Big(\dfrac{n}{\Delta^{1/8}}\Big)$ and hence $|R_1| = n - O\Big(\dfrac{n}{\Delta^{1/8}}\Big)$. By our choice of S_0 we have that $e(V_1, V \backslash R_0) \leq \Delta^{7/8}|V_1|$. (Throughout this proof, $e(U, W)$ means $e_G(U, W)$.) Since every vertex in $R_0 \backslash R_1$ has at most one neighbour in V_1, we have $e(V_1, R_0 \backslash R_1) \leq |R_0 \backslash R_1| = O\Big(\dfrac{n}{\Delta^{1/8}}\Big) \leq O(\Delta^{7/8})|V_1|$, where the final inequality uses (1). We obtain

$$e(V_1, V \backslash R_1) \;=\; e(V_1, V \backslash R_0) + e(V_1, R_0 \backslash R_1) \;\leq\; O(\Delta^{7/8})|V_1|. \tag{2}$$

We let F_1 be the bipartite graph formed by the edges between the vertices of R_1 and the vertices of V_1. We remind the reader that each vertex of R_1 has degree two in this graph. We let H_1 be the multigraph with vertex set V_1 from which F_1 is obtained by subdividing each edge exactly once (so F_1 is a copy of H_1^*).

We check if F_1 is a removable copy of H_1^* as in (b). The only reason that it might not be is that there is some subset $Z_1 \subseteq V_1$ of vertices of H_1 such that:

$$e(Z_1, V \backslash R_1) \;=\; \sum_{v \in Z_1} d_{G-R_1}(v) \;>\; e_{H_1}(Z_1, V_1 \backslash Z_1) + |Z_1|\Delta^{9/10}.$$

In this case, we set $V_2 = V_1 \backslash Z_1$, let R_2 be the set of vertices in R_1 with no neighbours in Z_1, let F_2 be the bipartite subgraph of G formed by the edges between the vertices of R_2 and the vertices of V_2, and let H_2 be the multigraph on V_2 from which F_2 is obtained by subdividing each edge exactly once.

Now we check if F_2 is empty or a removable copy of H_2^* as in (b). If not, we can proceed in the same fashion, deleting a set Z_2 of vertices from V_2 and a set of vertices from R_2, to obtain V_3, R_3 and a new bipartite graph F_3 with parts V_3 and R_3 and corresponding multigraph H_3. We continue this process until it stops. We have constructed new sets $V_1, R_1, V_2, R_2, \dots, \dots V_i, R_i$ such that letting $Z_j = R_j - R_{j+1}$ for $j = 1, \dots, i-1$ we have:

$$e(Z_j, V \backslash R_j) \;=\; \sum_{v \in Z_j} d_{G-R_j}(v) \;>\; e_{H_j}(Z_j, V_j \backslash Z_j) + |Z_j|\Delta^{9/10}. \tag{3}$$

We must show that $R_i \neq \varnothing$, since then the corresponding multigraph H_i has at least one edge and we are done. To this end, we note that for each $j = 1, \dots, i-1$

$$e(V_{j+1}, V \backslash R_{j+1}) \;=\; e(V_j \backslash Z_j, V \backslash R_{j+1}) \;=\; e(V_j, V \backslash R_j) - e(Z_j, V \backslash R_j) + e(V_{j+1}, R_j \backslash R_{j+1}). \tag{4}$$

Furthermore, for every vertex in $R_j \backslash R_{j+1}$ adjacent to a vertex in V_{j+1}, there also is a vertex in Z_j it is adjacent to. Hence $e(V_{j+1}, R_j \backslash R_{j+1})$ is precisely the number of edges of H_j with exactly one endpoint in Z_j: $e(V_{j+1}, R_j \backslash R_{j+1}) = e_{H_j}(Z_j, V_j \backslash Z_j)$. Now (3) and (4) give:

$$e(V_j, V \backslash R_j) \;>\; e(V_{j+1}, V \backslash R_{j+1}) + |Z_j|\Delta^{9/10}.$$

Let $Z' = \bigcup_{j=1}^{i-1} Z_j = V_1 \backslash V_i$. Summing the inequality above over $j = 1, \dots, i-1$ yields: $e(V_1, V \backslash R_1) \geq e(V_i, V \backslash R_i) + |Z'|\Delta^{9/10}$. Using (2), this implies

$$|Z'| \;\leq\; \frac{e(V_1, V \backslash R_1) - e(V_i, V \backslash R_i)}{\Delta^{9/10}} \;\leq\; \frac{O(\Delta^{7/8})\,|V_1|}{\Delta^{9/10}} \;=\; |V_1|\,O(\Delta^{-1/40}).$$

Hence $|V_i| \geq |V_1|(1 - O(\Delta^{-1/40}))$, which also gives

$$|V_1| \;\leq\; (1 + O(\Delta^{-1/40}))|V_i|. \tag{5}$$

Since $V_i \subseteq V_1$, it follows from (2) and (5) that:

$$e(V_i, V \backslash R_0) \;\leq\; e(V_1, V \backslash R_1) \;\leq\; O(\Delta^{7/8})|V_1| \;\leq\; O(\Delta^{7/8})|V_i|.$$

Finally, for each edge between V_i and $R_1 \backslash R_i$, we have an edge between $R_1 \backslash R_i$ and Z' as well. We find

$$e(V_i, R_1 \backslash R_i) \;\leq\; |Z'|\Delta \;\leq\; |V_1|O(\Delta^{39/40}) \;\leq\; O(\Delta^{39/40})|V_i|.$$

Combining these estimates we obtain

$$e(V_i, V \backslash R_i) \;=\; e(V_i, V \backslash R_0) + e(V_i, R_0 \backslash R_1) + e(V_i, R_1 \backslash R_i) \;\leq\; O(\Delta^{39/40})|V_i|.$$

But $V_i \neq \varnothing$, and each vertex in V_i has degree at least $\frac{1}{2}\Delta$. This means that $e(V_i, R_i) > 0$ for large enough Δ. In particular, it follows that R_i is non-empty. Thus, H_i contains an edge. We have shown that (b) holds.

This completes the proof of Lemma 3.3. $\square$

5 Proof of Lemma 3.2: Reducing using Line Graphs

It remains to prove Lemma 3.2, which we do in this section. A removable core satisfying the hypotheses of that lemma corresponds to the edge set of a multigraph H with maximum degree Δ, with each vertex of the core corresponding to a distinct edge. Having coloured $G^2 - R$ for such a removable core R, colouring the induced subgraph $G^2[R]$ translates to finding a list colouring of the line graph of H (where an edge inherits the list assigned to the corresponding vertex of H) so that certain side conditions are satisfied. Firstly, a vertex of R corresponding to an edge $e = vw$ of G may not use (a) a colour assigned to one of its neighbours in the square of G lying in $V - R$, (b) a colour within $\Delta^{1/4}$ of the colours assigned to its neighbours in G lying in $V \backslash R$. Secondly, two vertices of R cannot (c) use the same colour if they have a common neighbour in G which does not correspond to a vertex of H, or (d) use colours within $\Delta^{1/4}$ if they are adjacent in G.

To handle side constraints (a) and (b) we simply delete the forbidden colours from the list for e. Now, because the vertex corresponding to e is removable, it has at most $\Delta^{1/4}$ neighbours which are not v or w, and each such neighbour has degree at most $\Delta^{1/4}$. So even after these deletions, the list for e will have at least ℓ_e^* elements, where

$$\ell_e^* \;=\; \left\lceil \left(\tfrac{3}{2} + \epsilon\right)\Delta - (\Delta - d_H(v)) - (\Delta - d_H(w)) - 3\Delta^{1/2} \right\rceil. \tag{6}$$

To handle side constraints (c) and (d) we use two auxiliary graphs J_1 and J_2. The removability of the vertices of R ensures that these graphs have bounded degree (in terms of Δ). Thus, as we show later, if G is Δ-regular then Lemma 3.2 follows directly from the following result.

Lemma 5.1
For every $0 < \epsilon < \frac{1}{4}$ there is a Δ_ϵ such that the following holds for all $\Delta \geq \Delta_\epsilon$. Let H be a multigraph with vertex set V and maximum degree at most Δ. For every edge e, let $L(e)$ be a list of colours. Let J_1 be a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/2}$, and let J_2 be a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/4}$. Suppose that the following two conditions are satisfied.

(1) For every edge e: $|L(e)| = \ell_e^*$.

(2) For every set X of an odd number of vertices of H:

$$\sum_{v \in X} (\Delta - d(v)) - e(X, V \setminus X) \;\leq\; \tfrac{1}{30}\epsilon |X| \Delta.$$

Then we can find an acceptable edge-colouring of H such that any pair of edges of H joined by an edge of J_1 receive different colours, and any pair of edges of H joined by an edge of J_2 receive colours that differ by at least $\Delta^{1/4}$.

Remark 5.2 Condition (2) of Lemma 5.1 applied to the set $X = \{v\}$ implies that for each vertex v, $d(v) \geq \left(\frac{1}{2} - \frac{1}{60}\epsilon\right)\Delta$. Hence, for each edge e we have $\ell_e^* \geq \frac{1}{2}\Delta + \frac{29}{30}\epsilon\Delta - 3\Delta^{1/2} > \frac{1}{2}\Delta$ if Δ is sufficiently large.

As we show at the end of this section, a relatively simple trick allows us to reduce to the case when G is regular. So the bulk of the work in the section is in proving Lemma 5.1: we complete this task at the end of Subsection 5.4.

The way we prove Lemma 5.1 is by exploiting some beautiful work of Kahn, developed to show that, letting $\chi'_f(H)$ be the fractional chromatic index of H, we have that the list chromatic index of H is $(1+o(1))\chi'_f(H)$. We will do two things: (i) explain why Lemma 5.1 follows from Kahn's proof in the special case when J_1 and J_2 are empty (that is, J_1 and J_2 have no edges, and so are irrelevant), and (ii) discuss the modifications needed to Kahn's proof to deal with J_1 and J_2.

Kahn's proof analyses an iterative procedure which in each iteration, for each colour γ, randomly extends a matching in the spanning subgraph H_γ of H whose edges are those on which γ is available, to progressively colour more and more of the edges of H. The first step of his proof is to show that if each list has at least $(1+o(1))\chi'_f(H)$ elements, then there is a probability distribution on these matchings which ensures that: (a) for every edge e and colour γ in $L(e)$, the probability that e is in the matching of colour γ is $|L(e)|^{-1}$, and (b) other desirable properties hold. The second step is to show that for any family of lists for which there are probability distributions satisfying (a) and (b), this iterative procedure yields a colouring of $E(H)$ where each edge gets a colour from its own list.

It is natural to state Kahn's result precisely, before discussing our modification of it. Having done so, before delving into the details of Kahn's proof, we will show that for lists satisfying the conditions of Lemma 5.1 there are probability distributions on the matchings satisfying (a) and (b) above. We can then apply Kahn's work, as a black box, to prove Lemma 5.1 in the special case when J_1 and J_2 are empty.

We then turn to strengthening the result so that it can deal with J_1 and J_2. This has two parts. First we perform some straightforward preprocessing which allows us to reserve some colours which can be used to recolour vertices involved in conflicts caused by J_1 and J_2. Then we impose additional constraints which provide, for each iteration, an upper bound on the number of edges incident to each vertex which are involved in such conflicts because of a colour they are assigned in that iteration. Here we must get into the guts of Kahn's proof sufficiently, so as to be able to explain the (relatively straightforward) additions to it which allow us to do so. Then in a postprocessing phase, we recolour to eliminate such conflicts using the colours we reserved in the first phase.

We will actually discuss this preprocessing and postprocessing first. We do this in part because all of the rest of the discussion involves Kahn's proof, while the pre/postprocessing does not, so it is natural to hive it off; and in part because our discussion of the preprocessing introduces the Lovász Local Lemma, an important tool in Kahn's proof, in a simple setting.

5.1 Before and After

Our preprocessing consists of applying the following lemma (which we prove in this subsection).

Lemma 5.3
Suppose we are given a multigraph H with maximum degree Δ sufficiently large, satisfying the conditions of Lemma 5.1. Then we can find, for every list $L(e)$, two disjoint sublists $L'(e)$ and $R(e)$ such that:

(a) *no colour in $R(e)$ appears in $L'(f)$ for any edge f incident to e in H;*

(b) $|L'(e)| \geq |L(e)| - \frac{2}{3}\epsilon\Delta$; *and*

(c) $|R(e)| \geq \Delta^{9/10}$.

We then apply the following variant (to be proved later) of Lemma 5.1 to the family $L'(e)$ of lists, using $\frac{1}{2}\epsilon$ in place of ϵ. In this variant, condition (2) is weakened by replacing 30 by 10, and the conclusions are weakened by allowing some of the graph to remain uncoloured. We can apply this amended lemma because of our bound on $|L(e) - L'(e)|$ and the fact that conditions (1) and (2) of Lemma 5.1 hold before the preprocessing.

Lemma 5.4
For every $0 < \epsilon < \frac{1}{4}$ there is a Δ_ϵ such that the following holds for all $\Delta \geq \Delta_\epsilon$. Let H be a multigraph with vertex set V and maximum degree at most Δ. For every edge e we are given a list $L(e)$ of acceptable colours. Additionally, J_1 is a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/2}$ and J_2 is a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/4}$. Suppose that the following two conditions are satisfied.

(1) *For every edge e: $|L(e)| = \ell^*_e$.*

(2) *For every set X of an odd number of vertices of H:*

$$\sum_{v \in X} (\Delta - d(v)) - e(X, V \setminus X) \;\leq\; \tfrac{1}{10}\epsilon|X|\Delta.$$

Then we can find an acceptable edge-colouring of H such that by uncolouring a set of edges of H, including at most $\frac{1}{3}\Delta^{9/10}$ edges incident to any vertex v of H, we obtain a partial colouring such that any two coloured edges joined by an edge of J_1 receive different colours, and any two coloured edges joined by an edge of J_2 receive colours that differ by at least $\Delta^{1/4}$.

Remark 5.5 Much as in Remark 5.2, condition (2) in Lemma 5.4 gives $d(v) \geq \left(\frac{1}{2} - \frac{1}{20}\epsilon\right)\Delta$, and so $\ell^*_e > \frac{1}{2}\Delta$ for each edge e, for sufficiently large Δ.

To complete the proof of Lemma 5.1 (assuming Lemmas 5.3 and 5.4), we uncolour edges as specified in Lemma 5.4, and then recolour each such edge e using a colour from its reserve list $R(e)$. By conclusion (a) of Lemma 5.3, this colour cannot conflict with the colour of any edge incident to e which was not uncoloured. So, in colouring e we must avoid any colour from $R(e)$ assigned to an edge incident to it which we have uncoloured (and re-coloured), avoid any colour assigned to a neighbour in J_1, and avoid any colour within $\Delta^{1/4}$ of neighbours in J_2. But in total there are at most $3\Delta^{1/2} + \frac{2}{3}\Delta^{9/10}$ colours to avoid, so if Δ is large enough we can carry out the recolouring greedily.

So, to prove Lemma 5.1 it is enough to prove Lemmas 5.3 and 5.4. In the remainder of this section we prove Lemma 5.3; the proof of Lemma 5.4 will not be complete until the end of Section 5.4.

The key to the proof Lemma 5.3 is the following general lemma.

Lemma 5.6 (Erdős and Lovász [11]) (Local Lemma)
Suppose that $\mathcal{B}$ is a set of (bad) events in a probability space Ω. Suppose further that there are p and d such that:

(1) *for every event B in $\mathcal{B}$, there is a subset $\mathcal{S}_B$ of $\mathcal{B}$ of size at most d, such that the conditional probability of B, given any conjunction of occurrences or non-occurrences of events in $\mathcal{B}\setminus\mathcal{S}_B$, is at most p, and*

(2) $\mathrm{e}pd < 1$.

Then with positive probability, none of the events in $\mathcal{B}$ occur.

In our preprocessing step, we apply the Local Lemma to the (product) probability space obtained by, for each colour c and vertex v, independently assigning c to a list $R(v)$ with probability $\frac{1}{6}\epsilon$.

For an edge e with endpoints u and v, we set $R(e) = L(e) \cap R(u) \cap R(v)$ and $L'(e) = L(e) - (R(u) \cup R(v))$. Note that the sublists $R(e)$ and $L'(e)$ defined in this way must satisfy condition (a) of Lemma 5.3. We shall now prove that, with positive probability, conditions (b) and (c) are also satisfied for all edges e. Let B_e be the event that condition (b) is not satisfied for the edge e, i.e. $|L(e) \setminus L'(e)| = |L(e) \cap (R(u) \cup R(v))| > \frac{2}{3}\epsilon\Delta$. Let C_e be the event that $|R(e)| < \frac{1}{100}\epsilon^2\Delta$ for the edge e. For sufficiently large Δ, $\frac{1}{100}\epsilon^2\Delta \geq \Delta^{9/10}$, so if C_e does not hold then condition (c) is satisfied for edge e.

To conclude the proof of Lemma 5.3, we apply the Local Lemma to show that with positive probability none of the bad events B_e, C_e occurs. Now B_e and C_e are determined completely by the random assignments made at the endpoints of e, so letting $S_{B_e} = S_{C_e} = \{B_f, C_f \mid f = e \text{ or } f \text{ is incident to } e\}$, we see that condition (1) of the Local Lemma holds with $d = 4\Delta - 2$ and p the maximum of the unconditional probability of B_e and the unconditional probability of C_e.

Now, for any edge e with endpoints u and v, the number of colours in $R(e) = L(e) \cap R(u) \cap R(v)$ is the sum of $|L(e)|$ independent 0-1 variables, each of which is 1 with probability $\frac{1}{36}\epsilon^2$. So, the expected value of this random variable is $\frac{1}{36}\epsilon^2\ell_e^*$; and by Remark 5.5, for large Δ, this is at least $\frac{1}{72}\epsilon^2\Delta$. Standard concentration inequalities (e.g. the Chernoff bounds) tell us that the probability that this variable differs from its expected value by some $t > 0$ which is less than its expected value is $2^{-\Omega(t^2/\Delta)}$. So, the probability of C_e is $2^{-\Omega(\Delta)}$.

In the same vein, for any edge e with endpoints u and v, the number of colours in $L(e) \cap (R(u) \cup R(v))$ is the sum of $|L(e)|$ independent 0-1 variables, each of which is 1 with probability at most $\frac{2}{6}\epsilon$. We obtain that the expected value of this random variable is at most $\frac{1}{3}\epsilon\ell_e^*$, which is at most $\frac{1}{3}\epsilon\big(\frac{3}{2} + \epsilon\big)\Delta < \frac{7}{12}\epsilon\Delta$. Again applying standard concentration inequalities, we see that the probability of B_e is $2^{-\Omega(\Delta)}$.

Thus for large Δ the hypotheses of the Local Lemma hold with $p = {}^1\!/_{3d}$, and we have completed the proof of Lemma 5.3. $\square$

5.2 Kahn's Result as a Black Box

Kahn presents an algorithm in [19] which shows that the list chromatic index of a multigraph exceeds its fractional chromatic index by $o(\Delta)$. Actually, the algorithm implicitly contains a subroutine which does more than this, providing a proof (which we shall describe later, see Subsection 5.3.2) of the following result.

Theorem 5.7 (Kahn [19])
For every δ with $0 < \delta < 1$ and every $C > 0$, there exists a $\Delta_{\delta,C}$ such that the following holds for all $\Delta \geq \Delta_{\delta,C}$. Let H be a multigraph with maximum degree at most Δ, and with a list $L(e)$ of acceptable colours for every edge e. Suppose that the following conditions are satisfied:

(1) For every vertex v and edge e incident to v, $|L(e)| \geq d(v)(1+\delta)$.

(2) For every odd set X of vertices of H, the sum of $z_e = \dfrac{1+\delta}{|L(e)|}$ over the edges joining vertices of X is at most $\frac{1}{2}(|X|-1)$.

(3) For every edge e: $|L(e)| \geq {}^{\Delta}\!/_{C}$.

Then we can find an acceptable edge-colouring of H.

This theorem is not explicitly stated in Kahn's paper, although it follows in just a few pages from the proof of [19, Lemma 3.1], which forms the bulk of his paper. We pull the result out of his discussion, after showing that Theorem 5.7 implies the special case of Lemma 5.4 when J_1 and J_2 are empty (and we have just the simple conclusion "*Then we can find an acceptable edge-colouring of H.*").

To prove this implication, we need only show that for $0 < \epsilon < \frac{1}{4}$ and sufficiently large Δ, any family of lists satisfying conditions (1) and (2) of Lemma 5.4 must satisfy conditions (1) and (2) of Theorem 5.7 for $\delta = \frac{1}{2}\epsilon$, since we noted in Remark 5.5 that, for sufficiently large Δ, $\ell_e^* > \frac{1}{2}\Delta$ for each edge e, so condition (3) holds with $C = 2$. This is the content of the following lemma.

Lemma 5.8
Let $0 < \epsilon < \frac{1}{4}$. Then there is a Δ_ϵ such that for every $\Delta \geq \Delta_\epsilon$ the following holds. Let H be a multigraph with vertex set V and maximum degree at most Δ, and for every edge e let $L(e)$ be a list of acceptable colours. Suppose the following conditions are satisfied:

(1) For every edge e: $|L(e)| \geq \ell_e^*$.

(2) For every set X of an odd number of vertices of H:

$$\sum_{v \in X} (\Delta - d(v)) - e(X, V \setminus X) \;\leq\; \tfrac{1}{10}\epsilon |X| \Delta.$$

Then, the following properties hold:

(a) *For every vertex v and edge e incident to v: $|L(e)| \geq \big(1 + \frac{1}{2}\epsilon\big)d(v)$.*

(b) *For every odd set X of vertices of H: the sum of $z_e = \dfrac{1+\frac{1}{2}\epsilon}{|L(e)|}$ over the edges joining vertices of X is at most $\frac{1}{2}(|X|-1)$.*

Proof Whenever an inequality requires Δ to be large enough, we use "$\geq_*$".

To begin, we note that condition (2) of the lemma implies that every vertex w of H has degree at least $\big(\frac{1}{2} - \frac{1}{20}\epsilon\big)\Delta$. Hence, for any edge $e = vw$ of H, condition (1) of the lemma implies that $|L(e)| \geq d(v) + \frac{19}{20}\epsilon\Delta - 3\Delta^{1/2} \geq_* d(v) + \frac{3}{4}\epsilon\Delta \geq \big(1 + \frac{3}{4}\epsilon\big)d(v)$. Thus property (a) holds.

Now we check property (b) for the case $|X| = 3$. Consider a subgraph F of H with vertex set X consisting of three distinct vertices x, y, z, and with $\alpha\Delta > 0$ edges. Note that $\alpha \leq \frac{3}{2}$, since $2|E(F)| \leq d(x)+d(y)+d(z) \leq 3\Delta$ (where $d(\cdot)$ refers to the degree in the whole graph H). Applying the second condition gives

$$3\Delta - d(x) - d(y) - d(z) \ \leq \ e(X, V\setminus X) + \tfrac{3}{10}\epsilon\Delta.$$

Since we also have $3\Delta - d(x) - d(y) - d(z) = 3\Delta - 2\alpha\Delta - e(X, V\setminus X)$, we obtain

$$3\Delta - d(x) - d(y) - d(z) \ \leq \ \Big(\tfrac{1}{2}\big(3 + \tfrac{3}{10}\epsilon\big) - \alpha\Big)\Delta,$$

which we can rewrite as

$$\tfrac{3}{2}\Delta \ \geq \ 3\Delta - d(x) - d(y) - d(z) - \tfrac{3}{20}\epsilon\Delta + \alpha\Delta.$$

Substituting this into the first condition of the lemma yields that for any edge $e = uv$ in F:

$$|L(e)| \ \geq \ \Delta + (d(u) + d(v) - d(x) - d(y) - d(z)) + \big(\alpha + \tfrac{17}{20}\epsilon\big)\Delta - 3\Delta^{1/2}.$$

Since $\Delta - d(w)$ is non-negative for any w in X, and $\{u, v\} \subset \{x, y, z\}$, this yields

$$|L(e)| \ \geq \ \big(\alpha + \tfrac{17}{20}\epsilon\big)\Delta - 3\Delta^{1/2} \ \geq_* \ \big(\alpha + \tfrac{3}{4}\epsilon\big)\Delta.$$

Since $\alpha \leq \frac{3}{2}$, this gives that for any edge e in F, $z_e \leq \dfrac{1+\frac{1}{2}\epsilon}{\big(\alpha + \frac{3}{4}\epsilon\big)\Delta} \leq \dfrac{1}{\alpha\Delta}$.

We can conclude that $\sum_{e\in E(F)} z_e \leq (\alpha\Delta)\cdot\dfrac{1}{\alpha\Delta} = 1$. This shows that property (b) holds for all sets X of three vertices.

Given a multigraph G and a vertex v, we write $E_{G,v}$, or simply E_v, for the set of edges incident to v. Next consider any subgraph F of H with vertex set X, where $|X| \geq 5$ is odd. Throughout the rest of the proof, for each vertex v of F we write E_v for $E_{F,v}$. We partition the vertices of F into a set B of vertices with degree at least $\frac{3}{4}\Delta$ and a set S of vertices with degree less than $\frac{3}{4}\Delta$ (where degrees are in H).

Case 1: There is a vertex in B with degree at most $\frac{7}{8}\Delta$, or a vertex in S with degree at most $\frac{5}{8}\Delta$.

For any edge $e = vw$ with $w \in B$, applying the first condition of the lemma, we obtain $|L(e)| \geq d(v) + \frac{1}{4}\Delta + \epsilon\Delta - 3\Delta^{1/2} \geq_* \left(1 + \frac{1}{2}\epsilon\right)\frac{5}{4}d(v)$. Thus, $z_e \leq \dfrac{4}{5d(v)}$. Since (a) holds, we have that $z_e \leq {}^1\!/_{d(v)}$ for all edges e incident to v, and hence for each vertex $v \in B$:

$$\sum_{e \in E_v} z_e \;\leq\; \frac{4}{5d(v)}|E_v| + \frac{1}{5d(v)}e(\{v\}, S);$$

while for each vertex v in S:

$$\sum_{e \in E_v} z_e \;\leq\; \frac{1}{d(v)}|E_v| - \frac{1}{5d(v)}e(\{v\}, B).$$

We estimate, using that the vertices in S have smaller degree than the vertices in B,

$$\begin{aligned}
2\sum_{e \in E(F)} z_e \;&=\; \sum_{v \in X}\sum_{e \in E_v} z_e \\
&\leq\; \sum_{v \in B}\frac{4}{5d(v)}|E_v| + \sum_{v \in S}\frac{1}{d(v)}|E_v| + \sum_{\substack{e \in E(F) \\ e = vw,\, v \in B,\, w \in S}} \left(\frac{1}{5d(v)} - \frac{1}{5d(w)}\right) \\
&\leq\; \sum_{v \in B}\frac{4}{5d(v)}|E_v| + \sum_{v \in S}\frac{1}{d(v)}|E_v| \\
&\leq\; \tfrac{4}{5}|B| + |S| - \tfrac{4}{5}e(X, V \setminus X)\frac{1}{\Delta}.
\end{aligned}$$

Also, applying the second condition of the lemma and the assumption for this Case 1, we see that

$$e(X, V \setminus X) \;\geq\; \tfrac{1}{4}\Delta|S| + \tfrac{1}{8}\Delta - \tfrac{1}{10}\epsilon|X|\Delta.$$

Combining the two estimates, we obtain

$$2\sum_{e \in E(F)} z_e \;\leq\; \tfrac{4}{5}|B| + \tfrac{4}{5}|S| - \tfrac{1}{10} + \tfrac{2}{25}\epsilon|X| \;=\; |X|\left(\tfrac{4}{5} + \tfrac{2}{25}\epsilon\right) - \tfrac{1}{10}.$$

Since $\epsilon \leq \frac{1}{4}$ and $|X| \geq 5$, this yields that $2\sum\limits_{e \in E(F)} z_e \leq |X| - \frac{9}{50}|X| - \frac{1}{10} \leq |X| - 1$, as required for property (b).

Case 2: Every vertex in B has degree at least $\frac{7}{8}\Delta$ and every vertex in S has degree at least $\frac{5}{8}\Delta$.

Applying the first condition of the lemma as in Case 1, we see that for an edge e with endvertices v and w, we have $|L(e)| \geq d(v) + \frac{1}{8}\Delta + \epsilon\Delta - 3\Delta^{1/2} \geq_* \left(1 + \frac{1}{2}\epsilon\right) \cdot \frac{9}{8}d(v)$, and if $v \in B$, then we get $|L(e)| \geq d(v) + \frac{3}{8}\Delta + \epsilon\Delta - 3\Delta^{1/2} \geq_* \left(1 + \frac{1}{2}\epsilon\right) \cdot \frac{11}{8}d(v)$. So, for each vertex $v \in B$ we have

$$\sum_{e \in E_v} z_e \;\leq\; \frac{8}{11d(v)}|E_v| + \frac{16}{99d(v)}e(\{v\}, S);$$

while for each vertex v in S we can write

$$\sum_{e \in E_v} z_e \;\le\; \frac{8}{9d(v)}|E_v| - \frac{16}{99d(v)}e(\{v\}, B).$$

Following the same method as in Case 1, this leads to

$$\begin{aligned} 2\sum_{e \in E(F)} z_e \;=\; \sum_{v \in X}\sum_{e \in E_v} z_e \;&\le\; \sum_{v \in B}\frac{8}{11d(v)}|E_v| + \sum_{v \in S}\frac{8}{9d(v)}|E_v| \\ &\le\; \tfrac{8}{11}|B| + \tfrac{8}{9}|S| - \tfrac{8}{11}e(X, V\setminus X)\frac{1}{\Delta}. \end{aligned}$$

Also, applying the second condition of the lemma, we see that

$$e(X, V\setminus X) \;\ge\; \tfrac{1}{4}\Delta|S| - \tfrac{1}{10}\epsilon|X|\Delta.$$

Since $\frac{8}{9} - \frac{2}{11} < \frac{8}{11}$, we obtain

$$2\sum_{e \in E(F)} z_e \;\le\; |X|\big(\tfrac{8}{11} + \tfrac{8}{110}\epsilon\big).$$

Since $\epsilon < 1$ and $|X| \ge 5$, this yields that $2\sum\limits_{e \in E(F)} z_e \le \frac{4}{5}|X| \le |X| - 1$, as required. □

5.3 Opening the Lid

In this section, we discuss how Theorem 5.7 is implicitly proved in Kahn's paper. First however, we need to introduce the special type of probability distributions he considers.

5.3.1 Hard-core Probability Distributions on Matchings

For a probability distribution p, defined on the matchings of a multigraph H, we let $x^p(e)$ be the probability that e is in a matching chosen according to p. We call the value of $x^p(e)$ the *marginal of* p *at* e. The vector $x^p = (x^p(e))$ indexed by the edges e is called the *marginal of* p.

We are actually interested in using special types of probability distributions on the matchings of H. A probability distribution p on the matchings of H is *hard-core* if it is obtained by associating a non-negative real $\lambda^p(e)$ to each edge e of H so that the probability that we pick a matching M is proportional to $\prod_{e \in M} \lambda^p(e)$. I.e. setting $\lambda^p(M) = \prod_{e \in M} \lambda^p(e)$ and letting $\mathcal{M}(H)$ be the set of matchings of H, we have

$$p(M) \;=\; \frac{\lambda^p(M)}{\sum\limits_{N \in \mathcal{M}(H)} \lambda^p(N)}.$$

We call the values $\lambda^p(e)$ the *activities of* p.

We want to characterise for which families of lists, a multigraph H with maximum degree at most Δ has a hard-core probability distribution p on its matchings such that we have (i) $x^p(e) = |L(e)|^{-1}$ for each edge e, and (ii) for some $K > 0$, $\lambda^p(e) \le K/\Delta$ for each edge e.

Finding an arbitrary probability distribution on the matchings of H with marginals x is equivalent to expressing x as a convex combination of incidence vectors of matchings of H. So, we can use a seminal result due to Edmonds [10] to understand for which x this is possible.

The *matching polytope* $\mathcal{MP}(H)$ is the set of non-negative vectors x indexed by the edges of H which are convex combinations of incidence vectors of matchings.

Theorem 5.9 (Edmonds [10]) (Characterisation of the Matching Polytope)
For a multigraph H, a non-negative vector $x = (x_e : e \in E(H))$ is in $\mathcal{MP}(H)$ if and only if

(1) for every vertex v of H: $\sum_{e \in E_v} x_e \le 1$, and

(2) for every set X of vertices of H with $|X| \ge 3$ and odd: $\sum_{e \in E(X)} x_e \le \frac{1}{2}(|X| - 1)$.

Remark 5.10 It is easy to see that conditions (1) and (2) are necessary as they are satisfied by all the incidence vectors of matchings and hence by all their convex combinations. It is the fact that they are sufficient which makes the theorem so valuable.

It turns out that we can choose a hard-core distribution with marginals x provided all of the above inequalities are strict.

Lemma 5.11 (Lee [31]; Rabinovitch, Sinclair and Widgerson [40])
For a multigraph H, there is a hard-core distribution with marginals a given non-negative vector $x = (x_e : e \in E(H))$ if and only if

(a) for every vertex v of H: $\sum_{e \in E_v} x_e < 1$, and

(b) for every set X of vertices of H with $|X| \ge 3$ and odd: $\sum_{e \in E(X)} x_e < \frac{1}{2}(|X| - 1)$.

In order to ensure that the λ^p are bounded, it turns out that we just have to bound our distance from the boundary of the Matching Polytope.

Lemma 5.12 (Kahn and Kayll [20])
For all δ with $0 < \delta < 1$, there is a β such that, for every multigraph H, if p is a hard-core distribution whose marginals are in $(1-\delta)\mathcal{MP}(H)$, then

(a) for every edge e of H: $\lambda^p(e) < \beta x^p(e)$, and

(b) for every vertex v of H: $\sum_{e \in E_v} \lambda^p(e) < \beta$.

The material presented in this subsection is discussed in fuller detail in [37, Chapter 22].

5.3.2 The Proof of Theorem 5.7

As we are about to show, we can prove Theorem 5.7 by combining the results of the last section with the following result, which is also implicit in [19], but much easier to pull out of it. Recall that for a multigraph H with a list $L(e)$ of acceptable colours for every edge e, for each colour γ, we let H_γ be the spanning subgraph of H with edges those e such that $\gamma \in L(e)$.

Theorem 5.13 (Kahn [19])
For every δ with $0 < \delta < 1$ and every $K > 0$, there exists a $\Delta_{\delta,K}$ such that the following holds for all $\Delta \geq \Delta_{\delta,K}$. Let H be a multigraph with maximum degree at most Δ, and with a list $L(e)$ of acceptable colours for every edge e.

Suppose that for every colour γ there exists a hard-core distribution p_γ on the matchings of H_γ, with corresponding marginal x^{p_γ} on the edges, satisfying the following conditions:

(1) For every edge e: $\sum_{\gamma \in L(e)} x^{p_\gamma}(e) = 1$.

(2) For every edge e and colour γ: $\lambda^{p_\gamma}(e) \leq {}^{K}/_{\Delta}$.

Then we can find an acceptable edge-colouring of H.

Proof of Theorem 5.7, assuming Theorem 5.13 Consider a multigraph H and family of lists satisfying the conditions of Theorem 5.7. For each colour γ consider the vector x^{p_γ} indexed by the edges of H_γ, where $x_e^{p_\gamma} = |L(e)|^{-1}$. Then condition (1) of Theorem 5.13 holds. Conditions (1) and (2) of Theorem 5.7, combined with Edmonds's characterisation of the matching polytope, tell us that x^{p_γ} is in $(1-\delta)\mathcal{MP}(H_\gamma)$. Now let β be as in Lemma 5.12. Then there is a hard-core distribution on H_γ with marginals x^{p_γ} such that, for every edge e and colour γ, we have $\lambda^{p_\gamma}(e) \leq \beta x^{p_\gamma}(e)$. Thus, setting $K = \beta C$, by condition (3) of Theorem 5.7, for every edge e and colour γ, we have $\lambda^{p_\gamma}(e) \leq {}^{K}/_{\Delta}$. Hence condition (2) of Theorem 5.13 holds, and we can apply that result to complete the proof. $\square$

The proof of Kahn's main theorem, [19, Theorem 1.1], demonstrates that we can obtain an acceptable edge-colouring for a given family of lists on the edges of a multigraph H with maximum degree at most Δ, by first showing that there are hard-core distributions with marginals $|L(e)|^{-1}$ in each H_γ which satisfy the hypotheses of [19, Lemma 3.1] (this is done in the second paragraph of [19, page 127]), and then iteratively applying this lemma to reach a situation where we can finish off greedily.

To prove Theorem 5.13 following exactly the same scheme, we need simply ensure that hard-core distributions satisfying the hypotheses of [19, Lemma 3.1] with marginals $|L(e)|^{-1}$ at e exist for our family of lists. But the hypotheses of [19, Lemma 3.1] are precisely that conditions (1) and (2) of Theorem 5.13 hold, and thus we have established Theorem 5.13.

5.4 Modifying Kahn's Result

In this section, we will modify Kahn's result so that by taking J_1 and J_2 into account it proves Lemma 5.4. In order to do so, we consider the modification of Theorem 5.13, obtained by:

(i) Adding at the end of the first paragraph of that theorem:
"*Suppose furthermore that J_1 is a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/2}$, J_2 is a graph with vertex set $E(H)$ and maximum degree at most $\Delta^{1/4}$, and every list $L(e)$ has at most 2Δ elements.*"

(ii) And adding at the end of the last sentence of the theorem:
"*so that we can uncolour a set of edges of H containing at most $\frac{1}{3}\Delta^{9/10}$ edges incident to any vertex v of H, to obtain a partial edge-colouring of H such that any two coloured edges joined by an edge of J_1 receive different colours, and such that any two coloured edges joined by an edge of J_2 receive colours that differ by at least $\Delta^{1/4}$.*"

We call this strengthening Our Theorem. We first show that it implies (the full version of) Lemma 5.4 and then discuss its proof.

We set $\delta = \frac{1}{2}\epsilon$, let β be the corresponding value from Lemma 5.12, and define Δ_ϵ to be $\Delta_{\delta,2\beta}$ (as in Our Theorem). We set $x_e^{p_\gamma} = |L(e)|^{-1}$ for each colour γ and edge e in H_γ, and $x_e^{p_\gamma} = 0$ if e is not in H_γ. Thus, for each edge e we have that $\sum_\gamma x_e^{p_\gamma} = 1$. Applying Lemma 5.8 together with Theorem 5.9, we see that each of the edge-vectors x^{p_γ} is in $(1-\delta)\mathcal{MP}(H)$ and hence in $(1-\delta)\mathcal{MP}(H_\gamma)$. Now Lemmas 5.11 and 5.12 show that there are hard-core distributions on H_γ with marginals x^{p_γ} such that, for every edge e and colour γ, we have $\lambda^{p_\gamma}(e) \le \beta x_e^{p_\gamma}$. Since, as we saw in Remark 5.5, for every edge e we have $|L(e)| \ge \frac{1}{2}\Delta$ (for Δ sufficiently large), setting $K = 2\beta$, for every edge e and colour γ, we have $\lambda^{p_\gamma}(e) \le {}^{K}/_{\Delta}$. Hence, conditions (1) and (2) of Our Theorem hold, and applying that result proves Lemma 5.4.

The key to Kahn's proof of Theorem 5.13 above is the following lemma, [19, Lemma 3.1].

Lemma 5.14 (Kahn [19])
For every $K, \delta > 0$, there exist $\xi = \xi_{\delta,K}$ with $0 < \xi \le \delta$ and $\Delta_{\delta,K}$ such that the following holds for all $\Delta \ge \Delta_{\delta,K}$. Let H be a multigraph with maximum degree at most Δ, and with a list $L(e)$ of acceptable colours for every edge e. Define the graphs H_γ as before.

Suppose that for every colour γ we are given a hard-core distribution p_γ on the matchings of H_γ with activities $\lambda^{p_\gamma} = \lambda_\gamma$ and marginals $x^{p_\gamma} = x_\gamma$, satisfying:

(1) for every edge e: $\sum_{\gamma \in L(e)} x_\gamma(e) > \mathrm{e}^{-\xi}$, and

(2) for every colour γ and edge e: $\lambda_\gamma(e) \le {}^{K}/_{\Delta}$.

Then there are matchings M_γ in H_γ for every colour γ, such that the following holds. If we set $H' = H - \bigcup_{\gamma^*} M_{\gamma^*}$ and $H'_\gamma = H_\gamma - V(M_\gamma) - \bigcup_{\gamma^*} M_{\gamma^*}$, we form a list $L'(e)$ for every edge e in H' by removing no longer allowed colours from $L(e)$, and we let x'_γ be the marginals corresponding to the activities λ_γ on H'_γ, then we have:

(a) for every edge e of H': $\sum_{\gamma \in L'(e)} x'_\gamma(e) > \mathrm{e}^{-\delta}$, and

(b) the maximum degree of H' is at most $\frac{1+\delta}{1+\xi}\,\mathrm{e}^{-1}\Delta$.

Here is a sketch of how we may use Lemma 5.14 to prove Theorem 5.13, following Kahn. First fix a suitable number s of iterations, where we take $s = \lceil \log(8K) \rceil$. Let $\delta_s = 1$, and define

$\delta_{s-1} \geq \cdots \geq \delta_1 > 0$ by setting $\delta_{i-1} = \xi_{\delta_i,K}$. Also, let $\delta_0 = 0$ and $\Delta^* = \Delta_{\delta_1,K}$. We start with a multigraph H with $\Delta \geq \mathrm{e}^s\Delta^*$, and with lists of acceptable colours and distributions satisfying the hypotheses in Theorem 5.13. For $i = 0, 1, \ldots, s$ let $\Delta_i = (1+\delta_i)\mathrm{e}^{-i}\Delta$ (so $\Delta_0 = \Delta$ and each $\Delta_i \geq \Delta^*$). Set $H^0 = H$, and $H^0_\gamma = H_\gamma$ for all γ. Once we have obtained H^{i-1} and H^{i-1}_γ, in iteration i we do the following.

I. Choose matchings M^i_γ in H^{i-1}_γ (for each colour γ) according to the lemma, with δ as δ_{i-1} and Δ as Δ_{i-1}.

II. For each edge e in some matching M^i_γ, chose γ independently and uniformly at random from those γ for which $e \in M^i_\gamma$, and assign colour γ to e.

III. Form H^i by removing from H^{i-1} all edges that were assigned a colour in step II. For each colour γ, form H^i_γ by removing from H^{i-1}_γ all edges that were assigned some colour in step II, and all vertices that are incident to any edge that was assigned colour γ in step II.

The key point is that the lemma ensures that if its hypotheses hold for H^{i-1} with δ as δ_{i-1} and Δ as Δ_{i-1}, then they hold for H^i with δ as δ_i and Δ as Δ_i. So we can indeed follow Kahn and iteratively apply the lemma in this way for s iterations, and colour all but an uncoloured subgraph with maximum degree at most $\Delta_s = 2\mathrm{e}^{-s}\Delta$, which is at most $\Delta/_{4K}$ by our choice of s.

On the other hand, in the last iteration we still have that for every edge e, the sum of the marginals at e is near 1. Furthermore, we are using the same activities, so by condition (2) of the theorem, for each γ we have $\lambda^\gamma(e) \leq K/_\Delta$. But since the distributions are hard-core, $x^{p_\gamma}(e) \leq \lambda_\gamma(e)$. (To see this, observe that

$$x^{p_\gamma}(e) \;=\; \sum_{M\colon e\in M} p(M) \;=\; \lambda_\gamma(e) \sum_{M\colon e\in M} p(M\backslash\{e\}) \;\leq\; \lambda_\gamma(e),$$

where the sums are over matchings M in H_γ containing e.) Taken together this implies that $|L(e)|$ is near $\Delta/_K$ and exceeds $\Delta/_{2K}$. Hence, we can finish off the colouring greedily.

This proof is given in [19, Section 3], and is fairly easy to extract from what is actually written there.

We shall modify this proof to obtain a proof of Our Theorem as follows. To deal with the conflicts caused by J_1 and J_2, we choose to uncolour the conflicting edge which was coloured last, uncolouring both edges if they were coloured in the same iteration. We need to ensure that the number of edges of H incident to any given vertex of H which need to be recoloured due to these conflicts is less than $\frac{1}{3}\Delta^{9/10}$.

To this end, we shall modify the statement of Lemma 5.14, but first we introduce some notation. In each iteration, for each edge e of H, we let $F(e)$ be the set of colours forbidden on e, either because they were assigned to a neighbour in J_1 in a previous iteration, or because they are too close to a colour assigned to a neighbour in J_2 in a previous iteration. For each vertex v of H, we let X_v be the number of edges e of H which are assigned a colour γ in this iteration such that: $\gamma \in F(e)$, or γ is assigned in this iteration to a neighbour of e in J_1, or γ is within $\Delta^{1/4}$ of a colour assigned in this iteration to a neighbour of e in J_2. For technical reasons, we count in X_v conflicts involving all colours assigned to edges in this iteration, not just the colours we finally choose to colour them.

We will use the variant of Lemma 5.14 in which we add:

(i) At the end of its first paragraph:
"*Let* $\tilde{\Delta} = 8K\Delta$. *Suppose further that we have a list* $F(e)$ *of at most* $3\tilde{\Delta}^{1/2}$ *colours for every edge* e, *and graphs* J_1 *and* J_2 *on* $E(H)$, *where* J_1 *has degree at most* $\tilde{\Delta}^{1/2}$ *and* J_2 *has degree at most* $\tilde{\Delta}^{1/4}$, *and that every* $L(e)$ *has at most* $2\tilde{\Delta} = 16K\Delta$ *elements.*"

(ii) At the very end an extra new conclusion:
"*(c) for every vertex* v, $X_v \leq \Delta^{4/5}$."

We call this variant Our Lemma. Since in proving Our Theorem we need only apply it when the maximum degree bound for H_i is between Δ and ${}^{\Delta}/_{8K}$, we see that we will always have the desired upper bound on the sizes of the lists, by applying the upper bound in Our Theorem. Also, since we carry out a constant number of iterations, Our Lemma tells us we need to uncolour only $O(\Delta^{4/5})$ edges of H which are incident with a specific vertex of H. So, provided Δ is large enough we can use Our Lemma to obtain Our Theorem, just as Kahn used Lemma 5.14 to prove his main theorem.

5.4.1 Proving Our Lemma

It remains to describe how to modify the proof of Lemma 5.14 to obtain a proof of Our Lemma.

Kahn proves Lemma 5.14 by applying the Local Lemma to an independent family of random matchings obtained by, for each γ, independently choosing a random matching M_γ according to the hard-core distribution p_γ. By doing so, he shows that he can avoid a set of bad events.

The *bad events* which he avoids by applying the Local Lemma are defined in the middle of [19, page 136]. There are two kinds: an event T_v such that its non-occurrence guarantees the degree of a vertex v drops sufficiently, and an event T_e such that its non-occurrence ensures that the marginals at an edge e of the hard-core distribution for the next iteration sum to a number close to 1.

Kahn defines a distance $t > 1$ which is a function of δ and K (and independent of Δ), and shows that the probability that a bad event occurs, *given all the edges of every matching* M_γ *which are at distance at least* t *in* H *from the vertex or edge indexing the event*, is at most p, for some p which is $\Delta^{-\omega(1)}$.

He can then apply the Local Lemma, where the set $\mathcal{S}_{T_z}$ (z a vertex or an edge) is the set of events indexed by an edge or vertex within distance $2t$ of z (this is done on [19, pages 136–137]). The key point is that this set has size at most $d = 2(\Delta + 1)\Delta^{2t}$, so we have $epd = o(1)$.

(A few remarks: Kahn uses D where we use Δ, and $\Delta_1 + \Delta_2$ where we use t. The result we have just stated is [19, Lemma 6.3]; the $\omega(1)$ here is with respect to Δ.)

To modify this proof to obtain Our Lemma, we introduce for each vertex v of H, a new bad event T_v' that X_v exceeds $\Delta^{4/5}$. In each iteration, along with insisting that all the T_e and T_v fail, we also insist that all the T_v' fail. In doing so we use the following claim. For each vertex v

of H, let $E^+(v)$ denote the set of edges of H consisting of the edges e incident to v together with the edges adjacent in J_1 or J_2 to edges e incident to v.

Claim 5.15
Let $v \in V(H)$. For every colour γ, let L_γ be a given matching in H_γ, and suppose that the event A that $M_\gamma \setminus E^+(v) = L_\gamma$ for each γ satisfies $\mathbb{P}(A) > 0$. Then $\mathbb{P}(T'_v \mid A)$ is $\Delta^{-\omega(1)}$.

Given the claim, to prove our variant of the lemma, we can use the Local Lemma, just as Kahn did. However, we have to use a slightly different dependency graph because the event T'_v depends on the neighbours of v in J_1 and J_2. Given an event U of the form T_x or T'_x indexed by a vertex or edge x, we let the set S_U consist of all the events indexed by some y at distance at most $4t$ from x in the graph H^+ formed by the union of H^*, J_1 and J_2 (where we identify edges of H and the vertices of H^* to which they correspond). Note that this graph has maximum degree at most 2Δ.

Just as with the other events, we have a $\Delta^{-\omega(1)}$ bound on the probability that any event T'_v holds, given the choice of all the matching edges at a suitable distance from v in H^+ (by applying our claim to all the choices of L_γ which extend this choice). Also, we need not worry further about the events T_v and T_e. We can therefore apply the Local Lemma iteratively as in the last section to prove Our Lemma.

Proof of Claim 5.15 To prove the claim we first bound the conditional expected value of X_v. We consider each edge e incident to v separately. We show that the conditional probability that e is in a conflict is $O(\Delta^{-1/2})$. Summing up over all edges e incident to v yields that the expected value of X_v is $O(\Delta^{1/2})$. We prove this bound for the conflicts involving edges coloured in a previous iteration and edges coloured in this iteration separately.

To begin we consider the colours in $F(e)$. We actually show that for any edge e, the conditional probability that e is assigned a colour from $F(e)$, given, for each colour γ, a matching N_γ not containing e such that M_γ is either N_γ or $N_\gamma + e$, is $O(\Delta^{-1/2})$. (We use $N_\gamma + e$ to denote $N_\gamma \cup \{e\}$.) Summing up over all the choices for the N_γ which extend the L_γ, then yields the desired result. If N_γ contains an edge incident to e, then $N_\gamma + e$ is not a matching, so $M_\gamma = N_\gamma$. Otherwise, by the definition of a hard-core distribution:

$$\mathbb{P}(e \in M_\gamma \mid M_\gamma \in \{N_\gamma, N_\gamma + e\}) \;=\; \frac{\lambda_\gamma(e)}{1 + \lambda_\gamma(e)} \;\leq\; \lambda_\gamma(e) \;\leq\; \frac{K}{\Delta}.$$

The conditional probability we want to bound is the sum over all colours γ in $F(e)$ of the conditional probability that e is coloured γ. For each of these colours, the conditional probability that a conflict actually occurs is at most the conditional probability that e is in M_γ. Since this is $O(\Delta^{-1})$, and $|F(e)| \leq 3\tilde{\Delta}^{1/2}$, the desired bound follows.

We next consider conflicts due to both e and a neighbour f in $J_1 \cup J_2$ being assigned the same colour in this iteration. It is enough to show that the conditional probability that e is assigned the same colour as any such uncoloured neighbour f is $O(\Delta^{-1})$. We actually show that for any such edge f, the conditional probability that e is assigned the same colour as f, given, for each colour γ, a matching N_γ not containing e or f such that M_γ is in the set $\mathcal{N}_\gamma^+ = \{N_\gamma, N_\gamma + e, N_\gamma + f, N_\gamma + e + f\}$, is $O(\Delta^{-1})$. Summing up over all the choices for the N_γ which extend the L_γ, then yields the desired result. We obtain our bound on the probability that e and f are both assigned the same colour by summing the probability they both get a specific colour γ over all the at most $2\tilde{\Delta}$ colours in $L(e)$. For each such colour, as in the previous paragraph, we obtain that

$$\mathbb{P}(\{e, f\} \subseteq M_\gamma \mid M_\gamma \in \mathcal{N}_\gamma^+) \;\leq\; \lambda_\gamma(e)\lambda_\gamma(f) \;\leq\; \left(\frac{K}{\Delta}\right)^2.$$

Summing over our choices for γ yields the desired result.

If f is adjacent to e in J_2, then having picked a colour γ in $L(e)$ we have at most $2\tilde{\Delta}^{1/4}$ choices for a colour $\gamma' \neq \gamma$ on f that causes a conflict. Proceeding as above with respect to γ' as well as γ, we can show that the conditional probability that e is coloured γ is at most ${}^{K}\!/_{\Delta}$, and the conditional probability that f is coloured γ', given that e is coloured γ, is at most ${}^{K}\!/_{\Delta}$. Thus the conditional probability that e is coloured γ and f is coloured γ' is at most $\left({}^{K}\!/_{\Delta}\right)^2$. Summing over the at most $2\tilde{\Delta}$ choices for γ, the corresponding choices for γ', and the at most $\tilde{\Delta}^{1/4}$ choices for f, we obtain the desired result.

We next bound the probability that X_v exceeds $\Delta^{4/5}$, by showing that it is concentrated. We note that if we change the choice of one M_γ, leaving all the other random matchings unchanged, then the only new J_1 or J_2 conflicts counted by X_v involve edges coloured with a colour within $\tilde{\Delta}^{1/4}$ of γ. There are at most $2\tilde{\Delta}^{1/4} + 1$ such edges incident to v. Thus, such a change can change X_v by at most $2\tilde{\Delta}^{1/4} + 1$. Furthermore, each conflict involves at most two of the matchings (only one if it also involves a previously coloured vertex). So, to certify that there were at least x conflicts involving edges incident to v in an iteration we need only produce at most $2x$ matchings involved in these conflicts. It follows by a result of Talagrand [41] (see also [37, Chapter 10]) that the probability that X_v exceeds its median M by more than t is at most

$$\exp\Big(-\Omega\frac{t^2}{\Delta^{1/2}M}\Big).$$

Since the median of X_v is at most twice its expectation, setting $t = \frac{1}{2}\Delta^{4/5}$ yields the desired result.

This completes the proof of the claim, and hence of Our Lemma. $\square$

Now that Our Lemma has been proved, we can deduce Our Theorem, Lemma 5.4 and Lemma 5.1. All that remains is to prove Lemma 3.2, which we will do now.

5.5 The Final Stage: Deriving Lemma 3.2

With Lemma 5.1 in hand, it is an easy matter to prove Lemma 3.2. In doing so we consider the natural bijection between the core R of H^* and $E(H)$, referring to these objects using whichever terminology is convenient. (We sometimes use both names for the same object in the same sentence.) Similarly, we use the same letter to denote a vertex of H and the corresponding vertex of G.

Before we really start, we make one observation concerning degrees. For a vertex v in H, the condition in Lemma 3.2, taking $X = \{v\}$, gives $d_{G-R}(v) - d_H(v) \le \frac{1}{30}\epsilon\Delta$. Since $d_{G-R}(v) = d_G(v) - d_H(v)$, this means that $d_H(v) \ge \frac{1}{2}d_G(v) - \frac{1}{60}\epsilon\Delta$, and hence

$$d_G(v) - d_H(v) \;\le\; \tfrac{1}{2}d_G(v) + \tfrac{1}{60}\epsilon\Delta \;\le\; \left(\tfrac{1}{2} + \tfrac{1}{60}\epsilon\right)\Delta.$$

This bound will guarantee that all the lists of colours we will consider below are not empty.

Starting with J_1 and J_2 empty, for every two vertices x, y from R, if x and y are adjacent in G, we add the edge xy to J_2, and if x and y are adjacent in G^2, but do not correspond to incident edges in H, then we add the edge xy to J_1. Since vertices in R have degree at most $\Delta^{1/4}$ in G, we get the required bounds on the degree for vertices in J_1 and J_2 in Lemma 5.1.

Now first suppose that every vertex v in H has degree Δ in G. Recall the definition of ℓ^*_e in equation (6). For an edge $e = vw$ in H, set $L'(e)$ to be a subset of ℓ^*_e colours in $L(e)$ which appear on no vertex of $V \setminus R$ which is a neighbour of e^* in G^2 and are not within $\Delta^{1/4}$ of any colour appearing on a neighbour of e^* in G. This is possible because e^* is adjacent in G^2 to at most $(\Delta - d_H(v)) + (\Delta - d_H(w))$ neighbours of v^* and w^* in $V - R$, and at most $\Delta^{1/2}$ other vertices of $V - R$ (since the vertex e^* in G is removable, hence has at most $\Delta^{1/4}$ neighbours other than v^* and w^*, and all these vertices have degree at most $\Delta^{1/4}$). Finally, condition (2) in Lemma 5.1 holds because of the corresponding condition for all sets X in the statement of Lemma 3.2. So applying Lemma 5.1, we are done in this case.

In general this approach does not work because for a vertex v of H with degree less than Δ, we do not have that $\Delta - d_H(v)$ is equal to the number of edges from v to $V - R$, so our two conditions are not quite equivalent. In order to fix this, we use a simple trick. Form $\widehat{G}$ by taking two disjoint copies $G^{(1)}$ and $G^{(2)}$ of G, with corresponding copies $H^{(i)}$, $R^{(i)}$, $J_1^{(i)}$, $J_2^{(i)}$, $i = 1, 2$, and copy the lists of colours on the vertices of G to the two copies of these vertices. For each vertex v of H, we add $\Delta - d_G(v)$ subdivided edges between its two copies $v^{(1)}$ and $v^{(2)}$. Give an arbitrary list of $\left\lceil\left(\frac{3}{2} + \epsilon\right)\Delta\right\rceil$ colours to the vertices at the middle of these new subdivided edges.

Let $\widehat{H}$ be the multigraph formed by the union of $H^{(1)}$ and $H^{(2)}$ together with multiple edges corresponding to the new subdivided edges between copies of vertices of H. Similarly, take $\widehat{R}$ the union of $R^{(1)}$, $R^{(2)}$ and all vertices in the middle of the new subdivided edges, and set $\widehat{J}_i = J_i^{(1)} \cup J_i^{(2)}$ for $i = 1, 2$. Note that the degrees in $\widehat{J}_1$ and $\widehat{J}_2$ haven't changed, so we can still use them in Lemma 5.1.

Recall that for $i \in \{1, 2\}$ and all $v \in H^{(i)}$, we have $\Delta - d_{\widehat{H}}(v) = d_G(v) - d_{H^{(i)}}(v)$. Now we choose lists of colours on the edges of $\widehat{H}$. Each new edge $v^{(1)}v^{(2)}$ gets an arbitrary list of $\left\lceil\left(\frac{3}{2}+\epsilon\right)\Delta - (\Delta - d_{\widehat{H}}(v^{(1)})) - (\Delta - d_{\widehat{H}}(v^{(2)})) - 3\Delta^{1/2}\right\rceil = \left\lceil\left(\frac{3}{2}+\epsilon\right)\Delta - 2(d_G(v) - d_H(v)) - 3\Delta^{1/2}\right\rceil$ colours from the $\left\lceil\left(\frac{3}{2}+\epsilon\right)\Delta\right\rceil$ colours we gave on the vertex in the middle of it. On the two copies of an edge $e = vw$ of H we take the same list of $\left\lceil\left(\frac{3}{2}+\epsilon\right)\Delta - (\Delta - d_{\widehat{H}}(v)) - (\Delta - d_{\widehat{H}}(w)) - 3\Delta^{1/2}\right\rceil$ colours. Since this is equal to $\left\lceil\left(\frac{3}{2}+\epsilon\right)\Delta - (d_G(v) - d_H(v)) - (d_G(w) - d_H(w)) - 3\Delta^{1/2}\right\rceil$, we can still choose this list to be disjoint from the colours used on the neighbours of this edge in $G^2 - R$.

We note that if we can find a proper colouring of $L(\widehat{H})$ using the chosen lists which avoids conflicts, then we get two (possibly identical) extensions of our colouring of $G^2 - R$ to G^2. We apply Lemma 5.1 to prove that we can indeed find such an acceptable colouring. To do so, we only need to show that for every odd set X of vertices of $\widehat{H}$, we have

$$\sum_{v\in X}(\Delta - d_{\widehat{H}}(v)) - e(X, V(\widehat{H})\backslash X) \;\le\; \tfrac{1}{30}\epsilon|X|\Delta.$$

In fact, we will do this for all subsets X of $V(\widehat{H})$. We set $X^{(i)} = X \cap V(H^{(i)})$, $i = 1, 2$. We immediately get that $e(X, V(\widehat{H})\backslash X) \;\ge\; e(X^{(1)}, V(H^{(1)})\backslash X^{(1)}) + e(X^{(2)}, V(H^{(2)})\backslash X^{(2)})$ (since on the right hand right we are ignoring the edges between the two copies of H). Recall that $\Delta - d_{\widehat{H}}(v) = d_{G-H}(v)$ for a vertex v in $\widehat{H}$. Using the condition in Lemma 3.2 for the two copies of H, this gives

$$\begin{aligned}
&\sum_{v\in X}(\Delta - d_{\widehat{H}}(v)) - e(X, V(\widehat{H})\backslash X)\\
&\qquad\le \sum_{v\in X^{(1)}} d_{G-H}(v) + \sum_{v\in X^{(2)}} d_{G-H}(v)\\
&\qquad\qquad - e(X^{(1)}, V(H^{(1)})\backslash X^{(1)}) - e(X^{(2)}, V(H^{(2)})\backslash X^{(2)})\\
&\qquad\le \tfrac{1}{30}\epsilon|X^{(1)}|\Delta + \tfrac{1}{30}\epsilon|X^{(2)}|\Delta \;=\; \tfrac{1}{30}\epsilon|X|\Delta,
\end{aligned}$$

and we are done. □

6 Conclusions and Discussion

In this paper, we showed that the chromatic number $\chi(G^2)$ of the square of a graph G from a fixed nice family is at most $\bigl(\frac{3}{2} + o(1)\bigr)\Delta(G)$. But many questions remain.

One can prove a bound of constant times the maximum degree for the chromatic number of the square of a graph from a minor-closed family. Krumke *et al.* [30] showed that if a graph G is *q-degenerate* (there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that every v_i has at most q neighbours in $\{v_1, \ldots, v_{i-1}\}$), then its square is $((2q-1)\Delta(G))$-degenerate — the same ordering does the job. But for every minor-closed family $\mathcal{F}$, there is a constant $C_{\mathcal{F}}$ such that every graph in $\mathcal{F}$ is $C_{\mathcal{F}}$-degenerate (see Theorem 2.1 and the first paragraph of Section 4). Hence G^2 is $((2C_{\mathcal{F}}-1)\Delta(G))$-degenerate for every $G \in \mathcal{F}$ and so its list chromatic number is at most $(2C_{\mathcal{F}}-1)\Delta(G) + 1$.

But it is unlikely that this is the best possible bound.

Question 6.1
For a given minor-closed family $\mathcal{F}$ graphs (not the set of all graphs), what is the smallest constant $D_{\mathcal{F}}$ so that $\chi(G^2) \le (D_{\mathcal{F}} + o(1))\Delta(G)$ for all $G \in \mathcal{F}$?

The following examples show that for $\mathcal{F}$ the class of $K_{4,4}$-minor-free graphs we must have $D_{\mathcal{F}} \ge 2$. Let $V_1, \ldots, V_4$ be four disjoint sets of m vertices, and let $X = \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$ be a further six vertices. Let G_m be the graph with vertex set $X \cup V_1 \cup \cdots \cup V_4$, and edges between any x_{ij} and all vertices in $V_i \cup V_j$, $1 \le i < j \le 4$. It is easy to check that G_m is $K_{4,4}$-minor-free. For $m \ge 2$ we have $\Delta(G_m) = d_{G_m}(x_{ij}) = 2m$. Moreover, all vertices in $V_1 \cup \cdots \cup V_4$ are adjacent in G_m^2, and hence $\chi(G_m^2) \ge 4m = 2\Delta(G_m)$. (Of course, G_m has $K_{3,m}$ as a minor, so we do not have a contradiction to Theorem 1.5.)

It is easy to generalise these examples to show that for $\mathcal{F}$ the class of $K_{k,k}$-minor-free graphs, $k \ge 3$, we must have $D_{\mathcal{F}} \ge \frac{1}{2}k$.

But even for nice classes of graphs, many open problems remain. Our proof of the upper bound on the (list) chromatic number does not provide an efficient algorithm. So, for a nice family $\mathcal{F}$, it would be interesting to find an efficient algorithm to find a colouring of the square of a graph $G \in \mathcal{F}$ with at most $\big(\frac{3}{2} + o(1)\big)\Delta(G)$ colours.

Moreover, our result suggests that Wegner's Conjecture(see Conjectures 1.1 and 1.2) should be generalised to nice families of graphs and to list colouring.

Conjecture 6.2
Let $\mathcal{F}$ be a nice family of graphs. Then for any graph $G \in \mathcal{F}$ with $\Delta(G)$ sufficiently large, $\chi(G^2) \le \mathrm{ch}(G^2) \le \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$

The results of Lih, Wang and Zhu [33] and Hetherington and Woodall [16] show that the conjecture is true when $\mathcal{F}$ is the family of K_4-minor-free graphs.

As $\omega(G^2) \le \chi(G^2)$, our result implies $\omega(G^2) \le \big(\frac{3}{2} + o(1)\big)\Delta(G)$ for G in a nice family. But does there exist a simple proof showing this inequality?

Hell and Seyffart [14] proved that for $\Delta \ge 8$, a planar graph with maximum degree Δ and diameter two has at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ vertices. Using their proof techniques it is not so hard to show that, for sufficiently large Δ, a planar graph G with maximum degree Δ satisfies $\omega(G^2) \le \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. (In fact, with some effort we expect this bound can be proved for all $\Delta \ge 8$.) Note that this last inequality is tight as shown by the examples of Figure 1.

Corollary 1.7 in [2] says that, for each fixed surface S, there is a constant c_S such that $\omega(G^2) \le \frac{3}{2}\Delta(G) + c_S$ for each graph G embeddable in S. More generally, can we prove that, for each nice family $\mathcal{F}$, there is a constant $c_{\mathcal{F}}$ such that $\omega(G^2) \le \frac{3}{2}\Delta(G) + c_{\mathcal{F}}$ for each graph G in $\mathcal{F}$? Can we take $c_{\mathcal{F}} = 1$ for sufficiently large Δ (depending on $\mathcal{F}$)?

A major part of the proof of our result is a reduction to list edge-colouring of line graphs. For edge-colourings, Kahn [19] proved that asymptotically the list chromatic number equals the fractional chromatic number. This may suggest that the same could be true for squares of planar graphs, or more generally for squares of graphs of a nice family.

Problem 6.3
Given a nice family $\mathcal{F}$ of graphs, is it true that $\mathrm{ch}(G^2) = (1 + o(1))\chi_f(G^2)$ for $G \in \mathcal{F}$?

As already mentioned in the introduction, we believe that for every square of a planar graph the list chromatic number equals the chromatic number.

Conjecture 6.4
If G is a planar graph, then $\mathrm{ch}(G^2) = \chi(G^2)$*.*

Since there are graphs G for which $\mathrm{ch}(G^2) > \chi(G^2)$, a natural problem is to determine the best possible upper bound on $\mathrm{ch}(G^2)$ in terms of $\chi(G^2)$ for general graphs. Since $\Delta(G)+1 \leq \chi(G^2) \leq \mathrm{ch}(G^2) \leq \Delta(G)^2+1$, we trivially have $\mathrm{ch}(G^2) \leq (\chi(G^2))^2$. However, this trivial quadratic upper bound is certainly not best possible. Kosar *et al.* [26] posed the following question.

Problem 6.5 (Kosar, Petrickova, Reiniger, and Yeager [26])
Is there a function $f(k) = o(k^2)$ such that for every graph G, $\mathrm{ch}(G^2) \leq f(\chi(G^2))$*?*

They also formulated the following more specific question.

Problem 6.6 (Kosar, Petrickova, Reiniger, and Yeager [26])
Does there exist a constant C such that every graph G satisfies $\mathrm{ch}(G^2) \leq C \cdot \chi(G^2) \log(\chi(G^2))$*?*

If the answer to Problem 6.6 is "yes", then the upper bound will be tight, up to the value of the constant C, as Kosar *et al.* [26] constructed an infinite family of graphs G with unbounded $\chi(G^2)$ such that $\mathrm{ch}(G^2) \geq C' \cdot \chi(G^2) \log(\chi(G^2))$ for some constant C'.

Finally, our proof uses Kahn's proof of his theorem that the list chromatic index $\mathrm{ch}'(G)$ of a graph G is $(1+o(1))\chi'_f(G)$, and that theorem of course implies that $\mathrm{ch}'(G) = (1+o(1))\chi'(G)$. This is an asymptotic version of the celebrated List Colouring Conjecture.

Conjecture 6.7 (List Colouring Conjecture)
For every graph G, $\mathrm{ch}'(G) = \chi'(G)$*.*

A more general conjecture was made by Gravier and Maffray[12], who conjectured that for every claw-free graph, the list chromatic number equals the chromatic number. It is possible that advances on the List Colouring Conjecture might be helpful towards Wegner's Conjecture.

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