Chromatic Edge Strength of Some Multigraphs

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Abstract

The edge strength $s'(G)$ of a multigraph G is the minimum number of colors in a minimum sum edge coloring of G. We give closed formulas for the edge strength of bipartite multigraphs and multicycles. These are shown to be classes of multigraphs for which the edge strength is always equal to the chromatic index.

Keywords: graph coloring, minimum sum coloring, chromatic strength

1 Introduction

During a banquet, n people are sitting around a circular table. The table is large, and each participant can only talk to her/his left and right neighbors. For each pair i, j of neighbors around the table, there is a given number m_{ij} of available discussion topics. Assuming that each participant can only discuss one topic at a time, and that each topic takes an unsplittable unit amount of time, what is the minimum duration of the banquet, after which all available topics have been discussed? What is the minimum average elapsed time before a topic is discussed?

In this paper, we show that there always exists a scheduling of the conversations such that these two minima are reached simultaneously. The underlying mathematical problem is that of coloring a *multicycle* with n vertices and m_{ij} parallel edges between consecutive vertices i and j.

Let $G = (V, E)$ be a finite undirected (multi)graph without loops. A *vertex coloring* of G is an application from V to a finite set of colors such that adjacent vertices are assigned different colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors that can be used in a coloring of G. An *edge coloring* of G is

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an application from E to a finite set of colors such that adjacent edges are assigned different colors. The minimum number of colors in an edge coloring is called the *chromatic index* $\chi'(G)$.

In the sequel, we assume that colors are positive integers. The *vertex chromatic sum* of G is defined as $\Sigma(G) = \min\left\{\sum_{v\in V} f(v)\right\}$ where the minimum is taken over all colorings f of G. Similarly, the *edge chromatic sum* of G, denoted by $\Sigma'(G)$, is defined as $\Sigma'(G) = \min\left\{\sum_{e\in E} f(e)\right\}$, where the minimum is taken over all edge colorings. In both case, a coloring yielding the chromatic sum is called a *minimum sum coloring*.

The chromatic sum is a useful notion in the context of parallel job scheduling. A *conflict graph* between jobs is a graph in which two jobs are adjacent if they share a resource, and therefore cannot be run in parallel. If we assume that each job takes a unit amount of time, then a scheduling that minimizes the makespan is a coloring of the conflict graph with a minimum number of colors. On the other hand, a minimum sum coloring of the conflict graph corresponds to a scheduling that minimizes the *average* time before a job is completed. In our example above, jobs are conversations, resources are the banqueters, and the conflict graph is the line graph of a multicycle.

We also define the minimum number of colors needed in a minimum sum coloring of G. This number is called the *strength* $s(G)$ in the case of vertex colorings, and the *edge strength* $s'(G)$ in the case of edge colorings. Trivially, we have $s(G) \geq \chi(G)$ and $s'(G) \geq \chi'(G)$.

Previous results

Chromatic sums have been introduced by Kubicka in 1989 [12]. The computational complexity of determining the vertex chromatic sum of a simple graph has been studied extensively. It is NP-hard even when restricted to some classes of graphs for which finding the chromatic number is easy, such as bipartite or interval graphs [2,19]. Approximability results for various classes of graphs were obtained in the last ten years [1,6,9,5]. Similarly, computing the edge chromatic sum is NP-hard for bipartite graphs [7], even if the graph is also planar and has maximum degree 3 [13]. Strong hardness results have also been given for the vertex and edge strength of a simple graph by Salavatipour [18], and Marx [14].

It has been known for long that the vertex strength can be arbitrarily larger than the chromatic number [4]. Nicoloso et al. however showed that $s(G) = \chi(G)$ for proper interval graphs [17], and that $s(G) \leq \min\{n, 2\chi(G) - 1\}$ for general interval graphs [16]. An analog of Brooks' theorem for the vertex strength of simple graphs has been proved by Hajiabolhassan, Mehrabadi, and Tusserkani [8]: $s(G) \leq \Delta(G)$ for any simple graph G that is not an odd cycle nor a complete graph, where $\Delta(G)$ is the maximum degree of a vertex in G.

Concerning the relation between the chromatic index and the edge strength, Mitchem, Morriss and Schmeichel [15] proved the following inequality, similar to Vizing's theorem : $s'(G) \leq \Delta(G) + 1$, for any simple graph G. Although it has been conjectured by Harary and Plantholt [20] that $s'(G) = \chi'(G)$ for any simple graph G, this has been disproved by Mitchem et al. [15] and Hajiabolhassan et al. [8].

Our results

We consider *multigraphs*, in which parallel edges are allowed, and give two classes of multigraphs for which we always have $s'(G) = \chi'(G)$. We prove the following two results:

(i) $s'(G) = \Delta(G)$ if G is a bipartite multigraph,

(ii) $s'(G) = \max\{\Delta(G), \lceil m/k \rceil\}$ if G is an odd multicycle, i.e. a cycle with parallel edges, of order $n = 2k+1$ with m edges.

These two statements are extensions of two classical results from König and Berge, respectively.

2 Bipartite Multigraphs

The following well-known result has been proved by König in 1916.

Theorem 1 (König's theorem [11]). *Let $G = (V, E)$ be a bipartite multigraph and let Δ denotes its maximum degree. Then $\chi'(G) = \Delta$.*

Let C be the set of colors used in an edge coloring of a multigraph G. We denote by C_x the subset of colors in C assigned to edges incidents with vertex x of G. We also denote by $d_G(x)$ the degree of vertex x in G.

We now show that in a bipartite multigraph, the edge chromatic sum can always be obtained with $\chi'(G)$ colors.

Theorem 2. *Let $G = (V, E)$ be a bipartite multigraph and let Δ denotes its maximum degree. Then $s'(G) = \chi'(G) = \Delta$.*

Proof. We proceed by contradiction. It is sufficient to assume that $s'(G) = \Delta + 1$. So, there is an edge coloring f for G using $\Delta+1$ colors such that $\sum_{e\in E} f(e) = \Sigma'(G)$. Let $C = \{1, \ldots, \Delta+1\}$ be the set of colors used by f. Choose an edge $[a, b]_0$ in G having color $\Delta + 1$. Clearly, $C_a \cup C_b = \{1, \ldots, \Delta\}$, otherwise, there exists a color $\alpha \in \{1, \ldots, \Delta\}$ not used by any edge adjacent to both vertices a and b which can be used to color edge $[a, b]_0$. We would obtain a new edge coloring f' such that $\sum_{e\in E} f'(e) < \sum_{e\in E} f(e)$ which is a contradiction to the minimality of f. Therefore, there exist colors $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$ such that $\alpha, \beta \leq \Delta$.

Let $P_{\alpha\beta}$ denotes a maximal (α, β)-path beginning at vertex a. Notice that such a path cannot end at vertex b, otherwise G contains an odd cycle contradicting the fact that G is bipartite. So, we can recolor the edges on $P_{\alpha\beta}$ by swapping colors α and β. Moreover, after such a color swap, color α is such that $\alpha \notin C_a$ and $\alpha \notin C_b$ and thus we can color edge $[a, b]_0$ with color $\alpha \leq \Delta$ obtaining a new edge coloring f'.

We now prove that after such a recoloring, $\sum_{e\in E} f'(e) < \sum_{e\in E} f(e)$ (*). First, note that if the length of $P_{\alpha\beta}$ is even, the recoloring only affects the value of the edge $[a, b]_0$, so (*) holds. Therefore, it is sufficient to consider the effects of such a recoloring when the length of $P_{\alpha\beta}$ is odd. Let $2s+1$ be the length of $P_{\alpha\beta}$, with $s \geq 0$. Thus, initially for f we have the sub-sum $(\Delta + 1) + (s + 1)\alpha + s\beta$ corresponding to edge $[a,b]_0$ and to the $2s+1$ edges on $P_{\alpha\beta}$. After the recoloring, we have for f' that such values have changed to $\alpha+(s+1)\beta+s\alpha$. The change value of f' w.r.t. f is $\beta-\Delta-1<0$ and so (*) always holds, contradicting in this way the minimality of f. Therefore, we have proved that if f is an edge coloring for G such that $\sum_{e\in E} f(e)=\Sigma'(G)$, then f uses at most Δ colors to color the edges of G. $\square$

3 Multicycles

Let G be a multigraph without loops with m edges. It is easy to deduce that $\chi'(G)\geq \max\left\{\Delta,\lceil\frac{m}{\tau}\rceil\right\}$, where Δ denotes the maximum degree and τ denotes the cardinality of a maximum matching in G. This lower bound is indeed tight for multicycles, defined as cycles in which we can have parallel edges between two consecutive vertices.

Theorem 3 ([3])**.** *Let $G=(V,E)$ be a multicycle on n vertices with m edges and degree maximum Δ. Let τ denotes the maximal cardinality of a matching in G. Then*

$$\chi'(G)=\begin{cases}\Delta, & \text{if } n \text{ is even,}\\ \max\left\{\Delta,\lceil\frac{m}{\tau}\rceil\right\}, & \text{if } n \text{ is odd.}\end{cases}$$

In order to determine the edge strength of a multicycle, we need the following lemma proved by Berge in [3].

Lemma 1 (Uncolored edge Lemma [3])**.** *Let G be a multigraph without loops with $\chi'(G)=r+1$. If a coloring of $G\setminus[a,b]_0$ using a set C of r colors cannot be extended to color the edge $[a,b]_0$, then the following identities are verified :*

(i) $|C_a\cup C_b|=r$,

(ii) $|C_a\cap C_b|=d_G(a)+d_G(b)-r-2$,

(iii) $|C_a\setminus C_b|=r-d_G(b)+1$,

(iv) $|C_b\setminus C_a|=r-d_G(a)+1$.

In order to prove the Lemma 1, it suffices to compute a linear system of 3 equations on 3 variables as follows : (i) $r=|C|=|C_a\cup C_b|=|C_a\cap C_b|+|C_a\setminus C_b|+|C_b\setminus C_a|$; (ii) $|C_a\setminus C_b|=d_G(a)-1-|C_a\cap C_b|$; and (iii) $|C_b\setminus C_a|=d_G(b)-1-|C_a\cap C_b|$.

Theorem 4. *Let $G=(V,E)$ be a multicycle on n vertices with m edges and maximum degree Δ. Let τ denotes the maximal cardinality of a matching in G. Then*

$$s'(G)=\chi'(G)=\begin{cases}\Delta, & \text{if } n \text{ is even,}\\ \max\left\{\Delta,\lceil\frac{m}{\tau}\rceil\right\}, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. If n is even, by Theorem 2, the result follows. So, we assume that $n=2k+1$ for a positive integer k. We proceed by induction on m. Let $r=\max\left\{\Delta,\lceil\frac{m}{\tau}\rceil\right\}$. Assume that $m=2k+1$. In this case, G is a simple odd cycle. Color the edges in G in such a way that there exist k edges colored with color 1, k edges colored with color 2 and one edge colored with color 3. Clearly, it is always possible. As $\chi'(G) = 3$ and k is the size of a maximum matching in G, it is easy to deduce that the theorem holds for this case. Therefore, we assume that $m > 2k+1$ and assume that the result holds for all multicycles on n vertices with fewer that m edges. Let $[a,b]_0$ be an edge in G and let $G' = G \setminus [a,b]_0$. By induction hypothesis, we have that there exists an edge coloring f' for G' using $r = \max\left\{\Delta, \lceil \frac{m}{\tau} \rceil\right\} \geq \max\left\{\Delta', \lceil \frac{m-1}{\tau} \rceil\right\} \geq \chi'(G')$ colors, such that $\sum_{e \in E'} f'(e) = \Sigma'(G')$, that is, under f' the multigraph G' verifies $s'(G') = \chi'(G') \leq r$. Assume, by contradiction, that $s'(G) = r+1$. Thus, there exists an edge coloring f for G which uses $r+1$ colors and verifies $\sum_{e \in E} f(e) = \Sigma'(G)$. Notice that the restriction of f to edges in G' verifies $\sum_{e \in E'} f(e) = \Sigma'(G')$, otherwise contradicting the optimality of f in G. So, the edge $[a,b]_0$ is the only edge in G colored by f with color $r+1$. So, let $C = \{1, \ldots, r\}$ be the set of colors used by f on the edges in G' and for each $1 \leq i \leq r$, let E_i denotes the set of edges in G' colored with color i. By induction hypothesis, we have the following claim.

Claim 1. *There exists a color $\sigma \in C$ such that $|E_\sigma| < k$.*

The claim holds, otherwise we would have that $m - 1 = \sum_{i=1}^{r} |E_i| = kr$, and $r = \frac{m-1}{k} < m/k$, a contradiction.
By Lemma 1, we know that $|C_a \cup C_b| = r$. Hence it is sufficient to analyze the cases $\sigma \in C_b \setminus C_a$ (or $\sigma \in C_a \setminus C_b$) and $\sigma \in C_a \cap C_b$.

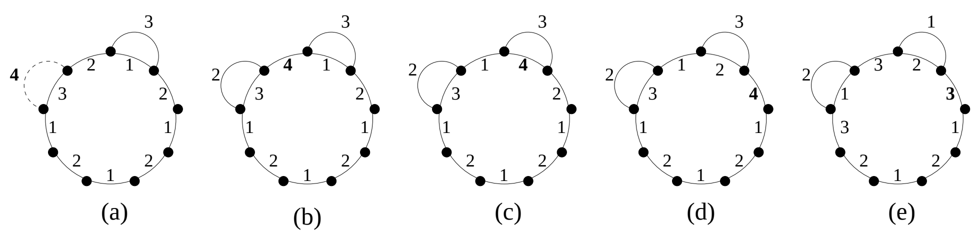

Fig. 1. **(a)** A multicycle G where $\chi'(G) = 3$ and having an edge colored with color 4. In the proof of Theorem 4, color 3 represents color σ. Figures **(a)-(c)** (resp. **(d)-(e)**) represent the case $\sigma \in C_a \cap C_b$ (resp. $\sigma \in C_b \setminus C_a$) in Theorem 4.

- Case $\sigma \in C_b \setminus C_a$. By Lemma 1, there exists a color $\alpha \in Ca \setminus C_b$. Let $G(\alpha, \sigma)$ denote the induced subgraph of G' by the edges colored by f with colors α and σ. Let $G_b(\alpha, \sigma)$ denote the connected component of $G(\alpha, \sigma)$ containing the vertex b. Clearly, $G_b(\alpha, \sigma)$ is a simple (σ, α)-path having b as end-vertex and not containing vertex a, otherwise, there is a contradiction to Claim 1. So, we can recolor the edges on the path $G_b(\alpha, \sigma)$ by swapping colors α and σ in such a way that $\sigma \notin C_b$. As color $\sigma \notin C_a$, we can color the edge $[a,b]_0$ with color σ obtaining in this way an edge coloring f'' for G which uses r colors. See Figure 1 (d)-(e) for an example of this case.

We want to show that $\sum_{e \in E} f''(e) < \sum_{e \in E} f(e)$ (**), contradicting $s'(G) > r$. If the length of the path $G_b(\alpha, \sigma)$ is even, then $\sum_{e \in E} f''(e) - \sum_{e \in E} f(e) = \sigma - r - 1 \leq r - r - 1 < 0$. If the length of the path $G_b(\alpha, \sigma)$ is odd (say $2s+1$, with $s \geq 0$), then such a difference is equal to $(\sigma + (s+1)\alpha + s\sigma) - (r + 1 + (s+1)\sigma + s\alpha) = \alpha - r - 1 \leq r - r - 1 < 0$. Thus, inequality (**) always holds.

- Case $\sigma \in C_a \cap C_b$. By Lemma 1, there exist colors $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$ with $\alpha \neq \beta \neq \sigma$. By induction hypothesis, the result holds for $G' = G \setminus [a,b]_0$ and G' has a minimum sum edge coloring using at most r colors. Thus, the edge $[a,b]_0$ in G is the only edge colored by f with color $r+1$.

Let us assume that vertices are ordered clockwise and let b be the right vertex of edge $[a,b]_0$. Recolor edge $[a,b]_0$ by color β and the edge of color β incident to b with color $r+1$ respectively. Notice that such a procedure does change neither the value of the sum of colors nor the number of colors used. Let $[x,y]_0$ be the current edge colored by such a recoloring with color $r+1$ such that x is its left vertex, and find a color $\beta_y \in C_y \setminus C_x$. By Lemma 1 such a color β_y exists, otherwise there is a color $\theta \leq r$ such that $\theta \notin C_x$ and $\theta \notin C_y$, and so we can recolor edge $[x,y]_0$ with color θ which gives a contradiction to the minimality of f. Repeat such a procedure until current edge $[x,y]_0$ in G colored with color $r+1$ is such that $\sigma \in C_x \setminus C_y$ or $\sigma \in C_y \setminus C_x$. Clearly it is always possible, because the cycle is odd. Moreover, notice that $|E_\sigma| < k$ always hold. Assume w.l.o.g. that $\sigma \in C_y \setminus C_x$. By relabeling the vertex set of G in such a way that x becomes a and y becomes b, we are back to the first case (Figure 1 (a)-(c) gives an example of this case). This concludes the proof. $\square$

4 Generalization

In the generalized *optimal cost chromatic partition* problem [10], each color has an integer cost, but this cost does not necessarily correspond to the index of the color. The cost of a vertex coloring is the $\sum_{v \in V} c(f(v))$, where $c(i)$ is the cost of color i. For any set of costs, our proofs can be generalized to show that the minimum number of colors needed in a minimum cost edge coloring of G is equal to $\chi'(G)$ when G is bipartite or a multicycle.

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