Kneser Colorings of Uniform Hypergraphs

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Abstract

For fixed positive integers r, k and ℓ with $\ell < r$, and an r-uniform hypergraph H, let $\kappa(H, k, \ell)$ denote the number of k-colorings of the set of hyperedges of H for which any two hyperedges in the same color class intersect in at least ℓ vertices. Consider the function $\mathrm{KC}(n, r, k, \ell) = \max_{H \in \mathcal{H}_n} \kappa(H, k, \ell)$, where the maximum runs over the family $\mathcal{H}_n$ of all r-uniform hypergraphs on n vertices. In this paper, we determine the asymptotic behavior of the function $\mathrm{KC}(n, r, k, \ell)$ and describe the extremal hypergraphs. This variant of a problem of Erdős and Rothschild, who considered colorings of graphs without a monochromatic triangle, is related to the Erdős–Ko–Rado Theorem [3] on intersecting systems of sets.

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1 Introduction

A *hypergraph* $H = (V, E)$ is given by its *vertex set* V and its set E of *hyperedges*, where $e \subseteq V$ for each hyperedge $e \in E$, and $H = (V, E)$ is said to be *r-uniform* if each $e \in E$ has cardinality r. For a fixed r-uniform hypergraph F, an r-uniform host-hypergraph H and an integer k, let $c_{k,F}(H)$ be the number of k-colorings of the set of hyperedges of H with no monochromatic copy of F and let $c_{k,F}(n) = \max_{H \in \mathcal{H}_n} c_{k,F}(H)$, where $\mathcal{H}_n$ is the family of all r-uniform hypergraphs on n vertices. Given an r-uniform hypergraph F, let $\mathrm{ex}(n, F)$ be the usual *Turán number* of F, i.e., the maximum number of hyperedges in an r-uniform n-vertex hypergraph that contains no copy of F. A hypergraph for which maximality is achieved is said to be an *extremal hypergraph for F*.

Every coloring of the set of hyperedges of any extremal hypergraph H for F trivially contains no monochromatic copy of F and, hence, $c_{k,F}(n) \geq k^{\mathrm{ex}(n,F)}$ for all $k \geq 2$. On the other hand, if $\mathrm{Forb}_F(n)$ denotes the family of all hypergraphs with vertex set $[n] = \{1, \ldots, n\}$ that contain no copy of F, every 2-coloring of the set of hyperedges of a hypergraph H on $[n]$ containing no monochromatic copy of F gives rise to a member of $\mathrm{Forb}_F(n)$; thus $c_{2,F}(n) \leq |\,\mathrm{Forb}_F(n)|$. The size of $\mathrm{Forb}_F(n)$ was first studied by Erdős, Kleitman, and Rothschild [2] for $F = K_3$, the triangle. This has been extended since to several other classes of graphs. For r-uniform hypergraphs, Nagle, Rödl, and Schacht [5] proved that $|\,\mathrm{Forb}_F(n)| \leq 2^{\mathrm{ex}(n,F)+o(n^r)}$. Thus, for 2-colorings of the set of hyperedges and any fixed r-uniform hypergraph F we have

$$2^{\mathrm{ex}(n,F)} \leq c_{2,F}(n) \leq 2^{\mathrm{ex}(n,F)+o(n^r)}. \tag{1}$$

For $r = 2$ and cliques $F = K_\ell$, Alon, Balogh, Keevash, and Sudakov [1] showed that the lower bound in (1) is the correct value of $c_{2,K_\ell}(n)$ for $n \geq n_0$. Moreover, for 3-colorings, they proved that $c_{3,K_\ell}(n) = 3^{\mathrm{ex}(n,K_\ell)}$ for $n \geq n_0$. In both cases, $k = 2$ and $k = 3$, equality is achieved only by the $(\ell - 1)$-partite Turán graph on n vertices. However, it was observed in [1] that $c_{k,K_\ell}(n) \gg k^{\mathrm{ex}(n,K_\ell)}$ for any fixed $k \geq 4$ and $n \geq n_0$.

An extension of these results to hypergraphs has been given recently in [4] for the Fano plane F. There it was shown in the case of k-colorings, $k \in \{2, 3\}$, that every 3-uniform hypergraph H on $n \geq n_0$ vertices satisfies $c_{k,F}(H) \leq k^{\mathrm{ex}(n,F)}$, with equality being attained by a unique extremal hypergraph.

Here, we investigate a variant of this problem, where we forbid pairs of hyperedges of the same color that share fewer than ℓ vertices, *thus forcing every color class to be ℓ-intersecting.* Formally, for fixed integers r, ℓ with $1 \leq \ell < r$, and $i \in [\ell]$, let $F_{r,i}$ be the r-uniform hypergraph on $2r - i + 1$ vertices with two hyperedges sharing exactly $i-1$ vertices, and let $\mathcal{B}_{r,\ell} = \{F_{r,i} : i \in [\ell]\}$. Following the notation above, let $c_{k,\mathcal{B}_{r,\ell}}(H)$ be the number of k-colorings of the set of hyperedges of a hypergraph H with no monochromatic copy of any $F \in \mathcal{B}_{r,\ell}$, which we call (k,ℓ)-*Kneser colorings*, and let $c_{k,\mathcal{B}_{r,\ell}}(n) = \max_{H \in \mathcal{H}_n} c_{k,\mathcal{B}_{r,\ell}}(H)$. We set $\mathrm{KC}(n,r,k,\ell) = c_{k,\mathcal{B}_{r,\ell}}(n)$ and $\kappa(H,k,\ell) = c_{k,\mathcal{B}_{r,\ell}}(H)$.

In the spirit of [1] and [4], we show that the *extremal* hypergraphs H on n vertices, i.e., those for which $\kappa(H,k,\ell) = \mathrm{KC}(n,r,k,\ell)$, for colorings with $k=2$ or $k=3$ colors are essentially determined by the well-known Erdős–Ko–Rado Theorem [3], while this does not hold for $k \geq 4$ colors.

2 Kneser Colorings with Two or Three Colors

The following result is a direct application of the Erdős–Ko–Rado Theorem and its generalizations.

Theorem 2.1 *Let $n \geq r > \ell$ be positive integers. Then it is $\mathrm{KC}(n,r,2,\ell) = 2^{\mathrm{ex}(\mathcal{B}_{r,\ell})}$. Moreover, equality is achieved by every r-uniform hypergraph H on $[n]$ whose hyperedges are given by an extremal configuration for $\mathcal{B}_{r,\ell}$. Conversely, unless $\ell = 1$ and $n = 2r$, all the extremal hypergraphs have this form.*

When looking at Kneser colorings with at least three colors, the following result plays an important role.

Lemma 2.2 *Let $k \geq 2$ be an integer. All optimal solutions $s = (s_1,\ldots,s_c)$ to the maximization problem*

$$\max \prod_{i=1}^{c} s_c, \tag{2}$$

where $c, s_1, \ldots, s_c \in \{1,2,\ldots\}$ and $s_1 + \cdots + s_c \leq k$, have the following form:

(a) If $k \equiv 0 \pmod 3$, then $c = k/3$ and all the components of s are equal to 3.

(b) If $k \equiv 1 \pmod 3$, then either $c = \lceil k/3 \rceil$, with exactly two components equal to 2 and all remaining components equal to 3, or $c = \lfloor k/3 \rfloor$, with exactly one component equal to 4 and all remaining components equal to 3.

(c) If $k \equiv 2 \pmod 3$, then $c = \lceil k/3 \rceil$ with exactly one component equal to 2 and all remaining components equal to 3.

As a consequence, the optimal value of (2) is $3^{k/3}$ if $k \equiv 0 \pmod 3$, $4 \cdot 3^{\lfloor k/3 \rfloor - 1}$ if $k \equiv 1 \pmod 3$, and $2 \cdot 3^{\lfloor k/3 \rfloor}$ if $k \equiv 2 \pmod 3$.

Aiming towards finding upper bounds on the function $\mathrm{KC}(n,r,k,\ell)$, we introduce a generalization of the concept of a vertex cover of a graph. For a positive integer ℓ, an ℓ*-cover* of a hypergraph H is a set C of ℓ-subsets of vertices of H such that every hyperedge of H contains an element of C. It may be shown that, for n sufficiently large, if $H^* = (V, E)$ is an r-uniform extremal hypergraph on $[n]$ with minimum ℓ-cover C, then the cardinality of C is equal to the number of components c of an optimal solution to the maximization problem (2). Moreover, H^* is *complete* with respect to the cover C, i.e., every r-subset of $[n]$ containing some set $t \in C$ is a hyperedge of H^*. If $k = 3$, this leads directly to the extremal hypergraph H^*: it has an ℓ-cover of size 1, since the single optimal solution to (2) is $s_1 = 3$, and it must be complete.

Theorem 2.3 *Let $r > \ell \geq 1$ be integers. Then, for every $n \geq n_0$, we have $\mathrm{KC}(n, r, 3, \ell) = 3^{\binom{n-\ell}{r-\ell}}$. Moreover, for $n \geq n_0$ equality is achieved only by the (n, r, ℓ)-star $S_{n,r,\ell}$.*

3 Colorings with at Least Four Colors

For $k \geq 4$, two additional questions arise. On the one hand, the structural result of the previous section does not determine precisely the size of a minimum ℓ-cover of the extremal hypergraph when $k \equiv 1 \pmod 3$, since there are two types of optimal solutions to (2). On the other hand, for $k \geq 5$, all optimal solutions to (2) have more than one component, which suggests that the way in which the cover elements intersect may play a role.

For positive integers k, $r \geq 2$, $\ell < r$, c and $n \geq \max\{r, c\ell\}$, let C be a set of cardinality c whose elements are ℓ-subsets of $[n]$. The (C, r)*-complete hypergraph* $H_{C,r}(n)$ has vertex set $[n]$ and the set of hyperedges is given by all r-subsets of $[n]$ containing some element of C as a subset. If C consists of exactly $\lceil k/3 \rceil$ mutually disjoint ℓ-subsets of $[n]$, then we denote the hypergraph $H_{C,r}(n)$ by $H_{n,r,k,\ell}$.

For $k = 4$ colors, we show that $\mathrm{KC}(n, r, k, \ell)$ is achieved only by hypergraphs with minimum ℓ-cover of size two. This leads to the following characterization.

Theorem 3.1 *Let $r > \ell \geq 1$ be integers. Given a positive integer n, let H^* be an r-uniform hypergraph on $[n]$ satisfying $\kappa(H^*, 4, \ell) = \mathrm{KC}(n, r, 4, \ell)$. Then, for $n \geq n_0$, H^* is isomorphic to $H_{C,r}(n)$ for some ℓ-cover $C = \{t_1, t_2\}$.*

If we have $k \geq 5$ colors available, the way in which the cover elements intersect affects the number of Kneser colorings significantly.

Theorem 3.2 *Let $r > \ell \geq 1$ and $k \geq 5$. Let H^* be an r-uniform hypergraph on $[n]$ with $\kappa(H^*, k, \ell) = \mathrm{KC}(n, r, k, \ell)$. Then, for $n \geq n_0$, the following holds.*

(a) If $r \geq 2\ell$, then H^* is isomorphic to $H_{n,r,k,\ell}$.

(b) If $r < 2\ell$, then H^* is isomorphic to $H_{C,r}(n)$ for a set $C = \{t_1, \dots, t_{c(k)}\}$ of ℓ-subsets of $[n]$ with $c(k) = \lceil k/3 \rceil$, and $|t_i \cup t_j| > r$ for all $i, j \in [c(k)]$, $i \neq j$.

For the case of arbitrary $k \geq 4$, we may derive the asymptotic behavior of $\mathrm{KC}(n, r, k, \ell)$ from a careful estimate on the number $\alpha(n, r, k, \ell)$ of a special class of Kneser colorings of the hypergraph $H_{n,r,k,\ell}$.

Theorem 3.3 *Let $r > \ell \geq 1$ and $k \geq 4$ be fixed integers. Then $\mathrm{KC}(n, r, k, \ell) = (1 + f(n))\, \alpha(n, r, k, \ell)$, where $f(n)$ is a function that tends to 0 as n tends to infinity, and*

(i) $\alpha(n, r, k, \ell) = N(k)D(k)^{\binom{n-\ell}{r-\ell}}$ if $k = 4$ or $r < 2\ell$,

(ii) $\alpha(n, r, k, \ell) \leq N(k)k^{\binom{\ell c(k)}{\ell+1}\binom{n-\ell-1}{r-\ell-1}}D(k)^{\binom{n-\ell}{r-\ell}}$ if $k \geq 5$ and $r \geq 2\ell$, where

$$\begin{cases} \text{if } k \equiv 0 \pmod 3,\ N(k) = \frac{k!}{(3!)^{\frac{k}{3}}} \text{ and } & D(k) = 3^{\frac{k}{3}} \\ \text{if } k \equiv 1 \pmod 3,\ N(k) = \binom{\lfloor \frac{k}{3} \rfloor + 1}{2} \frac{k!}{4 \cdot (3!)^{\lfloor \frac{k}{3} - 1 \rfloor}} \text{ and } & D(k) = 4 \cdot 3^{\lfloor \frac{k}{3} \rfloor - 1} \\ \text{if } k \equiv 2 \pmod 3,\ N(k) = \left(\lfloor \frac{k}{3} \rfloor + 1 \right) \frac{k!}{2 \cdot (3!)^{\lfloor \frac{k}{3} \rfloor}} \text{ and } & D(k) = 2 \cdot 3^{\lfloor \frac{k}{3} \rfloor}. \end{cases}$$

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