ON THE RECONSTRUCTION OF GRAPH INVARIANTS

T. KOTEK

Abstract. The reconstruction conjecture has remained open for simple undirected graphs since it was suggested in 1941 by Kelly and Ulam. In an attempt to prove the conjecture, many graph invariants have been shown to be reconstructible from the vertex-deleted deck, and in particular, some prominent graph polynomials. Among these are the Tutte polynomial, the chromatic polynomial and the characteristic polynomial. We show that the interlace polynomial, the U-polynomial, the universal edge elimination polynomial ξ and the colored versions of the latter two are reconstructible.

1. Introduction

1.1. **The Reconstruction Conjecture.** Let $G = (V(G), E(G))$ be a simple undirected graph. For a vertex $v \in V(G)$, we define the vertex-deleted subgraph G_v as the subgraph obtained from G by deleting v and all its incident edges. We define the *deck* of G, $\mathcal{D}(G)$, as the collection of subgraphs G_v for every $v \in V(G)$. The deck $\mathcal{D}(G)$ is a multiset which consists of isomorphism types of subgraphs. We say a graph H is a *reconstruction* of G if $V(G) = V(H)$ and for every $v \in V(G)$, $G_v \cong H_v$. A graph G is *reconstructible* if every reconstruction of it is isomorphic to it.

The reconstruction conjecture asserts that every simple undirected graph with at least three vertices is reconstructible. We say a graph invariant μ is reconstructible if for every H and G which are reconstructions of each other, $\mu(G) = \mu(H)$. If μ is the characteristic function of a class of graphs $\mathcal{C}$, we say that $\mathcal{C}$ is recognizable.

For graphs H and G we define $s(H, G)$ as the number of subgraphs of G which are isomorphic to H.

Lemma 1 (P. J. Kelly, 1957)**.** *For any H and G such that $|V(H)| < |V(G)|$, $s(H, G)$ is reconstructible from the deck of G.*

Another useful lemma by Tutte is the following (see [10]).

Lemma 2. *The number of disconnected spanning subgraphs of G having a specified number of components in each isomorphism class and the number of connected spanning subgraphs with a given number of edges are reconstructible.*

An edge-colored graph G is $G = (V(G), E(G), f)$ where $f : E(G) \to [\Lambda]$ is a coloring of the edges of G. Similarly to the deck of a simple undirected graph, we define the deck $\mathcal{D}(G)$ of a colored graph G to be the multiset of colored graphs G_v for all $v \in V(G)$ which consist of G with v and all incident edges deleted from it, and f restricted to $V(G) - \{v\}$. Reconstruction results on graphs can often

Partially supported by ISF-Grant "Model Theoretic Interpretations of Counting Functions", 2007-2010.

be adapted to colored graphs, and in particular, Kelly's lemma works for colored graphs, see [8, 11].

1.2. **Graph polynomials.** Graph polynomials are functions F from the class of simple undirected graphs to a polynomial ring $\mathcal{R}[\bar{x}]$ that are invariant under graph isomorphism. They are natural to the study of reconstruction. The characteristic polynomial $P(G,x)$ is the characteristic polynomial of the adjacency matrix A_G of the graph G, $P(G;x) = det(x \cdot \mathbf{1} - A_G) = \sum_{S\subseteq V(G)} (-x)^{|V(G)|-|S|} det(A_{G_S})$. Tutte showed a generalization of the characteristic polynomial is reconstructible in [10]. The Tutte polynomial is $T(G;x,y) = \sum_{M\subseteq E(G)} (x-1)^{r(E)-r(M)} (y-1)^{n(M)}$, where $k(M)$ is the number of connected components of the spanning subgraph $(V(G),M)$, $r(M) = |V(G)| - k(M)$ is its rank and $n(M) = |M| - |V(G)| + k(M)$ is its nullity. The Tutte polynomial was shown to be reconstructible by Tutte. The chromatic polynomial $\chi(G,\lambda)$, which counts the number of proper colorings of G with at most λ colors, is an evaluation of the Tutte polynomial, and so is reconstructible as well.

2. Main Results

2.1. **Reconstructibility of graph polynomials.** For a graph G and vertex set S, we denote by $G[S]$ the subgraph induced by S. The polynomial $q(G;x,y) = \sum_{S\subseteq V(G)} (x-1)^{rk(G[S])} (y-1)^{n(G[S])}$ where $rk(G[S])$ is the rank of $G[S]$'s adjacency matrix, and $n(G[S]) + rk(G[S]) = |S|$ is the 2-variable interlace polynomial ([1]).

Proposition 3. *The interlace polynomial is reconstructible.*

Proof. Proof (Sketch) The determinant of A_G is reconstructible (see [10]). $rk(V(G)) = |V(G)|$ iff $det(A_G) \neq 0$, and if $rk(V(G)) \neq |V(G)|$, $rk(G) = max_{v\in V(G)} rk(A_{G_v})$. So we can reconstruct the term for $S = V(G)$. The rest are reconstructible by Kelly's lemma. □

The U polynomial $U(G;\bar{x},y) = \sum_{A\subseteq E(G)} y^{|A|-r(A)} \prod_{D:\Phi_{cc}(D)} x_{|V(D)|}$, where the product iterates over the connected components of the graph (V,A), and $r(A) = |V(G)| - k(A)$, was introduced in [9]. The $U(G;\bar{x},y)$ polynomial generalizes both the Tutte polynomial and a graph polynomial related to Vassiliev invariants of knots. The graph polynomial on colored graphs $U_{lab}(G;\bar{x},\bar{y}) = \sum_{A\subseteq E} \prod_{e\in A} y_{c(e)} \cdot \prod_{D:\Phi_{cc}(D)} x_{|V(D)|}$ is a generalization of U to colored graphs. We show that U_{lab} is reconstructible (see also [7]):

Theorem 4. *The U_{lab}-polynomial is reconstructible.*

Proof. Proof (Sketch) Let G be a colored graph with color-set $[\Lambda]$. For $c \in [\Lambda]$ we denote by $f^{-1}(c)$ the set of edges colored c, and note $|f^{-1}(c)|$ is reconstructible by Kelly's lemma. For a tuple $(a_1, \dots a_\Lambda) \in \mathbb{N}^\Lambda$, we say a subgraph $A \subseteq E(G)$ is $(a_1, \dots a_\Lambda)$-colored if for every color $c \in [\Lambda]$, there are a_c many edges colored c. For every $(a_1, \dots a_\Lambda) \in \mathbb{N}^\Lambda$, the number of $(a_1, \dots a_\Lambda)$-colored subgraphs is $\prod_{c\in[\Lambda]} \binom{|f^{-1}(c)|}{a_c}$.

We reconstruct the terms of U_{lab} that correspond to subgraphs which are not spanning subgraphs by Kelly's lemma. For disconnected spanning subgraphs, we go over all tuples $(F_1, \dots, F_t)$, $t > 1$, of isomorphism types of colored graphs such that the number of vertices in all the F_i's is $\sum_{i=1}^{t} |V(F_i)| = |V(G)|$. For each such tuple we find the number of ways of choosing subgraphs $G_1, \ldots, G_t$ of G such that $G_i \cong F_i$ for every i, and their union $\bigcup_{i=1}^{t} G_i$ is G. We can do this by subtracting the number of ways to choose subgraphs as above such that their union does not span $V(G)$ from $s(F_1, G) \cdots s(F_t, G)$, the number of ways to choose the G_i without restriction.

For any $(a_1, \ldots a_{,\Lambda}) \in \mathbb{N}^{\Lambda}$, the number of $(a_1, \ldots a_{,\Lambda})$-colored spanning connected subgraphs is $\prod_{c\in[\Lambda]} \binom{|f^{-1}(c)|}{a_c}$ minus the number of $(a_1, \ldots a_{,\Lambda})$-colored subgraphs which are not spanning connected subgraphs. For them, the corresponding term is $x_{|V(G)|} \prod_{c\in[\Lambda]} y_c^{a_c}$.

□

Corollary 5. *Lemma 2 holds for colored graphs.*

The universal edge elimination polynomial $\xi(G; x, y, z)$ introduced in [2] is defined with respect to three basic operations on edges of the graph: edge contraction $G_{/e}$, edge deletion G_{-e} and edge elimination $G_{\dagger e}$. It is the most general graph polynomial which satisfies a linear recurrence relation with respect to the above operations which is multiplicative with respect to disjoint union and has initial conditions $\xi(E_1) = x$ and $\xi(\emptyset) = 1$. The ξ polynomial generalizes a number of graph polynomials and graph parameters which satisfy linear recurrences with respect to these operations, including the bivariate matching polynomial, the Tutte polynomial, and the bivariate chromatic polynomial (the latter defined in [6]). The authors of [2] also defined a version ξ_{lab} for colored graphs, given by $\xi_{lab}(G; x, z, \bar{t}) = \sum_{A\sqcup B\subseteq E(G)} x^{k(A\sqcup B)} \left(\prod_{e\in A\sqcup B} t_{c(e)}\right) z^{k_{cov}(B)}$, where the summation is over subsets A and B of $E(G)$ which cover disjoint subsets of vertices $V(A)$ and $V(B)$, $k_{cov}(B)$ denotes the number of connected components in the graph $(V(B), B)$, and $k(A \sqcup B)$ denotes the number of connected components in $(V(G), A \sqcup B)$, the graph which consists of the vertices of G and the edges of A and B.

Theorem 6. *ξ_{lab} and its evaluations, including the colored Sokal polynomial, Zaslavsky's normal function of the colored matroid, the chain polynomial and the multivariate matching polynomial, are all reconstructible.*

Proof. Proof (Sketch) A monomial of U_{lab} for some $M \subseteq E(G)$ corresponds in ξ_{lab} to monomials for which $A \sqcup B = M$. Since A and B are disjoint in vertices, every connected component must be either entirely in A or entirely in B. We can reconstruct the number of connected components spanned by M as the number of x variables in the monomial and the number of vertices spanned by M as the sum of the indices of the x variables. Now we need only choose for each connected component whether it is in A or in B. □

Restricting to uncolored graphs we have:

Theorem 7. *The U and ξ polynomials and all of their evaluations, including the bivariate chromatic polynomial, the chromatic symmetric function X_G and the polychromate polynomial, are reconstructible.*

Acknowledgement. I am grateful to my PhD supervisor J.A. Makowsky, for his advise and support, and to I. Averbouch, L. Lyaudet and N. Zewi for useful discussions on issues related to this work.

References

[1] R. Arratia, B. Bollobás, and G. B. Sorkin. The interlace polynomial of a graph. *J. Comb. Theory, Ser. B*, 92(2):199–233, 2004.

[2] I. Averbouch, B. Godlin, and J. A. Makowsky. A most general edge elimination polynomial. In *WG*, volume 5344 of *LNCS*, pages 31–42, 2008.

[3] J. A. Bondy. A graph reconstructor's manual. In D. Keedwell, editor, *Surveys in Combinatorics*, pages 221–252. Cambridge Univ. Press, 1991.

[4] J. A. Bondy and R. L. Hemminger. Graph reconstruction – a survey. *Journal of Graph Theory*, 1(3):227–268, 1977.

[5] Fagin, Stockmeyer, and Vardi. On monadic NP vs monadic co-NP. *INFCTRL: Information and Computation (formerly Information and Control)*, 120, 1995.

[6] A. Poenitz K. Dohmen and P. Tittmann. A new two-variable generalization of the chromatic polynomial. *Discrete Mathematics & Theoretical Computer Science*, 6(1):69–90, 2003.

[7] W. L. Kocay. An extension of kelly's lemma to spanning subgraphs. *Congressus Numeratium*, 31:109–120, 1981.

[8] B. Manvel and J. M. Weinstein. Nearly acyclic graphs are reconstructible. *Journal of Graph Theory*, 2(1):25–39, 1978.

[9] S. D. Noble and D. J. A. Welsh. A weighted graph polynomial from chromatic invariants of knots. *Ann. Inst. Fourier*, 49(3):1057–1087, 1999.

[10] W. T. Tutte. All the king's horses. In J. A. Bondy and U. S. R. Murty, editors, *Graph Theory and Related Topics*, pages 15–34, 1979.

[11] J. M. Weinstein. Reconstructing colored graphs. *Pacific Journal of Mathematics*, 57(1):307–314, 1975.

Current address: Department of Computer Science, Technion — Israel Institute of Technology, Haifa, Israel

E-mail address: `tkotek@cs.technion.ac.il`