Loebl–Komlós–Sós Conjecture: dense case

Jan Hladký* Diana Piguet†

Abstract

We prove a version of the Loebl–Komlós–Sós Conjecture for dense graphs. For each $q > 0$ there exists a number $n_0 \in \mathbb{N}$ such that for each $n > n_0$ and $k > qn$ the following holds: if G is a graph of order n with at least $\frac{n}{2}$ vertices of degree at least k, then each tree of order $k+1$ is a subgraph of G.

Keywords: Loebl–Komlós–Sós Conjecture, Ramsey number of trees.

1 Introduction

Embedding problems play a central role in Graph Theory. A variety of graph embeddings (subgraphs, minors, subdivisions, immersions, etc) have been studied extensively. A graph (finite, undirected, loopless, simple; here as well as in the rest of the paper) H *embeds* in a graph G if there exists an injective mapping $\phi : V(H) \to V(G)$ which preserves the edges of H, i. e., $\phi(x)\phi(y) \in E(G)$ for every edge $xy \in E(H)$. As a synonym we say that G *contains* H (*as a subgraph*) and write $H \subseteq G$. Let $\mathcal{H}$ be a family of graphs. The graph G is $\mathcal{H}$*-universal* if it contains every graph from $\mathcal{H}$. This fact is denoted by $\mathcal{H} \subseteq G$.

In this paper we investigate embeddings of trees. This topic has received considerable attention during the last 40 years. The class $\mathcal{T}_\ell$ consists of all trees of order ℓ. One can ask which properties force a graph G to be $\mathcal{T}_\ell$-universal. One sufficient condition for $\mathcal{T}_\ell$-universality can be given in terms of minimum degree.

Fact 1.1. *If a graph G has the minimum degree $\delta(G) \geq k$ then $\mathcal{T}_{k+1} \subseteq G$.*

To prove Fact 1.1 it suffices to embed a given tree $T \in \mathcal{T}_{k+1}$ greedily in the host graph G. Loebl, Komlós and Sós conjectured (see [10]) that the minimum degree condition can be relaxed to a median degree one.

Conjecture 1.2 (LKS Conjecture). *Let G be a graph of order n. If at least $\frac{n}{2}$ of the vertices of G have degree at least k, then $\mathcal{T}_{k+1} \subseteq G$.*

The bound on k of the minimal degree of large degree vertices cannot be decreased. Indeed, if G is a graph with maximum degree $k-1$, then it does not contain a star $K_{1,k}$. The graph shown in Figure 1 shows that the requirement on the number of large degree vertices cannot be relaxed substantially below $\frac{n}{2}$. See [28] and [13] for further discussions.

There have been several partial results concerning the LKS Conjecture. In [3], Bazgan, Li and Woźniak proved the conjecture for paths. Piguet and Stein [22] proved that the LKS Conjecture is true when restricted to the class of trees of diameter at most 5, improving upon results

*Institute of Mathematics, Czech Academy of Science. Žitná 25, 110 00, Praha, Czech Republic. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840. E-mail: honzahladky@gmail.com.

†Institute of Computer Science, Czech Academy of Sciences, Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic. With institutional support RVO:67985807.

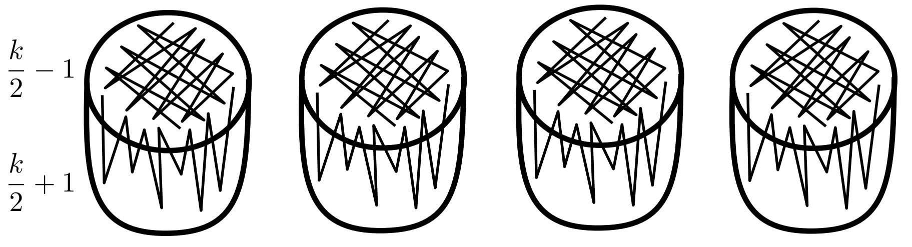


Figure 1: A graph with almost half of its vertices of degree k which does not contain a path of length k.

of Barr and Johansson [2] and of Sun [26]. There are several results proving the LKS Conjecture under additional assumptions on the host graph. Soffer [25] showed that the conjecture is true if the host graph has girth at least 7. Dobson [7] proved the conjecture when the complement of the host graph does not contain a $K_{2,3}$.

A special case of the LKS Conjecture is when $k = \frac{n}{2}$. This is often referred to as the $(\frac{n}{2}-\frac{n}{2}-\frac{n}{2})$ Conjecture, or Loebl's Conjecture. Zhao [28] proved the conjecture for large graphs.

Theorem 1.3. *There exists a number $n_0 \in \mathbb{N}$ such that if a graph G of order $n > n_0$ has at least $\frac{n}{2}$ of the vertices of degrees at least $\frac{n}{2}$, then $\mathcal{T}_{\lfloor \frac{n}{2} \rfloor +1} \subseteq G$.*

An approximate version of the LKS Conjecture for dense graphs was proven by Piguet and Stein [23].

Theorem 1.4. *For each $q, \varepsilon > 0$ there exists a number n_0 such that for each $n > n_0$ and $k > qn$ the following holds. If G is a graph of order n with at least $\frac{n}{2}$ vertices of degree at least $(1+\varepsilon)k$, then $\mathcal{T}_{k+1} \subseteq G$.*

In this paper we strengthen Theorem 1.4 by removing the ε term.

Theorem 1.5 (Main Theorem)**.** *For each $q > 0$ there exists a number $n_0 = n_0(q) \in \mathbb{N}$ such that for each $n > n_0$ and $k > qn$ the following holds. If G is a graph of order n with at least $\frac{n}{2}$ vertices of degree at least k, then $\mathcal{T}_{k+1} \subseteq G$.*

We can see from our proof of Theorem 1.5 that the requirement on the number of vertices of large degree can be relaxed in the case when $\frac{n}{k}$ is far from being an integer.

Theorem 1.6. *For each $q_2 > q_1 > 0$ such that the interval $[\frac{1}{q_2}, \frac{1}{q_1}]$ does not contain any integer, there exist $\varepsilon = \varepsilon(q_1, q_2) > 0$ and n_0 such that for each $n > n_0$ and $k \in (q_1 n, q_2 n)$ the following holds: if G is a graph of order n with at least $(\frac{1}{2}-\varepsilon)n$ vertices of degree at least k, then $\mathcal{T}_{k+1} \subseteq G$.*

In the paper, we explicitly prove only Theorem 1.5. In Section 2 we sketch how the proof method can be revised to give Theorem 1.6. However, determining the optimal value of $\varepsilon(q_1, q_2)$ remains open. Note also that Theorem 1.5 has slightly weaker assumptions on G than Theorem 1.3 when reduced to the case $k = \lfloor \frac{n}{2} \rfloor$ — when n is odd, the requirement on degrees of large vertices in Theorem 1.5 is smaller by one compared to Theorem 1.3.

The property which is considered in the LKS conjecture is given in terms of the median degree. If we consider the average degree instead we obtain a famous conjecture of Erdős and Sós which dates back to 1963.

Conjecture 1.7 (ES Conjecture, [8, p.30])**.** *Let G be a graph of order n with more than $\frac{1}{2}(k-2)n$ edges. Then $\mathcal{T}_k \subseteq G$.*

If true, the ES Conjecture is sharp. After several partial results on the problem, a breakthrough was achieved by Ajtai, Komlós, Simonovits and Szemerédi, who announced a proof of the Erdős–Sós Conjecture for large k.

Theorem 1.8. *There exists a number k_0 such that for each $k > k_0$ the following holds: if a graph G of order n has more than $\frac{1}{2}(k-2)n$ edges, then $\mathcal{T}_k \subseteq G$.*

A version of Theorem 1.8 for k linear in n could be obtained by an application of the Regularity Lemma; such a theorem would be a counterpart to Theorem 1.5. The proof of Theorem 1.8 by Ajtai et al. uses a decomposition technique which substantially generalizes the Regularity Lemma, and which is applicable even to sparse graphs. Hladký, Komlós, Piguet, Simonovits, Stein, and Szemerédi [14, 15, 16, 17] used this decomposition technique to prove an approximate version of the LKS Conjecture (see also [18] for a high-level overview of the proof).

Theorem 1.9. *For each $\varepsilon > 0$ there exists a number k_0 such that for each $k > k_0$ the following holds. If G is a graph of order n with at least $(\frac{1}{2}+\varepsilon)n$ vertices of degrees at least $(1+\varepsilon)k$, then $\mathcal{T}_{k+1} \subseteq G$.*

We believe that the techniques developed for Theorem 1.5 and for Theorem 1.9 can be utilized to proving the LKS Conjecture for k sufficiently large.

The current work builds on techniques of Zhao [28] and of Piguet and Stein [23]. We postpone a detailed discussion of similarities between our approach and theirs and of our own contribution until Section 2. After the first version of this manuscript was posted on the arXiv, Oliver Cooley [5] published an independent proof of Theorem 1.5.

1.1 Ramsey number of trees

In this section we show the connection between the LKS Conjecture and the Ramsey number of trees. For two graphs F and H we write $R(F, H)$ for the *Ramsey number* of the graphs F and H. This is the smallest number m such that in each red/blue edge-coloring of K_m there is a red copy of F or a blue copy of H. For two families of graphs $\mathcal{F}$ and $\mathcal{H}$ the Ramsey number $R(\mathcal{F}, \mathcal{H})$ is the smallest number m such that in each red/blue edge-coloring of K_m the graph induced by the red edges is $\mathcal{F}$-universal, or the graph induced by the blue edges is $\mathcal{H}$-universal. Theorem 1.5 implies an almost tight upper bound (up to an additive error of one) on the Ramsey number of pairs of families of trees of similar orders. This partially answers a question of Erdős, Füredi, Loebl and Sós [10]. For a fixed real $p \in (0, \frac{1}{2})$ consider two natural numbers ℓ_1 and ℓ_2 such that

$$n_0 < \ell_1 \le \ell_2 < \frac{\ell_1}{p} \;, \tag{1.1}$$

where $n_0 = n_0(\frac{p}{2})$ comes from Theorem 1.5. Consider any red/blue edge-coloring of the graph $K_{\ell_1+\ell_2}$. We color a vertex $v \in V(K_{\ell_1+\ell_2})$ red if it incident with at least ℓ_1 red edges, and blue otherwise (in which case it is incident with at least ℓ_2 blue edges). Thus at least half of the vertices of $K_{\ell_1+\ell_2}$ have the same color. Applying Theorem 1.5 to the graph whose edges are induced by this color, we conclude that $R(\mathcal{T}_{\ell_1+1}, \mathcal{T}_{\ell_2+1}) \le \ell_1 + \ell_2$.

For the lower bound, first consider the case when at least one of ℓ_1 and ℓ_2 is odd. It is a well-known fact that there exists a red/blue edge-coloring of $K_{\ell_1+\ell_2-1}$ such that the red degree of every vertex is $\ell_1 - 1$. Neither a red copy of K_{1,ℓ_1} nor a blue copy of K_{1,ℓ_2} is contained in $K_{\ell_1+\ell_2-1}$ with this coloring. Thus $R(\mathcal{T}_{\ell_1+1}, \mathcal{T}_{\ell_2+1}) > \ell_1 + \ell_2 - 1$. A construction in a similar spirit shows that $R(\mathcal{T}_{\ell_1+1}, \mathcal{T}_{\ell_2+1}) > \ell_1 + \ell_2 - 2$, if both ℓ_1 and ℓ_2 are even. Under the assumptions given by (1.1) we thus have

$$R(\mathcal{T}_{\ell_1+1}, \mathcal{T}_{\ell_2+1}) = \ell_1 + \ell_2 \ , \quad \text{if } \ell_1 \text{ is odd or } \ell_2 \text{ is odd, and} \tag{1.2}$$

$$\ell_1 + \ell_2 - 1 \leq R(\mathcal{T}_{\ell_1+1}, \mathcal{T}_{\ell_2+1}) \leq \ell_1 + \ell_2 \ , \quad \text{otherwise.} \tag{1.3}$$

The ES Conjecture, if true, shows that the lower bound in (1.3) is attained.

Ramsey numbers of several other classes of trees have been investigated; the reader is referred to a survey of Burr [4] and to newer results in [9, 11, 12].

2 Outline of the proof

We iterate the following procedure in steps $i = 1, 2, 3, \ldots$. At the beginning of step i we are given sets $V_1, \ldots, V_{i-1}$ that were obtained in previous steps. We then find a set $Q \subseteq V(G) \setminus \bigcup_{j<i} V_j$ such that at least about a half of the vertices in Q are *large* (i. e., of degree at least k). Furthermore, the set Q is almost isolated from the rest of the graph. Using the Regularity Lemma, we try to embed T in Q. If we do not succeed, then we can extract from Q a subset $V_i \subseteq Q$ of size approximately k which is nearly isolated from the rest of the graph, and for which at least half of the vertices are large. If we cannot embed T in any of the iterating steps (i. e., $V(G) \setminus \bigcup_i V_i \cong \emptyset$), we obtain a particular configuration of the graph G, called the *Extremal Configuration*. The structure of G is then very similar to that depicted in Figure 1. In this case, we prove that $T \subseteq G$, without the use of the Regularity Lemma.

In the remainder of the overview, we explain in more detail the proof of the part using the Regularity Lemma, as well as the part when G is in the Extremal configuration.

The Regularity Lemma Part. Before applying the Regularity Lemma, we first resolve two simple cases. The first one is when Q is close to a bipartite graph with one of its color classes being the large vertices (see Lemma 5.1). The second case (see Lemma 5.5) is when the tree T is locally unbalanced (see the definition on page 14). In both cases easy arguments show that $T \subseteq G$.

In other cases we use the Regularity Lemma on the graph G and obtain a cluster graph $\mathbf{G}$. We apply a matching lemma (Lemma 5.8) to the subgraph induced by the clusters in Q. This lemma guarantees the existence of one of two certain matching structures in $\mathbf{G}$. Each of these structures exposes a matching M in the cluster graph, and two clusters A and B that are adjacent in $\mathbf{G}$ and that have high average degree to the matching M. These structures are called Case I and Case II. The principle of the embedding is to use the edges of M to embed parts of the tree T in them, and use the clusters A and B to connect these parts.

The Extremal Case Configuration. In the Extremal case we are given disjoint sets $V_1, \ldots, V_i \subseteq V(G)$ such that each of them has size approximately k, contains at least nearly $\frac{k}{2}$ large vertices, and each set V_j is almost isolated from the rest of the graph.

If the sets $V_1, \ldots, V_i$ exhaust the whole graph G, we are able to show $T \subseteq G$ as follows. We find a set V_{i_0} so that most of T can be embedded in V_{i_0}. We may need to use a few edges that connect distinct sets V_j and embed some part of T outside V_{i_0}. The way of finding these "bridges" depends on the structure of the tree T.

If $V_1, \ldots, V_i$ do not exhaust G, the method remains the same. However, it has two possible outcomes. Either we show that $T \subseteq G$ or we are able to exhibit a set $Q \subseteq V \setminus \bigcup_{j<i} V_j$ with the properties as above allowing the next step of the iteration.

Strengthening of Theorem 1.5 — Theorem 1.6. The only place where we use the exact bound on the number of large vertices is the last step of the Extremal case. That is, the whole vertex set $V(G)$ is decomposed into sets $V_1, \dots, V_s$, each of size approximately k. Assume now that $k \in (q_1 n, q_2 n)$. We have $n = |V_1| + |V_2| + \ldots + |V_s| \approx ks \in (q_1 sn, q_2 sn)$, yielding that the the interval $(q_1 s, q_2 s)$ must contain 1 (or at least to be "close to 1"). Thus the Extremal case cannot occur when $[\frac{1}{q_2}, \frac{1}{q_1}] \cap \mathbb{N} = \emptyset$. This suffices to prove Theorem 1.6.

Relation to previous work. The proof of Theorem 1.5 is inspired by techniques used to prove Theorem 1.4 ([23]) and Theorem 1.3 ([28]). Both these papers build on a seminal paper of Ajtai, Komlós and Szemerédi [1] where an approximate version of the ($\frac{n}{2}$–$\frac{n}{2}$–$\frac{n}{2}$) Conjecture was proven. In [1] the basic strategy is outlined. It is worth noting that even though [1] addresses explicitly only the ($\frac{n}{2}$–$\frac{n}{2}$–$\frac{n}{2}$) Conjecture the proof actually yields Theorem 1.4 in the regime $\frac{k}{n} \geq \frac{1}{2}$. As in the proof overview above, the key step is a certain matching lemma applied to the cluster graph of the host graph.

The key ingredient in [28] was to identify — using the approach of Ajtai, Komlós and Szemerédi combined with the Stability method of Simonovits [24] — one extremal case. This extremal case was analysed and resolved by ad-hoc methods. The main contribution of [23] is a more general matching lemma, which is applicable even when $\frac{k}{n} < \frac{1}{2}$. In this paper we further strengthen the matching lemma from [23]. The Extremal case is an extensive generalization of the Extremal case from [28].

Algorithmic questions. Let us remark that our proof of Theorem 1.5 yields a polynomial time algorithm for finding an embedding of each tree $T \in \mathcal{T}_{k+1}$ in G, given that k and G satisfy the conditions of Theorem 1.5. Indeed, all the existential results we use (Regularity Lemma, and various matching theorems) are known to have polynomial-time constructive algorithmic counterparts. We omit details.

3 Notation and preliminaries

For $n \in \mathbb{N}$ we write $[n] = \{1, 2, \dots, n\}$. The symbol $\triangle$ means the symmetric difference of two sets. The function $\mathrm{ci} : \mathbb{R} \to \mathbb{Z}$ is the *closest integer function* defined by $\mathrm{ci}(x) = \lfloor x \rfloor$ if $x - \lfloor x \rfloor < 0.5$, and $\mathrm{ci}(x) = \lceil x \rceil$ otherwise.

We use standard graph theory terminology and notation, following Diestel's book [6]. We define here only symbols that are not used there. The order of a graph H and the number of its edges are denoted by $v(H)$ and $e(H)$, respectively. For two vertex sets X and Y we write $E(X, Y)$ for the set of edges with one end-vertex in X and the other in Y. We write $e(X, Y) = |E(X, Y)|$ (note that edges inside $X \cap Y$ get counted only once). When X and Y are disjoint, we write $H[X, Y]$ for the bipartite graph they induced. For a vertex x and a vertex set X we define $\deg(x, X) = \deg_X(x) = e(\{x\}, X)$. For two sets $X, Y \subseteq V(H)$ we define the *average degree* from X to Y by $\overline{\deg}_H(X, Y) = \frac{e(X, Y \setminus X)}{|X|}$. We write $\overline{\deg}_H(X)$ as a short for $\overline{\deg}_H(X, V(H))$. Let X and Y are arbitrary (not necessarily disjoint vertex sets). We define two variants of the minimum degree: $\delta(X) = \min_{v \in X} \deg(v)$, and $\delta(X, Y) = \min_{v \in X} \deg(v, Y)$. In this case, we may write H in the subscript (e.g. $\delta_H(X)$) to emphasize which graph we are dealing with. We denote by $\mathrm{N}(x)$ the set of neighbors of the vertex x, by $\mathrm{N}_X(x)$ the neighborhood of x restricted to a set X, i.e., $\mathrm{N}_X(x) = \mathrm{N}(x) \cap X$, and by $\mathrm{N}(X)$ the set of all vertices in H which are adjacent to at least one vertex from X, i.e., $\mathrm{N}(X) = \bigcup_{v \in X} \mathrm{N}(v)$.

Let $P = v_1 v_2 \dots v_\ell$ be a path. For arbitrary sets of vertices $X_1, X_2, \dots, X_\ell$ we say that P is an *$X_1 - X_2 - \ldots - X_\ell$-path* if $v_i \in X_i$ for every $i \in [\ell]$. An edge xy is an $X - Y$ edge if $x \in X$

and $y \in Y$ and a matching M is an $X-Y$ matching if its every edge is an $X-Y$ edge.

A pair (H,ω) is a *weighted graph* if H is a graph and $\omega : E(H) \to (0,+\infty)$ is a weight function. For two sets $X,Y \subseteq V(H)$ the *weight of the edges crossing from X to Y* is defined by $\omega(X,Y) = \sum_{xy\in E(X,Y)} \omega(xy)$. Denote also by ω the weighted degree, $\omega(v) = \sum_{u\in V(H),vu\in E(H)} \omega(vu)$. For a vertex v and a vertex set X we define $\omega(v,X)$ analogously to $\deg(v,X)$.

We omit rounding symbols when this does not effect the correctness of calculations.

3.1 Trees

Let T be a rooted tree with a root $r \in V(T)$. We define a partial order $\preceq$ on $V(T)$ by saying that $a \preceq b$ if and only if the vertex b lies on the (unique) path connecting a with r. If $a \preceq b$ and $a \neq b$ we say that *a is below b*. A vertex a is a *child of b* if $a \preceq b$ and $ab \in E(T)$. The vertex b is then the *parent of a*. $\mathrm{Ch}(b)$ denotes the set of children of b. The parent of a vertex a is denoted $\mathrm{Par}(a)$ (note that $\mathrm{Par}(a)$ is undefined if $a = r$). We extend the definitions of $\mathrm{Ch}(\cdot)$ and $\mathrm{Par}(\cdot)$ to an arbitrary set $U \subseteq V(T)$ by $\mathrm{Par}(U) = \bigcup_{u\in U} \mathrm{Par}(u)$ and $\mathrm{Ch}(U) = \bigcup_{u\in U} \mathrm{Ch}(u)$. We say that a tree $T_1 \subseteq T$ is *induced* by a vertex $x \in V(T)$ if $V(T_1) = \{v \in V(T) : v \preceq x\}$ and we write $T_1 = T(r, \downarrow x)$, or if the root is obvious from the context $T_1 = T(\downarrow x)$. Subtrees induced by a vertex are called *end subtrees*. Other subtrees are called *internal subtrees*. A subtree T_0 of T is a *full-subtree*, if there exists a vertex $y \in V(T)$ and a set $C \subseteq \mathrm{Ch}(y)$, $C \neq \emptyset$ such that $T_0 = T[\{y\} \cup \bigcup_{b\in C}\{v : v \preceq b\}]$. *Internal* vertices are simply non-leaf vertices.

We will want to find a full-subtree in such a way that we have some control over its order or over its number of leaves. To this end we will use the following fact.

Fact 3.1 ([28, Fact 7.9])**.** *Let (T,r) be a rooted tree of order m with ℓ leaves.*

(i) For each integer m_0, $0 < m_0 \leq m$, there exists a full-subtree T_0 of T of order $\tilde{m} \in [\frac{m_0}{2}, m_0]$.

(ii) For each integer ℓ_0, $0 < \ell_0 \leq \ell$, there exists a full-subtree T_0 of T with $\tilde{\ell}$ proper leaves (i.e. leaves of T), where $\tilde{\ell} \in [\frac{\ell_0}{2}, \ell_0]$.

For each tree F we write $F_\oplus$ and $F_\ominus$ for the vertices of its two color classes with $F_\oplus$ being the larger one. We define the *gap* of the tree F as $\mathrm{gap}(F) = |F_\oplus| - |F_\ominus|$. For a tree F, a partition of its vertices into sets U_1 and U_2 is called *semi-independent* if $|U_1| \leq |U_2|$ and U_2 is an independent set. Furthermore, the *discrepancy* of (U_1,U_2) is $\mathrm{disc}(U_1,U_2) = |U_2| - |U_1|$ and the discrepancy of F is defined as

$$\mathrm{disc}(F) = \max\{\mathrm{disc}(U_1,U_2) : (U_1,U_2) \text{ is semi-independent}\} .$$

Clearly, $\mathrm{gap}(F) \leq \mathrm{disc}(F)$.

The next three facts relate discrepancy to other properties of trees.

Fact 3.2 ([28, Fact 6.9])**.** *Let (U_1,U_2) be a semi-independent partition of a tree T of order $v(T) > 1$. Then U_2 contains at least $|U_2| - |U_1| + 1$ leaves.*

Fact 3.3. *Let r be a vertex of a tree F, and let (U_1,U_2) be any semi-independent partition of F. Let $\mathcal{K}$ be a subset of the components of the forest $F - \{r\}$ and let $V(\mathcal{K})$ denote all the vertices contained in the components of $\mathcal{K}$. Then*

(i) $||V(\mathcal{K}) \cap F_\oplus| - |V(\mathcal{K}) \cap F_\ominus|| \leq \mathrm{disc}(F) + 1$, and

(ii) $|V(\mathcal{K}) \cap U_2| - |V(\mathcal{K}) \cap U_1| \leq \mathrm{disc}(F) + 1$.

Proof. We focus first on (i). The statement is obvious when $|V(\mathcal{K}) \cap F_\oplus| - |V(\mathcal{K}) \cap F_\ominus| = 0$. Suppose that $|V(\mathcal{K}) \cap F_a| - |V(\mathcal{K}) \cap F_b| = \ell > 0$, where $a, b \in \{\oplus, \ominus\}$, $a \neq b$ is a choice of the color classes. It is enough to exhibit a semi-independent partition (W_1, W_2) of the tree F with $|W_2| - |W_1| \geq \ell - 1$. Partition the components of the forest $F - \{r\}$ that are not included in $\mathcal{K}$ into two families $\mathcal{A}$ and $\mathcal{B}$ so that $\mathcal{A}$ contains those components $K \notin \mathcal{K}$ for which $|V(K) \cap F_a| \geq |V(K) \cap F_b|$. Then the partition below satisfies the requirements.

$$\begin{aligned} W_1 &= \{r\} \cup (V(\mathcal{K}) \cap F_b) \cup (V(\mathcal{A}) \cap F_b) \cup (V(\mathcal{B}) \cap F_a) \ , \\ W_2 &= (V(\mathcal{K}) \cap F_a) \cup (V(\mathcal{A}) \cap F_a) \cup (V(\mathcal{B}) \cap F_b) \ . \end{aligned}$$

The proof of (ii) is similar, and we only sketch it. Again, we shall exhibit a semi-independent partition (W_1, W_2) with $|W_2| - |W_1| \geq |V(\mathcal{K}) \cap U_2| - |V(\mathcal{K}) \cap U_1| - 1$. We put r into W_1. On the components of $\mathcal{K}$ the partition into W_1 and W_2 is inherited from the partition (U_1, U_2). Every component $K \notin \mathcal{K}$ of $F - \{r\}$ is partitioned so that W_2 gets the majority color class of K. □

Fact 3.4. *Suppose that T is a tree with $\mathrm{disc}(T) \leq \ell$. Let $V(T) = U_1 \dot\cup U_2$ be a partition such that U_2 is independent. Then for the set X of the leaves in U_1 that have another leaf-sibling in U_1 we have $|X| \leq \ell + |U_1| - |U_2|$.*

Proof. We have $|X| \geq 2|\mathrm{Par}(X)|$. Thus, if $|X| > \ell + |U_1| - |U_2|$, we consider the partition

$$\big((U_1 \setminus X) \cup \mathrm{Par}(X) \ , \ (U_2 \setminus \mathrm{Par}(X)) \cup X\big) \ .$$

Even though we do not necessarily have $\mathrm{Par}(X) \subseteq U_2$ this is semi-independent partition of discrepancy at least $|U_2| - |U_1| + 2(|X| - |\mathrm{Par}(X)|) > \ell$, a contradiction. □

3.2 Greedy embeddings

Given a tree T and a graph H there are several situations when one can embed T in H greedily. The simplest such setting is given in Fact 1.1. An analogous procedure works if H is bipartite, $H = (V_1, V_2; E)$, and $\delta(V_1, V_2) \geq |T_\oplus|, \delta(V_2, V_1) \geq |T_\ominus|$. The facts stated below generalize the greedy procedure.

Fact 3.5 ([28, Fact 7.2(2)]). *Let (U_1, U_2) be a semi-independent partition of a tree T. If there are two disjoint sets of vertices V_1 and V_2 of a graph H such that $\min\{\delta(V_1, V_2), \delta(V_1, V_1), \delta(V_2, V_1)\} \geq |U_1|$ and $\delta(V_1) \geq v(T) - 1$, then $T \subseteq H$.*

Fact 3.6 ([28, Fact 7.2(1)]). *Suppose that H is a graph with a bipartite subgraph $K = (W_1, W_2; J)$. If $\delta(K) > \frac{\ell}{2}$ and $\delta_H(W_1) \geq \ell$ then $\mathcal{T}_{\ell+1} \subseteq H$.*

Fact 3.7. *Suppose $H' \subseteq H$ are two graphs. If $\delta(H') \geq x$ and $\delta_H(V(H')) \geq \ell$, then $F \subseteq H$ for each tree $F \in \mathcal{T}_{\ell+1}$ with at least $\ell - x$ leaves.*

Proof. We first embed the internal vertices of F in H' using the greedy procedure from Fact 1.1. We can then extend this embedding using the high degrees of $V(H')$. □

The next lemma allows us to embed a tree T into a graph containing a bipartite subgraph H which can *almost* accomodate T. So, additional connecting structures $\mathcal{M}$, $\mathcal{E}$ that will allow to divert small parts of T elsewhere are introduced. The main structures assumed in the lemma are shown in Figure 2.

Lemma 3.8. *Suppose that $\alpha \in (0, \frac{1}{10})$ is arbitrary. For each tree $T \in \mathcal{T}_{k+1}$ with less than αk leaves the following holds. Suppose that a bipartite graph $H = (A, B; E)$ and graphs $\{H_\kappa\}_{\kappa \in I}$ (where I is arbitrary) are pairwise vertex-disjoint subgraphs of a graph G on vertex set V. Suppose that the following properties are fulfilled.*

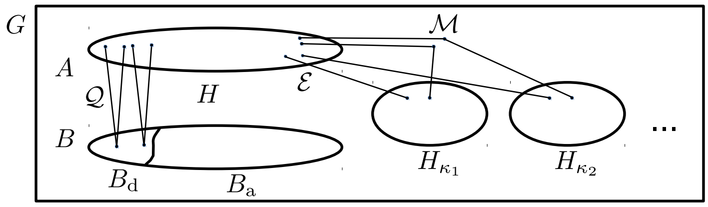


Figure 2: The situation in Lemma 3.8. Most of the set $T_\ominus$ is embedded in A, most of the set $T_\oplus$ will be embedded in B. The connections $\mathcal{E}$ and $\mathcal{M}$ are used to divert parts of T to the graphs H_κ.

(i) $\delta(H_\kappa) > 34\alpha k$ *for each* $\kappa \in I$.

(ii) $\delta_G(A) \geq k$.

(iii) *There exists an* $A - (\bigcup_\kappa(V(H_\kappa)))$*-matching* $\mathcal{E}$*, and a family* $\mathcal{M}$ *of pairwise vertex-disjoint* $A - (V \setminus V(H)) - (\bigcup_\kappa V(H_\kappa))$ *paths. Moreover,* $V(\mathcal{E}) \cap V(\mathcal{M}) = \emptyset$.

(iv) $|\mathcal{E}| + |\mathcal{M}| < \alpha k$.

(v) $|A| + |\mathcal{E}| \geq |T_\ominus|$.

(vi) $|B| + |\mathcal{E}| + |\mathcal{M}| \geq |T_\oplus| - 1$.

(vii) $\delta(A, B) \geq |B| - \alpha k$.

(viii) *The set* B *has a decomposition* $B = B_{\mathrm{a}} \dot\cup B_{\mathrm{d}}$, $|B_{\mathrm{d}}| \leq \alpha k$, $\delta(B_{\mathrm{a}}, A) \geq |A| - \alpha k$, *and there exists a family* $\mathcal{Q}$ *of* $|B_{\mathrm{d}}|$ *pairwise vertex-disjoint* $A - B_{\mathrm{d}} - A$ *paths. Moreover,* $V(\mathcal{Q}) \cap (V(\mathcal{E}) \cup V(\mathcal{M})) = \emptyset$.

Then, $T \subseteq G$.

The proof is given in the Appendix.

3.3 Specific notation

A graph H is said to have the *LKS-property* (with parameter k) if at least half of its vertices have degree at least k, i. e., we have $|L| \geq \frac{v(H)}{2}$, where $L = \{v \in V(H) \, : \, \deg_H(v) \geq k\}$.

When we refer to q, n_0, n, k or G in the rest of the paper, we always refer to the objects from the statement of Theorem 1.5. The vertex set of G is denoted by V. We partition $V = L \dot\cup S$, where $L = \{v \in V \, : \, \deg(v) \geq k\}$ and $S = \{v \in V \, : \, \deg(v) < k\}$. We call the vertices from L *large* and the vertices from S *small*. The hypothesis of Theorem 1.5 implies that $|L| \geq \frac{n}{2}$. Finally T denotes a tree of order $k+1$ that we want to embed in G.

We write $\alpha \ll \beta$ to express that α is sufficiently small compared to β.

4 Proof of the Main Theorem (Theorem 1.5)

The proof of Theorem 1.5 is based on an iterated application of Lemma 4.1 and 4.2 below. To state Lemma 4.1 we need to introduce the notion of (β, σ)-extremality. The (β, σ)-extremality says that a part of a graph resembles the extremal structure as in Figure 1. For two reals $\beta, \sigma \in (0, 1)$, a partition of the vertex set $V = V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_\ell \dot\cup \tilde{V}$ is (β, σ)-*extremal* if the following conditions are satisfied.

- $\ell \geq 1$.
- $(1-\beta)k \leq |V_i| \leq (1+\beta)k$ for each $i \in [\ell]$.
- $\tilde{V} = \emptyset$ or $|\tilde{V}| > \sigma k$.
- $e(V_i, V \setminus V_i) \leq \beta k^2$ for each $i \in [\ell]$, and $e(\tilde{V}, V \setminus \tilde{V}) \leq \beta k^2$.
- $(\frac{1}{2} - \beta)k \leq |V_i \cap L|$ for each $i \in [\ell]$.
- $|\tilde{V} \cap L| \leq (\frac{1}{2} - \sigma)|\tilde{V}|$.

Lemma 4.1 below, which will be proved in Section 7, deals with a graph that admits an extremal partition.

Lemma 4.1. *Given a number $q > 0$, there exists a constant $c_{\mathbf{E}} > 0$ such that the following holds. For each $\sigma \leq c_{\mathbf{E}}$ there exists a number $\beta \in (0, \sigma)$ such that if G is a graph satisfying the LKS-property with $k \geq qn$ that admits a (β, σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde{V}$, then $\mathcal{T}_{k+1} \subseteq G$, or there exists a set $Q \subseteq \tilde{V}$ such that*

(i) $|Q| > \frac{k}{2}$.

(ii) $|Q \cap L| > \frac{|Q|}{2}$.

(iii) $e(Q, V \setminus Q) < \sigma k^2$.

The next statement, which will be proved in Section 6, entails the regularity part of the proof of Theorem 1.5.

Lemma 4.2. *Given numbers $q, c, \rho > 0$ there are numbers $\lambda \in (0, \rho)$ and $n_0 = n_0(q, c, \rho) \in \mathbb{N}$ such that for each graph G on $n \geq n_0$ vertices satisfying the LKS-property with $k \geq qn$ with a subset $V_* \subseteq V$ having the following properties*

(i) $|V_*| > ck$,

(ii) $e(V_*, V \setminus V_*) \leq \lambda k^2$, *and*

(iii) $|L \cap V_*| \geq \frac{1}{2}(1-\lambda)|V_*|$,

there exists a subset $V' \subseteq V_*$ such that

- ⋄ $(1-\rho)k \leq |V'| \leq (1+\rho)k$,
- ⋄ $|V' \cap L| \geq \frac{1}{2}|V'|$, *and*
- ⋄ $e(V', V \setminus V') \leq \rho k^2$,

or $\mathcal{T}_{k+1} \subseteq G$.

Proof of Theorem 1.5. Given $q > 0$ let $c_{\mathbf{E}}$ be given by Lemma 4.1. Further let β be given by Lemma 4.1 with input parameters $q, c_{\mathbf{E}}$ and $\sigma = c_{\mathbf{E}}$. Set $c = \frac{q\beta}{2}$ and $C = \lceil \frac{1}{q} \rceil$. We find a sequence of parameters

$$0 < \sigma_1 \ll \rho_1 \ll \sigma_2 \ll \rho_2 \ll \cdots \ll \rho_{C-1} \ll \sigma_C \ll \rho_C \ , \tag{4.1}$$

constructed as follows. Set $\rho_C = c$. Inductively for each $i = C, \dots, 1$ let $\sigma_i = \lambda(q, c, \rho_i)$ be given by Lemma 4.2 for input parameters q, c and ρ_i. Further let β_i be given by Lemma 4.1 with input parameters $q, c_{\mathbf{E}}$ and $\frac{\sigma_i}{2}$. Finally for $i > 1$ set $\rho_{i-1} = \frac{\beta_i}{C}$. Set $n_0 = \max_{i=1,\dots,C}\{n_0(q, c, \rho_i)\}$, where the numbers $n_0(q, c, \rho_i)$ are from Lemma 4.2.

Let G be a graph satisfying the conditions of Theorem 1.5 (i.e., q is fixed, $n \geq n_0$, and $k > qn$).

Recall that $\mathrm{ci}(x)$ denotes the closest integer to x. Let $\vartheta = \mathrm{ci}(\frac{n}{k})$. We iterate the following process for at most ϑ steps. In step i, $i \leq \vartheta$, we prove that $\mathcal{T}_{k+1} \subseteq G$ or we define a set $V_i \subseteq V \setminus \bigcup_{j<i} V_j$ such that the following conditions are fulfilled for each $j \in [i]$.

(P1)$_i$ $(1 - \rho_i)k \leq |V_j| \leq (1 + \rho_i)k$,

(P2)$_i$ $|L \cap V_j| \geq (\frac{1}{2} - \rho_i)k$, and

(P3)$_i$ $e(V_j, V \setminus V_j) \leq \rho_i k^2$.

In step $i = 1$, we apply Lemma 4.2 with parameters q, c, ρ_1 and input set $V_* = V$. We obtain that $\mathcal{T}_{k+1} \subseteq G$, or there exists a set V_1 satisfying (P1)$_1$, (P2)$_1$, and (P3)$_1$. In step $i > 1$, suppose that we have sets $V_1, \dots, V_{i-1}$ satisfying (P1)$_{i-1}$, (P2)$_{i-1}$, and (P3)$_{i-1}$. Set $V^* = V \setminus \bigcup_{j<i} V_j$.

First assume that $|V^*| > ck$. If $|L \cap V^*| \geq \frac{1}{2}(1 - \sigma_i)|V^*|$, the graph G satisfies the conditions of Lemma 4.2 (with input parameters q, c, ρ_i and input set $V_* = V^*$). Indeed, $|V^*| > ck$ by assumption, $e(V^*, V \setminus V^*) \leq (i-1)\rho_{i-1}k^2 \leq \beta_i k^2 < \sigma_i k^2$ because $V_1, \dots, V_{i-1}$ satisfy (P3)$_{i-1}$, and $|L \cap V^*| \geq \frac{1}{2}(1 - \sigma_i)|V^*|$ by assumption.

If $|L \cap V^*| < \frac{1}{2}(1 - \sigma_i)|V^*|$, then the partition $V = V_1 \dot\cup \dots \dot\cup V_{i-1} \dot\cup V^*$ is $(C\rho_{i-1}, \frac{\sigma_i}{2})$-extremal. Indeed,

- $i > 1$;
- $(1 - C\rho_{i-1})k \leq (1 - \rho_{i-1})k \leq |V_j| \leq (1 + \rho_{i-1})k \leq (1 + C\rho_{i-1})k$ for each $j \leq i - 1$ by (P1)$_{i-1}$;
- $|V^*| > ck \geq \frac{\sigma_i k}{2}$ by assumption;
- $e(V_j, V \setminus V_j) \leq \rho_{i-1}k^2 \leq C\rho_{i-1}k^2$ for each $j \leq i - 1$ by (P3)$_{i-1}$ and
  $e(V^*, V \setminus V^*) \leq (i-1)\rho_{i-1}k^2 < C\rho_{i-1}k^2$;
- $|V_j \cap L| \geq (\frac{1}{2} - \rho_{i-1})k \geq (\frac{1}{2} - C\rho_{i-1})k$ for each $j \leq i - 1$ by (P2)$_{i-1}$;
- $|V^* \cap L| < \frac{1}{2}(1 - \sigma_i)|V^*| = (\frac{1}{2} - \frac{\sigma_i}{2})|V^*|$.

Therefore Lemma 4.1 with parameters $q, c_{\mathbf{E}}, \frac{\sigma_i}{2}$ applies. Thus $\mathcal{T}_{k+1} \subseteq G$, or there exists a set $Q \subseteq V^*$ satisfying Lemma 4.1 (i)–(iii). It is enough to assume the latter case. Here again, the graph G satisfies the conditions of Lemma 4.2 (with input parameters q, c, ρ_i and input set $V_* = Q$). Indeed, $|Q| > \frac{k}{2} \geq \frac{q\beta}{2}k = ck$, $e(Q, V \setminus Q) < \frac{\sigma_i}{2}k^2 < \sigma_i k^2$ and $|Q \cap L| > \frac{|Q|}{2} > \frac{1}{2}(1 - \sigma_i)|Q|$. Thus Lemma 4.2 yields that $\mathcal{T}_{k+1} \subseteq G$, or that there exists a set $V_i \subseteq Q$ satisfying Properties (P1)$_i$–(P3)$_i$.

It remains to deal with the case $|V^*| \leq ck$. The set V is decomposed into sets $V_1, \dots, V_{i-1}$, each of which is of size approximately k, and a little set V^*. Thus, $i - 1 = \theta$. Having found sets $V_1, \dots, V_\vartheta$ satisfying (P1)$_\vartheta$–(P3)$_\vartheta$, we set $V_1' = V_1 \cup V^*$ and $V_j' = V_j$ for $j \geq 2$. The thus defined partition $V = V_1' \dot\cup \dots \dot\cup V_\vartheta' \dot\cup \emptyset$ is $(\beta, c_{\mathbf{E}})$-extremal. Indeed, by (P1)$_\vartheta$–(P3)$_\vartheta$, we have

- $\vartheta \geq 1$;

- $(1-\beta)k \leq (1-\rho_\vartheta)k \leq |V_j| \leq |V_j'| \leq |V_j| + |V^*| \leq (1+\rho_\vartheta+c)k \leq (1+\beta)k$ for each $j \leq \vartheta$;
- $e(V_j', V \setminus V_j') \leq e(V_j, V \setminus V_j) + e(V^*, V \setminus V^*) \leq \rho_\vartheta k^2 + (\vartheta-1)\rho_\vartheta k^2 \leq \beta k^2$ for each $j \leq \vartheta$ (the summand $e(V^*, V \setminus V^*)$ is necessary only when $j=1$);
- $|V_j' \cap L| \geq |V_j \cap L| \geq (\frac{1}{2} - \rho_\vartheta)k \geq (\frac{1}{2} - \beta)k$ for each $j \leq \vartheta$.

Lemma 4.1 with parameters $q, c_{\mathbf{E}}$ and $\sigma = c_{\mathbf{E}}$ yields that $\mathcal{T}_{k+1} \subseteq G$ (as no new set Q can be found). $\square$

5 Tools for the proof of Lemma 4.2

5.1 Sparsity in the set of large vertices

Suppose that G is a graph with the LKS-property with parameter k such that its set L of large vertices is almost independent. In this section we provide an ad-hoc argument showing that in (a situation a bit more general than) the setting above, we have $\mathcal{T}_{k+1} \subseteq G$. Indeed, in this case G is close to a k-regular bipartite graph with color classes L and S, and thus we are roughly in the setting of Fact 3.6.

Lemma 5.1. *For every $q > 0$ there exists a real $c_{\mathbf{S}} > 0$ such that for each $c \in (0, c_{\mathbf{S}}]$ and each n-vertex graph $G = (V, E)$ with the LKS-property with parameter $k > qn$, and with a set $V_* \subseteq V$ satisfying*

(i) $|V_*| > \sqrt[4]{c}k$,

(ii) $e(V_*, V \setminus V_*) < ck^2$,

(iii) $(\frac{1}{2} - c)|V_*| < |V_* \cap L|$, *and*

(iv) $e(G[V_* \cap L]) < cn^2$,

we have $\mathcal{T}_{k+1} \subseteq G$.

Proof. Set $c_{\mathbf{S}} = q^9 10^{-8}$. Let $c \in (0, c_{\mathbf{S}}]$ be arbitrary. Let G be any graph satisfying the assumptions of the lemma. First observe that

$$|V_*| \geq \tfrac{3k}{4} \ . \tag{5.1}$$

Indeed, suppose the contrary. Assumptions (i) and (iii) imply that $|V_* \cap L| \geq (\frac{1}{2} - c)\sqrt[4]{c}k > \frac{1}{4}\sqrt[4]{c}k$. By the negation of (5.1), each vertex in $V_* \cap L$ emanates at least $\frac{k}{4}$ edges into $V \setminus V_*$. Therefore $e(V_* \cap L, V \setminus V_*) > \frac{1}{16}\sqrt[4]{c}k^2$, a contradiction to (ii).

Fix a set $L_1 \subseteq L \cap V_*$ of size $|L_1| = (\frac{1}{2} - c)|V_*|$. Define $L_2 = \{u \in L_1 \ : \ \deg(u, V_* \setminus L_1) \geq (1 - 2\sqrt{c})k\}$. For each vertex $x \in L_1 \setminus L_2$ we have that $\deg(x, L_1) + \deg(x, V \setminus V_*) > 2\sqrt{c}k$, otherwise x would have been included in L_2. Summing up (ii) and (iv), we have $e(G[L_1]) + e(L_1 \setminus L_2, V \setminus V_*) < 2cn^2$. Theorefore, we have that

$$|L_1 \setminus L_2| \leq \frac{4cn^2}{2\sqrt{c}k} \stackrel{(5.1)}{<} 3\sqrt{c}q^{-2}|V_*| \leq \frac{1}{2}\sqrt[4]{c}|V_*| \ .$$

Consequently,

$$|L_2| > (\tfrac{1}{2} - \sqrt[4]{c})|V_*| \ . \tag{5.2}$$

We verify that the set $\tilde{S} = \{u \in V_* \setminus L_1 \ : \ \deg(u, L_2) \geq (1 - \sqrt[8]{c})k\}$ covers almost the whole set $V_* \setminus L_1$. Define $L_* = \{y \in V_* \setminus L_1 \ : \ \deg(y, L_2) \geq k\}$. Observe that $L_* \subseteq L \cap \tilde{S}$. By (iv), less than cn^2 edges of $E[L_2, V_* \setminus L_1]$ are incident with a vertex from L_*. Hence the number of edges in the bipartite graph $B = G[L_2, V_* \setminus (L_1 \cup L_*)]$ is at least

$$e(B) \geq |L_2|(1-2\sqrt{c})k - cn^2 \overset{(5.2),(5.1)}{\geq} (\tfrac{1}{2} - 2\sqrt[4]{c})|V_*|k \ . \tag{5.3}$$

On the other hand, we upper-bound the number of edges in the graph B using the fact that for each $x \in \tilde{S} \setminus L_*$ and for each $y \in V_* \setminus (L_1 \cup \tilde{S})$ we have that $\deg_B(x) < k$ and $\deg_B(y) \leq (1 - \sqrt[8]{c})k$, respectively.

$$\begin{aligned} e(B) &\leq |\tilde{S} \setminus L_*|k + |V_* \setminus (L_1 \cup \tilde{S})|(1 - \sqrt[8]{c})k \qquad \left[\text{as } \tilde{S} \cup (V_* \setminus (L_1 \cup \tilde{S})) = V_* \setminus L_1\right] \\ &= |V_* \setminus L_1|k - \sqrt[8]{c}|V_* \setminus (L_1 \cup \tilde{S})|k \\ &= (\tfrac{1}{2} + c)|V_*|k - \sqrt[8]{c}|V_* \setminus (L_1 \cup \tilde{S})|k. \end{aligned} \tag{5.4}$$

Combining (5.3) with (5.4) we obtain

$$|V_* \setminus (L_1 \cup \tilde{S})| \leq 3\sqrt[8]{c}|V_*| \leq \frac{3\sqrt[8]{c}k}{q} \ . \tag{5.5}$$

By the choice of L_2 and $\tilde{S}$, the minimum degree of the vertices in L_2 in the bipartite graph $G_1 = G[L_2, \tilde{S}]$ is at least $(1 - 2\sqrt{c})k - |V_* \setminus (L_1 \cup \tilde{S})|$, and of those in $\tilde{S}$ at least $(1 - \sqrt[8]{c})k$. By (5.5) and the choice of $c_{\mathbf{S}}$ we have that $\delta(G_1) > \frac{k}{2}$.

Fact 3.6 applied on the graphs B and G yields that $\mathcal{T}_{k+1} \subseteq G$. □

5.2 Cutting trees, and (un)balanced trees

Definition 5.2. *An* ℓ-fine partition *of a tree $T \in \mathcal{T}_{k+1}$ rooted at a vertex $R \in V(T)$ is a quaternary $\mathcal{D} = (W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$ with the following properties.*

(i) W_A and W_B are sets of vertices in $V(T)$. $\mathcal{D}_A$ and $\mathcal{D}_B$ are sets of subtrees in T. Further, $V(T)$ is a disjoint union of W_A, W_B, and the sets $V(t)$, $t \in \mathcal{D}_A \dot\cup \mathcal{D}_B$.

(ii) The distance from each vertex in W_A to each vertex in W_B is odd. The distance between each pair of vertices in W_A or between each pair of vertices in W_B is even.

(iii) No tree from $\mathcal{D}_A$ is adjacent[1] *to any vertex in W_B. No tree from $\mathcal{D}_B$ is adjacent to any vertex in W_A.*

(iv) $v(t) \leq \ell$ for each tree $t \in \mathcal{D}_A \cup \mathcal{D}_B$.

(v) $R \in W_A \cup W_B$.

(vi) $\max\{|W_A|, |W_B|\} \leq \frac{12k}{\ell}$.

(vii) $\mathcal{D}_B$ contains no internal tree.

(viii) We have

$$\sum_{\substack{t \in \mathcal{D}_A \\ t \text{ end-tree}}} v(t) \geq \sum_{t \in \mathcal{D}_B} v(t) \ .$$

(ix) Each internal tree from $\mathcal{D}_A$ is adjacent to two vertices of W_A.

[1]a subtree t is *adjacent* to a vertex v if there is at least one edge from v to $V(t)$

For an ℓ-fine partition $\mathcal{D} = (W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$ the trees from $\mathcal{D}_A \cup \mathcal{D}_B$ are called *shrubs*. For a subset $\mathcal{F} \subseteq \mathcal{D}_A \cup \mathcal{D}_B$, we denote the vertices contained in $\mathcal{F}$ by $V(\mathcal{F})$ and we write $v(\mathcal{F}) = |V(\mathcal{F})|$.

It is proven in [23] that for every ℓ each tree can be cut up in a way which results in a partition that satisfies (i)–(viii) of Definition 5.2. Here we extend this result by the additional requirement of (ix) from Definition 5.2.

Lemma 5.3. *Let $T \in \mathcal{T}_{k+1}$ be a tree rooted at a vertex R and let $\ell \in \mathbb{N}, \ell < k$. Then the rooted tree (T, R) has an ℓ-fine partition.*

For the proof, we shall need the following easy claim.

Fact 5.4 ([28, Proposition 7.11])**.** *Let T be a tree with ℓ leaves. Then T has at most $\ell - 2$ vertices of degree at least three.*

Proof of Lemma 5.3. We first cut up the tree T into components of order at most ℓ. To this end we start with an empty set W_1 and place a token v on the root R. At each step we check whether all the components of $T - v$ possibly except the one containing R are of individual orders at most ℓ. If that is the case then we insert v into W_1, and we delete v as well as all the said components from T. We restart with the token v again on R. Otherwise, we move v one vertex down to any component of order more than ℓ. Obviously, at the stage when the process terminates, we have $|W_1| \leq \frac{k+1}{\ell+1}$. Last, we add R to W_1. Then $|W_1| \leq \frac{k+1}{\ell+1} + 1$.

Next, we want to refine the set of cut vertices W_1 in order to satisfy (ix) of Definition 5.2. To this end, consider the components $\mathcal{D}_{\geq 3}$ of $T - W_1$ that neighbour at least 3 vertices of W_1. Fix an arbitrary tree $t \in \mathcal{D}_{\geq 3}$. Let $X(t) \subseteq V(t)$ be the neighbors of W_1. Let $X'(t)$ be all the vertices of $X(t)$ with the $\preceq$-maximal element removed. We have $|X'(t)| = |X(t)| - 1$. Consider the tree $\mathrm{branch}(t) \subseteq t$ induced by the paths in t connecting all the pairs of vertices of $X(t)$. Let $Y(t)$ be the vertices of degree at least 3 in $\mathrm{branch}(t)$. By Fact 5.4, we have $|Y(t)| \leq |X(t)| - 2 < |X'(t)|$. Observe that a map assigning to each vertex of $\bigcup_{t \in \mathcal{D}_{\geq 3}} X'(t)$ any of its $\preceq$-minimal neighbors in W_1 is injective. Set $W_2 = W_1 \cup \bigcup_{t \in \mathcal{D}_{\geq 3}} Y(t)$. By the above, $|W_2| \leq |W_1| + \sum_{t \in \mathcal{D}_{\geq 3}} |X'(t)| \leq 2|W_1| \leq \frac{2(k+1)}{\ell+1} + 2$. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be a partition of all the components of $T - W_2$ where the respective membership of a component to $\mathcal{S}_A$ or to $\mathcal{S}_B$ is given by the parity of the distance of that component to R, and further such that

$$\sum_{\substack{t \in \mathcal{S}_A \\ t \text{ end-tree}}} v(t) \geq \sum_{\substack{t \in \mathcal{S}_B \\ t \text{ end-tree}}} v(t) \ .$$

In particular, we can write $W_2 = W_{2A} \dot\cup W_{2B}$ where W_{2A} are the parents of all the components of $\mathcal{S}_A$ and W_{2B} are the parents of all the components of $\mathcal{S}_B$.

It remains to add further cut vertices in order to satisfy (iii) and (vii) of Definition 5.2. Initially, set $W_3 = W_2$. For each internal tree $t_B \in \mathcal{S}_B$ we take its unique $\preceq$-maximal vertex and add it to the set W_3. Further, we add $\mathrm{Par}(W_{2B}) \cap V(t_B)$ to W_3. For each internal tree $t_A \in \mathcal{S}_A$ we add $\mathrm{Par}(W_{2B}) \cap V(t_A)$ to W_3. See Figure 3. As each vertex of W_{2B} has at most one parent lying in some internal tree from $S_A \cup S_B$, we have

$$|W_3| \leq |W_2| + |\{\text{internal trees in } \mathcal{S}_B\}| + |W_{2B}| \ .$$

As each internal tree can be associated with a unique vertex of W_2 lying directly below it we get $|W_3| \leq 3|W_2| \leq \frac{6(k+1)}{\ell+1} + 6 \leq \frac{12k}{\ell}$. It is straightforward to check that the set W_3 partitioned according to the bipartite colouring $W_3 = W_A \dot\cup W_B$ with the correspondingly partitioned components $\mathcal{D}_A \dot\cup \mathcal{D}_B$ of $T - W_3$ satisfies all requirements of Definition 5.2. $\square$

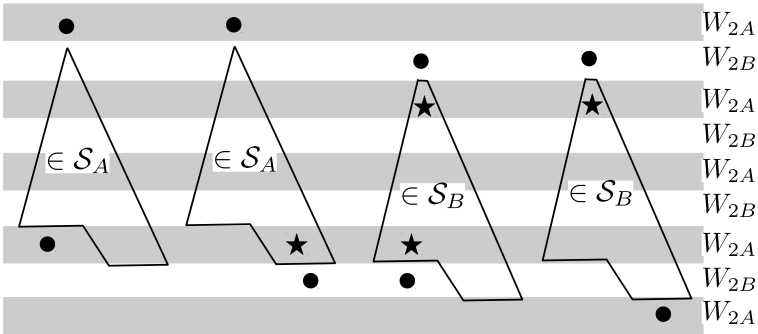


Figure 3: Obtaining the set W_3 from the set W_2 on examples of four internal trees depending on the parity of the neighbouring vertices of W_2 (which are denoted by dots). The newly added vertices are marked by stars.

The next lemma will allow us to remove trees which are locally unbalanced from further considerations in our proof of Theorem 1.5. Let us introduce the notion of (un)balanced forest now. For a real number $c \in (0, \frac{1}{2})$ we say that a family $\mathcal{C}$ of trees of total order at most $k+1$ is *c-balanced* if the forest formed by the trees $t \in \mathcal{C}$ with $|t_\ominus| > c \cdot v(t)$ is of order at least ck, i. e.,

$$\sum_{\substack{t \in \mathcal{C} \\ |t_\ominus| > cv(t)}} v(t) \geq ck \; .$$

Otherwise, we say that $\mathcal{C}$ is *c-unbalanced*.

Note that when $\mathcal{C}$ is c-balanced, then

$$\sum_{t \in \mathcal{C}} |t_\ominus| \geq c^2 k \; . \tag{5.6}$$

Lemma 5.5. *For each number $q > 0$ there exists a constant $c_{\mathbf{U}} > 0$ such that the following holds for each n-vertex graph G with the LKS-property with parameter $k > qn$. Suppose that $T \in \mathcal{T}_{k+1}$ is given. If there exists a set $W \subseteq V(T)$, $|W| < c_{\mathbf{U}}k$ such that the family $\mathcal{C}$ of all components of the forest $T - W$ is $c_{\mathbf{U}}$-unbalanced, then $T \subseteq G$.*

Proof. Set $c_{\mathbf{U}} = \frac{c_{\mathbf{S}}}{6}$, where $c_{\mathbf{S}}$ is given by Lemma 5.1.

If the set L induces less then $c_{\mathbf{S}} n^2$ edges then we have $T \subseteq G$ by Lemma 5.1 with $V_* = V$. In the rest we assume that $G[L]$ contains at least $c_{\mathbf{S}} n^2$ edges. A well-known fact asserts that there exists a graph $G' \subseteq G[L]$ with minimum degree at least half of the average degree of $G[L]$, i. e., $\delta(G') \geq c_{\mathbf{S}} n \geq 6c_{\mathbf{U}}(k+1)$.

Let $\mathcal{C}' \subseteq \mathcal{C}$ be those trees $t \in \mathcal{C}$ for which $|t_\ominus| \leq c_{\mathbf{U}} v(t)$. Since $\mathcal{C}$ is $c_{\mathbf{U}}$-unbalanced we have $\sum_{t \in \mathcal{C} \setminus \mathcal{C}'} v(t) < c_{\mathbf{U}} k$. Consequently,

$$\sum_{t \in \mathcal{C}'} v(t) = v(T) - |W| - \sum_{t \in \mathcal{C} \setminus \mathcal{C}'} v(t) > k + 1 - c_{\mathbf{U}} k - c_{\mathbf{U}} k > (1 - 2c_{\mathbf{U}})(k+1) \; . \tag{5.7}$$

Fact 3.2 gives that each tree $t \in \mathcal{C}'$, $v(t) > 1$ contains more than $(1 - 2c_{\mathbf{U}})v(t)$ leaves. The same property holds trivially for each tree $t \in \mathcal{C}'$, $v(t) = 1$. Employing (5.7), we get that there are at least $(1 - 2c_{\mathbf{U}}) \sum_{t \in \mathcal{C}'} v(t) \geq (1 - 4c_{\mathbf{U}})(k+1)$ leaves in the trees of $\mathcal{C}'$. A leaf of a tree $t \in \mathcal{C}'$ is either a leaf of T or it is adjacent to a vertex in W. We root T at an arbitrary vertex r, thus obtaining a partial order $\preceq$. Let X be the set of vertices that are leaves of some tree $t \in \mathcal{C}'$ but not leaves of T. Each vertex in X is either a $\preceq$-minimal or a $\preceq$-maximal vertex of some tree

$t \in \mathcal{C}$. Let $X_{\min} \subseteq X$ be the $\preceq$-minimal vertices and $X_{\max} = X \setminus X_{\min}$. (Note that the vertices which come out from 1-vertex trees of $\mathcal{C}'$ are included only in $X_{\min}$.) As each tree in $\mathcal{C}'$ has a unique $\preceq$-maximal vertex we get $|X_{\max}| \le h$, where h is the number of trees in $\mathcal{C}'$ which have order more than 1. Observe that each such tree has at least $\frac{1}{c_{\mathbf{U}}}$ vertices and thus $h \le c_{\mathbf{U}}(k+1)$. For each $v \in X_{\min}$ we have $|\mathrm{Ch}(v) \cap W| \ge 1$. Since for each $u \in W$ it holds $|\mathrm{Par}(u) \cap X_{\min}| \le 1$, we have $|X_{\min}| \le |W| < c_{\mathbf{U}}k$. Summing the bounds we get $|X| < 2c_{\mathbf{U}}(k+1)$. Thus T has at least $(1 - 6c_{\mathbf{U}})(k+1)$ leaves. Therefore, we can apply Fact 3.7 on $G' \subseteq G$ and conclude that $T \subseteq G$. $\square$

5.3 A matching structure

A graph H is said to be *factor critical* if for each its vertex v the graph $H - v$ has a perfect matching. The following statement is a fundamental result in Matching theory. See [6, Theorem 2.2.3], for example.

Theorem 5.6 (Gallai–Edmonds Matching Theorem)**.** *Suppose that H is a graph. Then there exist a set $Q \subseteq V(H)$ and a matching M of size $|Q|$ in H such that every component of $H - Q$ is factor critical and the matching M matches every vertex in Q to a different component of $H - Q$.*

The set Q in Theorem 5.6 is called a *separator*. In order to introduce the main result of this section, Lemma 5.8, we need the following setting.

Setting 5.7. *Let $s > 0$ and let (H, ω) be a weighted graph of order N, with $\omega : E(H) \to (0, s]$. Let σ, K be two positive reals with $\frac{1}{2N} < \sigma < \min\{\frac{K}{32Ns}, \frac{1}{30}\}$. Let $\mathcal{L}$ be a set of vertices such that*

(i) *$V(H) \setminus \mathcal{L}$ is an independent set,*

(ii) *$|\mathcal{L}| > \frac{N}{2} - \sigma N$,*

(iii) *$\omega(u) \ge K$ for every $u \in \mathcal{L}$,*

(iv) *the set $\mathcal{L}$ induces at least one edge in H,*

(v) *$\omega(u) < (1 + \sigma)K$ for every $u \in V(H) \setminus \mathcal{L}$.*

Lemma 5.8. *Let $s, N, \sigma, K, \mathcal{L}$, and a graph (H, ω) be as in Setting 5.7. Set $\mathcal{L}^* = \{u \in V(H) : \omega(u) \ge \frac{1}{2}(1 + \sigma)K\}$. Then there exist a matching M such that at least one of the following holds.*

Case I *There are two adjacent vertices $A, B \in V(H) \setminus V(M)$ with $A \in \mathcal{L}$, $\omega(A, V(M)) \ge K - s$, and $\omega(B, V(M) \cup \mathcal{L}^*) \ge \frac{1}{2}(1 + \sigma)K$. For each edge $e \in M$ we have $|\mathrm{N}(A) \cap e| \le 1$.*

Case II *There exists a set $\mathcal{O} \subseteq V(H)$ such that for each $x \in \mathcal{O}$ all but at most $2\sigma N$ neighbours of x are covered by M. Furthermore, the set $\mathcal{O} \cap \mathcal{L}$ induces at least one edge, and $|V(M') \setminus \mathcal{O}| \le 1$, where $M' = \{xy \in M : x, y \in \mathrm{N}(\mathcal{O})\}$.*

Moreover, observe that each edge $e \in M$ intersects the set $\mathcal{L}$.

Proof. Among all the matchings satisfying the conclusion of Theorem 5.6, choose a matching M_0 that covers the maximum number of vertices from $V(H) \setminus \mathcal{L}^*$. Let Q be the corresponding separator. By definition, M_0 is a $Q - (V(H) \setminus Q)$-matching. Set $\mathcal{L}_0 = \mathcal{L} \setminus Q$ and $\mathcal{S} = V(H) \setminus \mathcal{L}$. We distinguish three cases.

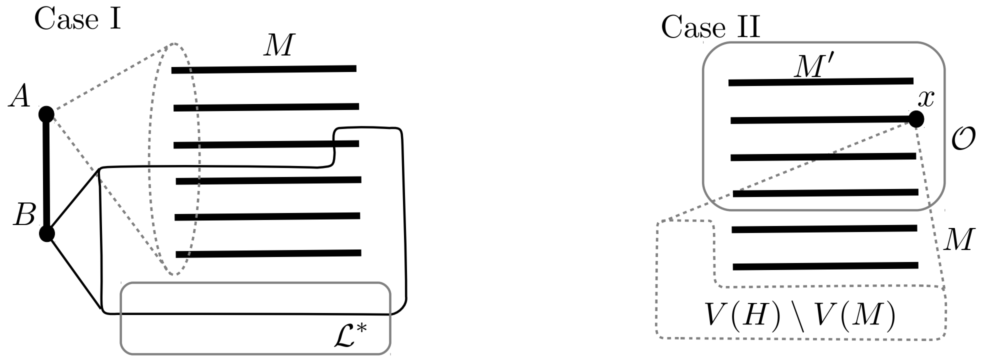


Figure 4: Two resulting matching structures from Lemma 5.8. Dashed lines represent no connections (in Case I), or sparse connections (in Case II).

- *There exists an $\mathcal{L}_0 - \mathcal{L}_0$ edge.* Let C be a component of $H - Q$ containing an $\mathcal{L}_0 - \mathcal{L}_0$ edge. If $V(M_0) \cap V(C) \neq \emptyset$, then we take $\{z\} = V(M_0) \cap V(C)$. Otherwise, we choose z arbitrarily in C. Since C is factor critical, there exists a perfect matching M_1 in $C - z$. We claim that the conditions of Case II are satisfied for $M = M_0 \cup M_1$, and $\mathcal{O} = V(C)$. Thus, $\mathcal{O} \cap \mathcal{L}$ induces an edge. Next, let $x \in \mathcal{O}$. We have $\mathrm{N}(x) \setminus \{z\} \subseteq V(M)$. Therefore, $\omega(x, V(M)) \geq \omega(x) - s \geq \omega(x) - 2\sigma N s$. Consequently, all but at most $2\sigma N$ neighbours of x are covered by M. To check that $|V(M') \setminus \mathcal{O}| \leq 1$, it is enough to observe that each edge of M' except at most one is contained entirely in C.

- *We have $\mathcal{L}_0 = \emptyset$.* Set $\mathcal{O} = V(H)$ and $M = M_0$. Setting 5.7 (iv) implies that there is an edge in $\mathcal{O} \cap \mathcal{L}$. It is clear that $V(M') \setminus \mathcal{O} = \emptyset$. Since $Q \supseteq \mathcal{L}$, $|\mathcal{L}| \geq \frac{N}{2} - \sigma N$, and $|V(M)| = 2|Q|$ it holds that all but at most $2\sigma N$ vertices of H are covered by M. The conditions of Case II are met.

- *$\mathcal{L}_0$ is an independent set and $\mathcal{L}_0 \neq \emptyset$.* We first derive some auxiliary properties of the graph H.

Claim 5.8.1. Each component C of $H - Q$ is a singleton.

Proof. Indeed, since $\mathcal{S}$ and $\mathcal{L}_0$ are independent, all the edges in each matching in C are in the form $\mathcal{S} - \mathcal{L}_0$. Since C is factor critical, we have $|V(C-u) \cap \mathcal{L}_0| = |V(C-u) \cap \mathcal{S}|$ for each vertex $u \in V(C)$. This is possible only when $v(C) = 1$. $\square$

Claim 5.8.1 implies that M_0 is a maximum matching in H. Define $\tilde{\mathcal{L}} = \{u \in \mathrm{N}(\mathcal{L}_0) : \omega(u) \geq K\}$. Observe that $\tilde{\mathcal{L}} \subseteq Q$. By Setting 5.7 (iii), we also have

$$\mathrm{N}(\mathcal{L}_0) \setminus \tilde{\mathcal{L}} \subseteq Q \setminus \mathcal{L} \,. \tag{5.8}$$

Claim 5.8.2. We have $\tilde{\mathcal{L}} \neq \emptyset$.

Proof. Assume for contradiction that $\tilde{\mathcal{L}} = \emptyset$. Then for every vertex $u \in \mathrm{N}(\mathcal{L}_0)$ we have $\omega(u) < K$. We get $|\mathcal{L}_0|K \leq \omega(\mathcal{L}_0, \mathrm{N}(\mathcal{L}_0)) < K|\mathrm{N}(\mathcal{L}_0)|$ (the second inequality is indeed strict because $\mathrm{N}(\mathcal{L}_0) \neq \emptyset$) implying

$$|\mathcal{L}_0| < |\mathrm{N}(\mathcal{L}_0)| \,. \tag{5.9}$$

On the other hand, from $\tilde{\mathcal{L}} = \emptyset$ it follows that $\mathrm{N}(\mathcal{L}_0) \cap \mathcal{L} = \emptyset$. Thus every vertex in $\mathrm{N}(\mathcal{L}_0)$ is matched by M_0 to a distinct vertex in $\mathcal{L}_0$, a contradiction to (5.9). $\square$

We show that the graph $V(H)$ fulfills the conditions of Case I. Suppose first that $B \in \mathrm{N}(\mathcal{L}_0)$ is such that $\omega(B, V(M_0) \cup \mathcal{L}^*) \geq \frac{1}{2}(1+2\sigma)K$ and let $A \in \mathrm{N}(B) \cap \mathcal{L}_0$ be arbitrary. Set $M = M_0 \setminus \{A, B\}$. It can then be easily shown that that pair (A, B) satisfies the conditions of Case I.

So assume that for every $B \in \tilde{\mathcal{L}} \subseteq \mathrm{N}(\mathcal{L}_0)$ we have

$$\omega(B, V(M_0) \cup \mathcal{L}^*) < \tfrac{1}{2}(1+2\sigma)K \;, \tag{5.10}$$

which yields

$$\omega(B, X) > \tfrac{1}{2}(1-2\sigma)K \;, \tag{5.11}$$

where $X = V(H) \setminus (V(M_0) \cup \mathcal{L}^*)$.

Claim 5.8.3. M_0 does not contain any edge with both end-vertices in $\mathcal{L}$.

Proof. Indeed, suppose that such an edge $xy \in M_0$ exists. Then $x \in \mathcal{L}_0$ and $y \in \tilde{\mathcal{L}}$. By (5.11), $\omega(y, X) > \frac{1}{2}(1-2\sigma)K$. In particular, there exists a vertex $p \in \mathrm{N}_X(y)$. The matching $\{yp\} \cup M_0 \setminus \{xy\}$ is a matching as in Theorem 5.6 (with separator Q) which covers more vertices of $V(H) \setminus \mathcal{L}^*$ than M_0. This contradicts the choice of M_0. □

Observe that for each vertex $u \in X$, we have $\omega(u, V(M)) = \omega(u) < \frac{1}{2}(1+\sigma)K$. As $\tilde{\mathcal{L}} \subseteq V(M_0)$, we have $\omega(u, \tilde{\mathcal{L}}) < \frac{1}{2}(1+\sigma)K$. We bound $\omega(\tilde{\mathcal{L}}, X)$ from both sides.

$$(1-2\sigma)|\tilde{\mathcal{L}}|\frac{K}{2} \overset{(5.11)}{\leq} \omega(\tilde{\mathcal{L}}, X) \leq (1+\sigma)|X|\frac{K}{2} \;,$$

which yields

$$|\tilde{\mathcal{L}}| \leq \frac{1+\sigma}{1-2\sigma}|X| \;. \tag{5.12}$$

We use (5.10) and $\mathcal{L}_0 \subseteq \mathcal{L}^*$ to get $\omega(\tilde{\mathcal{L}}, \mathcal{L}_0) \leq |\tilde{\mathcal{L}}|(1+2\sigma)K/2$. Also, by the definition of $\tilde{\mathcal{L}}$, we have $\omega(\mathrm{N}(\mathcal{L}_0) \setminus \tilde{\mathcal{L}}, \mathcal{L}_0) \leq K|\mathrm{N}(\mathcal{L}_0) \setminus \tilde{\mathcal{L}}|$. Therefore,

$$\begin{aligned}
|\mathcal{L}_0|K \leq \omega(Q, \mathcal{L}_0) &\leq \omega(\tilde{\mathcal{L}}, \mathcal{L}_0) + \omega(\mathrm{N}(\mathcal{L}_0) \setminus \tilde{\mathcal{L}}, \mathcal{L}_0) \\
&\leq (1+2\sigma)\frac{K}{2}|\tilde{\mathcal{L}}| + K|\mathrm{N}(\mathcal{L}_0) \setminus \tilde{\mathcal{L}}| \\
&\overset{(5.8)}{\leq} (1+2\sigma)\frac{K}{2}|\tilde{\mathcal{L}}| + K|Q \setminus \mathcal{L}| \;,
\end{aligned}$$

which gives

$$2|\mathcal{L}_0| \leq (1+2\sigma)|\tilde{\mathcal{L}}| + 2|Q \setminus \mathcal{L}| \;. \tag{5.13}$$

Every vertex in $Q \setminus \mathcal{L}$ is matched with a vertex in $\mathcal{L}_0$. The converse is true due to Claim 5.8.3: if a vertex in $\mathcal{L}_0$ is matched then it is matched with a vertex in $Q \setminus \mathcal{L}$. Therefore, $|Q \setminus \mathcal{L}| = |\mathcal{L}_0 \cap V(M_0)|$. Combined with (5.13) we get that $2|\mathcal{L}_0 \setminus V(M_0)| \leq (1+2\sigma)|\tilde{\mathcal{L}}|$. Plugging in (5.12) we obtain

$$2|\mathcal{L}_0 \setminus V(M_0)| \leq \frac{(1+2\sigma)^2}{1-2\sigma}|X| \;. \tag{5.14}$$

By Setting 5.7 (*ii*), we have $|\mathcal{L}| > |V(H) \setminus \mathcal{L}| - 2\sigma N$. By Claim 5.8.3, we get $|\mathcal{L}_0 \setminus V(M)| \geq |X| - 2\sigma N$. Combined with (5.14) we obtain

$$2|X| - 4\sigma N \leq \frac{(1+2\sigma)^2}{1-2\sigma}|X| \;.$$

We use the bounds $\sigma \leq \min\{\frac{K}{32Ns}, \frac{1}{30}\}$ to get

$$|X| \leq \frac{4\sigma N}{1-14\sigma} \leq 8\sigma N \leq \frac{8K}{32s} \,. \tag{5.15}$$

On the other hand, using (5.11) and Claim 5.8.2, we get $\omega(\tilde{\mathcal{L}}, X) > \frac{1}{2}(1-2\sigma)K|\tilde{\mathcal{L}}|$. As $\omega(e) \leq s$ for each $e \in E(H)$ we get $\omega(\tilde{\mathcal{L}}, X) \leq s|\tilde{\mathcal{L}}||X|$. Combining these two bounds we arrive at

$$|X| > \frac{(1-2\sigma)K}{2s} > \frac{K}{4s} \,,$$

a contradiction to (5.15). □

5.4 Regularity Lemma

In this section we recall briefly the Regularity Lemma [27] and establish related notation. The reader may find more on the Regularity Method in [20, 19, 21].

Let $H = (V(H); E(H))$ be a graph. For two nonempty disjoint sets $X, Y \subseteq V(H)$ we denote the *density* of the pair (A, B) by $\mathrm{d}(A, B) = \frac{e(A,B)}{|A||B|}$. The pair (A, B) is *ε-regular*, if for any subsets $X \subseteq A$, $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$. Such sets X and Y are called *significant.* We say that a vertex $v \in A$ is *typical* with respect to ("w. r. t.") a significant set $Y \subseteq B$, if $\deg(v, Y) \geq (\mathrm{d}(A, B) - \varepsilon)|Y|$. Analogously, if $\{(A, B_i)\}_{i=1}^{\ell}$ are ε-regular pairs, and $Y_i \subseteq B_i$ are significant, a vertex $v \in A$ is *typical* w. r. t. $\bigcup_{i-1}^{\ell} Y_i$, if $\deg(v, \bigcup_{i=1}^{\ell} Y_i) \geq \sum_{i=1}^{\ell}(d(A, B_i) - \varepsilon)|Y_i|$. Note that our definitions of typicality is only one-sided; this turns out to be sufficient for our proof.

Fact 5.9. *Let $X, Y_1, Y_2, \ldots, Y_\ell$ be disjoint sets of vertices, such that $(X, Y_1), (X, Y_2), \ldots, (X, Y_\ell)$ are ε-regular pairs. Suppose that sets $W_i \subseteq Y_i$ are significant.*

(i) All but at most $\varepsilon|X|$ vertices of X are typical w. r. t. $\bigcup_{i=1}^{\ell} W_i$.

(ii) All but at most $\sqrt{\varepsilon}|X|$ vertices of X are typical w. r. t. at least $\sqrt{\varepsilon}\ell$ sets W_i.

The proof of (ii) can be found in [28, Proposition 4.5]. We prove (i) in the Appendix. The next fact is the well-known "slicing property" of regular pairs.

Fact 5.10 ([20, Fact 1.5])**.** *Suppose that (X, Y) is an ε-regular pair of density d. Let $A \subseteq X$ and $B \subseteq Y$ be such that $|A| > \alpha|X|$, and $|B| > \alpha|Y|$ for $\alpha > 2\varepsilon$. Then the pair (A, B) is* $\max\left\{\frac{\varepsilon}{\alpha}, 2\varepsilon\right\}$*-regular of density at least $d - \varepsilon$.*

A partition $V(H) = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_N$ of the vertex set a graph H is called *(ε, N)-regular* if $|V_0| < \varepsilon v(H)$, $|V_i| = |V_j|$ for every $i, j \in [N]$, and for each $i \in [N]$ at most εN pairs (V_i, V_j) (where $j \in [N]$) are not ε-regular. The sets $V_1, \ldots, V_N$ are called *clusters.*

We are now ready to state a standard version Szemerédi's original result [27].

Theorem 5.11 ([27])**.** *For every $\varepsilon > 0$ and every $m_0, r \in \mathbb{N}$, there exist numbers $M_0, N_0 \in \mathbb{N}$ such that every graph H of order $m \geq N_0$ whose vertex sets is partitioned into r sets $V(H) = O_1 \dot\cup O_2 \dot\cup \ldots \dot\cup O_r$ admits an (ε, N)-regular partition $V(H) = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_N$ for some $m_0 \leq N \leq M_0$ such that for every $i \in [N]$ we have $V_i \subseteq O_j$ for some $j \in [r]$.*

In the above setting, let H_d denote the graph obtained from H by deleting the edges incident to V_0, contained in some V_i, or in pairs of clusters that are irregular or of density smaller than some fixed constant d. Let $\mathbf{H}$ denote the *cluster graph* induced by H_d. That is, $\mathbf{H}$ has order N, its vertices are $V(\mathbf{H}) = \{V_1, \ldots, V_N\}$ and edges are

$$E(\mathbf{H}) = \{V_i V_j \,:\, (V_i, V_j) \text{ is a } \varepsilon\text{-regular pair with density at least } d\} \,.$$

Set $\overline{\deg}_{\mathbf{H}}(C, D) := \overline{\deg}_{H_d}(C, D)$, for any disjoint sets $C, D \subseteq V(H)$. The function $\overline{\deg}_{\mathbf{H}}$ induces a weight function on $\mathbf{H}$.

5.5 Embedding lemmas

In this section, we introduce tools for embedding trees into regular pairs. Similar results are folklore. Here we give statements tailored to our needs; their proofs are included in the Appendix. The first lemma deals with embedding a tree into one regular pair.

Lemma 5.12. *Let (t, r) be a rooted tree, and $d > 2\varepsilon > 0$. Let (X, Y) be an ε-regular pair with $|X| = |Y| = s$ and density $\mathrm{d}(X, Y) \geq d$. Let $P' \subseteq P \subseteq X$ and $Q' \subseteq Q \subseteq Y$ be such that $\min\{|P|, |Q|\} \geq \Delta$ and $\max\{|P'|, |Q'|\} \geq \Delta$, where $\Delta \geq \frac{\varepsilon s + v(t)}{d-2\varepsilon}$. Then there exists an embedding ϕ of t to $P \cup Q$ such that the root r is mapped to $P' \cup Q'$. Moreover, if $|P'| \geq \Delta$, the vertex r can be mapped to P', and if $|Q'| \geq \Delta$, the vertex r can be mapped to Q'.*

The next lemma deals with embedding a tree using a matching structure in the underlying cluster graph. A simplified picture of the situation is given in Figure 5.

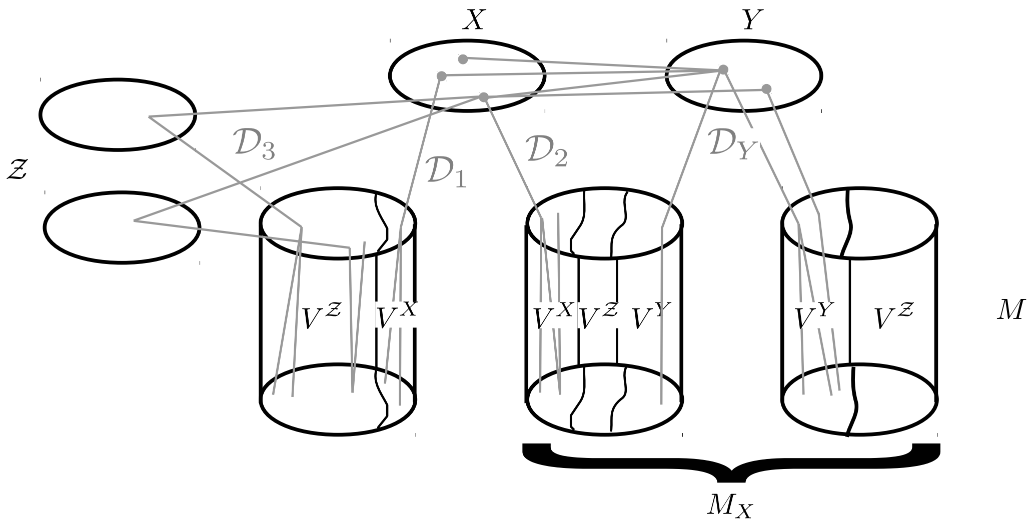


Figure 5: A simplified picture of an embedding provided by Lemma 5.13. The lemma provides with an embedding of a tree with a given fine partition $(W_X, W_Y, \mathcal{D}_X, \mathcal{D}_Y)$. The cut-vertices W_X and W_Y are mapped to X and Y, respectively. The shrubs $\mathcal{D}_Y$ are mapped to the part V^Y of the regular matching M. The shrubs $\mathcal{D}_X$ are embedded using one of three different ways which is indicated by the partition $\mathcal{D}_X = \mathcal{D}_1 \dot\cup \mathcal{D}_2 \dot\cup \mathcal{D}_3$. The shrubs of $\mathcal{D}_1$ are mapped to $V^X \setminus \bigcup V(M_X)$. The shrubs of $\mathcal{D}_2$ which are required to be balanced are mapped to $V^X \cap \bigcup V(M_X)$. Finally, the shrubs of $\mathcal{D}_3$ are accommodated to a set $V^{\mathcal{Z}}$ with their roots placed to an additional set of clusters $\mathcal{Z}$.

Lemma 5.13. *Let $0 < \varepsilon, \xi, d \leq 1$ and τ, s be such that $\tau/s \leq \varepsilon \leq \xi^2 d/400$. Let F be a tree of order at most $k+1$ with a τ-fine partition $(W_X, W_Y, \mathcal{D}_X, \mathcal{D}_Y)$ and let $\mathcal{D}_1 \dot\cup \mathcal{D}_2 \dot\cup \mathcal{D}_3$ be an arbitrary partition of $\mathcal{D}_X$. Let $\mathbf{H}$ be a cluster graph corresponding to an ε-regular partition of an n-vertex graph H, whose edges have density at least d and clusters have size s. Let $XY \in E(\mathbf{H})$, and $X' \subseteq X$, $Y' \subseteq Y$ such that $|X'|, |Y'| \geq (1 - d/2)s$. Let $\mathcal{Z} \subseteq V(\mathbf{H}) \setminus \{X, Y\}$. Further let $M_X \subseteq M$ be matchings in $\mathbf{H}$ disjoint from $\mathcal{Z} \cup \{X, Y\}$ such that for each edge of M_X contains at most one vertex of $\mathrm{N}_{\mathbf{H}}(X)$. Let $V^X, V^Y, V^{\mathcal{Z}} \subseteq \bigcup V(M)$ be pairwise disjoint sets. Suppose that*

(i) For all $CD \in M$, $|C \cap V^X| = |D \cap V^X|$, $|C \cap V^Y| = |D \cap V^Y|$, and $|C \cap V^{\mathcal{Z}}| = |D \cap V^{\mathcal{Z}}|$.

(ii) For all $C \in V(M)$, $|C \cap V^X|, |C \cap V^Y|, |C \cap V^{\mathcal{Z}}| \in \{0\} \cup (\varepsilon s, s]$.

(iii) *If* $\mathcal{D}_Y \neq \emptyset$, *then* $\overline{\deg}_{\mathbf{H}}(Y, V^Y) \geq v(\mathcal{D}_Y) + \xi n$.

(iv) *If* $\mathcal{D}_1 \neq \emptyset$, *then* $\overline{\deg}_{\mathbf{H}}(X, V^X \setminus \bigcup V(M_X)) \geq v(\mathcal{D}_1) + \xi n$.

(v) *If* $\mathcal{D}_2 \neq \emptyset$, *then* $\mathcal{D}_2$ *is* c*-balanced and* $\overline{\deg}_{\mathbf{H}}(X, V^X \cap \bigcup V(M_X)) \geq v(\mathcal{D}_2) - c^2 k + \xi n$.

(vi) *If* $\mathcal{D}_3 \neq \emptyset$, *then* $\overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}) \geq |V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)| + \xi n$.

(vii) *If* $\mathcal{D}_3 \neq \emptyset$, *then for each* $Z \in \mathcal{Z}$, $\overline{\deg}_{\mathbf{H}}(Z, V^{\mathcal{Z}}) \geq v(\mathcal{D}_3) + \xi n$.

Then there is an embedding φ *of* F *in* H *such that* $\varphi(W_X) \subseteq X'$, $\varphi(W_Y) \subseteq Y'$, $\varphi(V(\mathcal{D}_Y)) \subseteq V^Y$, *and* $\varphi(V(\mathcal{D}_X)) \subseteq (V^X \cup V^{\mathcal{Z}} \cup \bigcup \mathcal{Z})$.

6 Proof of Lemma 4.2

Suppose that q, c, and ρ are given. Let $c_{\mathbf{S}}$ be given by Lemma 5.1 for input parameter q. Further, let $c_{\mathbf{U}}$ be given by Lemma 5.5 for input parameter q. Set reals $\zeta, \alpha, \gamma, \beta, \vartheta, \lambda, \kappa$ so that

$$0 < \alpha \ll \beta \ll \gamma \ll \lambda \ll \zeta \ll \vartheta \ll \kappa \ll \min\{\rho, c, c_{\mathbf{S}}, c_{\mathbf{U}}, q\}.$$

Let n_0 (the minimal order of the graph) and Π_1 (the upper bound for the number of clusters) be the numbers given by the Regularity Lemma 5.11 for input parameters α (for precision), $\Pi_0 = \frac{2}{\alpha}$ (for the minimum number of clusters) and 4 (for the number of pre-partition classes).

Let G be a graph of order $n \geq n_0$ that has the LKS-property. We can assume that G is LKS-minimal, that is, there is no proper spanning subgraph $G' \subseteq G$ with the LKS-property. Then clearly,

$$\text{the set } S \text{ is independent .} \tag{6.1}$$

Let $V_* \subseteq V$ satisfy the assumptions of Lemma 4.2 and let $T \in \mathcal{T}_{k+1}$ be arbitrary. Our goal is to show that $T \subseteq G$. Root T at an arbitrary vertex R, and consider any τ-fine partition $(W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$ of (T, R), with $\tau = \frac{\alpha k}{\Pi_1}$. The existence of such a partition follows from Lemma 5.3.

Prepartition the vertex set V into $V_* \cap L, V_* \cap S, L \setminus V_*$, and $S \setminus V_*$. By the Regularity Lemma 5.11, there exists a partition $V = C_0 \dot\cup C_1 \dot\cup \ldots \dot\cup C_N$ satisfying the following.

(R1) $\Pi_0 \leq N \leq \Pi_1$,

(R2) $|C_i| = s$ for each $i \in [N]$,

(R3) $|C_0| \leq \alpha n$,

(R4) for each $i \in [N]$, all but at most αN pairs (C_i, C_j) (where $j \in [N]$) are α-regular,

(R5) for each $i \in [N]$, if $C_i \cap L \neq \emptyset$ then $C_i \subseteq L$, and if $C_i \cap V_* \neq \emptyset$ then $C_i \subseteq V_*$.

Let G_γ denote the graph obtained from G by deleting the edges incident to C_0, contained in some C_i, or in pairs of clusters that are irregular or of density smaller than γ and let $\mathbf{G}$ be the corresponding cluster graph with weight function $\overline{\deg}_{\mathbf{G}}$. Observe that by **(R1)**–**(R4)** we have

$$e(G_\gamma) \geq e(G) - 2\alpha n^2 - \gamma n^2 \geq e(G) - \lambda k^2 . \tag{6.2}$$

Denote by $\mathcal{L}$ the set of clusters contained in $L \cap V_*$ which have large average degree in V_*:

$$\mathcal{L} = \{C \in V(\mathbf{G}) \,:\, C \subseteq L \cap V_*,\ \overline{\deg}_{\mathbf{G}}(C, V_*) \geq k - \sqrt{\lambda} n\} .$$

Note that **(R5)** supports the definition and observation below. Let $\mathcal{V}_*$ be the set of clusters contained in V_*; we write $N_* = |\mathcal{V}_*|$. Observe that each cluster inside $L \cap V_*$ is in $\mathcal{L}$, unless it sends many edges to $V \setminus V_*$. To estimate the size of $\mathcal{L}$, we set $\mathcal{B} = \{C \in V(\mathbf{G}) : C \subseteq V_*, \overline{\deg}_{\mathbf{G}}(C, V \setminus V_*) \geq \frac{\sqrt{\lambda}n}{2}\}$. It follows from the assumptions of Lemma 4.2 that

$$|\mathcal{B}| \leq 3\sqrt{\lambda}N \ . \tag{6.3}$$

Further, observe that we have

$$\mathcal{L} \supset \{C \in V(\mathbf{G}) : C \subseteq V_* \cap L\} \setminus \mathcal{B} \ . \tag{6.4}$$

The ratio $|L \cap V_*| : |V_*|$ approximately corresponds to $|\mathcal{L}| : |\mathcal{V}_*|$. More precisely, we use later the following lower-bound on $|\mathcal{L}|$,

$$|\mathcal{L}| \geq \tfrac{1}{2}(1-2\lambda)N_* - |\mathcal{B}| \geq \tfrac{N_*}{2} - 4\sqrt{\lambda}N \geq \tfrac{N_*}{2} - \zeta N_* \ , \tag{6.5}$$

where we use $\lambda \ll \zeta \ll c, q$.

Let $\mathbf{H}$ be the subgraph of $\mathbf{G}$ induced by $\mathcal{V}_*$ such that all the edges induced by the set $\mathcal{V}_* \setminus \mathcal{L}$ are removed. The cluster graph $\mathbf{H}$ naturally inherits the function $\overline{\deg}_{\mathbf{G}}$ of $\mathbf{G}$ (which is denoted by $\overline{\deg}_{\mathbf{H}}$). The next lemma gives some simple properties of $\mathbf{H}$.

Lemma 6.1.

(i) For each $C \in \mathcal{L}$, we have $\sum_{D \in \mathrm{N}_{\mathbf{H}}(C)} \overline{\deg}_{\mathbf{H}}(C, D) = \sum_{D \in \mathrm{N}_{\mathbf{G}}(C, V_*)} \overline{\deg}_{\mathbf{G}}(C, D)$.

(ii) All but at most $3\sqrt{\lambda}N$ clusters $C \in \mathcal{V}_* \setminus \mathcal{L}$ satisfy

$$\sum_{D \in \mathrm{N}_{\mathbf{H}}(C)} \overline{\deg}_{\mathbf{H}}(C, D) \geq \Big(\sum_{D \in \mathrm{N}_{\mathbf{G}}(C, \mathcal{V}_*)} \overline{\deg}_{\mathbf{G}}(C, D)\Big) - 3\sqrt{\lambda}n \ .$$

Proof. Part (i) follows directly. Let us now deal with part (ii). By (6.3), for any cluster $C \in \mathcal{V}_* \setminus (\mathcal{L} \cup \mathcal{B})$, we have $\overline{\deg}_{\mathbf{H}}(C) \geq \overline{\deg}_{\mathbf{G}}(C, V_*) - |\mathcal{B}|s \geq \overline{\deg}_{\mathbf{G}}(C, V_*) - 3\sqrt{\lambda}n$, as edges of $\mathbf{G}$ sent by C go either to $\mathcal{B}$ or are kept in $\mathbf{H}$. At most $3\sqrt{\lambda}N$ clusters in $\mathcal{V}_* \setminus \mathcal{L}$ may be contained in $\mathcal{B}$. □

6.1 Matching structure in the cluster graph

Set $c' = \min\{c_{\mathbf{S}}, c^4\}$. If $e(G[V_* \cap L]) < c'n^2$, then the conditions of Lemma 5.1 are satisfied for the set V_* and parameter $c_{\mathrm{L}5.1} = c'$. Indeed, the assumptions *(i)*–*(iii)* of Lemma 5.1 follow from the assumptions of Lemma 4.2, and the fact that $\lambda \ll c'$. Then, by Lemma 5.1 we get $\mathcal{T}_{k+1} \subseteq G$. Therefore, we assume in the rest of the proof that $e(G[V_* \cap L]) \geq c'n^2$. By (6.2) as $\lambda \ll c'$, we get $e(G_\gamma[V_* \cap L]) \geq \frac{c'}{2}n^2$.

Lemma 6.2. *The set $\mathcal{L}$ induces at least one edge in* $\mathbf{H}$.

Proof. By (6.4) and (6.3) at most $4\sqrt{\lambda}n^2$ of the edges of $E(G_\gamma[V_* \cap L])$ are not induced by the vertices of $\bigcup \mathcal{L}$. As $e(G_\gamma[V_* \cap L]) \geq \frac{c'}{2}n^2 \gg 4\sqrt{\lambda}n^2$, $\mathcal{L}$ induces at least one edge in $\mathbf{G}$. This edge is also an edge in $\mathbf{H}$. □

The weighted graph $(\mathbf{H}, \overline{\deg}_{\mathbf{H}})$ satisfies the conditions of Lemma 5.8 with parameters s, $N = N_*$, $\sigma = \zeta$, and $K = k - \sqrt{\lambda}n$. Let us verify Conditions *(i)*–*(v)* of Setting 5.7. Condition *(i)* is satisfied by the way $\mathbf{H}$ was derived from $\mathbf{G}$. Condition *(ii)* follows from (6.5). Condition *(iii)* is given by the definition of $\mathcal{L}$. Condition *(iv)* was derived in Lemma 6.2. Finally, Condition *(v)* follows from the definitions of L and $\mathcal{L}$. Lemma 5.8 ensures that one of the two specific matching structures in $\mathbf{H}$ exists.

Case I: There are two adjacent clusters A, B and a matching M in $\mathbf{H} - \{A, B\}$ such that:

(a) We have $\overline{\deg}_{\mathbf{H}}(A, V(M)) \geq k - 2\sqrt{\lambda}n$.

(b) For each edge $e \in M$ we have $|\mathrm{N}_{\mathbf{H}}(A) \cap e| \leq 1$.

(c) There is a set $\mathcal{L}^* \in V(\mathbf{H})$ such that for all $C \in \mathcal{L}^*$ we have $\overline{\deg}_{\mathbf{H}}(C) \geq (1 + \frac{\zeta}{2})\frac{k}{2}$ and

$$\overline{\deg}_{\mathbf{H}}(B, V(M) \cup \mathcal{L}^*) \geq (1 + \tfrac{\zeta}{2})\frac{k}{2} \ . \tag{6.6}$$

Case II: There exist a set of clusters $\mathcal{O} \subseteq V(\mathbf{H})$ and a matching M in $\mathbf{H}$ such that:

(a) $\mathcal{O} \cap \mathcal{L}$ induces at least one edge in $\mathbf{H}$.

(b) $|V(M_{\mathcal{O}}) \setminus \mathcal{O}| \leq 1$, where $M_{\mathcal{O}} = \{CD \in M \ : \ C, D \in \mathrm{N}_{\mathbf{H}}(\mathcal{O})\}$.

(c) All clusters of $\mathcal{O} \cap \mathcal{L}$ and all but at most $3\sqrt{\lambda}N$ clusters $C \in \mathcal{O} \setminus \mathcal{L}$ satisfy

$$\overline{\deg}_{\mathbf{H}}(C, V(M)) \geq \overline{\deg}_{\mathbf{G}}(C, V_*) - 3\zeta n \ .$$

To see this, recall that by the assertion of Lemma 5.8 we have that

$$\sum_{D \in \mathrm{N}_{\mathbf{H}}(C) \cap V(M)} \overline{\deg}_{\mathbf{H}}(C, D) \geq \sum_{D \in \mathrm{N}_{\mathbf{H}}(C)} \overline{\deg}_{\mathbf{H}}(C, D) - 2\zeta n$$

for each $C \in \mathcal{O}$. Thus the assertion follows from Lemma 6.1.

(d) Each edge of M intersects $\mathcal{L}$.

We partition $\mathcal{D}_A = \mathcal{T}_F \dot\cup \mathcal{T}_A$, where $\mathcal{T}_F$ are the internal shrubs and by $\mathcal{T}_A$ are the end-shrubs of $\mathcal{D}_A$. Recall that $\mathcal{D}_B$ contains only end-shrubs and that $v(\mathcal{D}_B) \leq v(\mathcal{T}_A)$. We shall assume that $\mathcal{T}_F \cup \mathcal{T}_A \cup \mathcal{D}_B$ is $c_{\mathbf{U}}$-balanced, otherwise $T \subseteq G$ by Lemma 5.5.

As we shall show shortly, the proof of Lemma 4.2 follows from the following three statements, proofs of which are postponed to subsequent sections.

Lemma 6.3. *If we have Case I, then $T \subseteq G$.*

Lemma 6.4. *If we have Case II, then $T \subseteq G$, or for any two clusters $A, B \in \mathcal{O} \cap \mathcal{L}$ that are adjacent in $\mathbf{H}$, there exists a matching $M_A \subseteq M - \{A, B\}$ such that M_A and $V_A = \bigcup V(M_A)$ satisfy the following properties.*

(i) $\overline{\deg}_{\mathbf{H}}(A, C), \overline{\deg}_{\mathbf{H}}(A, D) > (1 - 2\vartheta)s$ *and* $\overline{\deg}_{\mathbf{H}}(A, CD) > (2 - 3\vartheta)s$*, for all* $CD \in M_A$.

(ii) $\overline{\deg}_{\mathbf{H}}(A, V(M_A)) \geq (1 - 8\vartheta)k$.

(iii) $(1 - 8\vartheta)k \leq |V_A| \leq k$.

(iv) $V(M_A) \subseteq \mathcal{O}$.

(v) *If* $v(\mathcal{D}_B) \geq \sqrt[4]{\zeta}k$*, then* $\overline{\deg}_{\mathbf{H}}(B, V(M_A)) \geq (1 - 9\vartheta)k$.

(vi) *If* $v(\mathcal{D}_B) < \sqrt[4]{\zeta}k$*, then there exists a matching* $M_B \subseteq M - (V(M_A) \cup \{A, B\})$ *such that* $|M_B| \leq \sqrt[4]{\zeta}N$ *and* $v(\mathcal{D}_B) + \lambda k \leq \overline{\deg}_{\mathbf{H}}(B, V(M_B)) \leq v(\mathcal{D}_B) + \lambda k + 2s$.

(vii) $|V_A \cap L| \geq \frac{1}{2}|V_A|$.

Lemma 6.5. *Suppose we have Case II and let $A, B \in \mathcal{O} \cap \mathcal{L}$, $AB \in E(\mathbf{H})$. Suppose that M_A, M_B and V_A satisfy (i)–(vii) from Lemma 6.4. (For convenience, we take $M_B = \emptyset$ if the assumption of Lemma 6.4 (vi) is not satisfied.) If $|e_{G_\gamma}(V_A, V \setminus V_A)| \geq \frac{\kappa n^2}{2}$, then $T \subseteq G$.*

Given Lemmas 6.3–6.5, Lemma 4.2 follows. Indeed, we get that $\mathcal{T}_{k+1} \subseteq G$, or $e_{G_\gamma}(V_A, V \setminus V_A) < \kappa^2 n^2/2$, with V_A from Lemma 6.4. In the latter case, the assertions of Lemma 4.2 are fullfilled with the set $V' := V_A$. Indeed, by Lemma 4.2 (ii) and by (6.2), we have $e_G(V_A, V \setminus V_A) \leq e_G(V_*, V \setminus V_*) + e_{G_\gamma}(V_A, V \setminus V_A) + e(G) - e(G_\gamma) \leq 2\lambda k^2 + \kappa n^2/2 \ll \rho k^2$.

6.2 Proof of Lemma 6.3

We shall partition each cluster $C \in V(\mathbf{H})$ so that the partition defines two disjoint sets $V^F, V^B \subseteq V(G)$. The embedding $\varphi : V(T) \to V(G)$ of T will be defined in three phases. In the first phase, we shall embed the subtree $T' = T[W_A \cup W_B \cup V(\mathcal{T}_F \cup \mathcal{T}_B^M)]$, where $\mathcal{T}_B^M \subseteq \mathcal{D}_B$ will be defined later. The trees $\mathcal{T}_F$ will be embedded in V^F and the trees $\mathcal{T}_B^M$ in V^B. In the second phase, we shall embed $\mathcal{T}_B^L = \mathcal{D}_B \setminus \mathcal{T}_B^M$ in V^B. In the last phase, we shall embed $\mathcal{T}_A$ in $V(G)$. From now on, we write φ for the partial embedding (at the current stage) of T.

The difference between the present proof of Theorem 1.5 and its approximate version Theorem 1.4 is that in the proof of Theorem 1.5 we have to fight to gain back small loses caused by the use of the Regularity Lemma. However, this is not necessary when we have the matching structure of Case I. Indeed, we can reduce this situation to the "approximate version", i.e., to a setting of similar nature as in Theorem 1.4.

Preparation. We partition each cluster $C \in V(\mathbf{H})$ into sets C^F and C^B in an arbitrary way so that $|C^F| = (1-y)|C|$ and $|C^B| = y|C|$, where

$$y = \frac{v(\mathcal{T}_A \cup \mathcal{D}_B)}{k} \cdot \frac{1}{1+\frac{\zeta}{4}} + \lambda \geq \frac{2v(\mathcal{D}_B)}{k} \cdot \frac{1}{1+\frac{\zeta}{4}} + \lambda. \tag{6.7}$$

Note that

$$\begin{aligned} 1 - y &\geq \frac{v(\mathcal{T}_F)}{k} + \frac{\zeta}{8} \cdot \frac{v(\mathcal{T}_A \cup \mathcal{D}_B)}{k} - \lambda & (6.8)\\ &\geq \frac{v(\mathcal{T}_F)}{k} - \lambda\,. & (6.9)\end{aligned}$$

Set

$$V^B = \bigcup_{C\in V(\mathbf{H})} C^B\,, \quad V^F = \bigcup_{C\in V(\mathbf{H}} C^F\,,$$
$$M^B = V^B \cap \bigcup V(M)\,, \quad M^F = V^F \cap \bigcup V(M)\,, \text{ and } \quad L^B = V^B \cap \bigcup(\mathcal{L}^* \setminus \{A,B\})\,.$$

Observe that (6.7) gives $y \in (\lambda, 1-\lambda)$. Indeed, the lower bound is trivial and the upper bound follows from $\frac{1}{1+\frac{\zeta}{4}} < 1 - 2\lambda$.

Let $\mathcal{T}_B^M \subseteq \mathcal{D}_B$ be a maximal subject to

$$\sum_{t\in\mathcal{T}_B^M} v(t) \leq \overline{\deg}_{\mathbf{H}}(B, M^B) - \lambda n. \tag{6.10}$$

Let $\mathcal{T}_B^L = \mathcal{D}_B \setminus \mathcal{T}_B^M$. From the maximality, we have

$$\sum_{t\in\mathcal{T}_B^M} v(t) \geq \overline{\deg}_{\mathbf{H}}(B, M^B) - \lambda n - \tau k \quad \text{or} \quad \mathcal{T}_B^L = \emptyset\,. \tag{6.11}$$

We now proceed with the three-phase embedding outlined above.

Phase 1 of the embedding. Let $A' \subseteq A$ be the set of typical vertices w. r. t. all but at most βN sets $C \in V(M)$ and let $B' \subseteq B$ be the set of typical vertices w. r. t. L^B. From Fact 5.9,

$$\min\{|A'|, |B'|\} \geq (1-\sqrt{\alpha})s\,. \tag{6.12}$$

We use Lemma 5.13 to embed the tree $T' = T[W_A \cup W_B \cup V(\mathcal{T}_F \cup \mathcal{T}_B^M)]$ with the following setting. The cluster graph is $\mathbf{H}$, with $AB \in E(\mathbf{H})$ and $A' \subseteq A$, $B' \subseteq B$. The set $\mathcal{Z}$ is empty. The tree T' has a τ-fine partition $(W_A, W_B, \mathcal{T}_F, \mathcal{T}_B^M)$. We have disjoint sets $M^F \dot\cup M^B \dot\cup \emptyset \subseteq \bigcup V(M)$. The sets M^F, M^B and $\emptyset$ play the roles of V^X, V^Y, and $V^{\mathcal{Z}}$ from Lemma 5.13. If $\mathcal{T}_F$ is $c_{\mathbf{U}}/2$-balanced, we set $\mathcal{D}_2 = \mathcal{T}_F$, $\mathcal{D}_1 = \mathcal{D}_3 = \emptyset$ and $M_X = M$. If $\mathcal{T}_F$ is not $c_{\mathbf{U}}/2$-balanced, we set $\mathcal{D}_1 = \mathcal{T}_F$, $\mathcal{D}_2 = \mathcal{D}_3 = \emptyset$, and $M_X = \emptyset$. In particular, note that

$$\varphi(V(T') \setminus V(\mathcal{T}_B^M)) \cap M^B = \emptyset \ . \tag{6.13}$$

We now verify the assumptions of Lemma 5.13, where we use $d_{\mathrm{L}5.13} = \gamma, \xi_{\mathrm{L}5.13} = \lambda, \varepsilon_{\mathrm{L}5.13} = \alpha$. The parameters $0 < \alpha \ll \lambda \ll \gamma < 1$ and τ, s satisfy $\tau/s < \alpha < \lambda^2\gamma/400$. The bound (6.12) guarantees that A' and B' have sizes as required by the lemma. Condition (i) follows from the way V^F and V^B were defined and Condition (ii) holds as $y \in (\lambda, 1-\lambda)$. Conditions (vi) and (vii) hold trivially. Condition (iii) follows from (6.10). If $\mathcal{T}_F$ is $c_{\mathbf{U}}/2$-balanced, Condition (iv) holds trivially and for Condition (v) observe that

$$\begin{aligned} \overline{\deg}_{\mathbf{H}}(A, M^F) &\geq (1-y)(\overline{\deg}_{\mathbf{H}}(A, V(M)) - \alpha n \qquad \text{[by (6.9) and Case I}(a)] \\ &\geq v(\mathcal{T}_F) - 3\sqrt{\lambda} n \geq v(\mathcal{T}_F) - \tfrac{c_{\mathbf{U}}^2}{4}k + \lambda n \ . \end{aligned}$$

As $\mathcal{T}_A \cup \mathcal{T}_F \cup \mathcal{D}_B$ is $c_{\mathbf{U}}$-balanced we get that if $\mathcal{T}_F$ is not $c_{\mathbf{U}}/2$-balanced then $\mathcal{T}_A \cup \mathcal{D}_B$ is $c_{\mathbf{U}}/2$-balanced. Condition (v) holds trivially and for Condition (iv) observe that

$$\begin{aligned} \overline{\deg}_{\mathbf{H}}(A, M^F) &\geq (1-y)(\overline{\deg}_{\mathbf{H}}(A, V(M)) - \alpha n \qquad \text{[by (6.8) and Case I}(a)] \\ &\geq v(\mathcal{T}_F) + v(\mathcal{T}_A \cup \mathcal{D}_B)\frac{\zeta}{8} - 3\sqrt{\lambda} n \geq v(\mathcal{T}_F) + \lambda n \ . \end{aligned}$$

Phase 2 of the embedding. Phase 2 is skipped when $\mathcal{T}_B^L = \emptyset$. We label the shrubs of $\mathcal{T}_B^L$ as $t_1, \ldots, t_{|\mathcal{T}_B^L|}$. In step $i \geq 1$, we define the embedding for the shrub t_i in a suitable edge $CD \in E(\mathbf{G})$. Set $U_i = \varphi(V(\mathcal{T}_F \cup \mathcal{T}_B^M) \cup \bigcup_{j<i} V(t_j))$. Let $x_i \in W_B$ be the parent of the root of the shrub t_i. The vertex $\varphi(x_i)$ is typical w. r. t. L^B and hence by (6.6), (6.7) and (6.11), we have

$$\begin{aligned} \deg(\varphi(x_i), L^B) &\geq \overline{\deg}_{\mathbf{H}}(B, L^B) - \alpha n \geq \overline{\deg}_{\mathbf{H}}(B, M^B \cup L^B) - \overline{\deg}_{\mathbf{H}}(B, M^B) - \alpha n \\ &\geq v(\mathcal{D}_B) + \tfrac{\zeta k}{4} - v(\mathcal{T}_B^M) - \lambda n - \tau k - 2\alpha n \geq v(\mathcal{T}_B^L) + \lambda n \ . \end{aligned}$$

Thus there is a cluster $C \in \mathcal{L}^*$ with

$$|\mathrm{N}(\varphi(x_i)) \cap C \setminus U_i| \geq \frac{\lambda n}{N} \geq \frac{\alpha s + \tau}{\gamma - 2\alpha} \ .$$

From the definition of $\mathcal{L}^*$, (6.7), and (6.13) we obtain

$$\overline{\deg}_{\mathbf{H}}(C, V^B \setminus U_i) \geq \overline{\deg}_{\mathbf{H}}(C, V^B) - |\varphi(V(\mathcal{D}_B)) \cap U_i| \geq \tfrac{\lambda k}{4} \ .$$

Therefore there is a cluster $D \in \mathrm{N}_{\mathbf{H}}(C)$ with $|D \setminus U_i| \geq \frac{\alpha s + \tau}{\gamma - 2\alpha}$. We use Lemma 5.12 to embed t_i in $(C \cup D) \setminus U_i$ so that the root of the shrub t_i is mapped to $\mathrm{N}(\varphi(x_i)) \cap C \setminus U_i$.

Phase 3 of the embedding. We label $\mathcal{T}_A$ as $t_1, \ldots, t_{|\mathcal{T}_A|}$. In step $i = 1, \ldots, |\mathcal{T}_A|$, we define the embedding for the shrub t_i. Let $x_i \in W_A$ be the parent of the root r_i of t_i. Set $U_i = \varphi(V(\mathcal{T}_F \cup \mathcal{D}_B) \cup \bigcup_{j<i} V(t_j))$. For an edge $CD \in M$ with $C \in \mathrm{N}_{\mathbf{H}}(A)$ we define

$$\Upsilon_{CD}^i = \min\{|\mathrm{N}(\varphi(x_i)) \cap C \setminus U_i|, |D \setminus U_i|\} \ .$$

By Lemma 5.12, the shrub t_i can be embedded in unused vertices of an edge $CD \in M$ so that r_i is mapped to a neighbor of $\varphi(x_i)$, whenever CD satisfies $\Upsilon^i_{CD} \geq \lambda s$. If $\mathcal{T}_F \cup \mathcal{D}_B$ is $\frac{c_{\mathbf{U}}}{2}$-balanced then by (5.6) we have

$$\sum_{\substack{CD \in M \\ C \in \mathrm{N}_{\mathbf{H}}(A)}} \max\{|C \cap U_i|, |D \cap U_i|\} \leq v(\mathcal{T}_A) + v(\mathcal{T}_F \cup \mathcal{D}_B) - \sum_{t \in \mathcal{T}_F \cup \mathcal{D}_B} |t_\ominus| \leq v(\mathcal{T}_A) + v(\mathcal{T}_F \cup \mathcal{D}_B) - \frac{c_{\mathbf{U}}^2 k}{4} \,.$$

By Fact 5.9 we have

$$\begin{aligned}\sum_{\substack{CD \in M \\ C \in \mathrm{N}_{\mathbf{H}}(A)}} \Upsilon^i_{CD} \geq & \sum_{\substack{CD \in M \\ C \in \mathrm{N}_{\mathbf{H}}(A)}} (|\mathrm{N}(\varphi(x_i)) \cap C| - \max\{|C \cap U_i|, |D \cap U_i|\}) \\ & \geq \overline{\deg}_{\mathbf{H}}(A, V(M)) - 2\sqrt{\alpha} n - (v(\mathcal{T}_F \cup \mathcal{D}_B) - \tfrac{c_{\mathbf{U}}^2 k}{4}) - v(\mathcal{T}_A) \geq \lambda n \,.\end{aligned}$$

If $\mathcal{T}_F \cup \mathcal{D}_B$ is $\frac{c_{\mathbf{U}}}{2}$-unbalanced, then $\mathcal{T}_A$ is $\frac{c_{\mathbf{U}}}{2}$-balanced. Then by (5.6), $\max\{|V(\mathcal{T}_A) \cap T_\oplus|, |V(\mathcal{T}_A) \cap T_\ominus|\} \leq v(\mathcal{T}_A) - (\frac{c_{\mathbf{U}}}{2})^2 k$. We get

$$\begin{aligned}\sum_{\substack{CD \in M \\ C \in \mathrm{N}_{\mathbf{H}}(A)}} \Upsilon^i_{CD} \geq & \sum_{\substack{CD \in M \\ C \in \mathrm{N}_{\mathbf{H}}(A)}} (|\mathrm{N}(\varphi(x_i)) \cap C| - \max\{|C \cap U_i|, |D \cap U_i|\}) \\ & \geq \overline{\deg}_{\mathbf{H}}(A, V(M)) - 2\sqrt{\alpha} n - v(\mathcal{T}_F \cup \mathcal{D}_B) - (v(\mathcal{T}_A) - \tfrac{c_{\mathbf{U}}^2 k}{4}) \geq \lambda n \,.\end{aligned}$$

In both cases, there is an edge $CD \in M$ with $\Upsilon^i_{CD} \geq \lambda s$.

6.3 Proof of Lemma 6.4

Let $\tilde{M} \subseteq M$ be the minimum matching covering clusters A and B. We claim that

$$\min\{\overline{\deg}_{\mathbf{H}}(A, V(M \setminus \tilde{M})), \overline{\deg}_{\mathbf{H}}(B, V(M \setminus \tilde{M}))\} \geq k - 4\zeta n \,. \tag{6.14}$$

As $A, B \in \mathcal{O} \cap \mathcal{L}$, $\min\{\overline{\deg}_{\mathbf{G}}(A, V_*), \overline{\deg}_{\mathbf{G}}(B, V_*)\} \geq k - \sqrt{\lambda} n$. From Case II ($c$) and the fact that $|V(\tilde{M})| \leq 4$, (6.14) follows.

The proof of *(i)–(vi)* corresponds to Lemma 6.11 from [28]. The hypotheses of [28, Lemma 6.11] and the present Lemma 6.4 are almost identical. We describe the correspondence and slight differences. Our Case II (b) implies hypothesis given by Claim 6.7(3) in [28]. Our Case II (c) is weaker than the corresponding hypothesis given in Claim 6.7(2). In his proof, Zhao only uses Claim 6.7(2) to deduce that the clusters A and B have a large weight to the matching $\mathcal{M}_{\mathsf{Zhao}}$ (which corresponds to our matching M). For the adaptation of the proof, we can use (6.14), instead. To help the reader comparing both statements, we indicate the differences in the notation

$$\begin{gathered}\lambda \approx 3\gamma_{\mathsf{Zhao}} \quad \zeta \approx d_{\mathsf{Zhao}} \quad \vartheta \approx \eta_{\mathsf{Zhao}} \quad N \approx 2k_{\mathsf{Zhao}} \quad s \approx N_{\mathsf{Zhao}} \quad k \approx n_{\mathsf{Zhao}} \quad n \approx 2n_{\mathsf{Zhao}} \\ M_A \approx \mathcal{M}_{\mathrm{in},\mathsf{Zhao}} \quad M \approx \mathcal{M}_{\mathsf{Zhao}} \quad V_A \approx \mathcal{V}_{1,\mathsf{Zhao}} \quad v(\mathcal{D}_B) \approx f_{b,\mathsf{Zhao}} \quad M_B \approx \mathcal{M}_{b,\mathsf{Zhao}} \,.\end{gathered} \tag{6.15}$$

The bound in *(iii)* is phrased in [28, Lemma 6.11(iii)] in terms of the cluster graph however this is an inessential difference.

It remains to prove *(vii)*. This follows from Case II (d) as $\bigcup \mathcal{L} \subseteq L$.

6.4 Proof of Lemma 6.5

Let $\tilde{M} \subseteq M$ be the minimum matching covering clusters A and B. Lemma 6.5 follows from the following Lemmas 6.6, 6.7 and 6.8. Set $\tilde{\mathcal{S}} = \{C : CD \in M_A,\ C \notin \mathcal{L}\}$, $\tilde{S} = \bigcup \tilde{\mathcal{S}}$, and $M_L = \{CD \in M_A : \{C, D\} \subseteq \mathcal{L}\}$.

Lemma 6.6. *If* $e_{G_\gamma}(\tilde{S}, V \setminus V_A) \geq 53\vartheta n^2$*, then* $T \subseteq G$ *.*

Lemma 6.7. *If* $e_{G_\gamma}(\tilde{S}, V \setminus V_A) < 53\vartheta n^2$*, then* $T \subseteq G$ *or* $|M_L| \geq 9\vartheta N$*.*

Lemma 6.8. *If* $|M_L| \geq 9\vartheta N$*, then* $T \subseteq G$ *.*

To prove Lemmas 6.6–6.8 we use auxiliary Lemmas 6.9, 6.10 and 6.11.

Lemma 6.9. *Let* $\mathcal{P} \subseteq V(M_A)$ *such that* $e_{G_\gamma}(\bigcup \mathcal{P}, V \setminus V_A) \geq \xi n^2$*. Then there exists* $\xi N/2 - 6\sqrt{\lambda}N$ *clusters* $C \in \mathcal{P}$ *with* $\overline{\deg}_{\mathbf{H}}(C, V(M \setminus (M_A \cup M_B))) \geq \xi n/2 - 2\sqrt[4]{\zeta}n$*.*

Set $\mathcal{T}^{\geq 3} = \{t \in \mathcal{D}_A : |V(t) \setminus \mathrm{N}_T(W_A)| \geq 2\}$. For $i = 1, 2$, set $\mathcal{T}^i = \{t \in \mathcal{T}_A : v(t) = i\}$.

Lemma 6.10. *Let* $M^- \subseteq M_A$ *and* $\mathcal{T}_A^* \subseteq \mathcal{D}_A$*. If* $v(\mathcal{T}_A^*) > 2|M^-|s + 10\vartheta n$*, then there exist disjoint matchings* $M_a, M_b \subseteq (M_A \cup M_B) \setminus M^-$ *such that*

$$\overline{\deg}_{\mathbf{H}}(A, V(M_a)) \geq v(\mathcal{D}_A) - v(\mathcal{T}_A^*) + \lambda k \ , \ \text{and} \tag{6.16}$$
$$\overline{\deg}_{\mathbf{H}}(B, V(M_b)) \geq v(\mathcal{D}_B) + \lambda k \ . \tag{6.17}$$

Lemma 6.11. *If* $v(\mathcal{T}^{\geq 3}) \geq 51\vartheta n$ *or* $v(\mathcal{T}_1) \geq 10\vartheta n$*, then* $T \subseteq G$ *.*

In the proof of Lemma 6.10 we use the following fact.

Fact 6.12 ([23, Lemma 9])**.** *Let J be a finite nonempty set, and let* $a, b, \Delta > 0$*. For* $i \in J$*, let* $\alpha_i, \beta_i \in (0, \Delta]$*. Suppose that*

$$\frac{a}{\sum_{i\in J}\alpha_i} + \frac{b}{\sum_{i\in J}\beta_i} \leq 1 \ .$$

Then J can be partitioned into two sets J_a and J_b so that $\sum_{i\in J_a}\alpha_i > a - \Delta$*, and* $\sum_{i\in J_b}\beta_i \geq b$*.*

Proof of Lemma 6.9. At least $\frac{\xi N}{2}$ clusters $C \in \mathcal{P}$ satisfy $\overline{\deg}_{\mathbf{G}}(C, V \setminus V_A) \geq \frac{\xi n}{2}$. From (6.3) we have that all but most $3\sqrt{\lambda}N$ clusters C of $\mathcal{P}$ satisfy $\overline{\deg}_{\mathbf{G}}(C, V \setminus V_*) < \sqrt{\lambda}n/2$. Therefore, all but at most $(\frac{\xi}{2} - 3\sqrt{\lambda})N$ clusters $C \in \mathcal{P}$ satisfy $\overline{\deg}_{\mathbf{G}}(C, V_* \setminus V_A) \geq \frac{\xi n}{2} - \frac{\sqrt{\lambda}n}{2}$.

By Case II (c) and by Lemma 6.4 (iv), all but at most $3\sqrt{\lambda}N$ clusters $C \in \mathcal{P}$ satisfy $\overline{\deg}_{\mathbf{H}}(C, V(M)) \geq \overline{\deg}_{\mathbf{G}}(C, V_*) - 3\zeta n$. As $\overline{\deg}_{\mathbf{H}}(C, V_A) \leq \overline{\deg}_{\mathbf{G}}(C, V_A)$, at least $\frac{\xi N}{2} - 6\sqrt{\lambda}N$ clusters $C \in \mathcal{P}$ satisfy $\overline{\deg}_{\mathbf{H}}(C, V(M) \setminus V_A) \geq \frac{\xi n}{2} - 4\zeta n$.

By Lemma 6.4 (v), (vi), for all clusters $C \in V(\mathbf{H})$ we have $\overline{\deg}_{\mathbf{H}}(C, V(M_B)) \leq \sqrt[4]{\zeta}n$. This proves the lemma. □

Proof of Lemma 6.10. If $v(\mathcal{D}_B) < \sqrt[4]{\zeta}k$, set $M_a = M_A \setminus M^-$ and $M_b = M_B$. From the assumption of the lemma, we have $M_B \cap M^- \subseteq M_B \cap M_A = \emptyset$. Condition (6.17) follows from Lemma 6.5 *(vi)*. For (6.16), Lemma 6.5 (ii) gives

$$\begin{aligned}\overline{\deg}_{\mathbf{H}}(A, V(M_a)) &\geq \overline{\deg}_{\mathbf{H}}(A, V(M_A)) - 2|M^-|s > k - 8\vartheta n - 2|M^-|s \\ &> v(\mathcal{D}_A) - v(\mathcal{T}_A^*) + \lambda k \ .\end{aligned}$$

If $v(\mathcal{D}_B) \geq \sqrt[4]{\zeta}k$, we get $M_a, M_b \subseteq M_A \setminus M^-$ satisfying (6.16) and (6.17) using Fact 6.12 with the following setting: $\Delta = 2s, a = v(\mathcal{D}_A) - v(T_A^*) + 2\lambda k, b = v(\mathcal{D}_B) + \lambda k, J = M_A \setminus M^-$ and for every $CD \in J$, $\alpha_{CD} = \overline{\deg}_{\mathbf{H}}(A, CD)$ and $\beta_{CD} = \overline{\deg}_{\mathbf{H}}(B, CD)$. By (ii) and (v) of Lemma 6.5,

$$\frac{v(\mathcal{D}_A) - v(\mathcal{T}_A^*) + 2\lambda k}{\overline{\deg}_{\mathbf{H}}(A, V(M_A \setminus M^-))} + \frac{v(\mathcal{D}_B) + \lambda k}{\overline{\deg}_{\mathbf{H}}(B, V(M_A \setminus M^-))} \leq \frac{k - v(\mathcal{T}_A^*) + 3\lambda k}{k - 9\vartheta n - 2|M^-|s} \leq 1 ,$$

as required for an application of Fact 6.12. □

Proof of Lemma 6.11.

Claim 6.11.1. If $v(\mathcal{T}^1) \geq 10\vartheta n$, then $T \subseteq G$.

Proof. By Lemma 6.10, with $\mathcal{T}^*_{A,\text{L5.13}} = \mathcal{T}^1$ and $M^-_{\text{L5.13}} = \emptyset$, there exists a partition $M_a \cup M_b = M \setminus \tilde{M}$ satisfying (6.16) and (6.17). We embed the tree $T' = T - V(\mathcal{T}^1)$ using Lemma 5.13 with $\mathcal{D}_{Y,\text{L5.13}} = \mathcal{D}_B$ and $\mathcal{D}_{1,\text{L5.13}} = \mathcal{D}_{X,\text{L5.13}} = \mathcal{D}_A \setminus \mathcal{T}^1$. It is easy to check that the conditions of Lemma 5.13 are met. The trees of $\mathcal{T}^1$ are leaves of T whose parent vertices are mapped to $A \subseteq L$, and can be then embedded greedily. □

We use Lemma 6.9 with setting $\mathcal{P} = V(M_A)$ and $\xi = \kappa/2$, and obtain a set $\mathcal{C} \subseteq V(M_A)$ with $|\mathcal{C}| = 20\vartheta N$ such that for all $C \in \mathcal{C}$ we have

$$\overline{\deg}_{\mathbf{H}}(C, V(M \setminus (M_A \cup M_B))) \geq \tfrac{\kappa n}{8} . \tag{6.18}$$

Set $M^- = \{CD \in M_A : \{C, D\} \cap \mathcal{C} \neq \emptyset\}$. Let $\mathcal{T}_A^* \subseteq \mathcal{T}^{\geq 3}$ be maximal, subject to $v(\mathcal{T}_A^*) \leq 50\vartheta n + \tau$. Hence, $v(\mathcal{T}_A^*) \geq 50\vartheta n > 2|M^-|s + 10\vartheta n$. By Lemma 6.10 there are disjoint matchings $M_a, M_b \subseteq (M_A \cup M_B) \setminus M^-$ satisfying (6.16) and (6.17).

We use Lemma 5.13 to embed the tree T with the τ-fine partition $(W_A, W_B, \mathcal{D}_A, \mathcal{D}_B)$ in G with the following setting: $\mathbf{H}_{\text{L5.13}} = \mathbf{H}$, $X'_{\text{L5.13}} = X_{\text{L5.13}} = A$, $Y_{\text{L5.13}} = Y'_{\text{L5.13}} = B$, $\mathcal{Z}_{\text{L5.13}} = \mathcal{C}$, $M_{X,\text{L5.13}} = \emptyset$, $M_{\text{L5.13}} = M \setminus (\tilde{M} \cup M^-)$, $\mathcal{D}_{1,\text{L5.13}} = \mathcal{D}_A \setminus \mathcal{T}_A^*$, $\mathcal{D}_{2,\text{L5.13}} = \emptyset$, and $\mathcal{D}_{3,\text{L5.13}} = \mathcal{T}_A^*$, $\mathcal{D}_{Y,\text{L5.13}} = \mathcal{D}_B$, $V^X_{\text{L5.13}} = \bigcup V(M_a)$, $V^Y_{\text{L5.13}} := \bigcup V(M_b)$, and $V^{\mathcal{Z}}_{\text{L5.13}} = \bigcup V(M \setminus (M_A \cup M_B))$. The parameters $\varepsilon_{\text{L5.13}} = \alpha, \xi_{\text{L5.13}} = \lambda q, d_{\text{L5.13}} = \gamma, \tau$, and s satisfy $\tau/s \leq \alpha \leq \lambda^2 q^2 \gamma/400$. Let us now verify the conditions of Lemma 5.13. Conditions (i), (ii), and (v) trivially hold. Conditions (iv) and (iii) follow from (6.16) and (6.17), respectively. Condition (vii) follows from (6.18).

For Condition (vi) first observe that $|\mathcal{T}_A^*| + |W_A| \geq |V(\mathcal{T}_A^*) \cap \mathrm{N}_T(W_A)|$. This is because each vertex in $V(\mathcal{T}_A^*) \cap \mathrm{N}_T(W_A)$ is either a root of a shrub, or a predecessor of a vertex in W_A. Moreover, each vertex in W_A is a predecessor of at most one such vertex. As $\mathcal{T}_A^* \subseteq \mathcal{T}^{\geq 3}$,

$$\overline{\deg}_{\mathbf{H}}(A, \bigcup \mathcal{C}) \geq (1 - 2\vartheta)20\vartheta n \geq v(\mathcal{T}_A^*)/3 + |W_A| + \lambda k \geq |V(\mathcal{T}_A^*) \cap \mathrm{N}_T(W_A)| + \lambda k .$$

□

Proof of Lemma 6.6. Using Lemma 6.9 with the setting $\mathcal{P} = \tilde{\mathcal{S}}$ and $\xi = 53\vartheta$ and obtain a set $\mathcal{C}' \subseteq \tilde{\mathcal{S}}$ of size $18\vartheta N$ such that for every $C \in \mathcal{C}'$,

$$\overline{\deg}_{\mathbf{H}}(C, M \setminus (M_A \cup M_B)) \geq 25\vartheta n , \tag{6.19}$$

At least $9\vartheta N$ such clusters are in different edges of M. Let $\mathcal{C}$ be the set of such clusters. Set $M^- = \{CD \in M_A : \{C, D\} \cap \mathcal{C} \neq \emptyset\}$ and $\mathcal{C}^- = V(M^-) \setminus \mathcal{C}$. Note that $|M^-| = 9\vartheta N$ and that $\mathcal{C}^- \subseteq \mathcal{L}$.

Lemma 6.11 tells us that $T \subseteq G$ if $v(\mathcal{T}^1) \geq 10\vartheta n$ or $v(\mathcal{T}^{\geq 3}) \geq 51\vartheta n$. Therefore, suppose that $v(\mathcal{T}^1) < 10\vartheta n$ and $v(\mathcal{T}^{\geq 3}) < 51\vartheta n$.

Observe that $\mathcal{D}_A \setminus (\mathcal{T}^{\geq 3} \cup \mathcal{T}^2 \cup \mathcal{T}^1)$ consists of those internal shrubs that have at most one vertex that is not adjacent to W_A. Consider a shrub t in $\mathcal{D}_A \setminus (\mathcal{T}^{\geq 3} \cup \mathcal{T}^2 \cup \mathcal{T}^1)$. Any vertex in t is either a predecessor of W_A, or the only vertex of t not adjacent to W_A, or the only root in t. Moreover, t always contains a predecessor of W_A, and each vertex in W_A is a predecessor of at most one vertex in such shrubs. Hence, $v(\mathcal{D}_A \setminus (\mathcal{T}^{\geq 3} \cup \mathcal{T}^2 \cup \mathcal{T}^1) \leq 3|W_A|$. Therefore

$$\begin{aligned} v(\mathcal{T}^2) &= v(\mathcal{D}_A) - v(\mathcal{T}^{\geq 3}) - v(\mathcal{T}^1) - v(\mathcal{D}_A \setminus (\mathcal{T}^{\geq 3} \cup \mathcal{T}^2 \cup \mathcal{T}^1)) \\ &\geq \tfrac{k}{2} - |W_A \cup W_B| - 51\vartheta n - 10\vartheta n - 3|W_A| > 29\vartheta n \,. \end{aligned}$$

Let $\mathcal{T}_A^* \subseteq \mathcal{T}^2$ be maximal subject to $v(\mathcal{T}_A^*) \leq 29\vartheta n$. Then $v(\mathcal{T}_A^*) \geq 28\vartheta n \geq 2|M^-|s + 10\vartheta n$ and that $T - V(\mathcal{T}_A^*)$ is a tree. By Lemma 6.10 there exist disjoint matchings $M_a, M_b \subseteq (M_A \cup M_B) \setminus M^-$ satisfying (6.16) and (6.17).

Set $B' = B$ and let $A' \subseteq A$ be the set of typical vertices w. r. t. $\bigcup V(M^-)$. By Fact 5.9 (i) $\min\{|A'|, |B'|\} \geq (1-\alpha)s$. We use Lemma 5.13 to embed $T - V(\mathcal{T}_A^*)$ in $A' \cup B' \cup \bigcup V(M_a \cup M_b)$ with $\mathcal{D}_{Y,\mathrm{L}5.13} = \mathcal{D}_B$ and $\mathcal{D}_{1,\mathrm{L}5.13} = \mathcal{D}_{X,\mathrm{L}5.13} = \mathcal{D}_A \setminus \mathcal{T}_A^*$. It is easy to check that the conditions of Lemma 5.13 are met. It remains to embed $\mathcal{T}_A^*$.

Let $\tilde{C} \subseteq \bigcup \mathcal{C}$ be the set of typical vertices w. r. t. $\bigcup V(M \setminus (M_A \cup M_B)$. By Fact 5.9 (i) $|\bigcup \mathcal{C} \setminus \tilde{C}| \leq \alpha n$. As the current embedding satisfies $\varphi(W_A) \subseteq A'$, we get for every $x \in W_A$,

$$\deg(\varphi(x), \tilde{C} \cup \bigcup \mathcal{C}^-) \geq (1 - 2\vartheta)2|M^-|s - 2\alpha n \geq 17\vartheta n \geq v(\mathcal{T}_A^*)/2 + \lambda n \,.$$

We map the roots of the trees in $\mathcal{T}_A^*$ to $\tilde{C} \cup \bigcup \mathcal{C}^-$. The rest of the trees in $\mathcal{T}_A^*$ can be then embedded greedily using the typicality of the vertices in $\tilde{C}$, (6.19) and that $\bigcup \mathcal{C}^- \subseteq L$. Thus, $T \subseteq G$ as needed. □

Proof of Lemma 6.8. The proof is similar (and actually simpler) to that of Lemma 6.6 and we provide only the needed adaptations. We use M_L instead of M^-. When $\mathcal{T}^1$ and $\mathcal{T}^{\geq 3}$ are small we use the property that $\bigcup V(M_L) \subseteq L$ instead of (6.19) to embed greedily $\mathcal{T}_A^*$. □

Proof of Lemma 6.7.

Claim 6.7.1. There exists a set $\mathcal{C} \subseteq \mathrm{N}_{\mathbf{H}}(A) \cap \mathcal{L} \cap \mathcal{O}$ of size $\frac{\kappa}{20}N$ such that for every $C \in \mathcal{C}$, we have $\overline{\deg}_{\mathbf{H}}(C, V_* \setminus V_A) \geq \frac{\kappa n}{8}$ and the clusters of $\mathcal{C}$ lie in different edges of M.

Proof. We have

$$\begin{aligned} e_{G_\gamma}(V_A \setminus \tilde{S}, V_* \setminus V_A) &\geq e_{G_\gamma}(V_A, V \setminus V_A) - e_{G_\gamma}(\tilde{S}, V \setminus V_A) - e_{G_\gamma}(V_*, V \setminus V_*) \\ &\geq \tfrac{\kappa n^2}{2} - 53\vartheta n^2 - \lambda k^2 > \tfrac{\kappa n^2}{4} \,. \end{aligned}$$

Thus, $\frac{\kappa N}{8}$ clusters C of $V(M_A) \setminus \tilde{\mathcal{S}}$ satisfy $\overline{\deg}_{G_\gamma}(C, V_* \setminus V_A) \geq \frac{\kappa n}{8}$. Pick $\frac{\kappa N}{16}$ of them in different edges of M, and denote them by $\mathcal{C}$. As $\mathcal{V}_* \setminus \mathcal{L}$ is independent, $\mathcal{C} \subseteq \mathcal{L}$. Moreover, by Lemma 6.4 (iv), we have $\mathcal{C} \subseteq \mathcal{O}$. By Lemma 6.1(i), we have $V(M_A) \subseteq N_{\mathbf{H}}(A)$ and thus $\mathcal{C}$ satisfies the assertion of the claim. □

For each $X \in V(\mathbf{H})$, we define $M_X^* = \{CD \in M \,:\, |\overline{\deg}_{\mathbf{H}}(X, C) - \overline{\deg}_{\mathbf{H}}(X, D)| \geq \vartheta s\}$.

Claim 6.7.2. For each cluster $X \in \mathcal{O} \cap \mathcal{L} \cap \mathrm{N}_{\mathbf{H}}(\mathcal{O} \cap \mathcal{L})$, we have $|M_X^*| < \vartheta N/2$, or $T \subseteq G$.

We do not prove Claim 6.7.2 here. The proof can be taken verbatim from [28, Lemma 6.15 (Case 1)]. There, Zhao considers two adjacent clusters $A_{\mathsf{Zhao}}, B_{\mathsf{Zhao}}$ with high average degree in a matching. He shows that if for some $X \in \{A_{\mathsf{Zhao}}, B_{\mathsf{Zhao}}\}$, the matching M_X^* is substantial, then $T \subseteq G$. (He uses notation $\mathcal{M}_{unbal,\mathsf{Zhao}} \approx M_X^*$; recall (6.15) for further vocabulary). The condition of Case II (c) is the counterpart of the property [28, (6.14)].

Let $\mathcal{C}$ be given by Claim 6.7.1. Set $\mathcal{D} = V(M \setminus M_A) \cap \mathcal{O} \cap \mathcal{L}$.

Claim 6.7.3. We have $T \subseteq G$ or $|\mathcal{D}| > \frac{\kappa N}{17}$ and $e_{G_\gamma}(\bigcup \mathcal{C}, \bigcup \mathcal{D}) \geq \frac{\kappa^2 n^2}{340}$.

Proof. For each $C \in \mathcal{C}$, we apply Claim 6.7.2. We get that $|M_C^*| \leq \vartheta N/2$ as otherwise $T \subseteq G$ and we are done. Hence, $\overline{\deg}_{\mathbf{H}}(C, V(M \setminus (M_A \cup M_C^*))) \geq \frac{\kappa n}{8} - \vartheta n$. Let

$$M_C^- = \{D_1 D_2 \in M \ : \ \overline{\deg}_{\mathbf{H}}(X, D_1) < \vartheta s \text{ or } \overline{\deg}_{\mathbf{H}}(X, D_2) < \vartheta s\} \ .$$

By the definition of M_C^*, the weight C sends to both end-clusters of $M \setminus M_C^*$ differs by at most ϑs. Thus, $\overline{\deg}_{\mathbf{H}}(C, V(M \setminus (M_A \cup M_C^* \cup M_C^-))) \geq \frac{\kappa n}{8} - 4\vartheta n$. By Case II (d), all edges in $M \setminus (M_A \cup M_C^* \cup M_C^-)$ meet $\mathcal{L}$. The definition of M_C^* tells us that

$$\begin{aligned}\overline{\deg}_{\mathbf{H}}(C, \mathcal{L} \cap V(M \setminus (M_A \cup M_C^* \cup M_C^-))) &\geq \frac{1}{2+\vartheta} \overline{\deg}_{\mathbf{H}}(C, V(M \setminus (M_A \cup M_C^* \cup M_C^-))) \\ &\geq (1-\vartheta)\tfrac{\kappa n}{16} - 4\vartheta n \ .\end{aligned}$$

Case II (b) gives that $|V(M \setminus M_C^-) \setminus \mathcal{O}| \leq 1$. Therefore, $\overline{\deg}_{\mathbf{H}}(C, \mathcal{D}) > \frac{\kappa n}{17}$, implying $|\mathcal{D}| \geq \frac{\kappa N}{17}$. The assertion follows from the bound on $|\mathcal{C}|$ given by Claim 6.7.1. □

Claim 6.7.4. We have $T \subseteq G$ or $|M_L| \geq \frac{\kappa^3 N}{2 \cdot 10^4}$.

Proof. Let us assume that $T \not\subseteq G$. In particular, the second assertion of Claim 6.7.3 applies. At least $\kappa N/680$ clusters $D \in \mathcal{D}$ satisfy $\overline{\deg}_{G_\gamma}(D, \mathcal{C}) \geq \kappa^2 n/680$. By Claim 6.7.2, we may assume that each of these chosen clusters satisfy $\overline{\deg}_{G_\gamma}(D, \mathcal{C} \setminus V(M_D^*)) \geq \frac{\kappa^2 n}{680} - \vartheta n$, as otherwise $T \subseteq G$. By Lemma 6.1(i), these clusters satisfy $\overline{\deg}_{\mathbf{H}}(D, \mathcal{C} \setminus V(M_D^*)) \geq \frac{\kappa^2 n}{690}$. Let $\mathcal{C}^- = V(M) \setminus \mathcal{C}$. By the definition of M_D^*, we get $\overline{\deg}_{\mathbf{H}}(D, \mathcal{C}^- \setminus V(M_D^*)) \geq \frac{\kappa^2 n}{690} - \vartheta n > \frac{\kappa^2 n}{700}$. Observe that $\mathcal{C}^- \setminus V(M_L) \subseteq \tilde{\mathcal{S}}$. As $|\mathcal{D}| \geq \frac{\kappa N}{17}$ we get,

$$\tfrac{\kappa^3 n^2}{12 \cdot 10^3} < e_{G_\gamma}\left(\bigcup \mathcal{D}, \bigcup \mathcal{C}^-\right) \leq e_{G_\gamma}\left(\bigcup \mathcal{D}, \tilde{S}\right) + e_{G_\gamma}\left(\bigcup \mathcal{D}, V(M_L)\right) \leq 53\vartheta n^2 + |M_L| sn \ ,$$

implying $|M_L| \geq \frac{\kappa^3 N}{2 \cdot 10^4}$. □

Claim 6.7.4 gives the statement of the lemma (recall that $\kappa \gg \vartheta$). □

This finishes the proof of the Lemma 4.2.

7 Proof of Lemma 4.1 (Extremal case)

Let $c_{\mathbf{E}}$ be sufficiently small compared to q. Given $\sigma \in (0, c_{\mathbf{E}}]$, let β and γ be chosen so that $\beta \ll \gamma \ll \sigma$. Given a (β, σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde{V}$ we show that $\mathcal{T}_{k+1} \subseteq G$, or there exists a set $Q \subseteq \tilde{V}$ satisfying Properties (i)–(iii) of Lemma 4.1.

The proof of Lemma 4.1 is split into two statements, Lemma 7.1 and Lemma 7.2, according to the number of leaves of the tree $T \in \mathcal{T}_{k+1}$ considered.

Lemma 7.1. *Let $T \in \mathcal{T}_{k+1}$ be a tree that has at most $60\gamma k$ leaves. Suppose that G admits a (β, σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde{V}$. Then $T \subseteq G$, or there exists a set $Q \subseteq \tilde{V}$ satisfying Properties (i)–(iii) of Lemma 4.1.*

Lemma 7.2. *Let $T \in \mathcal{T}_{k+1}$ be a tree that has more than $60\gamma k$ leaves. Suppose that G admits a (β, σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde{V}$. Then $T \subseteq G$.*

Lemma 4.1 follows Lemmas 7.1 and 7.2. The proofs of these lemmas occupy Sections 7.1, and 7.2. First however, we establish some basic properties of a (β,σ)-extremal partition. Throughout this section we write $m = \mathrm{ci}(\frac{n}{k})$ for the integer closest to $\frac{n}{k}$. The sets V_i, $i \in [\ell]$ are called *clumps*.

Suppose that G admits a (β,σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde V$. Then $\ell \le m$.

Lemma 7.3. *For each $i \in [\ell]$ the following holds.*

(i) For all but at most $\sqrt{\beta}k$ vertices $v \in V_i \cap L$, we have that $\deg(v, V_i) \ge k - \sqrt{\beta}k$.

(ii) For all but at most $2\sqrt{\beta}k$ vertices $v \in V_i \cap S$, we have that $\deg(v, V_i \cap L) \ge |V_i \cap L| - \sqrt{\beta}k$.

(iii) For all but at most $\sqrt{\beta}k$ vertices $v \in V \setminus V_i$, we have that $\deg(v, V_i) < \sqrt{\beta}k$.

Proof. (i) Let $U = \{v \in V_i \cap L : \deg(v, V_i) < k - \sqrt{\beta}k\}$. Since every vertex $v \in U$ sends at least $\sqrt{\beta}k$ edges outside V_i, we deduce from $e(V_i, V \setminus V_i) < \beta k^2$ that $|U| \le \sqrt{\beta}k$.

(ii) Let $W = \{v \in V_i \cap S : \deg(v, V_i \cap L) < |V_i \cap L| - \sqrt{\beta}k\}$. From

$$\begin{aligned} e(V_i \cap L, V_i \cap S) &> |V_i \cap L|k - |V_i \cap L|^2 - \beta k^2 > |V_i \cap L||V_i \cap S| - 2\beta k^2 \text{ , and} \\ e(V_i \cap L, V_i \cap S) &= e(V_i \cap L, W) + e(V_i \cap L, V_i \cap S \setminus W) \\ &\le (|V_i \cap L| - \sqrt{\beta}k)|W| + |V_i \cap L|(|V_i \cap S| - |W|) \\ &= |V_i \cap L||V_i \cap S| - \sqrt{\beta}k|W| \end{aligned}$$

we infer that $|W| < 2\sqrt{\beta}k$.

(iii) Let $Z = \{v \in V \setminus V_i : \deg(v, V_i) \ge \sqrt{\beta}k\}$. We have

$$\beta k^2 > e(V_i, V \setminus V_i) \ge \sum_{v \in Z} \deg(v, V_i) \ge |Z|\sqrt{\beta}k \text{ ,}$$

which proves the statement.

□

For each $i \in [\ell]$, we set $L^i = \{u \in L : \deg(u, V_i) > (1 - \frac{\gamma}{4})k\}$. For every $A \subseteq V_i$, Lemma 7.3(i) and the assumption $|V_i \cap L| \ge (\frac{1}{2} - \beta)k$ give that

$$|L^i| \ge (1 - \tfrac{\gamma}{2})\frac{k}{2} \quad \text{and} \quad \delta(L^i, A) \ge |A| - \frac{\gamma k}{2} \text{ .} \tag{7.1}$$

For each $i \in [\ell]$, we set $S_0^i = \{v \in S \cap V_i : \deg(v, L^i) > |L^i| - \frac{\gamma k}{2}\}$. As the sets V_i are pairwise disjoint, so are the sets $S_0^1, S_0^2, \ldots, S_0^\ell$. Any vertex $v \in S \cap V_i$ with $\deg(v, V_i \cap L) \ge |V_i \cap L| - \sqrt{\beta}k$ satisfies $\deg(v, L^i) \ge |V_i \cap L^i| - \sqrt{\beta}k - |(V_i \cap L) \setminus L^i| \ge |L^i| - \sqrt{\beta}k - |(V_i \cap L) \setminus L^i| - |L^i \setminus V_i|$. Therefore by Lemma 7.3(i),(iii) any such vertex v belongs to S_0^i. By Lemma 7.3(ii) and by (7.1) we have

$$|L^i \cup S_0^i| \ge (1 - \tfrac{\gamma}{2})k \text{ .} \tag{7.2}$$

The next lemma allows to discard trees with substantial discrepancy from further considerations.

Lemma 7.4. *Suppose that G admits a (β,σ)-extremal partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde V$. Then each tree $T \in \mathcal{T}_{k+1}$ with discrepancy at least $2\gamma k$ is a subgraph of G.*

Proof. Fix $i \in [\ell]$. Choose $L^* \subseteq L^i$ with $|L^*| = (1-\frac{\gamma}{2})\frac{k}{2}$, and set $S^* = (L^i \cup S_0^i) \setminus L^*$. By (7.2), $|S^*| \geq (1-\frac{\gamma}{2})\frac{k}{2}$. Using (7.1) and the definition of S_0^i, we have

$$\min\{\delta(L^*, S^*), \delta(S^*, L^*), \delta(L^*, L^*)\} \geq (1-\tfrac{3\gamma}{2})\frac{k}{2} .$$

Take a semi-independent partition (U_1, U_2) of T witnessing that $\mathrm{disc}(T) \geq 2\gamma k$. We apply Fact 3.5 to embed T in G using the sets L^* and S^*. $\square$

Lemma 7.5. *(i) The sets $\{L^i\}_{i\in[\ell]}$ are mutually disjoint, or $\mathcal{T}_{k+1} \subseteq G$.*

(ii) Suppose that $\tilde{V} = \emptyset$. If there exists a vertex $u \in L \setminus (\bigcup_i L^i)$, then $\mathcal{T}_{k+1} \subseteq G$.

Proof. For each $i \in [\ell]$, fix a set $A_i \subseteq L^i$ of size $(\frac{1}{2} - \frac{\gamma}{4})k$, and set $B_i = (L^i \cup S_0^i) \setminus A_i$. By (7.1), (7.2) and the definition of the set S_0^i we have

$$\delta(G[A_i, B_i]) \geq \left(\frac{1}{2} - \frac{5\gamma}{4}\right) k . \tag{7.3}$$

Proof of Part (i). Suppose that there exist distinct indices $i, j \in [\ell]$ and a vertex $u \in L^i \cap L^j$. Let $T \in \mathcal{T}_{k+1}$ be arbitrary. By Lemma 7.3(iii), we have

$$|L^i \cap L^j| < \frac{k}{100} . \tag{7.4}$$

By Lemma 7.4 we can assume in the following that $\mathrm{disc}(T) < 2\gamma k$. By Fact 3.1 there exists a full-subtree $\tilde{T} \subseteq T$ rooted at a vertex r such that $v(\tilde{T}) \in [\frac{k}{6}, \frac{k}{3}]$. We map r to u, and embed the tree $\tilde{T}$ in $G[A_i, B_i]$ greedily. This is possible since

$$\max\{|T_\oplus \cap V(\tilde{T})|, |T_\ominus \cap V(\tilde{T})|\} < \tfrac{v(\tilde{T})}{2} + 2\gamma k \leq \tfrac{k}{6} + 2\gamma k$$

by Fact 3.3, and the graph $G[A_i, B_i]$ satisfies (7.3). It remains to embed the tree $T - \tilde{T}$. By Fact 3.3, we have $\min\{|T_\oplus \cap V(T-\tilde{T})|, |T_\ominus \cap V(T-\tilde{T})\}| > \frac{v(T-\tilde{T})}{2} - 2\gamma k$, and thus $\max\{|T_\oplus \cap V(T-\tilde{T})|, |T_\ominus \cap V(T-\tilde{T})|\} < \frac{5k}{12} + 2\gamma k$. We embed $T - \tilde{T}$ in $G[A_j, B_j]$ greedily (avoiding the previously used vertices of $L^i \cap L^j$; we use (7.4) to bound the number of occupied vertices).

Proof of Part (ii). Suppose that there exists a vertex $u \in L \setminus \bigcup_i L^i$. By Part (i) of the lemma, we may assume that the sets L^i are pairwise disjoint. Let

$$X_i = \{u \in A_i \,:\, \deg(u, V_i) > (1 - \tfrac{\gamma}{13m})k\} \text{ , and}$$
$$Y_i = \{u \in B_i \,:\, \deg(u, L^i) > |L^i| - \tfrac{\gamma k}{13m}\} .$$

(In applications, we use that $\deg(u, X_i) > |X_i| - \frac{\gamma k}{13m}$ for every $u \in Y_i$.) Applying Lemma 7.3 (i)–(ii) to L^i, S_0^i, X_i and Y_i, we get that

$$|V_i \setminus (X_i \cup Y_i)| < \tfrac{\gamma k}{6m^2} . \tag{7.5}$$

As $X_i \subseteq L^i$ and $Y_i \subseteq S_0^i$, all the sets X_i and Y_i are pairwise disjoint. Without loss of generality, we assume that $\deg(u, X_1 \cup Y_1) \geq \ldots \geq \deg(u, X_m \cup Y_m)$. As $u \in L \setminus L^1$ we have

$$\begin{aligned} k \leq \deg(u, L) &\leq \sum_{i=1}^{m} \deg(u, X_i \cup Y_i) + \tfrac{\gamma k}{6m} \leq (1 - \tfrac{\gamma}{2})k + \sum_{i=2}^{m} \deg(u, X_i \cup Y_i) + \tfrac{\gamma k}{6m} \\ &\leq (1 - \tfrac{\gamma}{3})k + (m-1)\deg(u, X_2 \cup Y_2) . \end{aligned}$$

This yields that

$$\deg(u, X_1 \cup Y_1) \geq \deg(u, X_2 \cup Y_2) \geq \frac{\gamma k}{3(m-1)} \geq 2 . \tag{7.6}$$

Let $T \in \mathcal{T}_{k+1}$ be arbitrary. Analogously as in the proof of Lemma 7.4 we have $T \subseteq G$ if $\mathrm{disc}(T) \geq \frac{\gamma k}{6m}$. Therefore we assume that $\mathrm{disc}(T) < \frac{\gamma k}{6m}$. By Fact 3.1 there exists a full-subtree $\tilde{T} \subseteq T$ rooted at a vertex r such that $v(\tilde{T}) \in [0.3k, 0.6k]$. Let D be the set of leaves of T in $\mathrm{N}_T(r)$. We first embed the tree $T - D$, mapping r to u, as described below. The embedding is then extended to an embedding of T using the fact that $u \in L$.

A 2^+-*component* is a component of the forest $T - r$ of order at least two. Let $\mathcal{C}$ be the family of all 2^+-components. For each subfamily $\mathcal{C}' \subseteq \mathcal{C}$, we have by Fact 3.3 and by the assumption $\mathrm{disc}(T) \leq \frac{\gamma k}{6m}$ that

$$\max\{|V(\mathcal{C}') \cap T_\ominus|, |V(\mathcal{C}') \cap T_\oplus|\} < \tfrac{|V(\mathcal{C}')|}{2} + \tfrac{\gamma k}{12m} + 1 . \tag{7.7}$$

By (7.5) at most $\frac{\gamma k}{6m}$ vertices of the graph G are not contained in $\bigcup_i (X_i \cup Y_i)$. Thus, $\deg(u, \bigcup_i (X_i \cup Y_i)) \geq (1 - \frac{\gamma}{6m})k$. We assign each 2^+-component $C \in \mathcal{C}$ an index $i_C \in [m]$ such that C will be mapped to the clump V_{i_C}. For each $j \in [m]$ we shall require:

$$\deg(u, X_j \cup Y_j) \geq |\{C \in \mathcal{C} : i_C = j\}| \text{ , and} \tag{7.8}$$

$$\sum_{\substack{C \in \mathcal{C} \\ i_C = j}} v(C) \leq (1 - \tfrac{\gamma}{3})k . \tag{7.9}$$

Claim 7.5.1. There exists a family $\{i_C\}_{C \in \mathcal{C}}$ such that (7.8) and (7.9) are satisfied.

Proof. We order the 2^+-components as $C_1, \ldots, C_{|\mathcal{C}|}$ so that $v(C_1) \geq v(C_2) \geq \ldots \geq v(C_{|\mathcal{C}|})$. For $j = 1, \ldots, |\mathcal{C}|$, take the smallest index $i \in [m]$ with the property that after assigning $i_{C_j} = i$, the properties (7.8) and (7.9) are satisfied for the partial assignment $\{i_{C_{j'}}\}_{j' \leq j}$. If for a given j there exists no such value i we just mark C_j as unassigned and proceed with $j + 1$.

We thus need to check that actually each 2^+-component C_j was assigned. Suppose for a contradiction that C_g was not. We have $v(C_1) \leq 0.7k$, and for $\ell \geq 2$ we have $v(C_\ell) \leq \frac{k}{\ell}$. These bounds and (7.6) guarantee us that $C_1, \ldots, C_4$ can always be assigned; one assignment satisfying (7.8) and (7.9) is $i_{C_1} = i_{C_4} = 1, i_{C_2} = i_{C_3} = 2$. Thus $g > 4$, and consequently $v(C_g) \leq 0.2k$.

To finish the argument, we distinguish two cases. First, assume that $\deg(u, X_1 \cup Y_1) \geq 0.5k$. Since $v(C) \geq 2$ for each $C \in \mathcal{C}$, property (7.8) for $j = 1$ holds trivially. As C_g could not be assigned with $i_{C_g} = 1$, by (7.9) we get that $\sum_{i_C = 1} v(C) > (1 - \frac{\gamma}{3})k - v(C_g)$. In particular, the number of 2^+-components C that are unassigned, or have $i_C \neq 1$ is less than $1 + \frac{\gamma k}{6}$. Further, the total order of the 2^+-components to be assigned to other clumps is at most $v(C_g) + \frac{\gamma k}{3} < 0.4k$. Thus, (7.9) holds trivially for $j > 1$. The reason why the component C_g was not assigned is that it did not satisfy (7.8) for any $j > 1$. Hence, by (7.5) we have

$$1 + \tfrac{\gamma k}{6} > \sum_{j > 1} \deg(u, X_j \cup Y_j) \geq k - \deg(u, X_1 \cup Y_1) - \sum_{j=1}^{m} |V_j \setminus (X_j \cup Y_j)| \geq \frac{k}{3} ,$$

a contradiction with the choice of γ.

Now, consider the case that $\deg(u, X_1 \cup Y_1) < 0.5k$. Then $\deg(u, X_2 \cup Y_2) < 0.5k$. Observe that for $j = 1, 2$ we have $\sum_{i_C = j} v(C) \geq 2\deg(u, X_j \cup Y_j) - \frac{\gamma k}{3} - v(C_g)$, as otherwise we could have assigned $i_{C_g} = j$ without violating (7.8) and (7.9). For $j > 2$ by similar arguments we have $\sum_{i_C=j} v(C) \geq \min\{0.7k, 2\deg(u, X_j \cup Y_j)\}$. Summing these bounds, we get that

$$\sum_{C\in\mathcal{C}} v(C) \geq 2\deg(u, X_1\cup Y_1)+2\deg(u, X_2\cup Y_2)-2\frac{\gamma k}{3}-2v(C_g)+\sum_{j=3}^{m}\min\{0.7k, 2\deg(u, X_j\cup Y_j)\} \ . \tag{7.10}$$

Suppose that for some $j > 2$ we have $0.7k \leq 2\deg(u, X_j \cup Y_j)$. Then $2\deg(u, X_1 \cup Y_1) \geq 2\deg(u, X_2 \cup Y_2) \geq 0.7k$, and thus

$$\sum_{C\in\mathcal{C}} v(C) \geq 0.7k + 0.7k - 2\frac{\gamma k}{3} - 2v(C_g) + 0.7k \geq 1.6k > k \ ,$$

where we used that $v(C_g) \leq 0.2k$. This is a contradiction. Thus, we can assume that for all $j > 2$, $0.7k > 2\deg(u, X_j \cup Y_j)$. Plugging into (7.10) we get

$$\sum_{C\in\mathcal{C}} v(C) \geq \sum_{j=1}^{m} 2\deg(u, X_j \cup Y_j) - 2\frac{\gamma k}{3} - 2v(C_g) \geq 2\cdot 0.9k - 2\frac{\gamma k}{3} - 2\cdot 0.2k > k \ ,$$

which again gives a contradiction. □

We embed the tree $T - D$ as follows. Let us consider the indices $\{i_C\}_{C\in\mathcal{C}}$ from Claim 7.5.1. The vertex r is mapped to u. For each component $C \in \mathcal{C}$ we map its root $r_C \in V(C) \cap \mathrm{N}_T(r)$ to one vertex from $(X_{i_C} \cup Y_{i_C}) \cap \mathrm{N}_G(u)$ (so that distinct roots are mapped to distinct vertices). We denote the image of the root r_C by $\varphi(r_C)$. The mapping of the roots is extended to an embedding of all 2^+-components. This can be done greedily since each of the graphs $G[X_i, Y_i]$ has minimum degree at least $(\frac{1}{2} - \frac{\gamma}{12m})k + 1$, and we have by a double application of (7.7) that

$$\sum_{\substack{C\in\mathcal{C}\\ \varphi(r_C)\in X_i}} |V(C) \cap T_\oplus| + \sum_{\substack{C\in\mathcal{C}\\ \varphi(r_C)\in Y_i}} |V(C) \cap T_\ominus| < (1 - \tfrac{\gamma}{3})\tfrac{k}{2} + 2(\tfrac{\gamma k}{12m} + 1) \leq \delta(G[X_i, Y_i]) \ , \text{ and}$$

$$\sum_{\substack{C\in\mathcal{C}\\ \varphi(r_C)\in X_i}} |V(C) \cap T_\ominus| + \sum_{\substack{C\in\mathcal{C}\\ \varphi(r_C)\in Y_i}} |V(C) \cap T_\oplus| < (1 - \tfrac{\gamma}{3})\tfrac{k}{2} + 2(\tfrac{\gamma k}{12m} + 1) \leq \delta(G[X_i, Y_i]) \ .$$

□

Much of the work for proving Lemma 4.1 splits according to the following distinction. A (β, σ)-extremal partition is said to be *abundant* if there exists an index $i \in [\ell]$ with $|L^i| \geq \frac{k+1}{2}$. It is called *deficient* otherwise.

We now derive properties of G in the deficient case. First, we observe that G is decomposed into clumps.

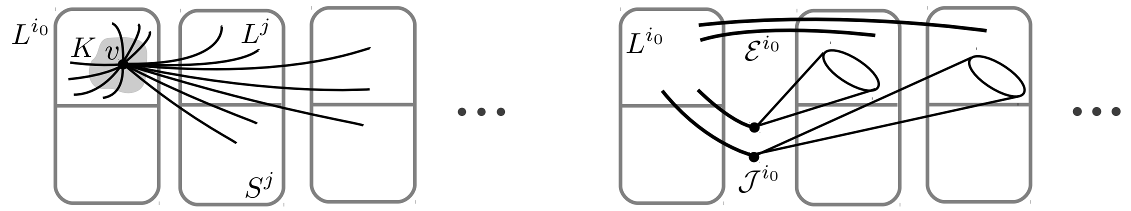
Lemma 7.6. *Suppose that G admits a (β, σ)-extremal deficient partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell \dot\cup \tilde{V}$. Then $\tilde{V} = \emptyset$, and $\ell = m$. Further,*

$$m(k + 1) > n \ . \tag{7.11}$$

Proof. Since the partition is deficient we have $|L \cap V_i| \leq \frac{k}{2}$ for all $i \in [\ell]$. Thus by the definition of (β, σ)-extremality, we have $|L| \leq \ell\frac{k}{2} + (\frac{1}{2} - \sigma)|\tilde{V}|$, and $|S| > \ell(1 - \beta)\frac{k}{2} + (\frac{1}{2} + \sigma)|\tilde{V}|$. Since $|L| \geq |S|$, we infer that $|\tilde{V}| < \frac{\gamma \ell k}{4\sigma}$. This in turn implies that $\tilde{V} = \emptyset$. Thus, $\ell = m$. To get the bound (7.11), we observe that

$$n = |L| + |S| \leq 2|L| = 2\sum_{i=1}^{m} |L \cap V_i| < 2m\frac{k+1}{2} \ .$$

□



(a) Connecting structure guaranteed by Lemma 7.7. (b) Connecting structure guaranteed by Lemma 7.8.

Figure 6: Structures in Lemma 7.7 and Lemma 7.8

Lemmas 7.7 and 7.8 deal with the deficient case. It may happen that none of the clumps is suitable for the embedding of the tree $T \in \mathcal{T}_{k+1}$. For this reason, we must find connecting structures that allow us to distribute parts of T to different clumps. Each lemma is used for a different type of trees.

For $j \in [m]$, set $S^j = \{v \in S \,:\, \deg(v, L^j) \geq \frac{k}{5m}\}$.

Lemma 7.7. *Suppose that G admits a (β,σ)-extremal deficient partition $V = V_1\dot{\cup}\ldots\dot{\cup}V_m$, such that $\{L^i\}_{i=1}^m$ is a partition of L. Then there exist an index $i_0 \in [m]$ such that we have $|K| \geq k/10$ for the set*

$$K = \left\{ v \in L^{i_0} \,:\, \deg(v, L^{i_0}) + \deg(v, \bigcup_{j \neq i_0} (L^j \cup S^j)) \geq \frac{k+1}{2} \right\} . \tag{7.12}$$

Proof. We partition $\bigcup_j S^j$ into sets $\tilde{S}^j$, $j \in [m]$ such that $\tilde{S}^j \subseteq S^j$. As $|L| \geq |S|$, there exists an index $i \in [m]$ such that $|\tilde{S}^i| \leq |L^i| \leq \frac{k}{2}$. Without loss of generality, assume that $\frac{k}{2} - |\tilde{S}^1|$ is the maximum value among all the values $\frac{k}{2} - |\tilde{S}^i|$ ($i \in [m]$); then $i_0 = 1$ is the index asserted by the lemma. We have that $\frac{k}{2} - |\tilde{S}^1|$ is non-negative. For each vertex $v \in L^1 \setminus K$, we have

$$\deg(v, S \setminus \bigcup_{j \neq 1} \tilde{S}^j) \geq \deg(v, S \setminus \bigcup_{j \neq 1} S^j) \geq \tfrac{k}{2}.$$

Thus $\deg(v, S^-) > \frac{k}{2} - |\tilde{S}^1|$, where $S^- = \{u \in S \,:\, \deg(u, L^i) < \frac{k}{5m}, \forall i = 1, \ldots, m\}$. We have

$$|S^-|\frac{k}{5m} > e(L^1 \setminus K, S^-) \geq |L^1 \setminus K| \left(\frac{k}{2} - |\tilde{S}^1| \right) . \tag{7.13}$$

On the other hand, as $\sum_j |L^j| = |L| \geq |S| = \sum_j |\tilde{S}^j| + |S^-|$, there exists an index $i \in [m]$ such that $|L^i| \geq |\tilde{S}^i| + \frac{|S^-|}{m}$. From the maximality of $\frac{k}{2} - |\tilde{S}^1|$ and from (7.13) we deduce that

$$\frac{k}{2} - |\tilde{S}^1| \geq \frac{k}{2} - |\tilde{S}^i| \geq |L^i| - |\tilde{S}^i| \geq \frac{|S^-|}{m} > \frac{5|L^1 \setminus K|}{k} \left(\frac{k}{2} - |\tilde{S}^1| \right).$$

This implies that $k > 5|L^1 \setminus K|$, and the asserted bound on $|K|$ follows from (7.1). □

Lemma 7.8. *Suppose that G admits a (β,σ)-extremal deficient partition $V = V_1\dot{\cup}\ldots\dot{\cup}V_m$. Furthermore, suppose that the sets $\{L^i\}_{i \in [m]}$ partition the set L.*

Then there exists an index $i_0 \in [m]$ and matchings $\mathcal{E}^{i_0}$, and $\mathcal{J}^{i_0}$ such that the following hold.

(i) *$\mathcal{E}^{i_0}$ is an $L^{i_0} - (L \setminus L^{i_0})$-matching, $\mathcal{J}^{i_0}$ is an $L^{i_0} - \bigcup_{i \neq i_0} S^i$-matching.*

(ii) *$V(\mathcal{E}^{i_0}) \cap V(\mathcal{J}^{i_0}) = \emptyset$.*

(iii) $|L^{i_0}| + |\mathcal{E}^{i_0}| + |\mathcal{J}^{i_0}| \geq \frac{k+1}{2}$.

(iv) $|\mathcal{E}^{i_0}| + |\mathcal{J}^{i_0}| < \gamma k$.

Proof. By Lemma 7.3 we have that $|S^i| > (\frac{1}{2} - \gamma)k$. We first find for each $i \in [m]$ two vertex-disjoint matchings $\mathcal{E}^i$ and $\mathcal{D}^i$, such that $\mathcal{E}^i$ is an $L^i - (L \setminus L^i)$-matching, $\mathcal{D}^i$ is an $L^i - (S \setminus S^i)$-matching, and such that the matchings $\{\mathcal{D}^i\}_{i\in[m]}$ are pairwise vertex-disjoint.

For each $i \in [m]$, take $\mathcal{E}^i$ to be a maximum $L^i - (L \setminus L^i)$ matching. If $|L^i| + |S^i| + |\mathcal{E}^i| > k+1$, we truncate $\mathcal{E}^i$ so that $|L^i| + |S^i| + |\mathcal{E}^i| = \max\{k+1, |L^i| + |S^i|\}$. Let us assume that

$$|L^1| + |S^1| + |\mathcal{E}^1| \geq |L^2| + |S^2| + |\mathcal{E}^2| \geq \ldots \geq |L^m| + |S^m| + |\mathcal{E}^m| . \tag{7.14}$$

Start with $i = 1$, and increase the index i gradually. Take $\mathcal{D}^i$ to be a maximum $(L^i \setminus V(\mathcal{E}^i)) - (S \setminus (S^i \cup \bigcup_{j<i} V(\mathcal{D}^j)))$ matching and truncate it so that $|L^i| + |S^i| + |\mathcal{E}^i| + |\mathcal{D}^i| = \max\{k + 1, |L^i| + |S^i| + |\mathcal{E}^i|\}$. Such a matching $\mathcal{D}^i$ exists. Indeed, if $|L^i| + |S^i| + |\mathcal{E}^i| \geq k + 1$, then set $\mathcal{D}^i = \emptyset$. Otherwise, we find a matching $\mathcal{D}^i$ of size $d_i = k + 1 - |L^i| - |S^i| - |\mathcal{E}^i|$ as follows. Set $B_i = S \cap \bigcup_{j<i} V(\mathcal{D}^j)$. From the sizes of the matchings $\mathcal{D}^j$ $(j < i)$ and the ordering given by (7.14) we get $|B_i| < md_i$. Each vertex $u \in L^i$ has at least d_i neighbors outside $L^i \cup S^i \cup V(\mathcal{E}^i)$. Color arbitrary d_i edges emanating from each vertex $u \in L^i$ outside $L^i \cup S^i \cup V(\mathcal{E}^i)$ by black, and the remaining edges incident with u by grey. We have

$$e_{\mathrm{black}}\big(L^i \setminus V(\mathcal{E}^i), S \setminus (S^i \cup B_i)\big) > d_i(\tfrac{1}{2} - 3\gamma)k - md_i\frac{k}{5m} > \frac{d_i k}{5} . \tag{7.15}$$

Since the maximum degree in the graph $G_{\mathrm{black}}[L^i \setminus V(E^i), S \setminus (S^i \cup B_i)]$ is upper-bounded by $\max\{\frac{k}{5m}, d_i\} = \frac{k}{5m}$, there is no vertex cover of $G_{\mathrm{black}}[L^i \setminus V(E^i), S \setminus (S^i \cup B_i)]$ of size less than $(\frac{d_i k}{5})/(\frac{k}{5m}) \geq d_i$. Hence, by König's Matching Theorem, there exists a matching $\mathcal{D}^i$ of size d_i with the desired properties. We set $X_i = V(\mathcal{D}^i) \setminus L^i$.

Let us summarize the properties of the obtained structure. For each $i \in [m]$ we have

$$|L^i| + |S^i| + |\mathcal{E}^i| + |X_i| \geq k + 1, \text{ and} \tag{7.16}$$

$$X_i \cap \bigcup_{j\neq i} X_j = \emptyset \quad \text{and} \quad S^i \cap X_i = \emptyset . \tag{7.17}$$

There is an index $i_0 \in [m]$ such that sufficiently many vertices from $S^{i_0} \cup X^{i_0}$ are contained in $\bigcup_{j\neq i_0} S^j$, giving the desired bridges from the clump V_{i_0}. Indeed,

$$\begin{aligned} n - |L| \geq \left|\bigcup_i (S^i \cup X_i)\right| &\overset{(7.17)}{\geq} \sum_i |S^i| + \sum_i |X_i| - \sum_i \left|(S^i \cup X_i) \cap \bigcup_{j\neq i} S^j\right| \\ &\overset{(7.16)}{\geq} m(k+1) - |L| - \sum_i \left|(S^i \cup X_i) \cap \bigcup_{j\neq i} S^j\right| - \sum_i |\mathcal{E}^i| , \end{aligned}$$

which yields

$$\begin{aligned} \sum_i \left(|L^i| + |\mathcal{E}^i| + \left|(S^i \cup X_i) \cap \bigcup_{j\neq i} S^j\right|\right) &\geq |L| + m(k+1) - n \geq m(k+1) - \frac{n}{2} \\ &\overset{(7.11)}{\geq} \frac{m(k+1)}{2} . \end{aligned}$$

By averaging, there exists an index $i_0 \in [m]$ such that

$$|L^{i_0}| + |\mathcal{E}^{i_0}| + \left|(S^{i_0} \cup X_{i_0}) \cap \bigcup_{j \neq i_0} S^j\right| \geq \frac{k+1}{2} . \tag{7.18}$$

It remains to define $\mathcal{J}^{i_0}$. Let $\mathcal{J}_1 = \{e \in \mathcal{D}^{i_0} : e \cap \bigcup_{j \neq i_0} S^j \neq \emptyset\}$. Set $Q = S^{i_0} \cap \bigcup_{j \neq i_0} S^j$. Let $\mathcal{J}_2$ be any matching in $G[Q, L^{i_0} \setminus V(\mathcal{E}^{i_0} \cup \mathcal{J}_1)]$ that covers Q. Since $|Q| < \gamma k$, we can find such a matching greedily. Set $\mathcal{J}^{i_0} = \mathcal{J}_1 \cup \mathcal{J}_2$.

Properties (i)–(ii) of the lemma are clear from the construction. Property (iii) follows from (7.18), and using that $|\mathcal{J}_1| = |X_{i_0} \cap \bigcup_{j \neq i_0} S^j|$ and $|\mathcal{J}_2| = |S^{i_0} \cap \bigcup_{j \neq i_0} S^j|$.

Last, (7.1) tells us that we can truncate $\mathcal{E}^{i_0}$ and $\mathcal{J}^{i_0}$ so that (iv) is satisfied without violating (iii). This truncation preserved properties (i)–(ii). □

7.1 Proof of Lemma 7.1

Suppose that T and G satisfy the hypothesis of Lemma 7.1. By Lemma 7.4, we can assume that T has discrepancy less than $2\gamma k$. In particular,

$$|T_\oplus| \leq \frac{k}{2} + \gamma k . \tag{7.19}$$

Recall that if G is deficient then by Lemma 7.6 we have $\tilde{V} = \emptyset$. For each $i \in [\ell]$ we define $X^i = \{v \in V_i : \deg(v, L^i) > \frac{k}{5m}\}$. If G is abundant, we set $\Lambda \subseteq [\ell]$ to be the set of indices i_0 such that $|L^{i_0}| \geq \frac{k+1}{2}$, and set $\mathcal{E}^{i_0} = \mathcal{J}^{i_0} = \emptyset$. If G is deficient, we apply Lemma 7.8 to obtain sets S^i, an index i_0, and two matchings $\mathcal{E}^{i_0}$ and $\mathcal{J}^{i_0}$ such that

$$|L^{i_0}| + |\mathcal{E}^{i_0}| + |\mathcal{J}^{i_0}| \geq \tfrac{k+1}{2} \geq |T_\ominus| . \tag{7.20}$$

We then set $\Lambda = \{i_0\}$.

For each $i_0 \in \Lambda$ individually, we shall try to embed the tree T so that most of the vertices of T are embedded in V_{i_0}. We show that if all the attempts fail, then there exists a set Q satisfying the assertions of Lemma 4.1.

Fix $i_0 \in \Lambda$. Let $F^{i_0} = V(\mathcal{E}^{i_0}) \cup V(\mathcal{J}^{i_0})$. By Lemma 7.8$(iv)$, $|F^{i_0} \cap L^{i_0}| \leq \gamma k$. Take a maximum family $\mathcal{P}$ of vertex-disjoint $(L^{i_0} \setminus F^{i_0}) - (V \setminus (L^{i_0} \cup S_0^{i_0} \cup F^{i_0})) - (L^{i_0} \setminus F^{i_0})$-paths.

Claim 1. *If $|L^{i_0} \cup S_0^{i_0} \cup F^{i_0}| + |\mathcal{P}| \geq k - 1$ then $T \subseteq G$.*

Proof. Consider a family of paths $\mathcal{P}' \subseteq \mathcal{P}$ by truncating $\mathcal{P}$ so that $|\mathcal{P}'| = \min\{|\mathcal{P}|, 30\gamma k\}$. By (7.2), $|L^{i_0} \cup S_0^{i_0} \cup F^{i_0}| + |\mathcal{P}'| \geq k - 1$. Observe that $V(\mathcal{P}') \setminus L^{i_0}$ are the middle vertices of $\mathcal{P}'$. Fix a set $A \subseteq L^{i_0}$ of size $|T_\ominus| - |\mathcal{J}^{i_0}| - |\mathcal{E}^{i_0}|$ and which contains $(F^{i_0} \cup V(\mathcal{P}')) \cap L^{i_0}$. This is possible by (7.20) and by

$$|(F^{i_0} \cup V(\mathcal{P}')) \cap L^{i_0}| \overset{\text{L7.8}(iv)}{<} 31\gamma k < \tfrac{k}{2} - 2\gamma k \overset{(7.19)}{<} |T_\ominus| - |\mathcal{J}^{i_0}| - |\mathcal{E}^{i_0}| .$$

We apply Lemma 3.8, setting the parameters of the lemma as $\alpha = 60\gamma$, $A, B_\text{a} = (L^{i_0} \setminus A) \cup S_0^{i_0}, B_\text{d} = V(\mathcal{P}') \setminus L^{i_0}, \mathcal{Q} = \mathcal{P}', \mathcal{E} = \mathcal{E}^{i_0} \cup \mathcal{J}^{i_0}, \mathcal{M} = \emptyset, I = [\ell] \setminus \{i_0\}$, and $H_\kappa = G[L^\kappa \cup S^\kappa]$ (for each $\kappa \in I$) to get $T \subseteq G$. To check Condition (vii) of Lemma 3.8, let us consider an arbitrary vertex $v \in A$.

$$\begin{aligned} \deg(v, B_\text{a} \cup B_\text{d}) &\overset{(7.1)}{\geq} |(B_\text{a} \cup B_\text{d}) \cap V_i| - \gamma k \\ &\geq |B_\text{a} \cup B_\text{d}| - |L^{i_0} \setminus V_{i_0}| - |S_0^{i_0} \setminus V_{i_0}| - |\mathcal{P}'| - \gamma k \\ &\geq |B_\text{a} \cup B_\text{d}| - \sqrt{\beta}k - \sqrt{\beta}k - 30\gamma k - \gamma k \geq |B_\text{a} \cup B_\text{d}| - 60\gamma k , \end{aligned} \tag{7.21}$$

where the last line follows from Lemma 7.3(iii) combined with the definition of L^{i_0}, $S_0^{i_0}$, and (7.1). Other conditions of Lemma 3.8 are easy to check. $\square$

It remains to consider the case that $|L^{i_0} \cup S_0^{i_0} \cup F^{i_0}| + |\mathcal{P}| \le k-2$. From (7.2), we have $|\mathcal{P}| < \gamma k$. Consider an arbitrary vertex $u \in L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}))$. Since $\deg(u) \ge k$, there are at least two edges ux_u^1 and ux_u^2 that emanate into $V \setminus (L^{i_0} \cup S_0^{i_0} \cup F^{i_0})$. By the maximality of $\mathcal{P}$ all the vertices x_u^1, x_u^2, $u \in L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}))$, are pairwise distinct. Set $R_{i_0} = \bigcup_{u \in L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}))} \{x_u^1, x_u^2\}$ and $\tilde{R}_{i_0} = R_{i_0} \cap \tilde{V}$.

Claim 2. *For an arbitrary set $U \subseteq R_{i_0}$ there exists a $U - (L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}))$ matching $\mathcal{F}_{i_0}$ with $|\mathcal{F}_{i_0}| \ge \frac{|U|}{2}$.*

Proof. For $q = 1,2$, let $U_q = \{u \in L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}) \,:\, x_u^q \in U\}$. There exists $q \in [2]$ such that $|U_q| \ge \frac{|U|}{2}$. The desired matching $\mathcal{F}_{i_0}$ is then $\{ux_u^q\}_{u \in U_q}$. $\square$

Claim 3. *If $|\tilde{R}_{i_0}| \le 2|L^{i_0}| - 7m\gamma k$ then $T \subseteq G$.*

Proof. Observe that

$$\begin{aligned}
\left| R_{i_0} \cap \bigcup_{i \in [\ell]} (L^i \cup X^i) \right| &\ge |R_{i_0}| - |\tilde{R}_{i_0}| - \left| V \setminus \left(\tilde{V} \cup \bigcup_{i \in [\ell]} X^i \right) \right| \\
&\overset{\text{L7.3}}{\ge} 2|L^{i_0} \setminus (F^{i_0} \cup V(\mathcal{P}))| - (2|L^{i_0}| - 7m\gamma k) - m\sqrt{\beta}k \\
&\ge 2|L^{i_0}| - 4\gamma k - 2|L^{i_0}| + 7m\gamma k - m\sqrt{\beta}k \ge 2\gamma k \,.
\end{aligned}$$

By Claim 2, there exists an $(L^{i_0} \setminus F^{i_0}) - (R_{i_0} \cap \bigcup_{i \in [\ell]} (L^i \cup X^i))$ matching $\mathcal{N}$ of size γk. Fix a set $A \subseteq L^{i_0}$ of size $|T_\ominus| - |\mathcal{J}^{i_0}| - |\mathcal{E}^{i_0}|$ and which contains $(F^{i_0} \cup V(\mathcal{N})) \cap L^{i_0}$. We apply Lemma 3.8 with parameters $\alpha = 60\gamma$, $A, B_{\mathrm{a}} = (L^{i_0} \setminus A) \cup S_0^{i_0}, B_{\mathrm{d}} = \mathcal{Q} = \mathcal{M} = \emptyset, \mathcal{E} = \mathcal{E}^{i_0} \cup \mathcal{J}^{i_0} \cup \mathcal{N}, I = [\ell] \setminus \{i_0\}$, and $H_\kappa = G[L^\kappa \cup S^\kappa]$ (for each $\kappa \in I$) and get that $T \subseteq G$. Condition (vi) of Lemma 3.8 follows from (7.2). Condition (vii) is checked analogously as in (7.21). Other conditions are easy to verify. $\square$

Putting Claim 1 and Claim 3 together, we can assume that for each $i \in \Lambda$, we have

$$|\tilde{R}_i| > 2|L^i| - 7m\gamma k \,. \tag{7.22}$$

Suppose that there exists an index $i_0 \in \Lambda$ such that

$$\left| \tilde{R}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{R}_i \right| \ge 8\gamma k \,. \tag{7.23}$$

Claim 2 gives an $(L^{i_0} \setminus F^{i_0}) - (\tilde{R}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{R}_i)$ matching $\mathcal{M}_1$ of size $4\gamma k$. Further applications of Claim 2 for indices in $i \in \Lambda \setminus \{i_0\}$ and sets $U = V(M_1) \cap \tilde{R}_{i_0} \cap \tilde{R}_i$ yield a $\left(V(\mathcal{M}_1) \cap \tilde{R}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{R}_i \right) - \left(\bigcup_{i \in \Lambda \setminus \{i_0\}} L^i \right)$ matching $\mathcal{M}_2$ of size $2\gamma k$. From this matching choose a matching $\mathcal{M}_3$ of size γk that is disjoint from F^{i_0}. Extend the edges of $\mathcal{M}_3$ by edges of $\mathcal{M}_1$. This leads to γk vertex-disjoint $L^{i_0} - (\tilde{R}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{R}_i) - \left(\bigcup_{i \in \Lambda \setminus \{i_0\}} L^i \right)$-paths, denoted by $\mathcal{M}$. Fix a set $A \subseteq L^{i_0}$ of size $|T_\ominus| - |\mathcal{J}^{i_0}| - |\mathcal{E}^{i_0}|$ and which contains $(F^{i_0} \cup V(\mathcal{M})) \cap L^{i_0}$. This is possible by (7.20) and by

$$|(F^{i_0} \cup V(\mathcal{M})) \cap L^{i_0}| \overset{\text{L7.8}(iv)}{<} 2\gamma k \overset{(7.19)}{<} |T_\ominus| - |\mathcal{J}^{i_0}| - |\mathcal{E}^{i_0}| \,.$$

We apply Lemma 3.8, setting the parameters of the lemma as $\alpha = 60\gamma$, $A, B_{\mathrm{a}} = (L^{i_0} \setminus A) \cup S_0^{i_0}, B_{\mathrm{d}} = \mathcal{Q} = \emptyset, \mathcal{E} = \mathcal{E}^{i_0} \cup \mathcal{J}^{i_0}, \mathcal{M}, I = [\ell] \setminus \{i_0\}$, and $H_\kappa = G[L^\kappa \cup S^\kappa]$ (for each $\kappa \in I$) to get $T \subseteq G$. Condition (vi) of Lemma 3.8 follows from (7.2). Condition (vii) is checked as in (7.21). Consequently, $T \subseteq G$.

We assume in the rest that no index i_0 satisfies (7.23). We have

$$\left|\bigcup_{i\in\Lambda} \tilde{R}_i\right| \geq \sum_{i\in\Lambda}(|\tilde{R}_i| - |\tilde{R}_i \cap \bigcup_{j\in\Lambda\setminus\{i_0\}} \tilde{R}_j|) \overset{(7.22),\neg(7.23)}{\geq} 2\sum_{i\in\Lambda}|L^i| - 15m^2\gamma k \ . \tag{7.24}$$

Set $Y = \bigcup_{i\in\Lambda} \tilde{R}_i$.

We distinguish three cases:

(♣1) *We have that* $|L \cap Y| \leq \frac{k}{8}$ *and* $e(Y, \tilde{V} \setminus Y) < \sigma k^2/2$.
Solution of (♣1): We show that the set $Q = \tilde{V} \setminus Y$ satisfies the assertions of Lemma 4.1.

First, we prove the property of Lemma 4.1 (ii). By the hypothesis of (♣1), not many vertices in Y are large. Thus the ratio of the large vertices in the graph $G[\bigcup_{i\in\Lambda} V_i \cup Y]$ is substantially smaller than one half. Then there must be substantially more than half of the large vertices in the complementary set Q, and the property follows. We make the idea rigorous by the following calculations. For each $i \in \Lambda$ set $l_i = |L^i|$.

$$\begin{aligned}\frac{1}{2}n \leq |L| &\leq (\ell - |\Lambda|)\tfrac{k}{2} + \sum_{i\in\Lambda} l_i + |L \cap Y| + |L \cap Q| + |L \setminus (\tilde{V} \cup \bigcup_{j\in[\ell]} L^j)| \\ &< (\ell - |\Lambda|)\tfrac{k}{2} + \sum_{i\in\Lambda} l_i + \tfrac{k}{8} + |L \cap Q| + \gamma n \ .\end{aligned}$$

Thus,

$$\begin{aligned}|L \cap Q| &> \frac{1}{2}n - (\ell - |\Lambda|)\tfrac{k}{2} - \sum_{i\in\Lambda} l_i - \tfrac{k}{8} - \gamma n \\ &> \frac{1}{2}\left(|\tilde{V}| - 2\sum_{i\in\Lambda} l_i\right) + |\Lambda|\tfrac{k}{2} - \tfrac{k}{8} - 2\gamma n \\ &\overset{(7.24)}{>} \frac{1}{2}|Q| + |\Lambda|\tfrac{k}{2} - \tfrac{k}{7} \geq \frac{1}{2}|Q| + \frac{5}{14}k \ ,\end{aligned} \tag{7.25}$$

which was needed to show the property of Lemma 4.1 (ii). Looking back at (7.25), we see that $|Q| \geq \frac{1}{2}|Q| + \frac{5}{14}k$, and thus also the property of Lemma 4.1 (i) follows.

Finally, to infer the property of Lemma 4.1 (iii) we write

$$e(Q, V \setminus Q) \leq e(Y, \tilde{V} \setminus Y) + e(\tilde{V}, V \setminus \tilde{V}) < \sigma k^2/2 + \beta k^2 \leq \sigma k^2 \ .$$

The bound on the first summand follows from the hypothesis of (♣1), the bound on the second summand follows from the (β, σ)-extremality.

(♣2) *We have that* $|L \cap Y| > \frac{k}{8}$ *and* $e(Y, \tilde{V} \setminus Y) < \sigma k^2/2$.
Solution of (♣2): We show that $T \subseteq G$. The hypothesis of (♣2) gives $e(G[Y]) \geq \frac{1}{2}|L \cap Y|k - e(Y, \tilde{V} \setminus Y) \geq \frac{k^2}{20}$. The average degree in $G[Y]$ is $\frac{2e(G[Y])}{|Y|} \geq \frac{k^2}{10n} \geq \frac{qk}{10}$. There exists a subgraph $H_* \subseteq G[Y]$ with $\delta(H_*) \geq \frac{qk}{20}$. By averaging, there exists an index $i_0 \in \Lambda$ such that

$$|\tilde{R}_{i_0} \cap V(H_*)| > \tfrac{qk}{20m} \ . \tag{7.26}$$

Fix such an index i_0. By (7.26) there exists an $L^{i_0} - V(H_*)$-matching $\mathcal{E}$ of size $30\gamma k$. Fix a set $A \subseteq L^{i_0}$ of size $|T_\ominus| - |\mathcal{E}|$ containing $V(\mathcal{E}) \cap L^{i_0}$. Such a set exists by (7.1). By Lemma 3.8 (with $\alpha = 60\gamma$, $A, B_{\mathrm{a}} = S_0^{i_0}, B_{\mathrm{d}} = \mathcal{Q} = \mathcal{M} = \emptyset, \mathcal{E}$, and H_*, $I = \{*\}$) we get that $T \subseteq G$. To check Condition (vi), observe that, by (7.2) and the fact that we are the deficient case, we have $|S_0^{i_0}| + |\mathcal{E}| \geq \frac{k}{2} - \gamma k + 30\gamma k \geq |T_\oplus|$. Condition (vii) follows from (7.1). Other conditions are straightforward.

(♣3) *We have that* $e(Y, \tilde{V} \setminus Y) \geq \sigma k^2/2$.
Solution of (♣3): We show that $T \subseteq G$. The average degree of the bipartite graph $G[Y, \tilde{V} \setminus Y]$ is at least $q\sigma k$. Thus there exists a graph $H_* \subseteq G[Y, \tilde{V} \setminus Y]$ with $\delta(H_*) \geq \frac{q\sigma k}{2}$. There must be an index $i_0 \in \Lambda$ such that $|\tilde{R}_{i_0} \cap V(H_*)| > \frac{\sigma q k}{2m}$. Fix such an index i_0, find a matching $\mathcal{E}$ and set A as in (♣2). We apply Lemma 3.8 as in (♣2).

7.2 Proof of Lemma 7.2

In order to prove Lemma 7.2 we need the following auxiliary lemma.

Lemma 7.9. *Let F be a rooted forest with a partition $V(F) = O_1 \cup O_2$, such that O_2 is independent. Let W be the set of leaves of F and set $P = \{u \in O_2 : |W \cap \mathrm{Ch}(u)| = 1\}$. Let H be a graph and let $A, B \subseteq V(H)$ be two disjoint sets such that for some $f \geq 0$ we have $|A| \geq |O_1|$, $\min\{\delta(A, A), \delta(B, A)\} > |O_1| - f$, $\delta(A, B) > |B| - f$, $|B| \geq |O_2 \setminus W|$, and $\delta(A) \geq v(F) - 1$. If $|P| \geq 2f$, then there exists an embedding φ of F in H such that $\varphi(O_1) \subseteq A$.*

Proof. Choose a subset $P' \subseteq P$ of size $|P'| = 2f$. Consider the subtree $F' = F - W'$, where $W' = W \cap (O_2 \cup \mathrm{N}(P'))$. We embed greedily the tree F' in $A \cup B$, so that $V(F') \cap O_1$ maps to A and $V(F') \cap O_2$ maps to B. Denote this embedding by φ'. Next we want to embed the leaves $W' \cap O_1$ in A. Let $A' = A \setminus \varphi(V(F'))$. We have $|A'| \geq 2f = |\varphi'(P')|$, $\delta(\varphi(P'), A') > f = \frac{|P'|}{2}$, and $\delta(A', \varphi(P')) > f = \frac{|P'|}{2}$. By König's matching theorem, there exists a matching M in $H[A', \varphi'(P')]$ that covers $\varphi'(P')$.

We extend φ' to an embedding φ of F, by embedding $W' \cap O_1$ according to the matching M, and by embedding $W \cap O_2$ greedily (this is guaranteed by the minimum degree condition of the set A). □

A semi-independent partition (U_1, U_2) of a tree F is *p-ideal* if each of the vertex sets U_1 and U_2 contains at least p leaves of F. If $\mathrm{disc}(T) \geq 2\gamma k$, then Lemma 7.4 ensures that $T \subseteq G$. Therefore, the proof of Lemma 7.2 follows from Lemma 7.10 and 7.11 below.

Lemma 7.10. *If we are in the setting of Lemma 7.2 and* $\mathrm{disc}(T) < 2\gamma k$, *then T has an $8\gamma k$-ideal semi-independent partition, or $T \subseteq G$.*

Lemma 7.11. *If we are in the setting of Lemma 7.2,* $\mathrm{disc}(T) < 2\gamma k$, *and T has an $8\gamma k$-ideal semi-independent partition then $T \subseteq G$.*

Proof of Lemma 7.10. We partition the set W of leaves of T into $W_\oplus = W \cap T_\oplus$ and $W_\ominus = W \cap T_\ominus$. Set $w_\oplus = |W_\oplus|$ and $w_\ominus = |W_\ominus|$. We have that $w_\oplus + w_\ominus \geq 60\gamma k$. We distinguish three cases based on the values of $w_\oplus$ and $w_\ominus$.
• *We have $w_\oplus \geq 8\gamma k$ and $w_\ominus \geq 8\gamma k$.*
Then $(T_\ominus, T_\oplus)$ is an $8\gamma k$-ideal semi-independent partition.
• *We have $w_\oplus < 8\gamma k$.*
Then we have $w_\ominus \geq 52\gamma k$. We distinguish two subcases.

- If $|\mathrm{Par}(W_\ominus)| \le 16\gamma k$, we consider the sets $U_1 = T_\ominus \triangle (W_\ominus \cup \mathrm{Par}(W_\ominus))$ and $U_2 = T_\oplus \triangle (W_\ominus \cup \mathrm{Par}(W_\ominus))$. The partition (U_1, U_2) is semi-independent with $|U_2| - |U_1| \ge 72\gamma k$, a contradiction with the assumption $\mathrm{disc}(T) < 2\gamma k$.

- If $|\mathrm{Par}(W_\ominus)| > 16\gamma k$, we choose an arbitrary subset $P' \subseteq \mathrm{Par}(W_\ominus)$ with $|P'| = 8\gamma k$ and set $W'_\ominus = \mathrm{N}(P') \cap W_\ominus$. The partition (U_1, U_2), defined by $U_1 = T_\ominus \triangle (W'_\ominus \cup P')$ and $U_2 = T_\oplus \triangle (W'_\ominus \cup P')$, is an $8\gamma k$-ideal semi-independent partition.

- *We have* $w_\ominus < 8\gamma k$.

We use Fact 3.1 *(ii)* to find a full-subtree $\tilde{T} \subseteq T$ rooted at a vertex r with p proper leaves, where $p \in [20\gamma k, 40\gamma k]$. The choice of $\tilde{T}$ has the property that

$$\min\{|W_\oplus \cap V(\tilde{T})|, |W_\oplus \setminus V(\tilde{T})|\} \ge 12\gamma k \ . \tag{7.27}$$

Set $d = |V(\tilde{T}) \cap T_\oplus| - |V(\tilde{T}) \cap T_\ominus|$. We distinguish six subcases.

(C1) $r \in T_\oplus$ and $d \le \frac{\mathrm{gap}(T)}{2}$, **(C2)** $r \in T_\ominus$ and $d \ge \frac{\mathrm{gap}(T)}{2}$,
(C3) $r \in T_\oplus$ and $d \ge \frac{\mathrm{gap}(T)}{2} + 1$, **(C4)** $r \in T_\ominus$ and $d \le \frac{\mathrm{gap}(T)}{2} - 1$,
(C5) $r \in T_\oplus$ and $d = \frac{\mathrm{gap}(T)+1}{2}$, **(C6)** $r \in T_\ominus$ and $d = \frac{\mathrm{gap}(T)-1}{2}$.

In cases **(C1)**–**(C4)** we obtain a semi-independent partition by flipping either $V(\tilde{T})$ (in cases **(C1)** and **(C2)**) or $V(\tilde{T}) \setminus \{r\}$ (in cases **(C3)** and **(C4)**) in the original partition $(T_\ominus, T_\oplus)$. In these cases, the obtained partition is indeed $8\gamma k$-ideal by (7.27).

In the rest, we consider only the case **(C5)**, the case **(C6)** being analogous. Notice that $\mathrm{gap}(T)$ has the same parity as $v(T) = k+1$. Thus, the integrality of d gives that k is even. We set $O_1 = T_\ominus \triangle V(\tilde{T})$ and $O_2 = T_\oplus \triangle V(\tilde{T})$. We have that $|O_1| = \frac{k+2}{2}$, and $|O_2| = \frac{k}{2}$.

Claim 7.10.1. We have $\mathrm{Par}(O_1 \cap W) \subseteq O_2$, or T has an $8\gamma k$-ideal semi-independent partition.

Proof. The existence of a vertex $u \in O_1 \cap W$ whose parent lies in O_1 would yield a semi-independent partition $(O_1 \setminus \{u\}, O_2 \cup \{u\})$, which would be by (7.27) $8\gamma k$-ideal. □

Claim 7.10.2. If there exist two distinct leaves $z_1, z_2 \in O_1$ with a common neighbor $\{x\} = \mathrm{Par}(\{z_1, z_2\})$, then T has an $8\gamma k$-ideal semi-independent partition.

Proof. By Claim 7.10.1 we can assume that $x \in O_2$. Set $U_1 = O_1 \triangle \{x, z_1, z_2\}$ and $U_2 = O_2 \triangle \{x, z_1, z_2\}$. Then $|U_1| = \frac{k}{2}$, $|U_2| = \frac{k}{2}+1$, $|U_1 \cap W| = |O_1 \cap W| - 2$, and $|U_2 \cap W| = |O_2 \cap W| + 2$. By (7.27), the partition (U_1, U_2) is $8\gamma k$-ideal semi-independent. □

By the two claims above, we restrict ourselves to the case that $\mathrm{Par}(O_1 \cap W) \subseteq O_2$, and the leaves in O_1 have pairwise distinct parents.

Claim 7.10.3. For the set $O_1^* = \{y \in O_1 \cap W \ : \ \deg(\mathrm{Par}(y)) = 2\}$, we have $|O_1^*| > 1.5\gamma k$.

Proof. Recall that every vertex in $\mathrm{Par}(O_1 \cap W)$ has exactly one leaf-child in O_1. Set $W_* = V(\tilde{T}) \cap W_\oplus$ and $T_* = \tilde{T} - W_*$. By (7.27), we have $|W_*| \ge 12\gamma k$.

$$\begin{aligned} |V(T_*) \cap T_\ominus| = |V(\tilde{T}) \cap T_\ominus| &\overset{\text{Fact 3.3}}{>} |V(\tilde{T}) \cap T_\oplus| - 2.5\gamma k \\ &= |V(T_*) \cap T_\oplus| + |W_*| - 2.5\gamma k \ge |V(T_*) \cap T_\oplus| + 9.5\gamma k \ . \end{aligned}$$

By Fact 3.2, the tree T_* contains at least $9.5\gamma k$ leaves from $T_\ominus$. These leaves are also leaves of $\tilde{T}$, with $|O_1^*|$ exceptions caused by $\mathrm{Par}(O_1^*)$. Since $w_\ominus < 8\gamma k$, we must have $|O_1^*| > 1.5\gamma k$. □

We show that $T \subseteq G$ in two cases **($\diamondsuit$1)** and **($\diamondsuit$2)** separately, based on whether G is in the abundant or deficient configuration.

($\diamondsuit$1) If G admits an abundant partition, then there exists an index $i \in [\ell]$ such that $|L^i| \geq \frac{k+1}{2}$. As k is even, $|L^i| \geq \frac{k+2}{2}$. Choose $L_* \subseteq L^i$ such that $|L_*| = \frac{k+2}{2}$. Define $Z = \{u \in W \cap O_1 : \mathrm{Par}(u) \in O_2\}$. Suppose that $|(W \cap O_1) \setminus Z| > \gamma k$. Then consider the partition (U_1, U_2) with $U_1 = O_1 \setminus ((W \cap O_1) \setminus Z)$ and $U_2 = O_2 \cup ((W \cap O_1) \setminus Z)$. We have $|U_2| - |U_1| > 2\gamma k$, a contradiction to $\mathrm{disc}(T) \leq 2\gamma k$. Thus $|(W \cap O_1) \setminus Z| \leq \gamma k$. Let $Z' \subseteq Z$ be the set of leaves in Z with no sibling in Z. Observe that Fact 3.4 gives $|Z \setminus Z'| \leq 2\gamma k$. We can now use Lemma 7.9 with $A = L_*$, $B = S_0^i \cup (L^i \setminus L_*)$, $f = \gamma k$, and the partition (O_1, O_2) of T to get $T \subseteq G$. Indeed, the above bounds imply that the set P (as defined in Lemma 7.9) is large.

($\diamondsuit$2) Suppose that G is in the deficient configuration. Consider the index $i \in [m]$ and the sets S^j, and $K \subseteq L^i$ given by Lemma 7.7. Let us discard from O_1^* arbitrary vertices so that we have $|O_1^*| = 1.5\gamma k$ (cf. Claim 7.10.3). We embed greedily the tree $T^- = T - (O_1^* \cup \mathrm{Par}(O_1^*))$ in $G[L^i \cup S_0^i]$ using L^i to host $O_1 \setminus O_1^*$ and S_0^i to host $O_2 \setminus \mathrm{Par}(O_1^*)$, and so that the vertices of $\mathrm{Par}(\mathrm{Par}(O_1^*))$ are always mapped to K. Such an embedding exists by (7.1) and (7.2), and because $|\mathrm{Par}(\mathrm{Par}(O_1^*))| \leq |\mathrm{Par}(O_1^*)| = |O_1^*| \leq 1.5\gamma k$, and $|K| \geq \frac{k}{10}$. It remains to extend the embedding of T^- first to $\mathrm{Par}(O_1^*)$ and then to O_1^*. For any vertex in $\mathrm{Par}(\mathrm{Par}(O_1^*))$ mapped to a vertex in K, we embed its child from $\mathrm{Par}(O_1^*)$ greedily to $L^i \cup \bigcup_{j \neq i}(L^j \cup S^j)$. This way, only vertices of $(O_1 \setminus O_1^*) \cup \mathrm{Par}(O_1^*)$ could be embedded in L^i. As $|(O_1 \setminus O_1^*) \cup \mathrm{Par}(O_1^*)| = |O_1| = \frac{k}{2} + 1 = \lceil \frac{k+1}{2} \rceil$, we can extend the embedding to $\mathrm{Par}(O_1^*)$ by (7.12). In the last step, we extend the embedding to O_1^*. Consider an arbitrary vertex $x \in \mathrm{Par}(O_1^*)$. The vertex x was embedded to L^i, or to $\bigcup_{j \neq i}(L^j \cup S^j)$. If x is mapped to $\bigcup_j L^j$, we use the high degree of those vertices to extend the embedding to the child of x. In the case x was mapped to $v \in S^j$ for some $j \neq i$, observe that only vertices from $O_1^* \cup \mathrm{Par}(O_1^*)$ could have been mapped to L^j. As $|O_1^* \cup \mathrm{Par}(O_1^*)| = 2|O_1^*| = 3\gamma k$, the definition of S^j tells us that $\deg(v, L^j) \geq \frac{k}{5m}$ and we can map the child of x to L^j. $\square$

Proof of Lemma 7.11. We assume that T has an $8\gamma k$-ideal semi-independent partition (U_1, U_2). Let W_2 be the leaves in U_2, and let W_1^* be those leaves in U_1 which have no leaf-sibling in U_1. By Fact 3.4, we have $|W_1^*| \geq 6\gamma k$.

First, we show how to resolve the situation in the abundant case. Let i be such that $|L^i| \geq \frac{k+1}{2}$. We first embed $T - (W_1^* \cup W_2)$ in $G[L^i \cup S_0^i]$, using L^i to host $U_1 \setminus W_1^*$, and S_0^i to host $U_2 \setminus W_2$. Properties (7.1) and (7.2) tell us that such an embedding exists.

Next, we map W_1^* to the set $L^* \subseteq L^i$ of unused vertices of L^i. To this end, consider an auxiliary bipartite graph H whose two colour classes are L^* and $\mathrm{Par}(W_1^*)$. A pair vx, $v \in L^*$, $x \in \mathrm{Par}(W_1^*)$ forms an edge in H if x was mapped to a vertex that is adjacent to v in G. By the definition of S_0^i, and by (7.1), we have $\delta_H(\mathrm{Par}(W_1^*), L^*) \geq |L^*| - \gamma k/2$, and $\delta_H(L^*, \mathrm{Par}(W_1^*)) \geq |\mathrm{Par}(W_1^*)| - \gamma k/2 = |W_1^*| - \gamma k/2$. We conclude that H has no vertex cover of size less than $\min\{|W_1^*|, |L^*|\} = |W_1^*|$. By König's Theorem, there exists a matching covering $\mathrm{Par}(W_1^*)$ in H. This matching tells us how to embed W_1^*. In the last step, we embed W_2. This can be done greedily as $\mathrm{Par}(W_2)$ were mapped to L.

It remains to resolve the situation in the deficient case. Consider the index $i \in [m]$ and the sets S^j, and $K \subseteq L^i$ given by Lemma 7.7. Set $W_1^{**} = \{x \in W_1^* : \deg(\mathrm{Par}(x)) \leq \gamma(k+1)\}$. The degree sum formula for trees gives $|W_1^{**}| \geq |W_1^*| - 2/\gamma > 5.9\gamma k$. Let $\tilde{T} \subseteq V(T)$ be a full-subtree rooted at a vertex $r \in V(T)$, such that $v(\tilde{T}) \in [k/4, k/2]$. The existence of $\tilde{T}$ is guaranteed by Fact 3.1. Let $W_1^{***} \subseteq W_1^{**} \setminus \mathrm{N}(r)$ be a set of size $5.8\gamma k$. This is possible, as by the definition of W_1^* we have $|W_1^{**} \cap \mathrm{N}(r)| \leq 1$. Observe that $|W_1^{***} \cap V(\tilde{T})| \geq 2.9\gamma k$ or $|W_1^{***} \setminus V(\tilde{T})| \geq 2.9\gamma k$.

First assume the former case. Let $X = \{x \in \mathrm{Par}(W_1^{***} \cap V(\tilde{T})) \;:\; \mathrm{Par}(x) \in U_1\}$. Observe that $X \subseteq V(\tilde{T}) \setminus \{r\}$. Thus

$$\sum_{x \in X} v(T(\downarrow x)) \le v(\tilde{T}) \le \frac{k}{2}\ . \tag{7.28}$$

We begin embedding greedily the tree $T' = T - W_2 - \bigcup_{x\in X} T(\downarrow x)$ so that U_1 is mapped to L^i, $\mathrm{Par}(X)$ is mapped to K, and U_2 is mapped to S_0^i. We can do so, as $|\mathrm{Par}(X)| \le |W_1^{***}| = 5.8\gamma k \le |K|$ (c.f. Lemma 7.7). Such an embedding exists by (7.1) and (7.2).

For every $x \in X$, we sequentially assign an index $j_x \in [m]$ to denote where $T(\downarrow x)$ will be embedded, according to the following rule. Let $X' \subseteq X$ be the set of those y's for which the index j_y has been assigned in previous rounds. Let $v_x \in K$ be the image of $\mathrm{Par}(x)$. If there exists any index $j \neq i$ such that

(*i*) $\deg(v_x, (L^j \cup S^j)) > |\{y \in X' \;:\; j_y = j\}|$, and

(*ii*) $\sum_{y \in X' \,:\, j_y = j} v(T(\downarrow y)) \le k/(5m) - 2\gamma k$,

then choose such an index j and fix $j_x = j$. If no such index $j \neq i$ exists, than set $j_x = i$.

The assignment finished, for every $x \in X$ with $j_x \neq i$ we map x to $\mathrm{N}(v_x) \cap (L^{j_x} \cup S^{j_x})$. This is possible thanks to Condition (*i*). Having mapped all $X_{\neq i} = \{y \in X \;:\; j_y \neq i\}$, we embed $\{\mathrm{Ch}(x), x \in X_{\neq i}\}$ in $L^{j_x} \cup S_0^{j_x}$ (see Figure 7.2). Even if x is mapped to S^{j_x}, the at most $\gamma(k+1)$ children of x (cf. definition of W_1^{**}) can be embedded thanks to Condition (*ii*) and the definition of S^j. Having embedded all the vertices $\bigcup_{x \in X_{\neq i}} \mathrm{Ch}(x)$, we continue as follows. For each $x \in X_{\neq i}$ we embed the rest of $T(\downarrow x)$ greedily in $L^{j_x} \cup S_0^{j_x}$, which is possible by (7.28). We are finished with embedding T in the case that $j_x \neq i$ for all $x \in X$. Thus, assume that

$$j_x = i \text{ for some } x \in X\ . \tag{7.29}$$

Suppose that $\sum_{y \in X \,:\, j_y = j} v(T(\downarrow y)) > k/(5m) - 2\gamma k$ for some $j \neq i$. Set $\mathcal{D} = \{T(\downarrow y) \;:\; j_y = j\}$.

Claim 7.11.4. We have $|U_1 \cap V(\mathcal{D})| \ge 500\gamma k$.

Proof. First, consider the case that the total order of *small components* of $\mathcal{D}$, defined as with at most 10 vertices, is at least $|V(\mathcal{D})|/2$. In each such component, there is at least one vertex of $W_1^{***} \subseteq U_1$. Hence $|U_1 \cap V(\mathcal{D})| \ge \frac{1}{10} \cdot \frac{|V(\mathcal{D})|}{2} \ge \frac{k}{200m} > 500\gamma k$.

Next, consider the case that the total order of *large components* of $\mathcal{D}$ (those having more than 10 vertices) is more than $|V(\mathcal{D})|/2$. Let $\mathcal{D}_1$ be those large components $D \in \mathcal{D}$ with $|U_2 \cap V(D)| < 10|U_1 \cap V(D)|$, and let $\mathcal{D}_2$ be those large components $D \in \mathcal{D}$ with $|U_2 \cap V(D)| \ge 10|U_1 \cap V(D)|$. Consider the tree $T'' = T - \mathcal{D}_2$, and its colour classes $T''_\oplus$ and $T''_\ominus$. Let R be the roots of the trees in $\mathcal{D}_2$. We have $|R| \le |V(\mathcal{D}_2)|/10$. Set the partition $V(T) = U_1' \dot\cup U_2'$, where $U_2' = T''_\oplus \cup (U_2 \cap V(\mathcal{D}_2)) \setminus R$ and $U_1' = V(T) \setminus U_2' = T''_\ominus \cup (U_1 \cap V(\mathcal{D}_2)) \cup R$. Observe that U_2' is an independent set. As $\mathrm{disc}(T) < 2\gamma k$, we have

$$2\gamma k > |U_2'| - |U_1'| \ge |U_2 \cap V(\mathcal{D}_2)| - |U_1 \cap V(\mathcal{D}_2)| - 2|R| \ge (\tfrac{9}{11} - \tfrac{1}{5})|V(\mathcal{D}_2)|\ .$$

We conclude that $|V(\mathcal{D}_2)| < 4\gamma k$. In particular, we have $|V(\mathcal{D}_1)| \ge |V(\mathcal{D})|/4$. Then

$$|U_1 \cap V(\mathcal{D})| \ge |U_1 \cap V(\mathcal{D}_1)| \ge \tfrac{1}{11} \cdot \tfrac{|V(\mathcal{D})|}{4} > 500\gamma k\ ,$$

as needed. □

Recall that we have embedded the entire tree T except the set $M = \bigcup_{x\in X\setminus X_{\neq i}} V(T(\downarrow x))$. Let $Q \subseteq U_2 \cap M$ be a set of size $\max\{400\gamma k, |U_2\cap M|\}$. To finish the embedding of T, we embed greedily the vertices of $Q \cup (U_1 \cap M)$ in L^i and the vertices of $(U_2 \cap M) \setminus Q$ in S_0^i. Prior to this embedding, by Claim 7.11.4, at least $500\gamma k$ vertices of U_1 had been embedded outside of L^i. Thus, the minimum-degree conditions (7.1) guarantee that such a greedy embedding indeed exists.

Thus, it remains to consider that $\sum_{y\in X\,:\,j_y=j} v(T(\downarrow y)) \leq k/(5m) - 2\gamma k$ for all $j \neq i$. At the same time, we had not been able to satisfy Condition (i) for any $j \neq i$ for the vertex x from (7.29). Then $\deg(v_x, \bigcup_{j\neq i}(L^j \cup S^j)) \leq |\{y \in X \,:\, j_y \neq i\}|$.

At least

$$(k+1)/2 - |L^i| \geq |U_1| - |L^i| \tag{7.30}$$

vertices of U_1 were embedded outside L^i. Indeed, at least $\deg(v_x, \bigcup_{j\neq i}(L^j \cup S^j))$ vertices $x \in X$ were assigned some $j_x \neq i$. Each corresponding tree $T(\downarrow x)$ contains at least one child of x, belonging to $W_1^{***} \subseteq U_1$, which was thus embedded outside L^i. Using (7.12), we get (7.30).

It remains to embed the trees $\{T(\downarrow x) \,:\, x \in X \setminus X_{\neq i}\}$. We first embed all the trees $T(\downarrow x) \setminus (W_1^{***} \cup W_2)$, $x \in X \setminus X_{\neq i}$. The extension to $W_1^{***} \cup W_2$ will be done at the very end.

Set $W_X = W_1^{***} \cap \left(\bigcup_{x\in X\setminus X_{\neq i}} V(T(\downarrow x))\right)$ and $W_Y = W_1^{***} \cap \left(\bigcup_{x\in X_{\neq i}} V(T(\downarrow x))\right)$.

Claim 7.11.5. We have $|W_X \cup W_Y| \geq 1.9\gamma k$.

Proof. Let $\tilde{W}$ be all vertices in $W_1^{***} \cap V(\tilde{T}) \subseteq U_1$ whose parent lies in U_2. For $y \in \tilde{W}$, the independence of U_2 implies that $\mathrm{Par}(\mathrm{Par}(y)) \in U_1$. Thus by the definition of X, we have that $\mathrm{Par}(y) \in X$ and thus $y \in W_1^{***} \cap \left(\bigcup_{x\in X} V(T(\downarrow x))\right) = W_X \cup W_Y$. Hence, $\tilde{W} \subseteq W_X \cup W_Y$.

The semi-independent partition

$$\left(U_1 \setminus ((W_1^{***} \cap V(\tilde{T})) \setminus \tilde{W})\,,\, U_2 \cup ((W_1^{***} \cap V(\tilde{T})) \setminus \tilde{W})\right)$$

has gap

$$\begin{aligned}&\left(|U_2| + |W_1^{***} \cap V(\tilde{T})| - |\tilde{W}|\right) - \left(|U_1| + |\tilde{W}| - |W_1^{***} \cap V(\tilde{T})|\right)\\ &\geq \left(|W_1^{***} \cap V(\tilde{T})| - |\tilde{W}|\right) - \left(|\tilde{W}| - |W_1^{***} \cap V(\tilde{T})|\right) \geq 2\cdot 2.9\gamma k - 2|\tilde{W}|\,.\end{aligned}$$

Since $\mathrm{disc}(T) < 2\gamma k$, we get $|\tilde{W}| \geq 1.9\gamma k$. □

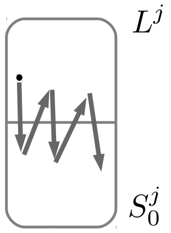
Set $N = \bigcup_{x\in X_{\neq i}} V(T(\downarrow x))$. By Claim 7.11.5, we have that $|W_Y| \geq 0.75\gamma k$, or $|W_X| \geq 0.75\gamma k$. Hence,

$$|U_1 \setminus (W_X \cup N)| \leq \frac{k+1}{2} - 0.75\gamma k \overset{(7.1)}{\leq} |L^i| - \frac{\gamma k}{2}\,. \tag{7.31}$$

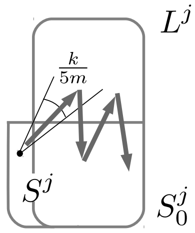
For a fixed $x \in X \setminus X_{\neq i}$, we proceed embedding $T(\downarrow x) \setminus W_X$ greedily in $G[L^i \cup S_0^i]$, using L^i to host $(U_1 \cap V(T(\downarrow x))) \setminus W_X$, and S_0^i to host $(U_2 \cap V(T(\downarrow x))) \setminus W_2$. By (7.31), (7.1), and the definition of S_0^i, we have $\deg(v_x, L^i) \geq |L^i| - \gamma k/2$ and $\delta(S_0^i, L^i) \geq |L^i| - \gamma k/2$ is sufficient to accommodate the vertices from $U_1 \setminus (W_X \cup N)$ in L^i. As for the vertices that need to be mapped to S_0^i, recall that the fact that (U_1, U_2) is $8\gamma k$-ideal yields $|W_2| \geq 8\gamma k$. Together with the fact that we are considering the deficient case, we get that at most

$$|U_2 \setminus W_2| \leq (\frac{k}{2} + \gamma k) - 8\gamma k \leq |S_0^i| - 7\gamma k$$

vertices are mapped to S_0^i. Hence, the minimum degree of vertices of L^i to S_0^i is sufficient for a greedy embedding.



(a) In case x is mapped to L^j we can embed the tree $T(\downarrow x)$ greedily in $G[L^j, S_0^j]$.



(b) In case x is mapped to $v \in S^j$ (but not necessarily in S_0^j) we first embed all its children to L^j. To this end we make use of Condition (ii). The rest of the embedding goes in $G[L^j, S_0^j]$.

Figure 7: Embedding the tree $T(\downarrow x)$ in Lemma 7.11. The placement of x is denoted by a black dot. The embedding the proceeds following the arrows.

The next stage is to embed the vertices of W_X. Let $L^* \subseteq L^i$ be the set of unused vertices. We consider a bipartite graph H whose two colour classes are L^* and $\mathrm{Par}(W_X)$. A pair vx, $v \in L^*$, $x \in \mathrm{Par}(W_X)$ forms an edge in H if x was mapped to a vertex that is adjacent to v in G. By the definition of S_0^i, and by (7.1), we have $\delta_H(\mathrm{Par}(W_X), L^*) \geq |L^*| - \gamma k/2$, and $\delta_H(L^*, \mathrm{Par}(W_X) \geq |\mathrm{Par}(W_X)| - \gamma k/2 = |W_X| - \gamma k/2$. We conclude that H has no vertex cover of size less than $\min\{|W_X|, |L^*|\}$. As we did not embed any vertex from W_X yet, and by (7.30) we mapped to L^i at most $|U_1| - (|U_1| - |L^i|) - |W_X| = |L^i| - |W_X|$ vertices, we get $|L^*| \geq |W_X|$ and thus the minimum vertex cover has size at least $|W_X|$. By König's Theorem, there exists a matching covering $\mathrm{Par}(W_X)$ in H. This matching tells us how to embed W_X. In the last step, we embed W_2. This can be done greedily as $\mathrm{Par}(W_2)$ were mapped to L.

The case $|W_1^{***} \setminus V(\tilde{T})| \geq 2.9\gamma k$ is treated similarly, the difference being that this time we start with $X = \{x \in \mathrm{Par}(W_1^{***} \setminus V(\tilde{T})) \, : \, \mathrm{Par}(x) \in U_1\}$. $\square$

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A Proofs of some auxiliary facts

Proofs of several auxiliary statements were omitted in the main body of the paper. Here we give these proofs.

A.1 Proof of Lemma 3.8

For the proof we need the following two statements. The first one is a simple corollary of Hall's Matching Theorem.

Lemma A.1. *Let $K = (W_1, W_2; J)$ be a bipartite graph such that $\delta(K) \geq \frac{|W_1|}{2}$ and $|W_1| \leq |W_2|$. Then K contains a matching covering W_1.*

Let ℓ be the number of leaves of T. Recall that $\ell < \alpha k$. Fact 3.2 gives that $\mathrm{disc}(F) < \alpha k$. In particular the lower bounds given in Properties (v) and (vi) of the lemma, combined with the upper bound in Property (iv) yield $|A|, |B| \geq \frac{4k}{10}$.

We write $r = |B_{\mathrm{d}}|$, and $\mathcal{Q} = \{P_1, \ldots, P_r\}$. Root T at an arbitrary vertex $v \in T_\ominus$. An *c-induced path* $a_1 \ldots a_{c+1} \subseteq T$ is a path whose internal vertices have degree two in T. Take a maximum family $\mathcal{F}$ of vertex-disjoint 7-induced paths in T. We show that $|V(\mathcal{F})| \geq k - 19\ell$.

Let $D_3 = \{u \in V(T) : \deg_T(u) \geq 3\}$ and $D_i = \{u \in V(T) : \deg_T(u) = i\}$ for $i = 1, 2$. By Fact 5.4, we have $|D_3| < \ell$ (and $|D_2| \geq k - 2\ell$). From

$$2k = \sum_{u \in V(T)} \deg(u) = |D_1| + 2|D_2| + \sum_{u \in D_3} \deg(u) \geq 2k - 3\ell + \sum_{u \in D_3} \deg(u) ,$$

we deduce that there are at most $3\ell + 1$ maximal (w. r. t. inclusion) paths formed by vertices of degree 2 or 1 not containing the root v. On each such maximal path, at most 7 vertices are not covered by $\mathcal{F}$. Thus the total number of vertices uncovered by $\mathcal{F}$ is at most $7(3\ell + 1) + |D_3| + |\{v\}| \leq 26\ell$. The order $\preceq_v$ naturally extends to an order on the paths of $\mathcal{F}$. For a family $\mathcal{F}' \subseteq \mathcal{F}$ we write $T(\downarrow \mathcal{F}')$ to denote all the vertices of $V(\mathcal{F}')$, and all vertices that are below some vertex of $V(\mathcal{F}')$, i.e.,

$$T(\downarrow \mathcal{F}') = \bigcup_{u \in V(\mathcal{F}')} V(T(\downarrow u)) .$$

There is a family $\mathcal{R} \subseteq \mathcal{F}$ satisfying the three properties below.

(P1) $|\mathcal{R}| \leq |\mathcal{E}| + |\mathcal{M}|$.

(P2) $|T(\downarrow \mathcal{R})| < 34\alpha k$, and $4(|\mathcal{E}| + |\mathcal{M}|) \leq \min\{|T_\oplus \cap T(\downarrow \mathcal{R})|, |T_\ominus \cap T(\downarrow \mathcal{R})|\}$.

(P3) $\mathcal{R}$ is a $\preceq_v$-antichain.

We describe a procedure how to obtain such a family $\mathcal{R}$. By an inductive construction, we first find an auxiliary family $\mathcal{R}'$, starting with $\mathcal{R}' = \emptyset$. While $|\mathcal{R}'| < |\mathcal{E}| + |\mathcal{M}|$ we take a $\preceq_v$-minimal path in $\mathcal{F}$ which is not included in $\mathcal{R}'$ and add it to $\mathcal{R}'$. From the bound $|V(T) \setminus V(\mathcal{F})| \leq 26\ell$, in each step we have that $|T(\downarrow \mathcal{R}')| < 8|\mathcal{R}'| + 26\alpha k$, and obviously $4|\mathcal{R}'| \leq \min\{|T_\oplus \cap T(\downarrow \mathcal{R}')|, |T_\ominus \cap T(\downarrow \mathcal{R}')|\}$. Let $\mathcal{R}$ be the $\preceq_v$-maximal elements of $\mathcal{R}'$. Hence $|T(\downarrow \mathcal{R})| = |T(\downarrow \mathcal{R}')|$. The properties **(P1)**, **(P2)**, and **(P3)** are satisfied.

Set $d = 5\alpha k$. Take a family $\mathcal{X} = \{X_1, \ldots, X_d\}$ of d 5-induced vertex-disjoint $T_\oplus - T_\ominus - T_\oplus - T_\ominus - T_\oplus$ paths that avoid $\{v\} \cup T(\downarrow \mathcal{R})$. For each path $R \in \mathcal{R}$ we write a_R to denote its $\preceq_v$-maximum vertex in $T_\ominus$, and set $b_R = \mathrm{Ch}(a_R)$, $c_R = \mathrm{Ch}(b_R)$, and $d_R = \mathrm{Ch}(c_R)$. We set $U = A \cap (V(\mathcal{E}) \cup V(\mathcal{M}))$ and $Q = A \cap V(\mathcal{Q})$.

We now describe the embedding ψ of T. We do not have to embed those leaves whose parents are embedded in A until the very end. Indeed, such a partial embedding easily extends to an embedding of T using Property (ii) of the lemma. We map the root v to an arbitrary vertex in $A \setminus (U \cup Q)$. We continue embedding T greedily, mapping vertices from $T_\ominus$ to $A \setminus (U \cup Q)$ and internal vertices of $T_\oplus$ to B_{a}. However, there are two exceptions in the greedy procedure:

(S1) If we are about to map a vertex b_R (for some $R \in \mathcal{R}$), we skip its embedding, as well as the embedding of $T(\downarrow b_R)$.

(S2) If we are about to map a vertex x_2 which was part of some path $x_1x_2x_3x_4x_5 \in \mathcal{X}$, we skip its embedding, as well as the embedding of the vertices x_3 and x_4. We continue with mapping x_5 to B_{a}.

Observe that we are able to finish the greedy part of the embedding since the two "skipping rules" guarantee that both in A and in B at least $d > \alpha k$ vertices of T remain unembedded.

In the next step, we build missing connections in the graph H caused by the skipping rules. We construct an auxiliary bipartite graph $K_1 = (O_{\mathrm{a}}, O_{\mathrm{b}}; E_1)$. We arbitrarily pair up $2(d-r)$ vertices of $A \setminus (U \cup Q)$ unused by ψ into pairs $\mu_1 = \{a_1^1, a_1^2\}, \dots, \mu_{d-r} = \{a_{d-r}^1, a_{d-r}^2\}$. The remaining r pairs are formed by endvertices of the paths in $\mathcal{Q}$. We set $\mu_{i+d-r} = A \cap V(P_i)$. Vertices of the color class O_{b} are formed by the pairs μ_i ($i \in [d]$). Vertices of the color class O_{a} are formed by the paths in $\mathcal{X}$. A path $x_1x_2x_3x_4x_5 \in \mathcal{X}$ is adjacent in K_1 to a pair μ_i if and only if there exists a perfect matching in the graph $H[\{\psi(x_1), \psi(x_5)\}, \mu_i]$. Since $|O_{\mathrm{a}}| = |O_{\mathrm{b}}|$ and $\delta(K_1) \geq |O_{\mathrm{a}}| - 2\alpha k \geq \frac{|O_{\mathrm{a}}|}{2}$, there exists, by Lemma A.1, a perfect matching M_1 in K_1. The matching M_1 tells us where to map the vertices x_2 and x_4 of each path $x_1x_2x_3x_4x_5 \in \mathcal{X}$. We extend ψ accordingly on the vertices $\bigcup_{x_1x_2x_3x_4x_5 \in \mathcal{X}} \{x_2, x_4\}$. If a path $x_1x_2x_3x_4x_5 \in \mathcal{X}$ was matched with μ_{i+d-r} (for some $i \in [r]$) in K_1, we map x_3 to the middle vertex of the path P_i. We write $\mathcal{X}'$ for those paths $x_1x_2x_3x_4x_5 \in \mathcal{X}$ whose vertex x_3 was not yet mapped. It holds $|\mathcal{X}'| \geq 4\alpha k$.

Let $\xi : \mathcal{R} \to U$ be an arbitrary injective mapping. We construct another bipartite graph $K_2 = (J_{\mathrm{a}}, J_{\mathrm{b}}; E_2)$. Vertices of the color class J_{a} are elements of $\mathcal{R} \cup \mathcal{X}'$ and vertices of the color class J_{b} are vertices of B_{a} unused by ψ.

Claim A.1.1. We have $|J_{\mathrm{a}}| \leq |J_{\mathrm{b}}|$.

Proof. Let W be the set of leaves of $T_\oplus \setminus V(\mathcal{R})$. Remember that the set W is mapped only at the very end of the embedding procedure. Further, for any path $x_1 \dots x_5 \in \mathcal{X} \setminus \mathcal{X}'$, the vertex $x_3 \in T_\oplus$ has been mapped to B_{d}, which is disjoint from B_{a}. Next for each path $x_1 \dots x_5 \in \mathcal{X}'$, the vertex $x_3 \in T_\oplus$ has not been embedded, yet. Each path in $\mathcal{R}$ has at most one vertex in $T_\oplus$ that has already been embedded. Therefore we have

$$\begin{aligned}
|J_{\mathrm{b}}| &\geq |B| - \big(|T_\oplus| - |W| - |X \setminus X'| - |X'| - (|T_\oplus \cap V(\downarrow \mathcal{R})| - |\mathcal{R}|)\big) \\
&\geq |B| - |T_\oplus| + |W| + |X| + 3(|\mathcal{E}| + |\mathcal{M}|) \\
&\geq |W| + |X| + 2|\mathcal{R}| - 1 \geq |J_{\mathrm{a}}| + |W| + |R| - 1 \geq |J_{\mathrm{a}}|\,,
\end{aligned}$$

where the last inequality follows from the fact that if $\mathcal{R} = \emptyset$ then $W \neq \emptyset$ by FactFact 3.2. $\square$

A path $R \in \mathcal{R}$ is adjacent in K_1 with a vertex $b \in J_{\mathrm{b}}$ if and only if $b\psi(a_R) \in E(H)$ and $b\xi(R) \in E(H)$. A path $x_1x_2x_3x_4x_5 \in \mathcal{X}'$ is adjacent to a vertex $b \in J_{\mathrm{b}}$ if and only if $b\psi(x_2) \in E(H)$ and $b\psi(x_4) \in E(H)$. Indeed, $\delta(K_1) \geq |J_{\mathrm{a}}| - 2\gamma k > \frac{|J_{\mathrm{a}}|}{2}$, and $|J_{\mathrm{a}}| \leq |J_{\mathrm{b}}|$. By Lemma A.1, there exists a matching M_2 in K_2 covering J_{a}. Such a matching tells us where to map unembedded vertices x_3 (in the case of a path $x_1x_2x_3x_4x_5 \in \mathcal{X}'$) and vertices b_R (in the case of a path $R \in \mathcal{R}$). For a path $R \in \mathcal{R}$ we finish embedding the part of the tree $T(\downarrow c_R)$, extending the mapping ψ. If $\psi(c_R) \in V(\mathcal{E})$ we just use the corresponding connecting edge of $\mathcal{E}$ to map d_R to H_κ (for some $\kappa \in I$) and continue embedding $T(\downarrow d_R)$ greedily in H_κ. If $\psi(c_R) \in V(\mathcal{M})$ we map d_R to the middle vertex of the corresponding connecting path $\mathcal{M}$ and embed the rest of $T(\downarrow d_R)$ greedily in H_κ (for some $\kappa \in I$).

A.2 Omitted proofs from Section 5

Proof of Fact 5.9 (i). Let $\tilde{X} \subseteq X$ be the set of vertices that are not typical w. r. t. $\bigcup_{i=1}^{\ell} W_i$, i.e., for every $v \in \tilde{X}$ we have $\deg(v, \bigcup_{i=1}^{\ell} W_i) < \sum_{i=1}^{\ell}(d(X,Y_i) - \varepsilon)|W_i|$. Thus $e(\tilde{X}, \bigcup_{i=1}^{\ell} W_i) < |\tilde{X}| \cdot \sum_{i=1}^{\ell}(d(X,Y_i) - \varepsilon)|W_i|$. Hence, there is an index $i \in [\ell]$ such that $d(\tilde{X}, W_i) < d(X,Y_i) - \varepsilon$. As W_i is significant and (X,Y_i) is ε-regular, we get that $|\tilde{X}| \leq \varepsilon|X|$. $\square$

Proof of Lemma 5.12. Without loss of generality assume that $|P'| \geq \Delta$. Let us fix an arbitrary set $S_P \subseteq P$ with $|S_P| = \Delta$ and another set $S_Q \subseteq Q$ with $|S_Q| = \Delta$. The sets S_P and S_Q are significant. Choose a vertex $v \in P'$ which is typical w. r. t. S_Q. There are at least $|P'| - \varepsilon s \geq 1$ such vertices. Set $\phi(r) = v$.

We inductively extend the embedding ϕ, so that every vertex of t that is mapped to S_P is typical w. r. t. S_Q, and so that every vertex that is mapped to S_Q is typical w. r. t. S_P. We illustrate the inductive step by describing how to embed the neighborhood of a vertex u that was already mapped to P. The case that $\phi(u) \in Q$ is analogous. Let $N \subseteq \mathrm{N}(u)$ be the yet unembedded neighbors of u. The vertex $\phi(u)$ has at least $(d-2\varepsilon)\Delta \geq \varepsilon s + v(t)$ neighbors in S_Q. At least $|N|$ of them are typical w. r. t. S_P and are not yet used by ϕ. We map N to these vertices.

For the moreover part, we only need to observe that if $|P'| \geq \Delta$, there is at least one vertex in P' which is typical w. r. t. S_Q. We map the root r to this vertex. The second condition of the moreover part is analogous. $\square$

For the proof of Lemma 5.13, we need to embed the shrubs of a given tree in an efficient way. To this end, we try to fill the clusters of a regular pair in a balanced way. The following definition of packedness formalizes this. Let X, Y, Z be three disjoint sets of vertices of a graph G. We say that $U \subseteq X \cup Y$ is (λ, τ)-*packed* with respect to the *head set* Z and the *embedding sets* X and Y,[2] if

$$\min\{|X \cap U|, |Y \cap U|\} \geq \min\{\overline{\deg}_{\mathbf{H}}(Z, X), \overline{\deg}_{\mathbf{H}}(Z, Y)\} - \lambda, \text{or} \tag{A.1}$$

$$||X \cap U| - |Y \cap U|| \leq \tau \tag{A.2}$$

Proof of Lemma 5.13. Assume that $\mathbf{H}$ has N clusters. Let $\tilde{X} \subseteq X'$ be the set of vertices that are typical w. r. t. all but at most $\sqrt{\varepsilon}N$ sets $C \cap V^X$, $C \in V(M)$, w. r. t. all but at most $\sqrt{\varepsilon}N$ clusters $Z \in \mathcal{Z}$, and w. r. t. the cluster Y. Let $\tilde{Y} \subseteq Y'$ be the set of vertices that are typical w. r. t. all but at most $\sqrt{\varepsilon}N$ sets $C \cap V^Y$, $C \in V(M)$ and w. r. t. the cluster X. Let $\tilde{Z} \subseteq \bigcup \mathcal{Z}$ be the set of vertices (viewed as vertices of individual clusters of $\mathcal{Z}$) that are typical w. r. t. all but at most $\sqrt{\varepsilon}N$ sets $C \cap V^{\mathcal{Z}}$, $C \in V(M)$ and w. r. t. the cluster X. Observe that by Fact 5.9, $|X' \setminus \tilde{X}| \leq 3\sqrt{\varepsilon}s$, $|Y' \setminus \tilde{Y}| \leq 2\sqrt{\varepsilon}s$, and for every $Z \in \mathcal{Z}$,

$$|Z \setminus \tilde{Z}| \leq 2\sqrt{\varepsilon}s\ . \tag{A.3}$$

Let Q_X be the set of vertices (viewed as vertices of individual clusters of $V(\mathbf{H})$) typical w. r. t. $\tilde{X}$. We define analogously Q_Y. For each $v \in \tilde{X} \cup \tilde{Y}$, let

$$M_v = \{CD \in M\ :\ v \text{ is typical w. r. t. both } C \cap V^X \text{ and } D \cap V^X\} \quad \text{if } v \in \tilde{X}\ ,$$
$$M_v = \{CD \in M\ :\ v \text{ is typical w. r. t. both } C \cap V^Y \text{ and } D \cap V^Y\} \quad \text{if } v \in \tilde{Y}\ .$$

For each cluster $C \in V(M)$ we have by Fact 5.9,

$$|C \setminus Q_X|, |C \setminus Q_Y| \leq \varepsilon s\ , \quad \text{and} \tag{A.4}$$

$$|M_v| \geq |M| - 2\sqrt{\varepsilon}N\ . \tag{A.5}$$

The embedding of F is divided into w steps, where $w = |W_X \cup W_Y|$. We label the vertices of $W_X \cup W_Y$ as $x_1, \ldots, x_w$, indexing from an arbitrary fixed root $R \in W_X \cup W_Y$ downwards, i.e., in such way that $j_1 \leq j_2$ whenever $x_{j_1} \succeq_R x_{j_2}$. We denote by φ the partial embedding

[2] the embedding sets will be typically clear, and then we only specify the head set

of F. For a set $U \subseteq V(F)$, $\varphi(U)$ refers to the image of the already embedded part of U at that moment.[3] In step $i \geq 1$, we embed the tree

$$F_i = F\Big[\{x_i\} \cup \bigcup_{\ell \in [c_i]} V(t_i^\ell)\Big] \,,$$

where $t_i^1, \ldots, t_i^{c_i}$ are the components $t \in \mathcal{D}_X \cup \mathcal{D}_Y$ such that $\mathrm{Ch}(x_i) \cap V(t) \neq \emptyset$. Set $V_i^\ell = \bigcup_{j<i} V(F_j) \cup \bigcup_{j<\ell} V(t_i^j)$, and $U_i^\ell = \varphi(V_i^\ell)$. We call the embedding φ *equable* at step i and substep ℓ, if for each $CD \in M$, we have $||U_i^\ell \cap V^{\mathcal{Z}} \cap C| - |U_i^\ell \cap V^{\mathcal{Z}} \cap D|| \leq \tau$. During the embedding procedure, we use an auxiliary set $\mathcal{Z}' \subseteq \mathcal{Z}$ of "active" clusters in $\mathcal{Z}$.

For $i = 1$, set $N_i = \tilde{X} \cup \tilde{Y}$ and $\mathcal{Z}' = \mathcal{Z}$. For $i > 1$, let $p_i = \mathrm{Par}(x_i)$ and set $N_i = \mathrm{N}_H(\varphi(p_i)) \cap (\tilde{X} \cup \tilde{Y})$. During the embedding process we will keep the following three properties in every step $i \in [w]$, and every substep $j \in [c_i]$.

(I1) For each $CD \in M$, the set $U_i^j \cap V^X \cap (C \cup D)$ is $(\frac{8\varepsilon s}{d}, \tau)$-packed w. r. t. the head set X and the set $U_i^j \cap V^Y \cap (C \cup D)$ is $(\frac{8\varepsilon s}{d}, \tau)$-packed w. r. t. the head set Y.

(I2) $|N_i \cap X| \geq |W_X|$ and $|N_i \cap Y| \geq |W_Y|$.

(I3) $\varphi(W_X) \subseteq \tilde{X}$, $\varphi(W_Y) \subseteq \tilde{Y}$, $\varphi(\mathcal{D}_Y) \subseteq V^Y$, $\varphi(\mathcal{D}_1) \subseteq V^X \setminus V(M_X)$, $\varphi(\mathcal{D}_2) \subseteq V^X \cap V(M_X)$, $\varphi(\mathcal{D}_3 \setminus \mathrm{N}_F(W_X)) \subseteq V^{\mathcal{Z}}$, and $\varphi(\mathcal{D}_3 \cap \mathrm{N}_F(W_X)) \subseteq \bigcup \mathcal{Z}$.

(I4) Either the embedding φ is equable and $\mathcal{Z}' = \mathcal{Z}$, or for every $CD \in M$ and every $Z \in \mathcal{Z}'$ we have

$$\min\{\overline{\deg}_{\mathbf{H}}(Z, C \cap V^{\mathcal{Z}}), \overline{\deg}_{\mathbf{H}}(Z, D \cap V^{\mathcal{Z}})\} \leq \min\{|(C \cap \varphi(\mathcal{D}_3)|, |(D \cap \varphi(\mathcal{D}_3)|\} + \frac{8\varepsilon s}{d} \,,$$

and $\overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}') \geq |(V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)) \setminus V_i^j| + |U_i^j \cap \bigcup \mathcal{Z}'| + \frac{\xi n}{2}$.

For $i = 1$ and $j = 1$, **(I1)**, **(I3)**, and **(I4)** hold trivially. Further, $\max\{|W_X|, |W_Y|\} \leq \frac{12k}{\tau} \ll \varepsilon s \leq \min\{|\tilde{X}|, |\tilde{Y}|\}$ by Definition 5.2 (vi), yielding **(I2)**.

We now proceed with a general step. We first give two claims which we then make use of for the embedding itself.

Claim A.1.2.

(a) Suppose that $\mathcal{D}_Y \neq \emptyset$. Then for every $v \in \tilde{Y}$, there is an edge $CD \in M_v$ such that

$$\deg(v, (C \cup D) \cap V^Y) \geq |\varphi(\mathcal{D}_Y) \cap (C \cup D)| + 2\tau + \frac{\xi s}{2} \,.$$

(b) Suppose that $\mathcal{D}_1 \neq \emptyset$. Then for every $v \in \tilde{X}$, there is an edge $CD \in M_v \setminus M_X$ such that

$$\deg(v, (C \cup D) \cap V^X \cap Q_X) \geq |\varphi(\mathcal{D}_1) \cap (C \cup D)| + 2\tau + \frac{\xi s}{2} \,.$$

(c) Suppose that $\mathcal{D}_2 \neq \emptyset$. Then for every $v \in \tilde{X}$, there is an edge $CD \in M_X \cap M_v$ such that

$$\deg(v, C \cap V^X \cap Q_X) \geq |\varphi(\mathcal{D}_2) \cap C| + \frac{\xi s}{2} \quad \text{and} \quad |D \cap V^X| \geq |\varphi(\mathcal{D}_2) \cap D| + \frac{\xi s}{2} \,. \tag{A.6}$$

[3] In particular, one may have $|\varphi(U)| < |U|$.

(d) Suppose that $\mathcal{D}_3 \neq \emptyset$. Then for every $v \in \tilde{X}$, there is a cluster $Z \in \mathcal{Z}'$ such that

$$\deg(v, Z \cap \tilde{Z}) \geq |\varphi(\mathcal{D}_3) \cap Z| + \frac{\xi s}{4} \,. \tag{A.7}$$

Proof. (a) Using the typicality of v, we get

$$\sum_{CD \in M_v} \deg(v, (C \cup D) \cap V^Y) \overset{(A.5)}{\geq} \overline{\deg}_{\mathbf{H}}(Y, V^Y) - 2\sqrt{\varepsilon} N s - \varepsilon n \overset{(iii)}{\geq} v(\mathcal{D}_Y) + \frac{3\xi n}{4} \,,$$

which implies the statement.

The proof of (b) is analogous, using (A.4) and (iv).

(c) By (A.5) and by the typicality of v, we have

$$\begin{aligned}\sum_{C \in V(M_X \cap M_v) \cap \mathrm{N}_{\mathbf{H}}(X)} \deg(v, C \cap V^X) &\geq \sum_{CD \in M_X} \deg(v, (C \cup D) \cap V^X) - 2\sqrt{\varepsilon} N s \\ &\geq \overline{\deg}_{\mathbf{H}}(X, V^X \cap \bigcup V(M_X)) - \varepsilon n - 2\sqrt{\varepsilon} N s \\ &\overset{(v)}{\geq} v(\mathcal{D}_2) - c^2 k + \xi n - 3\sqrt{\varepsilon} N s \,.\end{aligned}$$

As $\mathcal{D}_2$ is c-balanced, we get $v(\mathcal{D}_2) \geq c^2 k + \sum_{CD \in M_X \cap M_v} \max\{|\varphi(\mathcal{D}_2) \cap C|, |\varphi(\mathcal{D}_2) \cap D|\}$. So, there is an edge $CD \in M_X \cap M_v$ such that

$$|D \cap V^X| \overset{(i)}{\geq} \deg(v, C \cap V^X) \geq \max\{|\varphi(\mathcal{D}_2) \cap C|, |\varphi(\mathcal{D}_2) \cap D|\} + \xi s - 3\sqrt{\varepsilon} s \,.$$

Together with (A.4), we get (A.6).

(d) The vertex v is typical w. r. t. all but at most $\sqrt{\varepsilon} N$ clusters $Z \in \mathcal{Z}'$.

First assume that φ is equable and $\mathcal{Z}' = \mathcal{Z}$. We have

$$\begin{aligned}\deg(v, \tilde{Z} \cap \bigcup \mathcal{Z}') &\overset{\mathcal{Z}' = \mathcal{Z}}{\geq} \deg(v, \bigcup \mathcal{Z}) - \left|\bigcup \mathcal{Z} \setminus \tilde{Z}\right| \overset{(A.3)}{\geq} \overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}) - \varepsilon n - (1+2)\sqrt{\varepsilon} N s \\ &\overset{(vi)}{\geq} |V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)| + \xi n - 4\sqrt{\varepsilon} n \,.\end{aligned}$$

As by **(I3)** only $V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)$ is mapped to $\bigcup \mathcal{Z} = \bigcup \mathcal{Z}'$, there exists a cluster $Z \in \mathcal{Z}'$ satisfying (A.7).

If φ is not equable, we get

$$\begin{aligned}\deg(v, \tilde{Z} \cap \bigcup \mathcal{Z}') &\geq \deg(v, \bigcup \mathcal{Z}') - \left|\bigcup \mathcal{Z} \setminus \tilde{Z}\right| \geq \overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}') - \varepsilon n - (1+2)\sqrt{\varepsilon} N s \\ &\overset{\mathbf{(I4)}}{\geq} |V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X) \setminus V_i^j| + |U_i^j \cap \bigcup \mathcal{Z}'| + \xi n/2 - 4\sqrt{\varepsilon} n \,.\end{aligned}$$

As by **(I3)** only $V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)$ is mapped to $\bigcup \mathcal{Z}'$, there exists a cluster $Z \in \mathcal{Z}'$ satisfying (A.7). □

Claim A.1.3. Suppose that $\mathcal{D}_3 \neq \emptyset$. Then for every vertex $v \in \tilde{Z}$, there is an edge $CD \in M_v$ such that

$$\deg(v, (C \cup D) \cap V^{\mathcal{Z}}) \geq |\varphi(\mathcal{D}_3) \cap (C \cup D)| + 2\tau + 2\varepsilon s + \frac{\xi s}{2} \,. \tag{A.8}$$

Proof. Suppose that v lies in a cluster Z. Using the typicality of v, we get

$$\sum_{CD \in M_v} \deg(v, (C \cup D) \cap V^{\mathcal{Z}}) \overset{(A.5)}{\geq} \overline{\deg}_{\mathbf{H}}(Z, V^{\mathcal{Z}}) - 4\sqrt{\varepsilon} N s - \varepsilon n \overset{(vii)}{\geq} v(\mathcal{D}_3) + \frac{3\xi n}{4} \,.$$

□

Assume that we are in step $i \geq 1$ and that we want to embed the forest F_i. By **(I2)**, we can map x_i to an unused vertex in N_i (in $N_i \cap X$ if $x_i \in W_X$, and in $N_i \cap Y$ if $x_i \in W_Y$). Observe that $\varphi(x_i)$ has at least $(d-\varepsilon)s - ds/2 - 3\sqrt{\varepsilon}s \geq \varepsilon s \geq |W_X \cup W_Y|$ neighbors in $\tilde{X}$ or in $\tilde{Y}$ (depending whether $\varphi(x_i) \in \tilde{Y}$ or $\varphi(x_i) \in \tilde{X}$). This ensures that **(I2)** still holds. Assume that we are in substep $j \in [c_i]$, i.e., we have already embedded the components $t_i^1, \ldots, t_i^{j-1}$ and that we want to embed the component t_i^j.

(1) If $t_i^j \in \mathcal{D}_Y$, pick an edge $CD \in M_{\varphi(x_i)}$ as in Claim A.1.2 (a). We use Lemma 5.12 to embed t_i^j (where the root of t_i^j is the neighbor of x_i) with the following setting.

$$P' = \mathrm{N}_H(\varphi(x_i)) \cap C \cap V^Y \setminus U_i^j \qquad P = C \cap V^Y \setminus U_i^j \subseteq C\,,$$
$$Q' = \mathrm{N}_H(\varphi(x_i)) \cap D \cap V^Y \setminus U_i^j \qquad Q = D \cap V^Y \setminus U_i^j \subseteq D\,,$$

and $\Delta = \frac{4\varepsilon s}{d}$. We have that

$$\max\{|P'|, |Q'|\} \geq \frac{1}{2}\deg(\varphi(x_i), (C \cup D) \cap V^Y \setminus U_i^j) \geq \frac{1}{2}(2\tau + \tfrac{\xi s}{2}) \geq \tfrac{4\varepsilon s}{d}\,,$$

which verifies one of the assumption of Lemma 5.12. We use Lemma 5.12 differently in cases

$$\min\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} < \min\{\overline{\deg}_{\mathbf{H}}(X, C \cap V^Y), \overline{\deg}_{\mathbf{H}}(X, D \cap V^Y)\} - \tfrac{8\varepsilon s}{d} \quad \text{and} \tag{A.9}$$

$$\min\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} \geq \min\{\overline{\deg}_{\mathbf{H}}(X, C \cap V^Y), \overline{\deg}_{\mathbf{H}}(X, D \cap V^Y)\} - \tfrac{8\varepsilon s}{d} \tag{A.10}$$

Suppose first that we do not have (A.9). Thus in particular, the packedness of $U_i^j \cap V^X \cap (C \cup D)$ in **(I1)** has the form of (A.2). Then

$$\begin{aligned}
\min\{|P'|, |Q'|\} &= \min\{\deg(\varphi(x_i), C \cap V^Y \setminus U_i^j), \deg(\varphi(x_i), D \cap V^Y \setminus U_i^j)\} \\
&\overset{\textbf{(I3)}}{\geq} \min\{\deg(\varphi(x_i), C \cap V^Y), \deg(\varphi(x_i), D \cap V^Y)\} - \max\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} \\
&\overset{\textbf{(I1)}}{\geq} \min\{\overline{\deg}_{\mathbf{H}}(X, C \cap V^Y), \overline{\deg}_{\mathbf{H}}(X, D \cap V^Y)\} - \varepsilon s \\
&\quad - \min\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} - \tau \\
&\overset{\text{(A.9)}}{\geq} \tfrac{8\varepsilon s}{d} - \varepsilon s - \tau \geq \tfrac{4\varepsilon s}{d}\,,
\end{aligned}$$

which allows us to use the "moreover" part of Lemma 5.12. We can then choose in this case to which set P' or Q' we map the root of t_i^j. We thus can ensure that $\big||\varphi(\mathcal{D}_Y) \cap C| - |\varphi(\mathcal{D}_Y) \cap D|\big| \leq \tau$ still holds after embedding t_i^j, yielding **(I1)**.

Suppose now that (A.9) holds. Then

$$\begin{aligned}
\min\{|P|, |Q|\} &= \min\{|C \cap V^Y \setminus U_i^j|, |D \cap V^Y \setminus U_i^j|\} \\
&\geq \max\{\deg(\varphi(x_i), C \cap V^Y), \deg(\varphi(x_i), D \cap V^Y)\} - \max\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} \\
&\geq \deg(\varphi(x_i), (C \cup D) \cap V^Y) - |\varphi(\mathcal{D}_Y) \cap (C \cup D)| \\
&\quad - \min\{\deg(\varphi(x_i), C \cap V^Y), \deg(\varphi(x_i), D \cap V^Y)\} + \min\{|\varphi(\mathcal{D}_Y) \cap C|, |\varphi(\mathcal{D}_Y) \cap D|\} \\
&\overset{\text{(A.9)}}{\geq} 2\tau + \tfrac{\xi s}{2} - \tfrac{8\varepsilon s}{d} - \varepsilon s \geq \tfrac{4\varepsilon s}{d}\,,
\end{aligned}$$

which indeed allows us to embed t_i^j using Lemma 5.12 in this case. After the embedding of t_i^j in this case, **(I1)** holds trivially.

In both cases, **(I2)** holds, as $\mathcal{D}_Y$ contains only end-shrubs. The tree t_i^j was embedded in $(C \cup D) \cap V^Y$, ensuring **(I3)**. **(I4)** is immaterial in this step as nothing was done regarding $V^{\mathcal{Z}}$ or $\mathcal{D}_3$.

(2) If $t_i^j \in \mathcal{D}_1$, pick an edge $CD \in M_{\varphi(x_i)} \setminus M_X$ as in Claim A.1.2 (b). The embedding is done analogously to the case **(1)**, setting

$$\begin{aligned} P' &= \mathrm{N}_H(\varphi(x_i)) \cap C \cap V^X \cap Q_X \setminus U_i^j \qquad & P &= C \cap V^X \cap Q_X \setminus U_i^j \subseteq C\ , \\ Q' &= \mathrm{N}_H(\varphi(x_i)) \cap D \cap V^X \cap Q_X \setminus U_i^j \qquad & Q &= D \cap V^X \cap Q_X \setminus U_i^j \subseteq D\ . \end{aligned}$$

As $\varphi(V(t_i^j)) \subseteq Q_X$, every vertex in $V(t_i^j) \cap \mathrm{N}_F(W_X)$ is mapped to a vertex that has at least $(d-\varepsilon)|\tilde{X}| \geq |W_X|$ neighbours in $\tilde{X}$, ensuring **(I2)**. Conditions **(I1)** and **(I3)** are maintained as in case **(1)**. Again, **(I4)** is maintained automatically.

(3) If $t_i^j \in \mathcal{D}_2$, we pick an edge $CD \in M_{\varphi(x_i)} \cap M_X$ as in Claim (c). We use Lemma 5.12 with the following setting.

$$\begin{aligned} &P' = \mathrm{N}_H(\varphi(x_i)) \cap C \cap V^X \cap Q_X \setminus U_i^j \subseteq C \cap V^X \cap Q_X \setminus U_i^j \subseteq C\ , \\ &Q' = \emptyset \subseteq D \cap V^X \setminus U_i^j \subseteq D\ , \end{aligned}$$

and $\Delta = \frac{4\varepsilon s}{d}$. The requirements on $\max\{|P'|, |Q'|\}$, and $\min\{|P|, |Q|\}$ are fulfilled by (A.6). We get an embedding of t_i^j in $(C \cup D) \cap V^X$ (ensuring **(I3)**) such that every vertex at even distance to the root of t_i^j is mapped to Q_X. Therefore its image sends at least $(d-\varepsilon)|\tilde{X}| \geq |W_X|$ edges to $\tilde{X}$ (ensuring **(I2)**). The condition **(I1)** trivially holds by the property of M_X.

(4) Suppose that $t_i^j \in \mathcal{D}_3$.

First we consider the case, when there is a cluster $Z \in \mathcal{Z}$ such that

(*) $\deg(\varphi(x_i), Z \cap \tilde{Z}) \geq |U_i^j \cap Z| + \frac{\xi s}{4}$, and

(**) there is an edge $CD \in M$ such that $\overline{\deg}(Z, C \cap V^{\mathcal{Z}}) \geq |U_i^j \cap C \cap V^{\mathcal{Z}}| + \tau + \varepsilon s + \frac{3\varepsilon s + \tau}{d - 2\varepsilon}$, and $\overline{\deg}(Z, D \cap V^{\mathcal{Z}}) \geq |U_i^j \cap D \cap V^{\mathcal{Z}}| + \tau + \varepsilon s + \frac{3\varepsilon s + \tau}{d - 2\varepsilon}$.

Then we embed t_i^j in $Z \cup C \cup D$ as follows. We map the root r of t_i^j to an unused vertex $v \in Z \cap \tilde{Z}$ that is typical w.r.t. $C \cap V^{\mathcal{Z}}$ and typical w.r.t. $D \cap V^{\mathcal{Z}}$. By Fact 5.9 there are at least $\frac{\xi s}{4} - 2\varepsilon s > 0$ such vertices. By (**), the vertex v satisfies

$$\begin{aligned} \deg(v, (C \cap V^{\mathcal{Z}}) \setminus U_i^j) &\geq \tau + \frac{3\varepsilon s + \tau}{d - 2\varepsilon}\ , \text{ and} \\ \deg(v, (D \cap V^{\mathcal{Z}}) \setminus U_i^j) &\geq \tau + \frac{3\varepsilon s + \tau}{d - 2\varepsilon}\ . \end{aligned} \tag{A.11}$$

Let $K \subseteq C \cup D$ be the set of vertices that are typical (where typicality refers to C or D, respectively) w.r.t. $(Z \cap \tilde{Z}) \setminus U_i^j$. Note that the set $(Z \cap \tilde{Z}) \setminus U_i^j$ is significant by (*). By Fact 5.9,

$$|C \setminus K|, |D \setminus K| \leq \varepsilon s\ . \tag{A.12}$$

Let t_{even} be the set of vertices in $V(t_i^j) \setminus \{r\}$ of even distance from r, and let t_{odd} be the ones of odd distance. If $|t_{odd}| < |t_{even}|$ and $|(C \cap V^{\mathcal{Z}}) \setminus U_i^j| \leq |(D \cap V^{\mathcal{Z}}) \setminus U_i^j|$, or $|t_{odd}| \geq |t_{even}|$ and $|(C \cap V^{\mathcal{Z}}) \setminus U_i^j| > |(D \cap V^{\mathcal{Z}}) \setminus U_i^j|$, set $X_{\mathrm{L5.12}} = D$, and $Y_{\mathrm{L5.12}} = C$. Otherwise set $X_{\mathrm{L5.12}} = C$, and $Y_{\mathrm{L5.12}} = D$.

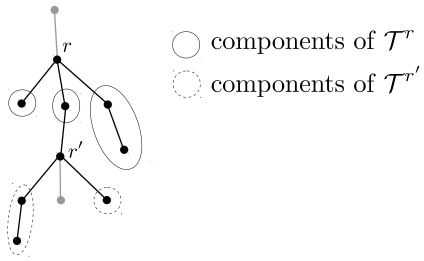

Figure 8: The components $\mathcal{T}^r$ and $\mathcal{T}^{r'}$. Vertices of W_X are shown in gray.

Consider the set $\mathcal{T}^r$ of components of $t_i^j - \mathrm{N}_F(W_X)$ that are incident to r. By Definition 5.2(ix), $V(t_i^j) \cap \mathrm{N}_F(W_X)$ has one or two elements. If r is the only element in $V(t_i^j) \cap \mathrm{N}_F(W_X)$ then $\mathcal{T}^r$ contains all the components of $t_i^j \setminus \{r\}$. We embed the elements $t \in \mathcal{T}^r$ one after the other using Lemma 5.12. At each application of Lemma 5.12 we use $P = (X_{\mathrm{L}5.12} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, $P' = P \cap \mathrm{N}(v)$, $Q = (Y_{\mathrm{L}5.12} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, and $Q' = Q \cap \mathrm{N}(v)$. By (A.11) and (A.12) we have $\min\{|P'|, |Q'|\} \geq \frac{3\varepsilon s+\tau}{d-2\varepsilon} - \varepsilon s \geq \frac{\varepsilon s+\tau}{d-2\varepsilon s}$. By the "moreover" part of Lemma 5.12 we can ensure that the root of t (i.e. the unique vertex in $V(t) \cap \mathrm{N}_F(r)$) is mapped to the set P'.

If r is the only element in $V(t_i^j) \cap \mathrm{N}_F(W_X)$, then we are done with embedding t_i^j. Otherwise, let r' be the second vertex in $V(t_i^j) \cap \mathrm{N}_F(W_X)$. The predecessor of r' is mapped on a vertex $u \in K$. Since u is typical w.r.t. the set $(Z \cap \tilde{Z}) \setminus U_i^j$, we have

$$\deg(u, (Z \cap \tilde{Z}) \setminus U_i^j) \geq (d - \varepsilon)|(Z \cap \tilde{Z}) \setminus U_i^j| \overset{(*)}{\geq} (d - \varepsilon)\frac{\xi s}{4} > 3\varepsilon s \,. \tag{A.13}$$

We can thus map the vertex r' to an unused vertex $v' \in (Z \cap \tilde{Z}) \cap \mathrm{N}(u)$ that is typical w.r.t. $C \cap V^{\mathcal{Z}}$ and typical w.r.t. $D \cap V^{\mathcal{Z}}$.

Consider the set $\mathcal{T}^{r'}$ of components of $t_i^j - \{r'\}$ that are incident to r' and does not contain r. See Figure 8. We embed the elements $t \in \mathcal{T}^{r'}$ one after the other using Lemma 5.12. At each application of Lemma 5.12 we use $P = (X_{\mathrm{L}5.12} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, $P' = P \cap \mathrm{N}(v)$, $Q = (Y_{\mathrm{L}5.12} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, and $Q' = Q \cap \mathrm{N}(v)$. By (**), the vertex v' satisfies

$$\deg(v', (C \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)) \geq \frac{3\varepsilon s + \tau}{d - 2\varepsilon}\ , \text{ and}$$
$$\deg(v', (D \cap V^{\mathcal{Z}}) \setminus \mathcal{D}_3)) \geq \frac{3\varepsilon s + \tau}{d - 2\varepsilon}\ .$$

By (A.12), we have that $\min\{|P'|, |Q'|\} \geq \frac{3\varepsilon s+\tau}{d-2\varepsilon} - \varepsilon s \geq \frac{\varepsilon s+\tau}{d-2\varepsilon s}$. We can thus use the "moreover" part of Lemma 5.12 to ensure that the root of t (i.e. the unique vertex in $V(t) \cap \mathrm{N}_F(r')$) is mapped to the set P'.

As we embedded $V(t_i^j) \cap \mathrm{N}(W_X)$ in Z and the rest of t_i^j in $(C \cup D) \cap V^{\mathcal{Z}}$, properties **(I1)**, **(I2)**, and **(I3)** trivially hold. As for **(I4)**, we did not alter the set $\mathcal{Z}'$ and the set $\varphi(\mathcal{D}_3)$ may have only increased. Also observe that by **(I3)** we have that $|V(t_i^j) \cap \mathrm{N}_F(W_X)| \geq |\varphi(t_i^j) \cap \bigcup \mathcal{Z}'|$. Therefore, it is enough to show that if φ was equable at the substep j, it is still equable at substep $j+1$ (i.e. after we embedded t_i^j). This was guaranteed by the choice of $X_{\mathrm{L}5.12}$ and $Y_{\mathrm{L}5.12}$, so that to minimise the difference between $|\varphi(\mathcal{D}_3) \cap (C \cap V^{\mathcal{Z}})|$ and $|\varphi(\mathcal{D}_3) \cap (D \cap V^{\mathcal{Z}})|$ together with the fact that $v(t_i^j) \leq \tau$.

Now consider the case when there is no cluster $Z \in \mathcal{Z}$ that satisfies (*) and (**). If φ is equable and $\mathcal{Z}' = \mathcal{Z}$ then we redefine $\mathcal{Z}'$ to be the set of clusters in $\mathcal{Z}$ with respect to which the vertex $\varphi(x_i)$ is typical and for which (*) holds. We want to check **(I4)** for this new set $\mathcal{Z}'$.

As $\varphi(x_i) \in \tilde{X}$ by **(I2)**, we have that $\varphi(x_i)$ is typical to all but at most $\sqrt{\varepsilon}N$ clusters of $\mathcal{Z}$. Therefore,

$$\begin{aligned}
\overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}') &\geq \overline{\deg}_{\mathbf{H}}(X, \bigcup \mathcal{Z}) - \sqrt{\varepsilon}n - \left(|U_i^j \cap \bigcup (\mathcal{Z} \setminus \mathcal{Z}')| + \frac{\xi n}{4}\right) \\
&\geq |V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)| - |U_i^j \cap \bigcup (\mathcal{Z} \setminus \mathcal{Z}')| + \xi n - \sqrt{\varepsilon}n - \frac{\xi n}{4} \\
&\overset{\textbf{(I3)}}{\geq} |(V(\mathcal{D}_3) \cap \mathrm{N}_F(W_X)) \setminus V_i^j| + |U_i^j \cap \bigcup \mathcal{Z}'| + \frac{\xi n}{2} \ .
\end{aligned}$$

By the definition of $\mathcal{Z}'$, (**) does not hold for any cluster $Z \in \mathcal{Z}'$. Then, as φ is equable, we have

$$\begin{aligned}
&\min\{\overline{\deg}(Z, C \cap V^{\mathcal{Z}}), \overline{\deg}(Z, D \cap V^{\mathcal{Z}})\} \\
&\quad \leq \max\{|U_i^j \cap C \cap V^{\mathcal{Z}}|, |U_i^j \cap D \cap V^{\mathcal{Z}}|\} + \tau + \varepsilon s + \frac{3\varepsilon s + \tau}{d - 2\varepsilon} \\
&\quad \leq \min\{|U_i^j \cap C \cap V^{\mathcal{Z}}|, |U_i^j \cap D \cap V^{\mathcal{Z}}|\} + 2\tau + \varepsilon s + \frac{3\varepsilon s + \tau}{d - 2\varepsilon} \\
&\quad \overset{\textbf{(I3)}}{\leq} \min\{|\varphi(\mathcal{D}_3) \cap C|, |\varphi(\mathcal{D}_3) \cap D|\} + \frac{8\varepsilon s}{d} \ ,
\end{aligned}$$

showing that the newly created set $\mathcal{Z}'$ satisfies **(I4)**.

So we may assume that the second condition of **(I4)** is satisfied. Let $Z \in \mathcal{Z}'$ be a cluster as in Claim A.1.2 (d) and map the root of t_i^j to a vertex $v \in Z \cap \tilde{Z}$. Then pick an edge $CD \in M_v$ as in Claim A.1.3. Let $K \subseteq C \cup D$ be the set of vertices that are typical (where typicality refers to C or D, respectively) w.r.t. $(Z \cap \tilde{Z}) \setminus U_i^j$.

Without loss of generality, assume that $|\mathrm{N}(v, K \cap D \cap V^{\mathcal{Z}} \setminus U_i^j)| \leq |\mathrm{N}(v, K \cap C \cap V^{\mathcal{Z}} \setminus U_i^j)|$. Let $X_{\mathrm{L5.12}} = C$ and $Y_{\mathrm{L5.12}} = D$. Consider the set $\mathcal{T}^r$ of components of $t_i^j - \mathrm{N}_F(W_X)$ that are incident to r. We embed the elements $t \in \mathcal{T}^r$ one after the other using Lemma 5.12 with the following setting,

$$\begin{aligned}
P &= (X_{\mathrm{L5.12}} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3) \ , &\qquad P' &= P \cap \mathrm{N}(v) \ , \\
Q &= (Y_{\mathrm{L5.12}} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3) \ , &\qquad Q' &= Q \cap \mathrm{N}(v) \ .
\end{aligned}$$

By (A.8) we have $|(P' \cup Q')| \geq \frac{\xi s}{4} - 2\varepsilon s$. As by assumption we have $|P'| \geq |Q'|$, we get $|P'| \geq \frac{\xi s}{8} - \varepsilon s \geq \frac{\varepsilon s + \tau}{d - 2\varepsilon}$.

From (A.8) we derive

$$
\begin{aligned}
|Q| &\geq |\tilde{D} \cap V^{\mathcal{Z}}| - |(D \cap V^{\mathcal{Z}}) \cap U_i^j| = |\tilde{D} \cap V^{\mathcal{Z}}| - |((C \cup D) \cap V^{\mathcal{Z}}) \cap U_i^j| + |(C \cap V^{\mathcal{Z}}) \cap U_i^j| \\
&\geq |\tilde{D} \cap V^{\mathcal{Z}}| - \left(\deg(v, (C \cup D) \cap V^{\mathcal{Z}}) - 2\tau - 2\varepsilon s - \frac{\xi s}{2}\right) + |(C \cap V^{\mathcal{Z}}) \cap U_i^j| \\
&= |\tilde{D} \cap V^{\mathcal{Z}}| - \max\left\{\deg(v, C \cap V^{\mathcal{Z}}), \deg(v, D \cap V^{\mathcal{Z}})\right\} + 2\tau + 2\varepsilon s + \frac{\xi s}{2} \\
&\quad - \min\left\{\deg(v, C \cap V^{\mathcal{Z}}), \deg(v, D \cap V^{\mathcal{Z}})\right\} + \min\left\{|(C \cap V^{\mathcal{Z}}) \cap U_i^j|, |(D \cap V^{\mathcal{Z}}) \cap U_i^j|\right\} \\
&\geq 2\tau + \varepsilon s + \frac{\xi s}{2} - \left(\min\left\{\overline{\deg}(Z, C \cap V^{\mathcal{Z}}), \overline{\deg}(Z, D \cap V^{\mathcal{Z}})\right\} + \varepsilon s\right) \\
&\quad + \min\left\{|(C \cap V^{\mathcal{Z}}) \cap U_i^j|, |(D \cap V^{\mathcal{Z}}) \cap U_i^j|\right\} \\
&\overset{\textbf{(I4)}}{\geq} 2\tau + \varepsilon s + \frac{\xi s}{2} - \varepsilon s \geq \frac{\varepsilon s + \tau}{d - 2\varepsilon}\,.
\end{aligned}
$$

If r is the only element in $V(t_i^j) \cap \mathrm{N}_F(W_X)$, then we are done with embedding t_i^j. Otherwise, let r' be the second vertex in $V(t_i^j) \cap \mathrm{N}_F(W_X)$. The predecessor of r' is mapped on a vertex $u \in K$ that is typical w.r.t. the set $(Z \cap \tilde{Z}) \setminus U_i^j$ and hence satisfies (A.13). We can thus mapped the vertex r' to an unused vertex $v' \in (Z \cap \tilde{Z}) \cap \mathrm{N}(u)$ that is typical w.r.t. $C \cap V^{\mathcal{Z}}$ and typical w.r.t. $D \cap V^{\mathcal{Z}}$. Let $X_{\mathrm{L5.12}} = C$ and $Y_{\mathrm{L5.12}} = D$. Consider the set $\mathcal{T}^{r'}$ of components of $t_i^j - \{r'\}$ that are incident to r' and does not contain r. We embed the elements $t \in \mathcal{T}^{r'}$ one after the other using Lemma 5.12 similarly as we did for the elements of $\mathcal{T}^r$. At each application of Lemma 5.12 we use $P = (X_{\mathrm{L5.12}} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, $P' = P \cap \mathrm{N}(v')$, $Q = (Y_{\mathrm{L5.12}} \cap K \cap V^{\mathcal{Z}}) \setminus \varphi(\mathcal{D}_3)$, and $Q' = Q \cap \mathrm{N}(v')$.

Observe that the embedding φ satisfies **(I1)**–**(I4)**.

□