On the diameter of reconfiguration graphs for vertex colourings

Marthe Bonamy [a,1] Matthew Johnson [b,2] Ioannis Lignos [b,2]
Viresh Patel [b,2] Daniël Paulusma [b,2]

[a] *École Normale Supérieure de Lyon, France*

[b] *School of Engineering and Computing Sciences,*
Durham University, United Kingdom

Abstract

The reconfiguration graph of the k-colourings of a graph G contains as its vertex set the proper vertex k-colourings of G, and two colourings are joined by an edge in the reconfiguration graph if they differ in colour on just one vertex of G. We prove that for a graph G on n vertices that is chordal or chordal bipartite, if G is k-colourable, then the reconfiguration graph of its ℓ-colourings, for $\ell \geq k+1$, is connected and has diameter $O(n^2)$. We show that this bound is asymptotically tight up to a constant factor.

Keywords: reconfiguration, graph colouring, graph diameter

1 Introduction

For any pair α and β of k-colourings of a graph, one can ask whether it is possible to transform one into the other by finding a sequence of proper

[1] Email: `marthe.bonamy@lirmm.fr`

[2] Email: `{matthew.johnson2,i.m.lignos,viresh.patel,daniel.paulusma}@durham.ac.uk`

colourings where the first is α, the last is β, and colourings that are adjacent in the sequence differ on just one vertex. One can also ask for bounds on the length of the sequence, or whether such a sequence exists for all possible pairs of colourings of a graph. In this paper, we investigate these questions for a number of graph classes. These are examples of what are known as *reconfiguration problems*.

For any search problem, the corresponding reconfiguration graph contains as vertices all solutions to the problem; the edge set is defined by a symmetric adjacency relation on the solutions which is normally chosen to represent a smallest possible change in the solution. In particular, the reconfiguration graph of the k-colourings of a graph G contains as its vertex set the proper vertex k-colourings of G, and two colourings are joined by an edge in the reconfiguration graph if they differ in colour on just one vertex of G. To date, the study of reconfiguration graphs has focussed on the computational complexity of the problems of deciding whether the reconfiguration graph is connected, and deciding whether it contains a path between two given solutions. Problems studied include boolean satisfiability [5], graph colouring [2,3,4], shortest path [1,9], and independent set, clique and others [8].

A fundamental problem is to characterise the relationship between the complexity of reconfiguration problems and search problems. Considering the problem of finding paths between solutions, previous results often follow a certain pattern: problems in P beget reconfiguration problems that are also in P; NP-complete problems have PSPACE-complete reconfiguration problems. There are exceptions such as the shortest path reconfiguration problem being PSPACE-hard [1].

Also of interest is finding *shortest* paths between solutions. The diameter of the reconfiguration graph provides an upper bound. This is also related to the complexity of finding paths in the reconfiguration graph between given solutions since paths of polynomial length in the reconfiguration graph are certificates for the problem being in NP. For any graph, the diameter of the reconfiguration graph of its 3-colourings has been shown to be quadratic (in the number of vertices of the input graph) if the reconfiguration graph is connected [4]. There is no known example of a family of graphs for which the reconfiguration graph of k-colourings is connected but does not have quadratic diameter, although there are cases where the reconfiguration graph is not connected but contains components of superpolynomial diameter [2].

In this paper, we study the reconfiguration problem for graph colouring and determine sufficient conditions for the reconfiguration graph to have a quadratic diameter. We give two examples of graph classes, namely chordal graphs and chordal bipartite graphs, that satisfy these conditions and show that our quadratic bound is tight by showing that the diameter of the reconfiguration graph for 3-colourings of a path is quadratic in the length of the path.

2 Sufficient Conditions for Quadratic Diameter

A *separator* of a graph G is a set $S \subseteq V$ such that $G - S$ has more connected components than G. We say that we *identify* two vertices u and v if we replace them by a new vertex adjacent to all neighbours of u and v. The set of neighbours of a vertex u is denoted $N(u)$.

Fix a positive integer k. A class $\mathcal{G}$ of k-colourable graphs is called *k-colour-dense* if for every $G \in \mathcal{G}$ either

(i) G is the disjoint union of cliques, each of which has at most k vertices, or

(ii) G has a separator S and $G - S$ has components D and D' with vertices $u \in D$ and $v \in D'$ such that
 (a) $|V_D| \leq \max\{1, k - |S|\}$,
 (b) $S \subseteq N(v)$, and
 (c) identifying u and v in G results in a graph $G' \in \mathcal{G}$.

We define the *ℓ-colour diameter* of a graph G to be the diameter of the reconfiguration graph of ℓ-colourings of G.

Theorem 2.1 *For an integer $k \geq 1$, let $\mathcal{G}$ be a k-colour-dense graph class. Then, for all $\ell \geq k+1$, the ℓ-colour diameter of every n-vertex graph $G \in \mathcal{G}$ is at most $2n^2$.*

Proof. Let $k \geq 1$ be an integer and let $\mathcal{G}$ be a k-colour-dense graph class. We assume $\ell = k+1$; the proof for $\ell > k+1$ is similar. We prove the following claim.

Claim 1. Let α and β be two $(k+1)$-colourings of an n-vertex graph $G \in \mathcal{G}$. Then we can transform α to β by recolouring every vertex of G at most $2n$ times.

The statement of the theorem follows immediately from the claim since we recolour each of n vertices at most $2n$ times, resulting in at most $2n^2$ recolourings when transforming any $(k+1)$-colouring to any other.

In order to prove Claim 1 we shall use condition (ii) in our definition of a k-colour-dense graph class to apply induction. The graphs of $\mathcal{G}$ for which this condition does not hold are the disjoint union of cliques each of size at most

k, and these graphs form the base case of our induction. It is easy to check these graphs satisfy the claim.

Suppose $G \in \mathcal{G}$ is an n-vertex graph that is not a disjoint union of cliques, and suppose α and β are two $(k+1)$-colourings of G. Thus, by definition, G has a separator S that satisfies condition (ii) of the definition of k-colour-dense with respect to sets D, D' and vertices $u \in D$ and $v \in D'$.

We first show how to transform α into some $(k+1)$-colouring α' satisfying $\alpha'(u) = \alpha'(v)$, where each vertex of G is recoloured at most once. Suppose that $\alpha(u) \neq \alpha(v)$. We can recolour u with $\alpha(v)$ unless there is a nonempty set W of neighbours of u such that each vertex in W has colour $\alpha(v)$. Because v is adjacent to all vertices in S, we find that $W \subseteq D \setminus \{u\}$. Because $|V_D| \leq \max\{1, k - |S|\}$ and W is nonempty, we find that $|V_D| \leq k - |S|$. Hence every vertex in W has at most $k - |S| - 1$ neighbours in D, and consequently, at most $k - 1$ neighbours in G. Hence, each vertex of W can be successively recoloured with some colour not used in its neighbourhood. This leads to a new $(k+1)$-colouring α' with $\alpha'(u) = \alpha'(v)$. By the same argument, we can modify β to a $(k+1)$-colouring β' with $\beta'(u) = \beta'(v)$. Changing α to α' and β to β' together require that each vertex of D is recoloured at most twice.

We identify u and v. This leads to a new vertex u' and a graph G' that belongs to $\mathcal{G}$ by definition. We can consider α' and β' to be colourings of G' by giving u' the colour and $\alpha'(u) = \alpha'(v)$. We apply the induction hypothesis and find that we can transform α' into β' on G' using at most $2(n-1)$ recolourings for each vertex. Thus we can transform α' into β' on G by simulating a recolouring of u' by a recolouring of u and v in G, i.e., every time we recolour u' in G' we apply the same recolouring to u and then immediately to v in G. Thus transforming α' to β' in G requires that each vertex of G be recoloured at most $2(n-1)$ times, and transforming α to α' and β' to β requires at most two additional recolourings of each vertex, resulting in a total of at most $2n$ recolourings of each vertex, as required. $\square$

3 Graph Classes

We present two graph classes for which Theorem 2.1 is applicable and show that the bound in Theorem 2.1 is asymptotically tight up to a constant factor.

3.1 Chordal graphs

A *chordal* graph is a graph with no induced cycle of length more than 3. A clique tree of a (connected) graph G is a tree that has as vertices the maximal cliques of G and has edges such that each graph induced by those cliques that contain a particular vertex of G is a subtree. The next lemma is well known.

Lemma 3.1 **([7])** *A connected graph is chordal if and only if it has a clique tree.*

Theorem 3.2 *For any integer $k \geq 1$, the class of k-colourable chordal graphs is k-colour-dense.*

Proof. (sketch) Let $\mathcal{G}$ denote the class of k-colourable chordal graphs. Let $G = (V, E)$ be a graph in $\mathcal{G}$ that is not the disjoint union of cliques. We must check that G satisfies condition (ii) of the definition of k-colour-dense. We may assume without loss of generality that G is connected; otherwise we can apply the argument below to any one of the components.

By Lemma 3.1, G has a clique tree $\mathcal{T}$. Let K be a leaf of $\mathcal{T}$, and let K' be the unique neighbour of K. In order to demonstrate condition (ii) we let $S = K \cap K'$ and we define $D := G[K \setminus S]$ and $D' := G[V \setminus K]$; here we use the notation $G[U]$ for the subgraph of G induced by a subset $U \subseteq V$. We observe that V_D and $V_{D'}$ are non-empty because otherwise one of K or K' would be contained in the other, contradicting that they are maximal cliques. Consequently, S is a separator. Because G is k-colourable, K has at most k vertices. This means that $|V_D| = |K| - |S| \leq k - |S| = \max\{1, k - |S|\}$, demonstrating condition (ii)(a). Recall that V_D and $V_{D'}$ are non-empty. For u and v we take arbitrary vertices in $K \backslash K' = V_D$ and $K' \backslash K \subseteq V_{D'}$, respectively. Then, because K' is a clique and $S \subseteq K'$, we find that v is adjacent to every vertex in S, demonstrating condition (ii)(b). Finally, we identify u and v. It is not hard to check that the resulting graph is a k-colourable chordal graph, demonstrating condition (ii)(c). This completes the proof of Theorem 3.2. □

3.2 Chordal bipartite graphs

A *chordal bipartite* graph is a bipartite graph with no induced cycle of length *strictly* more than 4. It is a misnomer since chordal bipartite graphs are only chordal if they are trees. We need the following terminology. A vertex u in a bipartite graph G is *weakly simplicial* if its neighbours can be labelled $v_1, \ldots, v_t$ such that $N(v_i) \subseteq N(v_{i+1})$ for $i = 1, \ldots, t-1$.

Lemma 3.3 **([6])** *A bipartite graph is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex.*

Because chordal bipartite graphs are bipartite, they are 2-colourable. The proof of Theorem 3.4 is based on Lemma 3.3; we use it to show that in a chordal bipartite graph we can always find two vertices u and v with $N(u) \subseteq N(v)$. Consequently, we can define $S = N(u)$ in order to satisfy conditions (i) and (ii). We omit the details.

Theorem 3.4 *The class of chordal bipartite graphs is 2-colour-dense.*

3.3 Lower Bounds

We can show that the 3-colour diameter of a path on n vertices is $\Theta(n^2)$ and have an example of a k-colourable chordal graph with $(k+1)$-colour diameter of order $\Theta(n^2)$ for every $k \geq 4$. We omit the proofs.

References

[1] P. Bonsma, Shortest path reconfiguration is PSPACE-hard, manuscript (ArXiv 2010).

[2] P. Bonsma and L. Cereceda, Finding paths between graph colourings: PSPACE-completeness and superpolynomial distances, *Theoret. Comput. Sci.* **410** (2009) 5215–5226.

[3] L. Cereceda, J. van den Heuvel, and M. Johnson, Mixing 3-colourings in bipartite graphs, *European J. Combin.* **30** (2009) 1593–1606.

[4] L. Cereceda, J. van den Heuvel, and M. Johnson, Finding Paths Between 3-Colourings, *J. Graph Theory* **67** (2011) 69–82.

[5] P. Gopalan, P.G. Kolaitis, E.N. Maneva, and C.H. Papadimitriou, The connectivity of boolean satisfiability: computational and structural dichotomies, *SIAM J. Comput.* **38** (2009) 2330–2355.

[6] P.L. Hammer, F. Maffray, and M. Preismann, A characterization of chordal bipartite graphs, tech. rep. (Rutgers University, New Brunswick, NJ, 1989).

[7] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, *J. Combin. Theory Ser. B* **16** (1974) 47–56.

[8] T. Ito, E.D. Demaine, N.J.A. Harvey, C.H. Papadimitriou, M. Sideri, R. Uehara, and Y. Uno, On the Complexity of Reconfiguration Problems, *Theoret. Comput. Sci.* **412** (2010) 1054–1065.

[9] M. Kaminski, P. Medvedev, M. Milanic, Shortest paths between shortest paths and independent sets, in: C.S. Iliopolous and W.F. Smyth, eds., *Proceedings IWOCA 2010* (Springer-Verlag, 2011) 56–67.