# Families of small regular graphs of girth 7 

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#### Abstract

The first known families of cages arised from the incidence graphs of generalized polygons of order $q, q$ a prime power. In particular, $(q+1,6)$-cages have been obtained from the projective planes of order $q$. Morever, infinite families of small regular graphs of girth 5 have been constructed performing algebraic operations on $\mathbb{F}_{q}$.

In this paper, we introduce some combinatorial operations to construct new infinite families of small regular graphs of girth 7 from the ( $q+1,8$ )-cages arising from the generalized quadrangles of order $q, q$ a prime power.


Keywords: Cages, girth, generalized quadrangles, latin squares.

## 1 Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [13], [20] and [25].

Let $G$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$. The girth of a graph $G$ is the number $g=g(G)$ of edges in a smallest cycle. For every $v \in V, N_{G}(v)$ denotes the neighbourhood of $v$, that is, the set of all vertices adjacent to $v$. The degree of a vertex $v \in V$ is the cardinality of $N_{G}(v)$. A graph is called regular if all the vertices have the same degree.

[^0]A $(k, g)$-graph is a $k$-regular graph with girth $g$. Erdős and Sachs [15] proved the existence of $(k, g)$-graphs for all values of $k$ and $g$ provided that $k \geq 2$. Thus most work carried out has focused on constructing a smallest one [2, 3, 4, 5, 6, 7, 19, 14, 16, 18, 19, 23, 24, 27, 28, 32]. A $(k, g)$-cage is a $k$-regular graph with girth $g$ having the smallest possible number of vertices. Cages have been studied intensely since they were introduced by Tutte [33] in 1947. Counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound $n_{0}(k, g)$ with the precise form of the bound depending on whether $g$ is even or odd:

$$
n_{0}(k, g)= \begin{cases}1+k+k(k-1)+\cdots+k(k-1)^{(g-3) / 2} & \text { if } g \text { is odd }  \tag{1}\\ 2\left(1+(k-1)+\cdots+(k-1)^{g / 2-1}\right) & \text { if } g \text { is even }\end{cases}
$$

Biggs [11] calls the excess of a $(k, g)$-graph $G$ the difference $|V(G)|-n_{0}(k, g)$. The construction of graphs with small excess is a difficult task. Biggs is the author of a report on distinct methods for constructing cubic cages [12]. More details about constructions of cages can be found in the survey by Wong [36] or in the book by Holton and Sheehan [22] or in the more recent dynamic cage survey by Exoo and Jajcay [17].

A $(k, g)$-cage with $n_{0}(k, g)$ vertices and even girth exist only when $g \in\{4,6,8,12\}$ [18]. If $g=4$ they are the complete bipartite graph $K_{k, k}$, and for $g=6,8,12$ these graphs are the incidence graphs of generalized $g / 2$-gons of order $k-1$. This is the main reason for $(k, g)$-cages with $n_{0}(k, g)$ vertices and even girth $g$ are called generalized polygon graphs [11]. In particular a 3 -gon of order $k-1$ is also known as a projective plane of order $k-1$. The 4 -gons of order $k-1$ are called generalized quadrangles of order $k-1$, and, the 6 -gons of order $k-1$, generalized hexagons of order $k-1$. All these objets are known to exist for all prime power values of $k-1$ [8, 20, 25], and no example is known when $k-1$ is not a prime power.

In this article we focus on the case $g=8$. Let $q$ be a prime power. Our main objective is to give an explicit construction of small $(q+1,7)$-graphs for $k=q+1$. Next we present the contributions of this paper and in the following sections we do the corresponding proofs.

## 2 Preliminaries

It is well known [30, 26] that $Q(4, q)$ and $W(3, q)$ are the only two classical generalized quadrangles with parameters $s=t=q$. The generalized quadrangle $W(3, q)$ is the dual generalized of $Q(4, q)$, and they are selfdual for $q$ even.

In 1966 Benson [9] constructed $(q+1,8)$-cages from the generalized quadrangle $Q(4, q)$. He defined the point/line incidence graph $\Gamma_{q}$ of $Q(4, q)$ which is a $(q+1)$-regular graph of girth 8 with $n_{0}(q+1,8)$ vertices. Hence, $\Gamma_{q}$ is a $(q+1,8)$-cage. Note that, $\Gamma_{q}$ is isomorphic to the point/line incidence graph of $W(3, q)$.

For any generalized quadrangle $Q$ of order $(s, t)$ and every point $x$ of $Q$, let $x^{\perp}$ denote the set of all points collinear with $x$. Note that in the incidence graph $x^{\perp}=N_{2}(x)$, with an abuse
of notation supposing that $x \in \Gamma_{q}$ corresponds to the point $x \in Q$
If $X$ is a nonempty set of vertices of $Q$, then we define $X^{\perp}:=\bigcap_{x \in X} x^{\perp}$. The span of the pair $(x, y)$ is $s p(x, y)=\{x, y\}^{\perp \perp}=\left\{u \in P: u \in z^{\perp} \forall z \in x^{\perp} \cap y^{\perp}\right\}$, where $P$ denotes the set of points in $Q$. If $x$ and $y$ are not collinear, then $\{x, y\}^{\perp \perp}$ is also called the hyperbolic line through $x$ and $y$. If the hyperbolic line through two noncollinear points $x$ and $y$ contains precisely $t+1$ points, then the pair $(x, y)$ is called regular. A point $x$ is called regular if the pair $(x, y)$ is regular for every point $y$ not collinear with $x$. It is important to recall that the concept of regular also exists for a graph to avoid confusion. Hence we will emphasize when regular refers to a point or a graph.

Remark 2.1 [31] Every point in $W(q)$ is regular.

There are several equivalent coordinatizations of these generalized quadrangles (cf. [29], [34], [35], see also [26]) each giving a labeling for the graph $\Gamma_{q}$. Now we present a further labeling of $\Gamma_{q}$, equivalent to previous ones (cf. [1]), which will be central for our constructions since it allows us to keep track of the properties (such as regularity and girth) of the small regular graphs of girth 7 obtained from $\Gamma_{q}$.

Definition 2.2 Let $\mathbb{F}_{q}$ be a finite field with $q \geq 2$ a prime power. Let $\Gamma_{q}=\Gamma_{q}\left[V_{0}, V_{1}\right]$ be a bipartite graph with vertex sets $V_{r}=\left\{(a, b, c)_{r},(q, q, a)_{r}: a \in \mathbb{F}_{q} \cup\{q\}, b, c \in \mathbb{F}_{q}\right\}, r=0,1$, and edge set defined as follows:

$$
\begin{aligned}
& \text { For all } a \in \mathbb{F}_{q} \cup\{q\} \text { and for all } b, c \in \mathbb{F}_{q}: \\
& N_{\Gamma_{q}}\left((a, b, c)_{1}\right)= \begin{cases}\left\{\left(x, a x+b, a^{2} x+2 a b+c\right)_{0}: x \in \mathbb{F}_{q}\right\} \cup\left\{(q, a, c)_{0}\right\} & \text { if } a \in \mathbb{F}_{q} \\
\left\{(c, b, x)_{0}: x \in \mathbb{F}_{q}\right\} \cup\left\{(q, q, c)_{0}\right\} & \text { if } a=q\end{cases} \\
& N_{\Gamma_{q}}\left((q, q, a)_{1}\right)=\left\{(q, a, x)_{0}: x \in \mathbb{F}_{q}\right\} \cup\left\{(q, q, q)_{0}\right\}
\end{aligned}
$$

Note that, in the labeling introduced in Definition 2.2, the second $q$ in $\mathbb{F}_{q} \cup\{q\}$, usually denoted by $\infty$, is meant to be just a symbol and no operations will be performed with it.

To finish, we define a Latin square as an $n \times n$ array filled with $n$ different symbols, each occurring exactly once in each row and exactly once in each column.

In the following two sections we present our results only for $(q+1,8)$-cages, but all preliminary results are valid for all $(k, 8)$-cages with any number $k$ given that they have the required combinatorial properties.

## 3 Constructions of small $(q+1 ; 7)$-graphs, for an even prime power $q$

In this section we will consider a $(q+1,8)$-cage $\Gamma_{q}$ with $q+1 \geq 5$ an odd integer, since the only known $(q+1,8)$-cages are obtained as the incidence graph of a Generalized Quadrangles, we let $q \geq 4$ a power of two.

Let $x \in V\left(\Gamma_{q}\right)$ and let $N(x)=\left\{x_{0}, \ldots, x_{q}\right\}$, label $N\left(x_{i}\right)=\left\{x_{i 0}, x_{i 1}, \ldots, x_{i q}=x\right\}$, for all $i \in\{0, \ldots q\}$, in the following way. Take $x_{0 j}$ and $x_{1 j}$ arbitrarily for $j=0, \ldots, q-1$ and let $N_{2}\left(x_{0 j}\right) \cap N_{2}\left(x_{1 j}\right)-x=W_{j}$, note that $\left|W_{j}\right|=q$. Let $x_{i j}=\left(\bigcap_{w \in W_{j}} N_{2}(w)\right) \cap N\left(x_{i}\right)$, these vertices exist and are uniquely labeled since the generalized quadrangle $W(q)$ is regular.

Let $H=x \cup N(x) \cup\left\{x_{q-1}, x_{q}\right\} \cup \bigcup_{0}^{q-2} N\left(x_{i}\right) \subset V\left(\Gamma_{q}\right)$.
We will delete the set $H$ of vertices of $\Gamma_{q}$ and add matchings $M_{Z}$ between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7 . In order to define the sets $M_{Z}$, we denote $X_{i}=N\left(x_{i}\right) \backslash\{x\}$ and $X_{i j}=N\left(x_{i j}\right) \backslash\left\{x_{i}\right\}$, for $i \in\{0, \ldots, q\}$ and $j \in\{0, \ldots, q-1\}$.

Let $\mathcal{Z}$ be the family of all $X_{q-1} X_{q}, X_{i j}$ for $i \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\}$. For each $Z \in \mathcal{Z}, M_{Z}$ will denote a perfect matching of $V(Z)$, which will eventually be added to $\Gamma_{q}$.

Definition 3.1 Let $\Gamma_{q}$ be a ( $\left.q+1,8\right)$-cage, with odd degree $q+1 \geq 5$.
Let $\Gamma_{q} 1$ be the graph with: $V\left(\Gamma_{q} 1\right):=V\left(\Gamma_{q}-H\right)$ and $E\left(\Gamma_{q} 1\right):=E\left(\Gamma_{q}-H\right) \cup \bigcup_{Z \in \mathcal{Z}} M_{Z}$.
Observe that the graph $\Gamma_{q} 1$ has order $\left|V\left(\Gamma_{q}\right)\right|-\left(q^{2}+2\right)$ and all its vertices have degree $q+1$.
Next proposition states a condition for the graph $\Gamma_{q} 1$ to have girth 7, for this it is useful to state the following remark.

Remark 3.2 Let $u, v \in V\left(\Gamma_{q}\right)$ a graph of girth 8 , such that there is a uv-path $P$ of length $t<8$. Then every uv-path $P^{\prime}$ such that $E(P) \cap E\left(P^{\prime}\right)=\emptyset$ has length $\left|E\left(P^{\prime}\right)\right| \geq 8-t$.

Proposition 3.3 Let $\Gamma_{q}$ be a $(q+1,8)$-cage, with odd degree $q+1 \geq 5$ and $\Gamma_{q} 1$ as in Definition 3.1. Then $\Gamma_{q} 1$ has girth 7 if given $u_{1} v_{1} \in M_{X_{i j}}$ and $u_{2}, v_{2} \in X_{k l}$ such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$, it holds $u_{2} v_{2} \notin M_{X_{k l}}$, for $i \neq k \in\{0, \ldots, q-2\}$ and $j, l \in\{0, \ldots, q-1\}$.

## Proof

Let us consider the distances (in $\Gamma_{q}-H$ ) between the elements in the sets $Z \in \mathcal{Z}$. There are five possible cases:
(1) Two vertices in the same set $u, v \in Z$ have a common neighbor $w$ in $\Gamma_{q}$, therefore $d_{\Gamma_{q}-H}(u, v) \geq 6$.
(2) If $u \in X_{q-1}$ and $v \in X_{q}$, then $d_{\Gamma_{q}-H}(u, v) \geq 4$, since $x_{q-1}, x_{q}$ have $x$ as a common neighbor in $\Gamma_{q}$.
(3) If $u \in X_{i}$ for $i \in\{q-1, q\}$ and $v \in X_{k j}$ for $k \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\}$ then $d_{\Gamma_{q}}\left(u, x_{i}\right)=1, d_{\Gamma_{q}}\left(v, x_{k}\right)=2$, and $x_{i}, x_{k}$ have a common neighbor $x \in V\left(\Gamma_{q}\right)$, hence there is a $u v$-path of length 5 in $\Gamma_{q}$, concluding from Remark 3.2 that $d_{\Gamma_{q}}(u, v) \geq 3$.
(4) If $u \in X_{i j}$ and $v \in X_{i k}$ for $i \in\{0, \ldots, q-2\}$ and $j, k \in\{0, \ldots, q-1\}$, then $u x_{i j} x_{i} x_{i k} v$ is a path of length 4 and from Remark $3.2 d_{\Gamma_{q}-H}(u, v) \geq 4$.
(5) If $u \in X_{i j}$ and $v \in X_{l k}$ for $i \neq l, i, l \in\{0, \ldots, q-2\}$ and $j, k \in\{0, \ldots, q-1\}$, then it is possible that there exist $w \in \Gamma_{q}-H$ such that $u, v \in N(w)$, that is $d_{\Gamma_{q}-H}(u, v) \geq 2$.

Let us consider $C$ a shortest cycle in $\Gamma_{q} 1$. If $E(C) \subset E\left(\Gamma_{q}-H\right)$ then $|C| \geq 8$. Suppose $C$ contains edges in $M=\bigcup_{Z \in \mathcal{Z}} M_{Z}$. If $C$ contains exactly one such edge, then by (1) $|C| \geq 7$. If $C$ contains exactly two edges $e_{1}, e_{2} \in M$, the following cases arise.
. If both $e_{1}, e_{2}$ lie in the same $M_{Z}$ then by (1) $|C| \geq 14>7$.
. If $e_{1} \in M_{X_{q-1}}$ and $e_{2} \in M_{X_{q}}$ then by (2) $|C| \geq 10>7$.
. If $e_{1} \in M_{X_{i}}$ and $e_{2} \in M_{X_{k j}}$ then by (3) $|C| \geq 8>7$.
. If $e_{1} \in M_{X_{i j}}$ and $e_{2} \in M_{X_{i k}}$ then by (4) $|C| \geq 10>7$.
. If $e_{1} \in M_{X_{i j}}$ and $e_{2} \in M_{X_{l k}}$, for $i \neq l$, by hypothesis $|C| \geq 7$.

If $C$ contains at least three edges of $M$, since $d(u, v) \geq 2$ for all $u, v \in\left\{X_{q-1}, X_{q}, X_{i j}\right\}$ with $i \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\},|C| \geq 9>7$.
Hence $\Gamma_{q} 1$ has girth 7 and we have finished the proof.

The following lemma gives sufficient conditions to define the matchings $M_{X_{i j}}$ for the sets $X_{i j}$, for $i \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\}$, in order to fulfill the condition from Proposition 3.3 .

Lemma 3.4 There exist $q^{2}-q$ matchings $M_{X_{i j}}$, for each $i \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\}$ with the following property:

Given $u_{1} v_{1} \in M_{X_{i j}}$ and $u_{2}, v_{2} \in X_{k j}$ such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$ then $u_{2} v_{2} \notin$ $M_{X_{k j}}$.

Proof By definition $\bigcap_{i=0}^{q-2} N\left(X_{i j}\right)=W_{j}$. Let $W_{j}=\left\{w_{j 1}, \ldots, w_{j q}\right\}$. Note that every vertex $w_{j h}$ is adjacent to exactly one vertex in $N\left(X_{i j}\right)$ that we will denote as $x_{i j h}$, for each $i \in\{0, \ldots, q-2\}$ and $j \in\{0, \ldots, q-1\}$.

Observe that $x_{i j h}$ is well defined, because if $x_{i j h}$ had two neighbors $w_{h}, w_{h^{\prime}} \in \bigcap_{i=0}^{q-2} N\left(X_{i j}\right)$, $\Gamma_{q}$ would contain the cycle $x_{i j h} w_{j h^{\prime}} x_{i^{\prime} j h^{\prime}} x_{i^{\prime} j} x_{i^{\prime} j h} w_{j h}$ of length 6.

Therefore, take the complete graph $K_{q}$ label its vertices as $h=1, \ldots, q$. We know that it has a 1-factorization with $q-1$ factors $F_{1}, \ldots, F_{q-1}$. For each $i=0, \ldots, q-2$, let $x_{i j h} x_{i j h^{\prime}} \in M_{X_{i j}}$ if and only if $h h^{\prime} \in F_{i}$.

To prove that the matchings $M_{X_{i j}}$ defined in this way fulfill the desired property suppose that $x_{i j h} x_{i j h^{\prime}} \in M_{X_{i j}}$ and $x_{i^{\prime} j h} x_{i^{\prime} j h^{\prime}} \in M_{X_{i^{\prime} j}}$ for $i^{\prime} \neq i$, then $F_{i}$ and $F_{i^{\prime}}$ would have the edge $h h^{\prime}$ in common contradicting that they are a factorization.

Therefore, there exist $q^{2}-q$ matchings $M_{X_{i j}}$ with the desired property.

To finish, notice that for $u_{1} v_{1} \in M_{X_{i j}}$ and $u_{2}, v_{2} \in X_{i^{\prime} j^{\prime}}$ with $j \neq j^{\prime}$ and possibly $i=i^{\prime}$, the distances $d\left(u_{1}, u_{2}\right)$ and $d\left(v_{1}, v_{2}\right)$ are at least 4. Then, counting the number of vertices of $\Gamma_{q} 1$ and using the Proposition 3.3 we have the following theorem.

Theorem 3.5 Let $q \geq 4$ be a power of two. Then there is a $(q+1)$-regular graph of girth 7 and order $2 q^{3}+q^{2}+2 q$.

## 4 Constructions of small $(q+1 ; 7)$-graphs for and odd prime power $q$.

In this section we will consider cages of even degree, that $\Gamma_{q}$ is a $(q+1,8)$-cage with $q$ an odd prime power. We proceed as before, but as will be evident from the proofs, the result is not as good as in the previous section.

We will delete a set $H$ of vertices of $\Gamma_{q}$ and add matchings $M_{Z}$ between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7. The sets $H$ and $M_{Z}$ are defined as follows.

Let $V=\{x, y\} \cup\left\{s_{0}, \ldots, s_{q}\right\}$ be the vertices of $K_{2, q+1}$.
Let $\widehat{K_{2, q+1}}$ be the graph obtained subdividing each edge of $K_{2, q+1}$.
Let $\Gamma_{q}$ be a graph containing a copy of $\widehat{K_{2, q+1}}$ as a subgraph and label its vertices as $H^{\prime}=\left\{x, y, s_{0}, \ldots, s_{q}\right\} \cup N(x) \cup N(y)$ where $N(x)=\left\{x_{0}, \ldots, x_{q}\right\}$ and $N(y)=\left\{y_{0}, \ldots, y_{q}\right\}$. Note
that $N\left(x_{i}\right) \cap N\left(y_{i}\right)=s_{i}$ for $i=0, \ldots, q$. Define:

$$
\begin{aligned}
H & =\left\{x, y, s_{3}, s_{4} \cdots, s_{q}\right\} \cup N(x) \cup N(y) \subset V\left(\Gamma_{q}\right) ; \\
X_{i} & =N\left(x_{i}\right) \cap V\left(\Gamma_{q}-H\right), \quad i=0, \ldots, q ; \\
Y_{i} & =N\left(y_{i}\right) \cap V\left(\Gamma_{q}-H\right), \quad i=0, \ldots, q ; \\
S_{i} & =N\left(s_{i}\right) \cap V\left(\Gamma_{q}-H\right), \quad i=3, \ldots, q .
\end{aligned}
$$

Notice that the vertices of $\Gamma_{q}-H$ have degrees $q-1, q$ and $q+1$. The vertices $s_{0}, s_{1}, s_{2}$ of degree $q-1$, those in $X_{i} \cup Y_{i} \cup S_{i}$ of degree $q$ and all the remaining vertices of $\Gamma_{q}-H$ have degree $q+1$. Therefore, in order to complete the degrees to such vertices its necessary to add edges to $\Gamma_{q}-H$, we define such edges next.

Let $\mathcal{Z}$ be the family of all $X_{i}, Y_{i}, S_{i}$. For each $Z \in \mathcal{Z}, M_{Z}$ will denote a perfect matching of $V(Z)$, which will eventually be added to $\Gamma_{q}$.


Definition 4.1 Let $\Gamma_{q}$ be a $(q+1,8)$-cage, with even degree $q+1 \geq 6$.

- Let $\Gamma_{q} 1$ be the graph with: $V\left(\Gamma_{q} 1\right):=V\left(\Gamma_{q}-H\right)$ and $E\left(\Gamma_{q} 1\right):=E\left(\Gamma_{q}-H\right) \cup \bigcup_{Z \in \mathcal{Z}} M_{Z}$.
- Define $\Gamma_{q} 2$ as $V\left(\Gamma_{q} 2\right):=V\left(\Gamma_{q} 1\right)$ and
$E\left(\Gamma_{q} 2\right):=\left(E\left(\Gamma_{q} 1\right) \backslash\left\{u_{0} v_{0}, u_{1} v_{1}, u_{2} v_{2}\right\}\right) \cup\left\{s_{0} u_{0}, s_{0} v_{0}, s_{1} u_{1}, s_{1} v_{1}, s_{2} u_{2}, s_{2} v_{2}\right\}$,
where $s_{i} \in H^{\prime}-H$, the deleted edges $u_{i} v_{i}$ belong to $M_{X_{i}}$ in $\Gamma_{q} 1$ and they are replaced by the paths of length two $u_{i} s_{i} v_{i}, i=0,1,2$.

By an immediate counting argument we know that the graph $\Gamma_{q} 1$ has order $\left|V\left(\Gamma_{q}\right)\right|-3(q+$ 1) +1 , and observe that all vertices in $\Gamma_{q} 1$ have degree $q+1$ except for $s_{0} s_{1}, s_{2}$ which remain of degree $q-1$. Hence, by the definition of $E\left(\Gamma_{q} 2\right)$, all vertices in $\Gamma_{q} 2$ are left with degree $q+1$.

Proposition 4.2 Let $\Gamma_{q}$ be a $(q+1,8)$-cage, with even degree $q \geq 5$ and $\Gamma_{q} 1, \Gamma_{q} 2$ be as in Definition 4.1.
(i) $\Gamma_{q} 1$ has girth 7 if the matchings $M_{S_{i}}, M_{X_{i}}$ and $M_{Y_{i}}$ have the following properties:
(a) Given $u_{1} v_{1} \in M_{S_{i}}$ and $u_{2}, v_{2} \in S_{j}$ such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$, it holds that $u_{2} v_{2} \notin M_{S_{j}}$.
(b) Given $u_{1} v_{1} \in M_{X_{i}}$ and $u_{2}, v_{2} \in Y_{j}$ such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$, it holds that $u_{2} v_{2} \notin M_{Y_{j}}$.
(ii) If conditions (a) and (b) hold then the graph $\Gamma_{q} 2$ also has girth 7.

Proof To prove ( $i$ ) let us consider the distances (in $\Gamma_{q}-H$ ) between the elements in the sets $Z \in \mathcal{Z}$. There are six possible cases:
(1) Two vertices in the same set $u, v \in Z$ have a common neighbor $w$ in $\Gamma_{q}$, therefore $d_{\Gamma_{q}-H}(u, v) \geq 6$.
(2) If $u \in X_{i}$ and $v \in X_{j}$ then $d_{\Gamma_{q}-H}(u, v) \geq 4$, given that $x_{i}, x_{j}$ have $x$ as a common neighbor in $\Gamma_{q}$.
(3) If $u \in Y_{i}$ and $v \in Y_{j}$ then $d_{\Gamma_{q}-H}(u, v) \geq 4$, as before.
(4) If $u \in S_{i}$ and $v \in S_{j}$ then it is possible that there exist $w \in \Gamma_{q}-H$ such that $u, v \in N(w)$, that is, $d_{\Gamma_{q}-H}(u, v) \geq 2$.
(5) If $u \in S_{i}$ and $v \in X_{j} \cup Y_{j}$ then $d_{\Gamma_{q}-H}(u, v) \geq 3$, since $s_{i} \in N\left(x_{i}\right) \cap N\left(y_{i}\right)$.
(6) If $u \in X_{i}$ and $v \in Y_{j}$ then $d_{\Gamma_{q}-H}(u, v) \geq 2$.

Let us consider $C$ a shortest cycle in $\Gamma_{q} 1$. If $E(C) \subset E\left(\Gamma_{q}-H\right)$ then $|C| \geq 8$. Suppose $C$ contains edges in $M=\bigcup_{Z \in \mathcal{Z}} M_{Z}$. If $C$ contains exactly one such edge, then by (1) $|C| \geq 7$. If $C$ contains exactly two edges $e_{1}, e_{2} \in M$, the following cases arise:
. If both $e_{1}, e_{2}$ lie in the same $M_{Z}$, then by (1) $|C| \geq 14>7$.
. If $e_{1} \in M_{X_{i}}$ and $e_{2} \in M_{X_{j}}$ for $i \neq j$, by (2) $|C| \geq 10>7$.
. If $e_{1} \in M_{Y_{i}}$ and $e_{2} \in M_{Y_{j}}$ for $i \neq j$, by (3) $|C| \geq 10>7$.
. If $e_{1} \in M_{S_{i}}$ and $e_{2} \in M_{X_{j}} \cup M_{Y_{j}}$, by (5) $|C| \geq 8>7$.
. If $e_{1} \in M_{S_{i}}$ and $e_{2} \in M_{S_{j}}$ for $i \neq j$, by the first hypothesis in item (i)(b) $|C| \geq 7$.
. If $e_{1} \in M_{X_{i}}$ and $e_{2} \in M_{Y_{j}}$, by the second hypothesis in item (i)(b) $|C| \geq 7$.

If $C$ contains at least three edges of $M$, since $d(u, v) \geq 2$ for all $u, v \in\left\{X_{i} \cup Y_{i}\right\}_{i=1}^{k} \cup\left\{S_{i}\right\}_{i=4}^{k}$, $|C| \geq 9>7$.
Hence $\Gamma_{q} 1$ has girth 7, concluding the proof of (i).
To prove (ii), let $C$ be a shortest cycle in $\Gamma_{q}$. If $E(C) \subset E\left(\Gamma_{q}-H\right) \cup M$ then $|C| \geq 7$.
. If $C$ contains exactly one edge $s_{i} u_{i}$ or $s_{i} v_{i}$ then $|C| \geq 7$ since $d_{\Gamma_{q}}\left(s_{i}, u_{i}\right)=d_{\Gamma_{q}}\left(s_{i}, v_{i}\right)=2$ which implies $d_{\Gamma_{q} 1}\left(s_{i}, u_{i}\right) \geq 6$ and $d_{\Gamma_{q} 1}\left(s_{i}, v_{i}\right) \geq 6$.
. If $C$ contains a path $u_{i} s_{i} v_{i}$ then $\left(C \backslash u_{i} s_{i} v_{i}\right) \cup u_{i} v_{i}$ is a cycle in $\Gamma_{q} 1$ with one less vertex than $C$, therefore $|C| \geq 8$.
. If $C$ contains two edges $s_{i} u_{i}, s_{j} u_{j}$, for $i \neq j$. Their distances $d_{\Gamma_{q} 1}\left(s_{i}, u_{j}\right) \geq 4, d_{\Gamma_{q} 1}\left(s_{i}, s_{j}\right) \geq$ 4 , and $d_{\Gamma_{q} 1}\left(u_{i}, u_{j}\right) \geq 4$, therefore in any case $C$ has length greater than 7 concluding the proof.

The following lemma gives sufficient conditions to define the matchings $M_{S_{i}}$ for the sets $S_{i}$, in order that they fulfill condition (a) from Proposition 4.2 (i). Notice that in the incidence graph of a generalized quadrangle $\{x, y\}^{\perp \perp}=\bigcap_{s \in N_{2}(x) \cap N_{2}(y)} N_{2}(s)$, thus Remark 2.1 implies that $\left|\bigcap_{i=0}^{q} N\left(S_{i}\right)\right|=q-1$, recalling that $\left\{s_{i}\right\}_{i=0}^{q}=N_{2}(x) \cap N_{2}(y)$. Since $\left|\bigcap_{i=0}^{q} N\left(S_{i}\right)\right|$ is contained in $\left|\bigcap_{i=3}^{q} N\left(S_{i}\right)\right|$, and $\left|\bigcap_{i=3}^{q} N\left(S_{i}\right)\right| \leq\left|S_{i}\right|=q-1$ then the condition for the following lemma holds.

Lemma 4.3 If $\left|\bigcap_{i=3}^{q} N\left(S_{i}\right)\right|=q-1$ then there exist matchings $M_{S_{i}}$, for $i=3, \ldots, q$, such that:

- Given $u_{1} v_{1} \in M_{S_{i}}$ and $u_{2}, v_{2} \in S_{j}$ such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$, it holds that $u_{2} v_{2} \notin M_{S_{j}}$.

Proof Let us suppose that $\bigcap_{i=3}^{q} N\left(S_{i}\right)=\left\{w_{1}, \ldots, w_{q-1}\right\}$, and since $S_{i}$ has $q-1$ vertices, every vertex $w_{j}$ is adjacent to exactly one vertex in $s_{i j} \in S_{i}$.

Observe that $s_{i j}$ is well defined, because if $s_{i j}$ had two neighbors $w_{j}, w_{j^{\prime}} \in \bigcap_{i=1}^{q+1} N\left(S_{i}\right), \Gamma_{q}$ would contain the cycle $\left(s_{i j} w_{j} s_{k j} s_{k} s_{k j^{\prime}} w_{j^{\prime}}\right)$ of length 6.

Therefore, take the complete graph $K_{q-1}$, label its vertices as $j=1, \ldots, q-1$. We know that it has a 1 -factorization with $q-2$ factors $F_{1}, \ldots, F_{q-2}$. For each $i=3, \ldots, q+1$, let $s_{i j} s_{i l} \in M_{S_{i}}$ if and only if $j l \in F_{i-3}$.

To prove that the matchings $M_{S_{i}}$ defined in this way fulfill the desired property suppose that $s_{i j} s_{i l} \in M_{S_{i}}$ and $s_{i^{\prime} j} j_{i^{\prime} l} \in M_{S_{i}^{\prime}}$ for $i^{\prime} \neq i$. Then $F_{i}$ and $F_{i^{\prime}}$ would have the edge $j l$ in common contradicting that they were a factorization.

So far, the steps of our construction have been independent from the coordinatization of the chosen ( $q+1,8$ )-cage, however, in order to define $M_{X_{i}}$ and $M_{Y_{i}}$ satisfying condition (b) of Lemma 4.2, we need to fix all the elements chosen so far.

We will distinguish two cases, when $q$ is a prime or when $q$ is a prime power.
Choose $x=(q, q, q)_{1}, y=(0,0,0)_{1}$.
When $q$ is a prime then $x_{i}=(q, q, i)_{0}, y_{i}=(i, 0,0)_{0}$ for $i=0, \ldots, q$.
Therefore, $N\left(x_{i}\right)=\left\{(q, t, i)_{1}: t=0, \ldots, q-1\right\} \cup x$ and $N\left(x_{q}\right)=\left\{(q, q, t)_{1}: t=0, \ldots q-1\right\} \cup x$; $N\left(y_{i}\right)=\left\{\left(t,-i t, i+t^{2}\right)_{1}: t=0, \ldots q-2\right\} \cup(q, 0, i)_{1}$ and $N\left(y_{q}\right)=\left\{(0, t, 0)_{1}: t=0, \ldots q-1\right\} \cup$ $(q, q, 0)_{1}$.

Thus, the corresponding vertices $s_{i}$ are: $s_{i}=(q, 0, i)_{1}$ for $i=0, \ldots q-1$ and $s_{q}=(q, q, 0)_{1}$; $N\left(s_{i}\right)=\left\{(i, 0, t)_{0}: t=1, \ldots, q-1, i=0, \ldots, q\right\} \cup\left\{x_{i}, y_{i}\right\}$. Hence, $S_{i}=\left\{(i, 0, t)_{0}: t=\right.$ $1, \ldots, q-1, i=0, \ldots, q\}$.

Then $N\left(S_{i}\right)=\left\{(a, b, c)_{1}: b=-i a, c=t+a^{2} i, i=0, \ldots, q-1\right\}$, and $N\left(S_{q}\right)=\left\{(q, 0, t)_{1}: t=\right.$ $0, \ldots, q-1\}$.

Solving the equations we obtain $N\left(S_{i}\right) \cap N\left(S_{j}\right)=\left\{(0,0, t)_{1}: t=0, \ldots, q-1\right\}$, moreover $N(i, 0, t)_{0} \cap N(j, 0, t)_{0}=(0,0, t)_{1}$, for each $j \neq i$ and $t=0, \ldots, q-1$, or equivalently, $N(0,0, t)_{1}=$ $\left\{(x, 0, t)_{0}: t=0, \ldots, q-1, x=0, \ldots, q\right\}$. Hence the sets $S_{i}$ satisfy the hypothesis of Lemma 4.3, yielding that there exist the matchings $M_{S_{i}}$ with the desired property.

Notice that the sets $X_{i}$ and $Y_{i}$ are naturally defined as the sets $X_{i}=\left\{(q, t, i)_{1}: t=1, \ldots, q-\right.$ $1, i=0, \ldots, q-1\}, X_{0}=\left\{(q, t, 0)_{1}: t=1, \ldots, q-1\right\}$ and $X_{q}=\left\{(q, q, t)_{1}: t=1, \ldots, q-1\right\}$. The sets $Y_{i}=\left\{\left(t,-i t, i t^{2}\right)_{1}: t=1, \ldots, q-1, i=0, \ldots, q-1\right\}$, and $Y_{q}=\left\{(0, t, 0)_{1}: t=1, \ldots, q-1\right\}$.

In this way we have defined all the sets in Lemma 4.2, and from Lemma 4.3 we know that the matchings $M_{S_{i}}$ have the property that:

- If $u_{1} v_{1} \in M_{S_{i}}$ and $u_{2}, v_{2} \in S_{j}$ are such that $d\left(u_{1}, u_{2}\right)=2$ and $d\left(v_{1}, v_{2}\right)=2$ then $u_{2} v_{2} \notin$ $M_{S_{j}}$.

It remains to define the matchings $M_{X_{i}}$ and $M_{Y_{i}}$ and prove they have property (b) from Proposition 4.2 (i).

For this we must analyze the intersection of the second neighborhood of an $X_{j}$ with an $Y_{i}$, $N_{2}\left(X_{j}\right) \cap Y_{i}$. For each $w \in Y_{i}$, we know there is exactly one $z \in X_{q}$ such that $w \in N_{2}(z)$.

This allows us to define the following sets of latin squares: For each $j$, let the coordinate $i \ell$ of the $j$-th latin square to have the symbol $s_{i \ell_{j}}$ if there is a $w_{i \ell j}=(a, b, c)_{1}$ such that

$$
w_{i \ell j} \in N\left((i, 0,0)_{0}\right) \cap N_{2}\left((q, \ell, j)_{1}\right) \cap N_{2}\left(\left(q, q, s_{i \ell j}\right)_{1}\right),
$$

where $(i, 0,0)_{0}=y_{i},(q, \ell, j)_{1} \in X_{j}$ and $\left(q, q, s_{i \ell j}\right)_{1} \in X_{q}$.

Since $N\left((i, 0,0)_{0}\right)=\left\{\left(t,-i t, i+t^{2}\right)_{1}: t=0, \ldots q-2\right\} \cup(q, 0, i)_{1}$, then $a=t, b=-i t$, and $c=i+t^{2}$.

Observe that $w_{i \ell j} \in N_{2}\left((q, \ell, j)_{1}\right)$ is equivalent to $(j, \ell, t)_{0} \in N\left((a, b, c)_{1}\right)$, since $N\left((q, \ell, j)_{1}\right)=$ $\left\{(j, \ell, t)_{0}: t=0, \ldots q-1\right\} \cup\left\{(q, q, j)_{0}\right\}$. Hence, $a j+b=\ell$.

And $w_{i \ell j} \in N_{2}\left(\left(q, q, s_{i \ell j}\right)_{1}\right)$ implies $a=s_{i \ell j}$.
Therefore we obtain the following equation for $s_{i \ell j}$.

$$
s_{i \ell j}(j-i)=\ell
$$

Notice that this equation is undefined for $j=i$, otherwise it would mean that $y_{i}$ has a neighbor at distance 3 from $x_{j}$ and this would imply the existence of a cycle of length 6 in $\Gamma_{q}$.

Also from the equation we deduce that $-s_{i \ell j}=s_{i-\ell j}$, and $s_{i+\ell \ell_{j+1}}=s_{i \ell j}$. This means that the $i+1$-th row of the $j+1$-th latin square is equal to the $i$-th row of the $j$-th latin square, hence all the set of latin squares have the same rows. This also implies that if we put an edge between two vertices on $Y_{i},\left(s_{i \ell j},-i s_{i \ell j}, i s_{i \ell j}^{2}\right)_{1}$ and $\left(-s_{i \ell j}, i s_{i \ell j}, i s_{i \ell j}^{2}\right)_{1}$, it will have at distance two in $X_{j}$ only the vertices $(q, \ell, i)_{1}$ and $(q,-\ell, i)_{1}$.

Therefore, the matchings $M_{X_{i}}=\left\{(q, \ell, i)_{1}(q,-(\ell+2), i)_{1}: i=0, \ldots q-1, \ell=1, \ldots, q-3\right\} \cup$ $\left\{(q,-2, i)_{1}(q,-1, i)_{1}: i=0, \ldots q-1\right\}, M_{X_{q}}=\left\{(q, q, \ell)_{1}(q, q,-(\ell+2))_{1}: \ell=1, \ldots, q-3\right\} \cup$ $\left\{(q, q,-2)_{1}(q, q,-1)_{1}\right\}$, and $M_{Y_{i}}=\left\{\left(t,-i t, i t^{2}\right)_{1}\left(-t, i t, i t^{2}\right)_{1}: i=0, \ldots, q-1, t=1, \ldots, q-1\right\}$, have the property (b) from Proposition $4.2(i)$.

When $q$ is a prime power, let $\alpha$ a primitive root of unity in $G F(q)$. Then, $x_{i}=\left(q, q, \alpha^{i-1}\right)_{0}$, $y_{i}=\left(\alpha^{i-1}, 0,0\right)_{0}$ for $i=1, \ldots q-1, x_{0}=(q, q, 0)_{0}$, and $y_{0}=(0,0,0)_{0}$. Moreover, $x_{q}=(q, q, q)_{0}$ and $y_{q}=(q, 0,0)_{0}$.

Therefore, $N\left(x_{i}\right)=\left\{\left(q, \alpha^{t}, \alpha^{i-1}\right)_{1}: t=0, \ldots q-2\right\} \cup\left(q, 0, \alpha^{i-1}\right)_{1} \cup x$ and $N\left(x_{0}\right)=\left\{\left(q, \alpha^{t}, 0\right)_{1}:\right.$ $t=0, \ldots q-2\} \cup(q, 0,0)_{1} \cup x ; N\left(y_{i}\right)=\left\{\left(\alpha^{t},-\alpha^{i-1+t}, \alpha^{i-1+2 t}\right)_{1}: t=0, \ldots q-2\right\} \cup\left(q, 0, \alpha^{i-1}\right)_{1}$ and $N\left(y_{0}\right)=\left\{\left(\alpha^{t}, 0,0\right)_{1}: t=0, \ldots q-2\right\} \cup(q, 0,0)_{1} ; N\left(x_{q}\right)=\left\{\left(q, q, \alpha^{t}\right)_{1}: s=0, \ldots q-2\right\} \cup$ $(q, q, 0)_{1} \cup x$; and $N\left(y_{q}\right)=\left\{\left(0, \alpha^{t}, 0\right)_{1}: t=0, \ldots q-2\right\} \cup(q, q, 0)_{1} \cup y$.

Thus, the corresponding vertices $s_{i}$ are: $s_{i}=\left(q, 0, \alpha^{i-1}\right)_{1}$, for $i=1, \ldots q-1, s_{0}=(q, 0,0)_{1}$ and $s_{q}=(q, q, 0)_{1} ; N\left(s_{i}\right)=\left\{\left(\alpha^{i-1}, 0, \alpha^{t}\right)_{0}: t=0, \ldots q-2, i=1, \ldots, q-1\right\} \cup\left\{x_{i}, y_{i}\right\}$, and $N\left(s_{0}\right)=\left\{\left(0,0, \alpha^{t}\right)_{0}: t=0, \ldots q-2\right\} \cup\left\{x_{0}, y_{0}\right\}$. Hence $S_{i}=\left\{\left(\alpha^{i-1}, 0, \alpha^{t}\right)_{0}: t=0, \ldots q-2, i=\right.$ $0, \ldots, q\}$ and $S_{0}=\left\{\left(0,0, \alpha^{t}\right)_{0}: t=0, \ldots q-2\right\}$.

Then $N\left(S_{i}\right)=\left\{(a, b, c)_{1}: b=-\alpha^{i-1} a, c=\alpha^{t}+a^{2} \alpha^{i-1}, i=1, \ldots, q-1\right\}, N\left(S_{0}\right)=\left\{(a, b, c)_{1}:\right.$ $\left.b=0, c=\alpha^{t}\right\}$ and $N\left(S_{q}\right)=\left\{\left(q, 0, \alpha^{t}\right)_{1}: t=0, \ldots, q-2\right\} \cup(q, 0,0)_{1}$.

Solving the equations we obtain $N\left(S_{i}\right) \cap N\left(S_{j}\right)=\left\{\left(0,0, \alpha^{t}\right)_{1}: t=0, \ldots, q-2\right\}$. Moreover, $N\left(\alpha^{i-1}, 0, \alpha^{t}\right)_{0} \cap N\left(\alpha^{j-1}, 0, \alpha^{t}\right)_{0}=\left(0,0, \alpha^{t}\right)_{1}$, for each $j \neq i$ and $t=0, \ldots, q-2$, or equivalently, $N\left(0,0, \alpha^{t}\right)_{1}=\left\{\left(\alpha^{x}, 0, \alpha^{t}\right)_{0}: x=0, \ldots, q-2\right\} \cup\left(0,0, \alpha^{t}\right)_{0} \cup\left(q, 0, \alpha^{t}\right)_{0}$, for each $t=0, \ldots, q-2$. Hence the sets $S_{i}$ satisfy the hypothesis of Lemma 4.3 yielding that there exist the matchings
$M_{S_{i}}$ with the desired property.
Notice that the sets $X_{i}$ and $Y_{i}$ are naturally defined as the sets $X_{i}=\left\{\left(q, \alpha^{t}, \alpha^{i-1}\right)_{1}: t=\right.$ $0, \ldots, q-2, i=1, \ldots, q-1\}, X_{0}=\left\{\left(q, \alpha^{t}, 0\right)_{1}: t=0, \ldots, q-2\right\}$ and $X_{q}=\left\{\left(q, q, \alpha^{t}\right)_{1}: t=\right.$ $0, \ldots q-2\}$. The sets
$Y_{i}=\left\{\left(\alpha^{t},-\alpha^{i-1+t}, \alpha^{i-1+2 t}\right)_{1}: t=0, \ldots, q-2\right\}, Y_{0}=\left\{\left(\alpha^{t}, 0,0\right)_{1}: t=0, \ldots, q-2\right\}$ and $Y_{q}=\left\{\left(0, \alpha^{t}, 0\right)_{1}: t=0, \ldots, q-2\right\}$.

In order to define the matchings $M_{X_{i}}$ and $M_{Y_{i}}$ and prove that they have the property (b) from Proposition $4.2(i)$, we proceed as before, by defining the sets of latin squares:

For each $j$, let the coordinate $i \ell$ of the $j$-th latin square to have the symbol $s_{i \ell j} \in\{0, \ldots, q-2\}$ if there is a $w_{i \ell j}=(a, b, c)_{1}$ such that

$$
w_{i \ell j} \in N\left(\left(\alpha^{i-1}, 0,0\right)_{0}\right) \cap N_{2}\left(\left(q, \alpha^{\ell}, \alpha^{j-1}\right)_{1}\right) \cap N_{2}\left(\left(q, q, \alpha^{s_{i \ell j}}\right)_{1}\right) \text { for } i, j \geq 1 \text {, }
$$

where $\left(\alpha^{i-1}, 0,0\right)_{0}=y_{i},\left(q, \alpha^{\ell}, \alpha^{j-1}\right)_{1} \in X_{j}$ and $\left(q, q, \alpha^{s_{i} e_{j}}\right)_{1} \in X_{q}$.
Since $N\left(\left(\alpha^{i-1}, 0,0\right)_{0}\right)=\left\{\left(\alpha^{t},-\alpha^{i-1+t}, \alpha^{i-1+2 t}\right)_{1}: t=0, \ldots q-2\right\} \cup(q, 0, i)_{1}$, then $a=\alpha^{t}$, $b=-\alpha^{i-1+t}$, and $c=\alpha^{i-1+2 t}$.

Also $w_{i \ell j} \in N_{2}\left(\left(q, \alpha^{\ell}, \alpha^{j-1}\right)_{1}\right)$ is equivalent to $\left(\alpha^{j-1}, \alpha^{\ell}, \alpha^{t}\right)_{0} \in N\left((a, b, c)_{1}\right)$, since $N\left(\left(q, \alpha^{\ell}, \alpha^{j-1}\right)_{1}\right)=\left\{\left(\alpha^{j-1}, \alpha^{\ell}, \alpha^{t}\right)_{0}: t=0, \ldots q-2\right\}$. Hence $a \alpha^{j-1}+b=\alpha^{\ell}$.

And $w_{i \ell j} \in N_{2}\left(\left(q, q, \alpha^{s_{i \ell j}}\right)_{1}\right)$ implies $a=\alpha^{s_{i \ell j}}$.
Therefore we obtain the following equation for $s_{i \ell j}$.

$$
\alpha^{s_{i \ell j}}\left(\alpha^{j-1}-\alpha^{i-1}\right)=\alpha^{\ell}
$$

Notice that this equation is undefined for $j=i$, otherwise it would mean that $y_{i}$ has a neighbor at distance 3 from $x_{j}$ and this would imply the existence of a cycle of length 6 in $\Gamma_{q}$.

For $i=0$, we obtain the equation $\alpha^{s_{0 \ell j}}\left(\alpha^{j-1}\right)=\alpha^{\ell}$, and for $j=0$, we obtain $\alpha^{s_{i \ell 0}}\left(-\alpha^{i-1}\right)=$ $\alpha^{\ell}$. From the equation we obtain that $s_{i \ell+1 j}=s_{i \ell j}+1$, and each latin square is the sum table of the cyclic group $\mathbb{Z}_{q-1}$ with the rows permuted.

Multiplying by $\alpha$ the equation $\alpha^{s_{i-1 j}}\left(\alpha^{j-1}-\alpha^{i-1}\right)=\alpha^{\ell-1}$, we obtain that $s_{i+1 \ell j+1}=s_{i \ell-1 j}$. This implies that the row $i+1$ of the $j+1$-th latin square is equal to the row $i$ of the $j$-th latin square subtracting 1 to each symbol (i.e., $s_{i+1 \ell j+1}+1=s_{i \ell j}$ ). That is, all the set of latin squares have the same rows but in a different order.

This also implies that if we put an edge between two vertices on $Y_{i},\left(\alpha^{s_{i \ell j}},-\alpha^{i-1+s_{i \ell j}}, \alpha^{i-1+2 s_{i \ell j}}\right)_{1}$ and $\left(\alpha^{s_{i \ell j}+1},-\alpha^{i-1+\left(s_{i \ell j}+1\right)}, \alpha^{i-1+2\left(s_{i} \ell_{j}+1\right)}\right)_{1}$, it will have at distance two in $X_{j}$ only the vertices, $\left(q, \alpha^{\ell}, i\right)_{1}$ and $\left(q, \alpha^{\ell+1}, i\right)_{1}$ and the other way around.

Therefore, the matchings $M_{X_{i}}=\left\{\left(q, \alpha^{2 \ell}, i\right)_{1}\left(q, \alpha^{2 \ell+1}, i\right)_{1}: i=0, \ldots q-1, \ell=\right.$
$1, \ldots,(q-1) / 2\}, M_{X_{q}}=\left\{\left(q, q, \alpha^{2 \ell}\right)_{1}\left(q, q, \alpha^{2 \ell+1}\right)_{1}: \ell=1, \ldots,(q-1) / 2\right\}$, and $M_{Y_{i}}=$ $\left\{\left(\alpha^{2 t},-\alpha^{i-1+2 t}, \alpha^{i-1+4 t}\right)_{1}\left(\alpha^{2 t+3},-\alpha^{i-1+(2 t+3)}, \alpha^{i-1+2(2 t+3)}\right)_{1}: i=0, \ldots q-1, t=1, \ldots,(q-1) / 2\right\}$ have the property (b) from Proposition $4.2(i)$, proving the theorem for $q$ prime power.

Theorem 4.4 Let $q \geq 5$ be a prime power. Then there is a $q+1$-regular graph of girth 7 and order $2 q^{3}+2 q^{2}-q+1$.

Proof Finally, by applying Lemma 4.2(ii), we obtain a $q+1$-regular graph of girth 7 with $2\left(q^{3}+q^{2}+q+1\right)-(q-3+2(q+2))=2 q^{3}+2 q^{2}-q+1$ vertices.

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