Families of small regular graphs of girth 7

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Abstract

The first known families of cages arised from the incidence graphs of generalized polygons of order q, q a prime power. In particular, (q + 1, 6)-cages have been obtained from the projective planes of order q. Morever, infinite families of small regular graphs of girth 5 have been constructed performing algebraic operations on \mathbb{F}_q .

In this paper, we introduce some combinatorial operations to construct new infinite families of small regular graphs of girth 7 from the (q + 1, 8)-cages arising from the generalized quadrangles of order q, q a prime power.

Keywords: Cages, girth, generalized quadrangles, latin squares.

1 Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [13], [20] and [25].

Let G be a graph with vertex set V = V(G) and edge set E = E(G). The girth of a graph G is the number g = g(G) of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the neighbourhood of v, that is, the set of all vertices adjacent to v. The degree of a vertex $v \in V$ is the cardinality of $N_G(v)$. A graph is called regular if all the vertices have the same degree.

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A (k,g)-graph is a k-regular graph with girth g. Erdős and Sachs [15] proved the existence of (k,g)-graphs for all values of k and g provided that $k \ge 2$. Thus most work carried out has focused on constructing a smallest one [2, 3, 4, 5, 6, 7, 9, 14, 16, 18, 19, 23, 24, 27, 28, 32]. A (k,g)-cage is a k-regular graph with girth g having the smallest possible number of vertices. Cages have been studied intensely since they were introduced by Tutte [33] in 1947. Counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound $n_0(k,g)$ with the precise form of the bound depending on whether g is even or odd:

$$n_0(k,g) = \begin{cases} 1+k+k(k-1)+\dots+k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1+(k-1)+\dots+(k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$
(1)

Biggs [11] calls the *excess* of a (k, g)-graph G the difference $|V(G)| - n_0(k, g)$. The construction of graphs with small excess is a difficult task. Biggs is the author of a report on distinct methods for constructing cubic cages [12]. More details about constructions of cages can be found in the survey by Wong [36] or in the book by Holton and Sheehan [22] or in the more recent dynamic cage survey by Exoo and Jajcay [17].

A (k,g)-cage with $n_0(k,g)$ vertices and even girth exist only when $g \in \{4, 6, 8, 12\}$ [18]. If g = 4 they are the complete bipartite graph $K_{k,k}$, and for g = 6, 8, 12 these graphs are the incidence graphs of generalized g/2-gons of order k - 1. This is the main reason for (k,g)-cages with $n_0(k,g)$ vertices and even girth g are called generalized polygon graphs [11]. In particular a 3-gon of order k - 1 is also known as a projective plane of order k - 1. The 4-gons of order k - 1 are called generalized quadrangles of order k - 1, and, the 6-gons of order k - 1, generalized hexagons of order k - 1. All these objets are known to exist for all prime power values of k - 1 [8, 20, 25], and no example is known when k - 1 is not a prime power.

In this article we focus on the case g = 8. Let q be a prime power. Our main objective is to give an explicit construction of small (q + 1, 7)-graphs for k = q + 1. Next we present the contributions of this paper and in the following sections we do the corresponding proofs.

2 Preliminaries

It is well known [30, 26] that Q(4,q) and W(3,q) are the only two classical generalized quadrangles with parameters s = t = q. The generalized quadrangle W(3,q) is the dual generalized of Q(4,q), and they are selfdual for q even.

In 1966 Benson [9] constructed (q + 1, 8)-cages from the generalized quadrangle Q(4, q). He defined the point/line incidence graph Γ_q of Q(4, q) which is a (q + 1)-regular graph of girth 8 with $n_0(q + 1, 8)$ vertices. Hence, Γ_q is a (q + 1, 8)-cage. Note that, Γ_q is isomorphic to the point/line incidence graph of W(3, q).

For any generalized quadrangle Q of order (s,t) and every point x of Q, let x^{\perp} denote the set of all points collinear with x. Note that in the incidence graph $x^{\perp} = N_2(x)$, with an abuse

of notation supposing that $x \in \Gamma_q$ corresponds to the point $x \in Q$

If X is a nonempty set of vertices of Q, then we define $X^{\perp} := \bigcap_{x \in X} x^{\perp}$. The span of the pair (x, y) is $sp(x, y) = \{x, y\}^{\perp \perp} = \{u \in P : u \in z^{\perp} \forall z \in x^{\perp} \cap y^{\perp}\}$, where P denotes the set of points in Q. If x and y are not collinear, then $\{x, y\}^{\perp \perp}$ is also called the *hyperbolic line* through x and y. If the *hyperbolic line* through two noncollinear points x and y contains precisely t + 1 points, then the pair (x, y) is called *regular*. A point x is called *regular* if the pair (x, y) is regular for every point y not collinear with x. It is important to recall that the concept of regular also exists for a graph to avoid confusion. Hence we will emphasize when regular refers to a point or a graph.

Remark 2.1 [31] Every point in W(q) is regular.

There are several equivalent coordinatizations of these generalized quadrangles (cf. [29], [34], [35], see also [26]) each giving a labeling for the graph Γ_q . Now we present a further labeling of Γ_q , equivalent to previous ones (cf. [1]), which will be central for our constructions since it allows us to keep track of the properties (such as regularity and girth) of the small regular graphs of girth 7 obtained from Γ_q .

Definition 2.2 Let \mathbb{F}_q be a finite field with $q \ge 2$ a prime power. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_r = \{(a, b, c)_r, (q, q, a)_r : a \in \mathbb{F}_q \cup \{q\}, b, c \in \mathbb{F}_q\}, r = 0, 1, and edge set defined as follows:$

For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a,b,c)_1) = \begin{cases} \{(x, ax+b, a^2x+2ab+c)_0 : x \in \mathbb{F}_q\} \cup \{(q,a,c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c,b,x)_0 : x \in \mathbb{F}_q\} \cup \{(q,q,c)_0\} & \text{if } a = q. \end{cases}$$

 $N_{\Gamma_q}((q,q,a)_1) = \{(q,a,x)_0 : x \in \mathbb{F}_q\} \cup \{(q,q,q)_0\}.$

Note that, in the labeling introduced in Definition 2.2, the second q in $\mathbb{F}_q \cup \{q\}$, usually denoted by ∞ , is meant to be just a symbol and no operations will be performed with it.

To finish, we define a *Latin square* as an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

In the following two sections we present our results only for (q + 1, 8)-cages, but all preliminary results are valid for all (k, 8)-cages with any number k given that they have the required combinatorial properties.

3 Constructions of small (q + 1; 7)-graphs, for an even prime power q

In this section we will consider a (q+1, 8)-cage Γ_q with $q+1 \ge 5$ an odd integer, since the only known (q+1, 8)-cages are obtained as the incidence graph of a Generalized Quadrangles, we let $q \ge 4$ a power of two.

Let $x \in V(\Gamma_q)$ and let $N(x) = \{x_0, ..., x_q\}$, label $N(x_i) = \{x_{i0}, x_{i1}, ..., x_{iq} = x\}$, for all $i \in \{0, ..., q\}$, in the following way. Take x_{0j} and x_{1j} arbitrarily for j = 0, ..., q - 1 and let $N_2(x_{0j}) \cap N_2(x_{1j}) - x = W_j$, note that $|W_j| = q$. Let $x_{ij} = (\bigcap_{w \in W_j} N_2(w)) \cap N(x_i)$, these vertices

exist and are uniquely labeled since the generalized quadrangle W(q) is regular.

Let
$$H = x \cup N(x) \cup \{x_{q-1}, x_q\} \cup \bigcup_{0}^{q-2} N(x_i) \subset V(\Gamma_q).$$

We will delete the set H of vertices of Γ_q and add matchings M_Z between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7. In order to define the sets M_Z , we denote $X_i = N(x_i) \setminus \{x\}$ and $X_{ij} = N(x_{ij}) \setminus \{x_i\}$, for $i \in \{0, ..., q\}$ and $j \in \{0, ..., q - 1\}$.

Let \mathcal{Z} be the family of all $X_{q-1}X_q, X_{ij}$ for $i \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}$. For each $Z \in \mathcal{Z}, M_Z$ will denote a perfect matching of V(Z), which will eventually be added to Γ_q .

Definition 3.1 Let Γ_q be a (q+1, 8)-cage, with odd degree $q+1 \ge 5$.

Let
$$\Gamma_q 1$$
 be the graph with: $V(\Gamma_q 1) := V(\Gamma_q - H)$ and $E(\Gamma_q 1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.

Observe that the graph $\Gamma_q 1$ has order $|V(\Gamma_q)| - (q^2 + 2)$ and all its vertices have degree q + 1.

Next proposition states a condition for the graph $\Gamma_q 1$ to have girth 7, for this it is useful to state the following remark.

Remark 3.2 Let $u, v \in V(\Gamma_q)$ a graph of girth 8, such that there is a uv-path P of length t < 8. Then every uv-path P' such that $E(P) \cap E(P') = \emptyset$ has length $|E(P')| \ge 8 - t$.

Proposition 3.3 Let Γ_q be a (q+1,8)-cage, with odd degree $q+1 \ge 5$ and $\Gamma_q 1$ as in Definition 3.1. Then $\Gamma_q 1$ has girth 7 if given $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{kl}$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds $u_2v_2 \notin M_{X_{kl}}$, for $i \ne k \in \{0, ..., q-2\}$ and $j, l \in \{0, ..., q-1\}$.

Proof

Let us consider the distances (in $\Gamma_q - H$) between the elements in the sets $Z \in \mathcal{Z}$. There are five possible cases:

(1) Two vertices in the same set $u, v \in Z$ have a common neighbor w in Γ_q , therefore $d_{\Gamma_q-H}(u,v) \geq 6$.

(2) If $u \in X_{q-1}$ and $v \in X_q$, then $d_{\Gamma_q-H}(u,v) \ge 4$, since x_{q-1}, x_q have x as a common neighbor in Γ_q .

(3) If $u \in X_i$ for $i \in \{q-1,q\}$ and $v \in X_{kj}$ for $k \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}$ then $d_{\Gamma_q}(u, x_i) = 1$, $d_{\Gamma_q}(v, x_k) = 2$, and x_i, x_k have a common neighbor $x \in V(\Gamma_q)$, hence there is a *uv*-path of length 5 in Γ_q , concluding from Remark 3.2 that $d_{\Gamma_q}(u, v) \geq 3$.

(4) If $u \in X_{ij}$ and $v \in X_{ik}$ for $i \in \{0, ..., q-2\}$ and $j, k \in \{0, ..., q-1\}$, then $ux_{ij}x_ix_{ik}v$ is a path of length 4 and from Remark 3.2 $d_{\Gamma_q-H}(u, v) \ge 4$.

(5) If $u \in X_{ij}$ and $v \in X_{lk}$ for $i \neq l$, $i, l \in \{0, ..., q-2\}$ and $j, k \in \{0, ..., q-1\}$, then it is possible that there exist $w \in \Gamma_q - H$ such that $u, v \in N(w)$, that is $d_{\Gamma_q - H}(u, v) \geq 2$.

Let us consider C a shortest cycle in $\Gamma_q 1$. If $E(C) \subset E(\Gamma_q - H)$ then $|C| \ge 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (1) $|C| \ge 7$. If C contains exactly two edges $e_1, e_2 \in M$, the following cases arise.

- . If both e_1, e_2 lie in the same M_Z then by (1) $|C| \ge 14 > 7$.
- . If $e_1 \in M_{X_{q-1}}$ and $e_2 \in M_{X_q}$ then by (2) $|C| \ge 10 > 7$.
- . If $e_1 \in M_{X_i}$ and $e_2 \in M_{X_{kj}}$ then by (3) $|C| \ge 8 > 7$.
- . If $e_1 \in M_{X_{ij}}$ and $e_2 \in M_{X_{ik}}$ then by (4) $|C| \ge 10 > 7$.
- . If $e_1 \in M_{X_{ij}}$ and $e_2 \in M_{X_{lk}}$, for $i \neq l$, by hypothesis $|C| \geq 7$.

If C contains at least three edges of M, since $d(u, v) \ge 2$ for all $u, v \in \{X_{q-1}, X_q, X_{ij}\}$ with $i \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}, |C| \ge 9 > 7$. Hence $\Gamma_q 1$ has girth 7 and we have finished the proof.

The following lemma gives sufficient conditions to define the matchings $M_{X_{ij}}$ for the sets X_{ij} , for $i \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}$, in order to fulfill the condition from Proposition 3.3.

Lemma 3.4 There exist $q^2 - q$ matchings $M_{X_{ij}}$, for each $i \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}$ with the following property:

Given $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{kj}$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$ then $u_2v_2 \notin M_{X_{kj}}$.

Proof By definition $\bigcap_{i=0}^{q-2} N(X_{ij}) = W_j$. Let $W_j = \{w_{j1}, \dots, w_{jq}\}$. Note that every vertex w_{jh} is adjacent to exactly one vertex in $N(X_{ij})$ that we will denote as x_{ijh} , for each $i \in \{0, ..., q-2\}$ and $j \in \{0, ..., q-1\}$.

Observe that x_{ijh} is well defined, because if x_{ijh} had two neighbors $w_h, w_{h'} \in \bigcap_{i=0}^{q-2} N(X_{ij})$, Γ_q would contain the cycle $x_{ijh}w_{jh'}x_{i'jh'}x_{i'jh}w_{jh}$ of length 6.

Therefore, take the complete graph K_q label its vertices as $h = 1, \ldots, q$. We know that it has a 1-factorization with q - 1 factors F_1, \ldots, F_{q-1} . For each $i = 0, \ldots, q-2$, let $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ if and only if $hh' \in F_i$.

To prove that the matchings $M_{X_{ij}}$ defined in this way fulfill the desired property suppose that $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ and $x_{i'jh}x_{i'jh'} \in M_{X_{i'j}}$ for $i' \neq i$, then F_i and $F_{i'}$ would have the edge hh'in common contradicting that they are a factorization.

Therefore, there exist $q^2 - q$ matchings $M_{X_{ij}}$ with the desired property.

To finish, notice that for $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{i'j'}$ with $j \neq j'$ and possibly i = i', the distances $d(u_1, u_2)$ and $d(v_1, v_2)$ are at least 4. Then, counting the number of vertices of $\Gamma_q 1$ and using the Proposition 3.3 we have the following theorem.

Theorem 3.5 Let $q \ge 4$ be a power of two. Then there is a (q+1)-regular graph of girth 7 and order $2q^3 + q^2 + 2q$.

4 Constructions of small (q + 1; 7)-graphs for and odd prime power q.

In this section we will consider cages of even degree, that Γ_q is a (q+1, 8)-cage with q an odd prime power. We proceed as before, but as will be evident from the proofs, the result is not as good as in the previous section.

We will delete a set H of vertices of Γ_q and add matchings M_Z between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7. The sets H and M_Z are defined as follows.

Let $V = \{x, y\} \cup \{s_0, \dots, s_q\}$ be the vertices of $K_{2,q+1}$.

Let $\widehat{K_{2,q+1}}$ be the graph obtained subdividing each edge of $K_{2,q+1}$.

Let Γ_q be a graph containing a copy of $\widehat{K_{2,q+1}}$ as a subgraph and label its vertices as $H' = \{x, y, s_0, \ldots, s_q\} \cup N(x) \cup N(y)$ where $N(x) = \{x_0, \ldots, x_q\}$ and $N(y) = \{y_0, \ldots, y_q\}$. Note

that $N(x_i) \cap N(y_i) = s_i$ for $i = 0, \ldots, q$. Define:

$$\begin{split} H &= \{x, y, s_3, s_4 \cdots, s_q\} \cup N(x) \cup N(y) \subset V(\Gamma_q); \\ X_i &= N(x_i) \cap V(\Gamma_q - H), \quad i = 0, \dots, q; \\ Y_i &= N(y_i) \cap V(\Gamma_q - H), \quad i = 0, \dots, q; \\ S_i &= N(s_i) \cap V(\Gamma_q - H), \quad i = 3, \dots, q. \end{split}$$

Notice that the vertices of $\Gamma_q - H$ have degrees q - 1, q and q + 1. The vertices s_0, s_1, s_2 of degree q - 1, those in $X_i \cup Y_i \cup S_i$ of degree q and all the remaining vertices of $\Gamma_q - H$ have degree q + 1. Therefore, in order to complete the degrees to such vertices its necessary to add edges to $\Gamma_q - H$, we define such edges next.

Let \mathcal{Z} be the family of all X_i, Y_i, S_i . For each $Z \in \mathcal{Z}, M_Z$ will denote a perfect matching of V(Z), which will eventually be added to Γ_q .



Definition 4.1 Let Γ_q be a (q+1, 8)-cage, with even degree $q+1 \ge 6$.

- Let $\Gamma_q 1$ be the graph with: $V(\Gamma_q 1) := V(\Gamma_q H)$ and $E(\Gamma_q 1) := E(\Gamma_q H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.
- Define $\Gamma_q 2$ as $V(\Gamma_q 2) := V(\Gamma_q 1)$ and $E(\Gamma_q 2) := (E(\Gamma_q 1) \setminus \{u_0 v_0, u_1 v_1, u_2 v_2\}) \cup \{s_0 u_0, s_0 v_0, s_1 u_1, s_1 v_1, s_2 u_2, s_2 v_2\},$ where $s_i \in H' - H$, the deleted edges $u_i v_i$ belong to M_{X_i} in $\Gamma_q 1$ and they are replaced by the paths of length two $u_i s_i v_i$, i = 0, 1, 2.

By an immediate counting argument we know that the graph $\Gamma_q 1$ has order $|V(\Gamma_q)| - 3(q + 1) + 1$, and observe that all vertices in $\Gamma_q 1$ have degree q + 1 except for $s_0 s_1, s_2$ which remain of degree q - 1. Hence, by the definition of $E(\Gamma_q 2)$, all vertices in $\Gamma_q 2$ are left with degree q + 1.

Proposition 4.2 Let Γ_q be a (q + 1, 8)-cage, with even degree $q \ge 5$ and $\Gamma_q 1$, $\Gamma_q 2$ be as in Definition 4.1.

- (i) $\Gamma_q 1$ has girth 7 if the matchings M_{S_i}, M_{X_i} and M_{Y_i} have the following properties:
 - (a) Given $u_1v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2v_2 \notin M_{S_j}$.
 - (b) Given $u_1v_1 \in M_{X_i}$ and $u_2, v_2 \in Y_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2v_2 \notin M_{Y_j}$.
- (ii) If conditions (a) and (b) hold then the graph $\Gamma_q 2$ also has girth 7.

Proof To prove (i) let us consider the distances (in $\Gamma_q - H$) between the elements in the sets $Z \in \mathcal{Z}$. There are six possible cases:

(1) Two vertices in the same set $u, v \in Z$ have a common neighbor w in Γ_q , therefore $d_{\Gamma_q-H}(u,v) \ge 6$.

(2) If $u \in X_i$ and $v \in X_j$ then $d_{\Gamma_q-H}(u,v) \ge 4$, given that x_i, x_j have x as a common neighbor in Γ_q .

- (3) If $u \in Y_i$ and $v \in Y_j$ then $d_{\Gamma_q H}(u, v) \ge 4$, as before.
- (4) If $u \in S_i$ and $v \in S_j$ then it is possible that there exist $w \in \Gamma_q H$ such that $u, v \in N(w)$, that is, $d_{\Gamma_q H}(u, v) \ge 2$.
- (5) If $u \in S_i$ and $v \in X_j \cup Y_j$ then $d_{\Gamma_q H}(u, v) \ge 3$, since $s_i \in N(x_i) \cap N(y_i)$.
- (6) If $u \in X_i$ and $v \in Y_j$ then $d_{\Gamma_q H}(u, v) \ge 2$.

Let us consider C a shortest cycle in $\Gamma_q 1$. If $E(C) \subset E(\Gamma_q - H)$ then $|C| \geq 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (1) $|C| \geq 7$. If C contains exactly two edges $e_1, e_2 \in M$, the following cases arise:

- . If both e_1, e_2 lie in the same M_Z , then by (1) $|C| \ge 14 > 7$.
- . If $e_1 \in M_{X_i}$ and $e_2 \in M_{X_i}$ for $i \neq j$, by (2) $|C| \ge 10 > 7$.
- . If $e_1 \in M_{Y_i}$ and $e_2 \in M_{Y_j}$ for $i \neq j$, by (3) $|C| \ge 10 > 7$.
- . If $e_1 \in M_{S_i}$ and $e_2 \in M_{X_j} \cup M_{Y_j}$, by (5) $|C| \ge 8 > 7$.
- . If $e_1 \in M_{S_i}$ and $e_2 \in M_{S_i}$ for $i \neq j$, by the first hypothesis in item (i)(b) $|C| \geq 7$.
- . If $e_1 \in M_{X_i}$ and $e_2 \in M_{Y_j}$, by the second hypothesis in item (i)(b) $|C| \ge 7$.

If C contains at least three edges of M, since $d(u, v) \ge 2$ for all $u, v \in \{X_i \cup Y_i\}_{i=1}^k \cup \{S_i\}_{i=4}^k$, $|C| \ge 9 > 7$.

Hence $\Gamma_q 1$ has girth 7, concluding the proof of (i).

To prove (ii), let C be a shortest cycle in $\Gamma_q 2$. If $E(C) \subset E(\Gamma_q - H) \cup M$ then $|C| \geq 7$.

. If C contains exactly one edge $s_i u_i$ or $s_i v_i$ then $|C| \ge 7$ since $d_{\Gamma_q}(s_i, u_i) = d_{\Gamma_q}(s_i, v_i) = 2$ which implies $d_{\Gamma_q 1}(s_i, u_i) \ge 6$ and $d_{\Gamma_q 1}(s_i, v_i) \ge 6$.

. If C contains a path $u_i s_i v_i$ then $(C \setminus u_i s_i v_i) \cup u_i v_i$ is a cycle in $\Gamma_q 1$ with one less vertex than C, therefore $|C| \geq 8$.

. If C contains two edges $s_i u_i$, $s_j u_j$, for $i \neq j$. Their distances $d_{\Gamma_q 1}(s_i, u_j) \geq 4$, $d_{\Gamma_q 1}(s_i, s_j) \geq 4$, and $d_{\Gamma_q 1}(u_i, u_j) \geq 4$, therefore in any case C has length greater than 7 concluding the proof.

The following lemma gives sufficient conditions to define the matchings M_{S_i} for the sets S_i , in order that they fulfill condition (a) from Proposition 4.2 (i). Notice that in the incidence graph of a generalized quadrangle $\{x, y\}^{\perp \perp} = \bigcap_{s \in N_2(x) \cap N_2(y)} N_2(s)$, thus Remark 2.1 implies that $|\bigcap_{i=0}^{q} N(S_i)| = q - 1$, recalling that $\{s_i\}_{i=0}^{q} = N_2(x) \cap N_2(y)$. Since $|\bigcap_{i=0}^{q} N(S_i)|$ is contained in $|\bigcap_{i=3}^{q} N(S_i)|$, and $|\bigcap_{i=3}^{q} N(S_i)| \le |S_i| = q - 1$ then the condition for the following lemma holds.

Lemma 4.3 If $|\bigcap_{i=3}^{q} N(S_i)| = q - 1$ then there exist matchings M_{S_i} , for $i = 3, \ldots, q$, such that:

- Given $u_1v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2v_2 \notin M_{S_i}$.

Proof Let us suppose that $\bigcap_{i=3}^{q} N(S_i) = \{w_1, \ldots, w_{q-1}\}$, and since S_i has q-1 vertices, every vertex w_j is adjacent to exactly one vertex in $s_{ij} \in S_i$.

Observe that s_{ij} is well defined, because if s_{ij} had two neighbors $w_j, w_{j'} \in \bigcap_{i=1}^{q+1} N(S_i)$, Γ_q would contain the cycle $(s_{ij}w_js_{kj}s_ks_{kj'}w_{j'})$ of length 6.

Therefore, take the complete graph K_{q-1} , label its vertices as $j = 1, \ldots, q-1$. We know that it has a 1-factorization with q-2 factors F_1, \ldots, F_{q-2} . For each $i = 3, \ldots, q+1$, let $s_{ij}s_{il} \in M_{S_i}$ if and only if $jl \in F_{i-3}$.

To prove that the matchings M_{S_i} defined in this way fulfill the desired property suppose that $s_{ij}s_{il} \in M_{S_i}$ and $s_{i'j}s_{i'l} \in M_{S'_i}$ for $i' \neq i$. Then F_i and $F_{i'}$ would have the edge jl in common contradicting that they were a factorization.

So far, the steps of our construction have been independent from the coordinatization of the chosen (q + 1, 8)-cage, however, in order to define M_{X_i} and M_{Y_i} satisfying condition (b) of Lemma 4.2, we need to fix all the elements chosen so far.

We will distinguish two cases, when q is a prime or when q is a prime power.

Choose $x = (q, q, q)_1, y = (0, 0, 0)_1.$

When q is a prime then $x_i = (q, q, i)_0, y_i = (i, 0, 0)_0$ for i = 0, ..., q.

Therefore, $N(x_i) = \{(q, t, i)_1 : t = 0, \dots, q-1\} \cup x$ and $N(x_q) = \{(q, q, t)_1 : t = 0, \dots, q-1\} \cup x;$ $N(y_i) = \{(t, -it, i + t^2)_1 : t = 0, \dots, q-2\} \cup (q, 0, i)_1$ and $N(y_q) = \{(0, t, 0)_1 : t = 0, \dots, q-1\} \cup (q, q, 0)_1.$

Thus, the corresponding vertices s_i are: $s_i = (q, 0, i)_1$ for $i = 0, \dots, q-1$ and $s_q = (q, q, 0)_1$; $N(s_i) = \{(i, 0, t)_0 : t = 1, \dots, q-1, i = 0, \dots, q\} \cup \{x_i, y_i\}$. Hence, $S_i = \{(i, 0, t)_0 : t = 1, \dots, q-1, i = 0, \dots, q\}$.

Then $N(S_i) = \{(a, b, c)_1 : b = -ia, c = t + a^2i, i = 0, \dots, q-1\}$, and $N(S_q) = \{(q, 0, t)_1 : t = 0, \dots, q-1\}$.

Solving the equations we obtain $N(S_i) \cap N(S_j) = \{(0,0,t)_1 : t = 0, \ldots, q-1\}$, moreover $N(i,0,t)_0 \cap N(j,0,t)_0 = (0,0,t)_1$, for each $j \neq i$ and $t = 0, \ldots, q-1$, or equivalently, $N(0,0,t)_1 = \{(x,0,t)_0 : t = 0, \ldots, q-1, x = 0, \ldots, q\}$. Hence the sets S_i satisfy the hypothesis of Lemma 4.3, yielding that there exist the matchings M_{S_i} with the desired property.

Notice that the sets X_i and Y_i are naturally defined as the sets $X_i = \{(q, t, i)_1 : t = 1, \dots, q-1\}$ $1, i = 0, \dots, q-1\}$, $X_0 = \{(q, t, 0)_1 : t = 1, \dots, q-1\}$ and $X_q = \{(q, q, t)_1 : t = 1, \dots, q-1\}$. The sets $Y_i = \{(t, -it, it^2)_1 : t = 1, \dots, q-1, i = 0, \dots, q-1\}$, and $Y_q = \{(0, t, 0)_1 : t = 1, \dots, q-1\}$.

In this way we have defined all the sets in Lemma 4.2, and from Lemma 4.3 we know that the matchings M_{S_i} have the property that:

- If $u_1v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ are such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$ then $u_2v_2 \notin M_{S_i}$.

It remains to define the matchings M_{X_i} and M_{Y_i} and prove they have property (b) from Proposition 4.2 (i).

For this we must analyze the intersection of the second neighborhood of an X_j with an Y_i , $N_2(X_j) \cap Y_i$. For each $w \in Y_i$, we know there is exactly one $z \in X_q$ such that $w \in N_2(z)$.

This allows us to define the following sets of latin squares: For each j, let the coordinate $i\ell$ of the *j*-th latin square to have the symbol $s_{i\ell j}$ if there is a $w_{i\ell j} = (a, b, c)_1$ such that

$$w_{i\ell j} \in N((i,0,0)_0) \cap N_2((q,\ell,j)_1) \cap N_2((q,q,s_{i\ell j})_1),$$

where $(i, 0, 0)_0 = y_i, (q, \ell, j)_1 \in X_j$ and $(q, q, s_{i\ell j})_1 \in X_q$.

Since $N((i, 0, 0)_0) = \{(t, -it, i + t^2)_1 : t = 0, \dots, q - 2\} \cup (q, 0, i)_1$, then a = t, b = -it, and $c = i + t^2$.

Observe that $w_{i\ell j} \in N_2((q, \ell, j)_1)$ is equivalent to $(j, \ell, t)_0 \in N((a, b, c)_1)$, since $N((q, \ell, j)_1) = \{(j, \ell, t)_0 : t = 0, \dots, q-1\} \cup \{(q, q, j)_0\}$. Hence, $aj + b = \ell$.

And $w_{i\ell j} \in N_2((q, q, s_{i\ell j})_1)$ implies $a = s_{i\ell j}$.

Therefore we obtain the following equation for $s_{i\ell j}$.

$$s_{i\ell j}(j-i) = \ell$$

Notice that this equation is undefined for j = i, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q .

Also from the equation we deduce that $-s_{i\ell j} = s_{i-\ell j}$, and $s_{i+1\ell j+1} = s_{i\ell j}$. This means that the i + 1-th row of the j + 1-th latin square is equal to the i-th row of the j-th latin square, hence all the set of latin squares have the same rows. This also implies that if we put an edge between two vertices on Y_i , $(s_{i\ell j}, -is_{i\ell j}, is_{i\ell j}^2)_1$ and $(-s_{i\ell j}, is_{i\ell j}, is_{i\ell j}^2)_1$, it will have at distance two in X_j only the vertices $(q, \ell, i)_1$ and $(q, -\ell, i)_1$.

Therefore, the matchings $M_{X_i} = \{(q, \ell, i)_1(q, -(\ell+2), i)_1 : i = 0, \dots, q-1, \ell = 1, \dots, q-3\} \cup \{(q, -2, i)_1(q, -1, i)_1 : i = 0, \dots, q-1\}, M_{X_q} = \{(q, q, \ell)_1(q, q, -(\ell+2))_1 : \ell = 1, \dots, q-3\} \cup \{(q, q, -2)_1(q, q, -1)_1\}, \text{ and } M_{Y_i} = \{(t, -it, it^2)_1(-t, it, it^2)_1 : i = 0, \dots, q-1, t = 1, \dots, q-1\},$ have the property (b) from Proposition 4.2 (i).

When q is a prime power, let α a primitive root of unity in GF(q). Then, $x_i = (q, q, \alpha^{i-1})_0$, $y_i = (\alpha^{i-1}, 0, 0)_0$ for i = 1, ..., q - 1, $x_0 = (q, q, 0)_0$, and $y_0 = (0, 0, 0)_0$. Moreover, $x_q = (q, q, q)_0$ and $y_q = (q, 0, 0)_0$.

Therefore, $N(x_i) = \{(q, \alpha^t, \alpha^{i-1})_1 : t = 0, \dots, q-2\} \cup (q, 0, \alpha^{i-1})_1 \cup x \text{ and } N(x_0) = \{(q, \alpha^t, 0)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1 \cup x; N(y_i) = \{(\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2\} \cup (q, 0, \alpha^{i-1})_1 \text{ and } N(y_0) = \{(\alpha^t, 0, 0)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1; N(x_q) = \{(q, q, \alpha^t)_1 : s = 0, \dots, q-2\} \cup (q, q, 0)_1 \cup x; \text{ and } N(y_q) = \{(0, \alpha^t, 0)_1 : t = 0, \dots, q-2\} \cup (q, q, 0)_1 \cup y.$

Thus, the corresponding vertices s_i are: $s_i = (q, 0, \alpha^{i-1})_1$, for $i = 1, \ldots, q-1$, $s_0 = (q, 0, 0)_1$ and $s_q = (q, q, 0)_1$; $N(s_i) = \{(\alpha^{i-1}, 0, \alpha^t)_0 : t = 0, \ldots, q-2, i = 1, \ldots, q-1\} \cup \{x_i, y_i\}$, and $N(s_0) = \{(0, 0, \alpha^t)_0 : t = 0, \ldots, q-2\} \cup \{x_0, y_0\}$. Hence $S_i = \{(\alpha^{i-1}, 0, \alpha^t)_0 : t = 0, \ldots, q-2, i = 0, \ldots, q\}$ and $S_0 = \{(0, 0, \alpha^t)_0 : t = 0, \ldots, q-2\}$.

Then $N(S_i) = \{(a, b, c)_1 : b = -\alpha^{i-1}a, c = \alpha^t + a^2\alpha^{i-1}, i = 1, \dots, q-1\}, N(S_0) = \{(a, b, c)_1 : b = 0, c = \alpha^t\}$ and $N(S_q) = \{(q, 0, \alpha^t)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1.$

Solving the equations we obtain $N(S_i) \cap N(S_j) = \{(0, 0, \alpha^t)_1 : t = 0, \dots, q-2\}$. Moreover, $N(\alpha^{i-1}, 0, \alpha^t)_0 \cap N(\alpha^{j-1}, 0, \alpha^t)_0 = (0, 0, \alpha^t)_1$, for each $j \neq i$ and $t = 0, \dots, q-2$, or equivalently, $N(0, 0, \alpha^t)_1 = \{(\alpha^x, 0, \alpha^t)_0 : x = 0, \dots, q-2\} \cup (0, 0, \alpha^t)_0 \cup (q, 0, \alpha^t)_0$, for each $t = 0, \dots, q-2$. Hence the sets S_i satisfy the hypothesis of Lemma 4.3 yielding that there exist the matchings M_{S_i} with the desired property.

Notice that the sets X_i and Y_i are naturally defined as the sets $X_i = \{(q, \alpha^t, \alpha^{i-1})_1 : t = 0, \ldots, q-2, i = 1, \ldots, q-1\}, X_0 = \{(q, \alpha^t, 0)_1 : t = 0, \ldots, q-2\}$ and $X_q = \{(q, q, \alpha^t)_1 : t = 0, \ldots, q-2\}$. The sets

 $Y_i = \{ (\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2 \}, Y_0 = \{ (\alpha^t, 0, 0)_1 : t = 0, \dots, q-2 \}$ and $Y_q = \{ (0, \alpha^t, 0)_1 : t = 0, \dots, q-2 \}.$

In order to define the matchings M_{X_i} and M_{Y_i} and prove that they have the property (b) from Proposition 4.2 (i), we proceed as before, by defining the sets of latin squares:

For each j, let the coordinate $i\ell$ of the j-th latin square to have the symbol $s_{i\ell j} \in \{0, \ldots, q-2\}$ if there is a $w_{i\ell j} = (a, b, c)_1$ such that

$$w_{i\ell j} \in N((\alpha^{i-1}, 0, 0)_0) \cap N_2((q, \alpha^{\ell}, \alpha^{j-1})_1) \cap N_2((q, q, \alpha^{s_{i\ell j}})_1)$$
 for $i, j \ge 1$,

where $(\alpha^{i-1}, 0, 0)_0 = y_i, (q, \alpha^{\ell}, \alpha^{j-1})_1 \in X_j$ and $(q, q, \alpha^{s_{i\ell j}})_1 \in X_q$.

Since $N((\alpha^{i-1}, 0, 0)_0) = \{(\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2\} \cup (q, 0, i)_1$, then $a = \alpha^t$, $b = -\alpha^{i-1+t}$, and $c = \alpha^{i-1+2t}$.

Also $w_{i\ell j} \in N_2((q, \alpha^{\ell}, \alpha^{j-1})_1)$ is equivalent to $(\alpha^{j-1}, \alpha^{\ell}, \alpha^t)_0 \in N((a, b, c)_1)$, since $N((q, \alpha^{\ell}, \alpha^{j-1})_1) = \{(\alpha^{j-1}, \alpha^{\ell}, \alpha^t)_0 : t = 0, \dots, q-2\}$. Hence $a\alpha^{j-1} + b = \alpha^{\ell}$.

And $w_{i\ell j} \in N_2((q, q, \alpha^{s_{i\ell j}})_1)$ implies $a = \alpha^{s_{i\ell j}}$.

Therefore we obtain the following equation for $s_{i\ell j}$.

$$\alpha^{s_{i\ell j}}(\alpha^{j-1} - \alpha^{i-1}) = \alpha^{\ell}$$

Notice that this equation is undefined for j = i, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q .

For i = 0, we obtain the equation $\alpha^{s_{\ell\ell j}}(\alpha^{j-1}) = \alpha^{\ell}$, and for j = 0, we obtain $\alpha^{s_{i\ell 0}}(-\alpha^{i-1}) = \alpha^{\ell}$. From the equation we obtain that $s_{i\ell+1j} = s_{i\ell j} + 1$, and each latin square is the sum table of the cyclic group \mathbb{Z}_{q-1} with the rows permuted.

Multiplying by α the equation $\alpha^{s_{i\ell-1j}}(\alpha^{j-1}-\alpha^{i-1}) = \alpha^{\ell-1}$, we obtain that $s_{i+1\ell j+1} = s_{i\ell-1j}$. This implies that the row i + 1 of the j + 1-th latin square is equal to the row i of the j-th latin square subtracting 1 to each symbol (i.e., $s_{i+1\ell j+1} + 1 = s_{i\ell j}$). That is, all the set of latin squares have the same rows but in a different order.

This also implies that if we put an edge between two vertices on Y_i , $(\alpha^{s_{i\ell j}}, -\alpha^{i-1+s_{i\ell j}}, \alpha^{i-1+2s_{i\ell j}})_1$ and $(\alpha^{s_{i\ell j}+1}, -\alpha^{i-1+(s_{i\ell j}+1)}, \alpha^{i-1+2(s_{i\ell j}+1)})_1$, it will have at distance two in X_j only the vertices, $(q, \alpha^{\ell}, i)_1$ and $(q, \alpha^{\ell+1}, i)_1$ and the other way around.

Therefore, the matchings $M_{X_i} = \{(q, \alpha^{2\ell}, i)_1(q, \alpha^{2\ell+1}, i)_1 : i = 0, \dots, q - 1, \ell = 0\}$

1,..., (q-1)/2}, $M_{X_q} = \{(q,q,\alpha^{2\ell})_1(q,q,\alpha^{2\ell+1})_1 : \ell = 1,..., (q-1)/2\}$, and $M_{Y_i} = \{(\alpha^{2t}, -\alpha^{i-1+2t}, \alpha^{i-1+4t})_1(\alpha^{2t+3}, -\alpha^{i-1+(2t+3)}, \alpha^{i-1+2(2t+3)})_1 : i = 0, ..., q-1, t = 1, ..., (q-1)/2\}$ have the property (b) from Proposition 4.2 (i), proving the theorem for q prime power.

Theorem 4.4 Let $q \ge 5$ be a prime power. Then there is a q + 1-regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.

Proof Finally, by applying Lemma 4.2(ii), we obtain a q + 1-regular graph of girth 7 with $2(q^3 + q^2 + q + 1) - (q - 3 + 2(q + 2)) = 2q^3 + 2q^2 - q + 1$ vertices.

References

- M. Abreu, G. Araujo–Pardo, C. Balbuena, D. Labbate. Small k-regular graphs of girth 8, (submitted).
- [2] M. Abreu, G. Araujo–Pardo, C. Balbuena, D. Labbate, Families of Small Regular Graphs of Girth 5, *Discrete Math.*, 312 (18), 2012, 20832–2842.
- [3] M. Abreu, M. Funk, D. Labbate, V. Napolitano, On (minimal) regular graphs of girth 6, Australas. J. Combin. 35 (2006) 119–132.
- [4] G. Araujo, C. Balbuena, and T. Héger, Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages, Discrete Math. 310 (8) (2010) 1301–1306.
- [5] E. Bannai and T. Ito, On finite Moore graphs, J. Fac. Sci. Univ. Tokio, Sect. I A Math 20 (1973) 191–208.
- [6] C. Balbuena, Incidence matrices of projective planes and other bipartite graphs of few vertices, Siam J. Discrete Math. 22(4) (2008), 1351–1363.
- [7] C. Balbuena, A construction of small regular bipartite graphs of girth 8, Discrete Math. Theor. Comput. Sci. 11(2) (2009) 33–46.
- [8] L.M. Batten, Combinatorics of finite geometries, Cambridge University Press, Cambridge, UK, 1997.
- C.T. Benson, Minimal regular graphs of girth eight and twelve, Canad. J. Math. 18 (1966) 1091–1094.
- [10] L. Beukemann and K. Metsch, Regular Graphs Constructed from the Classical Generalized Quadrangle Q(4,q), J. Combin. Designs 19 (2010) 70–83.
- [11] N. Biggs, Algebraic Graph Theory, Cambridge University Press, New York, 1996.

- [12] N. Biggs, Construction for cubic graphs with large girth, Electron. J. Combin. 5 (1998) #A1.
- [13] J.A. Bondy, U.S.R. Murty, *Graph Theory*, Springer Series: Graduate Texts in Mathematics, Vol. 244, 2008.
- [14] G. Brinkmann, B. D. McKay and C. Saager, The smallest cubic graphs of girth nine, Combin. Prob. and Computing 5 (1995) 1–13.
- [15] P. Erdös and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, Wiss. Z. Uni. Halle (Math. Nat.), 12 (1963) 251–257.
- [16] G. Exoo, A Simple Method for Constructing Small Cubic Graphs of Girths 14, 15 and 16, Electron. J. Combin., 3, 1996.
- [17] G. Exoo and R. Jajcay, Dynamic cage survey, Electron. J. Combin. 15 (2008) #DS16.
- [18] W. Feit and G. Higman, The non-existence of certain generalized polygons, J. Algebra 1 (1964) 114–131.
- [19] A. Gács and T. Héger, On geometric constructions of (k, g)-graphs, Contrib. to Discrete Math. 3(1) (2008) 63–80.
- [20] C. Godsil and G. Royle, Algebraic Graph Theory, Springer, NY 2000.
- [21] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs, Monogr. Textbooks Pure Appl. Math., 208, Dekker, New York, (1998).
- [22] D.A. Holton and J. Sheehan, The Petersen Graph, Chapter 6: Cages, Cambridge University (1993).
- [23] F. Lazebnik and V.A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, Discrete Appl. Math. 60 (1995) 275–284.
- [24] F. Lazebnik, V.A. Ustimenko, and A.J. Woldar, New upper bounds on the order of cages, Electron. J. Combin. 4 (1997) # 2.
- [25] J. H. van Lint and R. M. Wilson, A course in Combinatorics, Cambridge University Press, UK 1994.
- [26] H. van Maldeghem, Generalized polygons, Monographs in Mathematics, 93, Birkhauser Verlag, Basel, 1998.
- [27] M. Meringer, Fast generation of regular graphs and construction of cages, J. Graph Theory 30 (1999) 137–146.
- [28] M. O'Keefe and P.K. Wong, The smallest graph of girth 6 and valency 7, J. Graph Theory 5(1) (1981), 79–85.

- [29] S.E. Payne, Affine representation of generalized quadrangles, J. Algebra 51, (1970), 473– 485.
- [30] S. E. Payne and J. A. Thas. Finite generalized quadrangles. Research Notes in Mathematics, 110. Pitman (Advanced Publishing Program), Boston, MA, 1984.
- [31] S.E.Payne and J.A. Thas. Finite Generalized Quadrangles, 2nd edn. European Mathematical Society (2009).
- [32] T. Pisanski, M. Boben, D. Marusic, A. Orbanic, A. Graovac, The 10-cages and derived configurations, Discrete Math. 275 (2004) 265–276.
- [33] W. T. Tutte, A family of cubical graphs. Proc. Cambridge Philos. Soc., (1947) 459–474.
- [34] V.A. Ustimenko, A linear interpretation of the flag geometries of Chevalley groups. Ukr. Mat. Zh., Kiev University, 42 (3), (1990), 383–387.
- [35] V.A. Ustimenko, On the embeddings of some geometries and flag systems in Lie algebras and superalgebras, in: Root Systems, Representation and Geometries, IM AN UkrSSR, Kiev, 1990, 3–16,
- [36] P. K. Wong, Cages-a survey, J. Graph Theory 6 (1982) 1–22.