# $(3,1)^*$-choosability of planar graphs without adjacent short cycles

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**Abstract**

A list assignment of a graph $G$ is a function $L$ that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $(L,d)^*$-coloring is a mapping $\pi$ that assigns a color $\pi(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most $d$ neighbors of $v$ receive color $\pi(v)$. A graph $G$ is said to be $(k,d)^*$-choosable if it admits an $(L,d)^*$-coloring for every list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$. In 2001, Lih et al. [6] proved that planar graphs without $4$- and $l$-cycles are $(3,1)^*$-choosable, where $l \in \{5,6,7\}$. Later, Dong and Xu [3] proved that planar graphs without $4$- and $l$-cycles are $(3,1)^*$-choosable, where $l \in \{8,9\}$.

There exist planar graphs containing $4$-cycles that are not $(3,1)^*$-choosable (Crown, Crown and Woodall, 1986 [1]). This partly explains the fact that in all above known sufficient conditions for the $(3,1)^*$-choosability of planar graphs the $4$-cycles are completely forbidden. In this paper we allow $4$-cycles nonadjacent to relatively short cycles. More precisely, we prove that every planar graph without $4$-cycles adjacent to $3$- and $4$-cycles is $(3,1)^*$-choosable. This is a common strengthening of all above mentioned results. Moreover as a consequence we give a partial answer to a question of Xu and Zhang [11] and show that every planar graph without $4$-cycles is $(3,1)^*$-choosable.

*Keyword:* Planar graphs; Improper choosability; Cycle.

## 1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. For a graph $G$, we use $V(G)$, $E(G)$, $|G|$, $|E(G)|$ and $\delta(G)$ to denote its vertex set, edge set, order, size and minimum degree, respectively. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of $v$ in $G$. If there is no confusion about the context, we write $N(v)$ for $N_G(v)$.

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A $k$-coloring of $G$ is a mapping $\pi$ from $V(G)$ to a color set $\{1, 2, \cdots, k\}$ such that $\pi(x) \neq \pi(y)$ for any adjacent vertices $x$ and $y$. A graph is $k$-colorable if it has a $k$-coloring. Cowen, Cowen, and Woodall [1] considered *defective* colorings of graphs. A graph $G$ is said to be *$d$-improper $k$-colorable*, or simply, *$(k, d)^*$-colorable*, if the vertices of $G$ can be colored with $k$ colors in such a way that each vertex has at most $d$ neighbors receiving the same color as itself. Obviously, a $(k, 0)^*$-coloring is an ordinary proper $k$-coloring.

A *list assignment* of $G$ is a function $L$ that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $L$-coloring with impropriety of integer $d$, or simply an *$(L, d)^*$-coloring*, of $G$ is a mapping $\pi$ that assigns a color $\pi(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most $d$ neighbors of $v$ receive color $\pi(v)$. A graph is *$k$-choosable* with impropriety of integer $d$, or simply *$(k, d)^*$-choosable*, if there exists an $(L, d)^*$-coloring for every list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$. Clearly, a $(k, 0)^*$-choosable is the ordinary $k$-choosability introduced by Erdős, Rubin and Taylor [5] and independently by Vizing [10].

The concept of list improper coloring was independently introduced by Škrekovski [7] and Eaton and Hull [4]. They proved that every planar graph is $(3, 2)^*$-choosable and every outerplanar graph is $(2, 2)^*$-choosable. These are both improvement of the results showed in [1] which say that every planar graph is $(3, 2)^*$-colorable and every outerplanar graph is $(2, 2)^*$-colorable. Let $g(G)$ denote the *girth* of a graph $G$, i.e., the length of a shortest cycle in $G$. The $(k, d)^*$-choosability of planar graph $G$ with given $g(G)$ has been studied by Škrekovski in [9]. He proved that every planar graph $G$ is $(2, 1)^*$-choosable if $g(G) \geq 9$, $(2, 2)^*$-choosable if $g(G) \geq 7$, $(2, 3)^*$-choosable if $g(G) \geq 6$, and $(2, d)^*$-choosable if $d \geq 4$ and $g(G) \geq 5$. Recently, Cushing and Kierstead [2] proved that every planar graph is $(4, 1)^*$-choosable. So it would be interesting to investigate the sufficient conditions of $(3, 1)^*$-choosability of subfamilies of planar graphs where some families of cycles are forbidden. Škrekovski proved in [8] that every planar graph without 3-cycles is $(3, 1)^*$-choosable. Lih et al. [6] proved that planar graphs without 4- and $l$-cycles are $(3, 1)^*$-choosable, where $l \in \{5, 6, 7\}$. Later, Dong and Xu [3] proved that planar graphs without 4- and $l$-cycles are $(3, 1)^*$-choosable, where $l \in \{8, 9\}$. Moreover, Xu and Zhang [11] asked the following question:

**Question 1** *Is it true that every planar graph without adjacent triangles is $(3, 1)^*$-choosable?*

Recall that there is a planar graph containing 4-cycles that is not $(3, 1)^*$-colorable [1]. Therefore, while describing $(3, 1)^*$-choosability planar graphs, one must impose these or those restrictions on 4-cycles. Note that in all previously known sufficient conditions for the $(3, 1)^*$-choosability of planar graphs, the $4$-cycles are completely forbidden. In this paper we allow $4$-cycles, but disallow them to have a common edge with relatively short cycles.

The purpose of this paper is to prove the following

**Theorem 1** *Every planar graph without* $4$*-cycles adjacent to* $3$*- and* $4$*-cycles is* $(3, 1)^*$*-choosable.*

Clearly, Theorem 1 implies Corollary 1 which is a common strengthening of the results in [6, 3].

**Corollary 1** *Every planar graph without* $4$*-cycles is* $(3, 1)^*$*-choosable.*

Moreover, Theorem 1 partially answers Question 1, since adjacent triangles can be regarded as a $4$-cycle adjacent to a $3$-cycle.

## 2 Notation

A vertex of degree $k$ (resp. at least $k$, at most $k$) will be called a $k$*-vertex* (resp. $k^+$*-vertex*, $k^-$*-vertex*). A similar notation will be used for cycles and faces. A *triangle* is synonymous with a 3-cycle. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of $f$ and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are the boundary vertices of $f$ in cyclic order. For any $v \in V(G)$, we let $v_1, v_2, \cdots, v_{d(v)}$ denote the neighbors of $v$ in a cyclic order. Let $f_i$ be the face with $vv_i$ and $vv_{i+1}$ as two boundary edges for $i = 1, 2, \cdots, d(v)$, where indices are taken modulo $d(v)$. Moreover, we let $t(v)$ denote the number of 3-faces incident to $v$ and let $n_3(v)$ denote the number of 3-vertices adjacent to $v$.

An $m$-face $f = [v_1v_2 \cdots v_m]$ is called an $(a_1, a_2, \cdots, a_m)$*-face* if the degree of the vertex $v_i$ is $a_i$ for $i = 1, 2, \cdots, m$. Suppose $v$ is a $4$-vertex incident to a $4^-$-face $f$ and adjacent to two $3$-vertices not on $b(f)$. If $d(f) = 3$, then we call $v$ a *light* $4$-vertex. Otherwise, we call $v$ a *soft* $4$-vertex if $d(f) = 4$. A vertex $v$ is called an $\mathcal{S}$*-vertex* if it is either a $3$-vertex or a light $4$-vertex. Moreover, we say a $3$-face $f = [v_1v_2v_3]$ is an $(a_1, *, a_3)$-face if $d(v_i) = a_i$ for each $i \in \{1, 3\}$ and $v_2$ is an $\mathcal{S}$-vertex. Suppose $v$ is a $5$-vertex incident to two 3-faces $f_1 = [vv_1v_2]$ and $f_3 = [vv_3v_4]$. Let $v_5$ be the neighbour of $v$ not belonging to the $3$-faces. If $d(v_5) = 3$ and $f_1$ is a $(5, *, 4)$-face, then we call $v$ a *bad* $5$-vertex.

For all figures in the following section, a vertex is represented by a solid circle when all of its incident edges are drawn; otherwise it is represented by a hollow circle. Moreover, we use a hollow square to denote an $\mathcal{S}$-vertex.

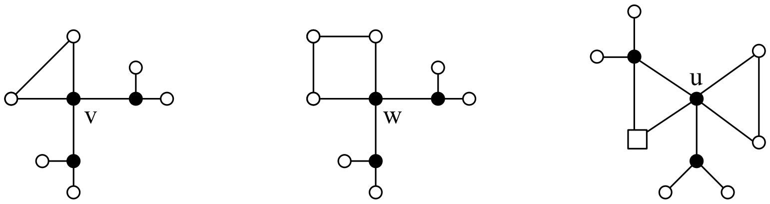

Figure 1: A light $4$-vertex $v$, a soft $4$-vertex $w$ and a bad $5$-vertex $u$.

# 3 Proof of Theorem 1

The proof of Theorem 1 is done by reducible configurations and discharging procedure. Suppose the theorem is not true. Let $G$ be a counterexample with the least number of vertices and edges embedded in the plane. Thus, $G$ is connected. We will apply a discharging procedure to reach a contradiction.

We first define a weight function $\omega$ on the vertices and faces of $G$ by letting $\omega(v) = 3d(v) - 10$ if $v \in V(G)$ and $\omega(f) = 2d(f) - 10$ if $f \in F(G)$. It follows from Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces is equal to

$$\sum_{v\in V(G)} (3d(v) - 10) + \sum_{f\in F(G)} (2d(f) - 10) = -20.$$

We then design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function $\omega^*$ is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new weight function satisfies $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction,

$$-20 = \sum_{x\in V(G)\cup F(G)} \omega(x) = \sum_{x\in V(G)\cup F(G)} \omega^*(x) \geq 0$$

and hence demonstrates that no such counterexample can exist.

## 3.1 Reducible configurations of $G$

In this section, we will establish structural properties of $G$. More precisely, we prove that some configurations are reducible. Namely, they cannot appear in $G$ because of the minimality of $G$. Since $G$ does not contain a $4$-cycle adjacent to an $i$-cycle, where $i = 3, 4$, by hypothesis, the following fact is easy to observe and will be frequently used throughout this paper without further notice.

**Observation 1** *$G$ does not contain the following structures:*

(a) *adjacent* $3$*-cycles;*

(b) *a* $4$*-cycle adjacent to a* $3$*-cycle;*

(c) *a* $4$*-cycle adjacent to a* $4$*-cycle.*

We first present Lemma 1, whose proof was provided in [6].

**Lemma 1** [6]

(A1) $\delta(G) \geq 3$.

(A2) *No two adjacent* $3$*-vertices.*

(A3) *There is no* $(3, 4, 4)$*-face.*

Before showing Lemmas 2-7, we need to introduce some useful concepts, which were firstly defined by Zhang in [12].

**Definition 1** For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$. We simply write $G - S = G[V(G) \setminus S]$. Let $L$ be an arbitrary list assignment of $G$, and $\pi$ be an $(L, 1)^*$-coloring of $G - S$. For each $v \in S$, let $L_\pi(v) = L(v) \setminus \{\pi(u) : u \in N_{G-S}(v)\}$, and we call $L_\pi$ an *induced assignment* of $G[S]$ from $\pi$. We also say that $\pi$ can be extended to $G$ if $G[S]$ admits an $(L_\pi, 1)^*$-coloring.

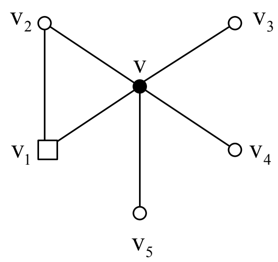


Figure 2: The configuration (Q) in Lemma 2.

**Lemma 2** *Suppose that* $G$ *contains the configuration* $(Q)$*, depicted in Figure 2. Let* $\pi$ *be an* $(L, 1)^*$*-coloring of* $G - S$*, where* $S = \{v, v_1, v_2, v_3, v_4\}$*. Denote by* $L_\pi$ *an induced list assignment of* $G[S]$*. If* $|L_\pi(v_i)| \geq 1$ *for each* $i \in \{1, \cdots, 4\}$*, then* $\pi$ *can be extended to the whole graph* $G$*.*

**Proof.** Since $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \cdots, 4\}$, we can color each $v_i$ with a color $\pi(v_i) \in L_\pi(v_i)$ properly. Note that $|L_\pi(v)| \geq 2$. If there exists a color in $L_\pi(v)$ which appears at most once on the set $\{v_1, v_2, v_3, v_4\}$, then we assign such a color to $v$. It is easy to check that the resulting coloring is an $(L,1)^*$-coloring and thus we are done. Otherwise, w.l.o.g., suppose $L(v)=\{1,2,3\}$, $\pi(v_5)=1$, and each color in $\{2,3\}$ appears exactly twice on the set $\{v_1,v_2,v_3,v_4\}$. W.l.o.g., suppose $\pi(v_1)=2$.

By definition, we see that $v_1$ is either a 3-vertex or a light 4-vertex. We label two steps in the proof for future reference.

(i) If $d(v_1)=3$, then $|L_\pi(v_1)|\geq 2$. We may assign color 2 to $v$ and then recolor $v_1$ with a color in $L_\pi(v_1)\setminus\{2\}$.

(ii) If $v_1$ is a light 4-vertex, denote by $x_1,y_1$ the other two neighbors which are different from $v$ and $v_2$. Erase the color of $v_1$, color $v$ with 2, and recolor $x_1$ and $y_1$ with a color different from its neighbors. We can do this since $d(x_1)=d(y_1)=3$ by definition. Next, we will show how to extend the resulting coloring, denoted by $\pi'$, to $G$. If $\pi'(v_2)\notin\{\pi'(x_1),\pi'(y_1)\}$, then color $v_1$ with a color in $L(v_1)\setminus\{2,\pi'(x_1)\}$. Otherwise, we color $v_1$ with a color in $L(v_1)\setminus\{2,\pi'(v_2)\}$. In each case, one can easily check that the obtained coloring of $G$ is an $(L,1)^*$-coloring.

Therefore, we complete the proof of Lemma 2. □

**Lemma 3** *$G$ satisfies the following.*

(B1) *A 4-vertex is adjacent to at most two 3-vertices.*

(B2) *There is no $(4^-,4^-,4^-)$-face.*

(B3) *There is no $(5^+,4,4)$-face which is incident to two light 4-vertices.*

(B4) *There is no 5-vertex incident to a $(5,*,4)$-face $f$ and adjacent to two 3-vertices not on $b(f)$.*

(B5) *There is no 6-vertex incident to two $(6,4^-,4^-)$-faces and one $(6,*,4)$-face.*

**Proof.** Let $L$ be a list assignment such that $|L(v)|=3$ for all $v\in V(G)$. We make use of contradiction to show (B1)-(B5).

(B1) Suppose that $v$ is adjacent to three 3-vertices $v_1,v_2$ and $v_3$. Denote $G'=G-\{v,v_1,v_2,v_3\}$. By the minimality of $G$, $G'$ admits an $(L,1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G-G'$. It is easy to deduce that $|L_\pi(v)|\geq 2$ and $|L_\pi(v_i)|\geq 1$ for each $i\in\{1,2,3\}$. So for each $v_i$, we assign the color $\pi(v_i)\in L_\pi(v_i)$ to it. Now we observe that there exists a color in $L_\pi(v)$ appearing at most once on the set $\{v_1,v_2,v_3\}$. We color $v$ with such a color. The obtained coloring is an $(L,1)^*$-coloring of $G$. This contradicts the choice of $G$.

(B2) It suffices to prove that $G$ does not contain a $(4,4,4)$-face by (A3). Suppose $f=[v_1v_2v_3]$ is a 3-face with $d(v_1)=d(v_2)=d(v_3)=4$. For each $i\in\{1,2,3\}$, let $x_i,y_i$ denote the other two neighbors of $v_i$ not on $b(f)$. Denote by $G'$ the graph obtained from $G$ by deleting edge $v_1v_2$. By the minimality of $G$, $G'$ has an $(L,1)^*$-coloring $\pi$. If $\pi(v_1) \neq \pi(v_2)$, then $G$ itself is $(L,1)^*$-colorable and thus we are done. Otherwise, suppose $\pi(v_1) = \pi(v_2)$. If $\pi$ is not an $(L,1)^*$-coloring of the whole graph $G$, then without loss of generality, assume that $\pi(v_1) = \pi(v_2) = \pi(x_1) = 1$ and $\pi(v_3) = 2$. Moreover, none of $x_1$'s neighbors except $v_1$ is colored with 1. First, we recolor each $v_i$ with a color $\pi'(v_i)$ in $L(v_i) \setminus \{\pi(x_i), \pi(y_i)\}$, where $i \in \{1,2,3\}$. We should point out that $\pi'(v_i)$ may be the same as $\pi(v_i)$, but it does not matter. Note that if at most two of $\pi'(v_1), \pi'(v_2), \pi'(v_3)$ are equal then the resulting coloring is an $(L,1)^*$-coloring and thus we are done. Otherwise, suppose that $\pi'(v_1) = \pi'(v_2) = \pi'(v_3)$. Since $\pi'(v_1) \neq 1$ and $1 \in L(v_1)$, we may further reassign color 1 to $v_1$ to obtain an $(L,1)^*$-coloring of $G$. This contradicts the choice of $G$.

(B3) Suppose $f = [v_1v_2v_3]$ is a $(5^+,4,4)$-face incident to two light 4-vertices $v_2$ and $v_3$. By definition, we see that each $v_i$ ($i \in \{2,3\}$) is incident to two other 3-vertices, denoted by $x_i$ and $y_i$, which are not on $b(f)$. Let $G'$ denote the graph obtained from $G$ by deleting edge $v_2v_3$. Obviously, $G'$ has an $(L,1)^*$-coloring $\pi$ by the minimality of $G$. Similarly, if $\pi(v_2) \neq \pi(v_3)$, then $G$ itself is $(L,1)^*$-colorable and thus we are done. Otherwise, suppose $\pi(v_2) = \pi(v_3)$. If $\pi$ is not an $(L,1)^*$-coloring of $G$, then w.l.o.g., assume that $\pi(v_2) = \pi(v_3) = \pi(x_2) = 1$ and $\pi(v_1) = 2$. Erase the color of $v_2$ and recolor $y_2$ with a color $a \in L(y_2)$ different from its neighbors. If $L(v_2) \neq \{1,2,a\}$, then color $v_2$ with a color in $L(v_2) \setminus \{1,2,a\}$. Otherwise, color $v_2$ with $a$. It is easy to verify that the resulting coloring is an $(L,1)^*$-coloring of $G$, which is a contradiction.

(B4) Suppose that a 5-vertex $v$ is incident to a $(5,*,4)$-face $f_1 = [vv_1v_2]$ and adjacent to two 3-vertices $v_3$ and $v_4$. Let $G' = G - \{v, v_1, v_2, v_3, v_4\}$. By the minimality of $G$, $G'$ has an $(L,1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. Obviously, $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \cdots, 4\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, $\pi$ can be extended to $G$, which is a contradiction.

(B5) Suppose that a 6-vertex $v$ is incident to two $(6,4^-,4^-)$-faces $f_1, f_3$ and one $(6,*,4)$-face $f_5$ such that $d(v_i) \leq 4$ for each $i = \{1,2,3,4\}$, $d(v_6) = 4$ and $v_5$ is an $\mathcal{S}$-vertex. Namely, $v_5$ is either a 3-vertex or a light 4-vertex. Let $G' = G - \{v, v_1, v_2, \cdots, v_6\}$. By minimality, $G'$ admits an $(L,1)^*$-coloring $\pi$. Denote by $L_\pi$ an induced list assignment of $G - G'$. It is easy to verify that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \cdots, 6\}$ and $|L_\pi(v)| \geq 3$. So we can color $v_i$ with $\pi(v_i) \in L_\pi(v_i)$ for each $i \in \{1, 2, \cdots, 6\}$. If there exists a color $a \in L_\pi(v)$ appearing at most once on the set $\{v_1, v_2, \cdots, v_6\}$, then we further assign color $a$ to $v$ and thus obtain an $(L,1)^*$-coloring of $G$.

Otherwise, each color in $L_\pi(v)$ appears exactly twice on the set $\{v_1, v_2, \cdots, v_6\}$. Since $v_5$ is an $\mathcal{S}$-vertex, we can apply versions of arguments (i) and (ii) in the proof of Lemma 2 to obtain an $(L,1)^*$-coloring of $G$. $\square$

**Lemma 4** *Suppose that* $f = [uvxy]$ *is a* $(3,4,m,4)$*-face. Then*
(F1) $m \neq 3$.
(F2) $x$ *cannot be a soft* $4$*-vertex.*

**Proof.** (F1) Suppose to the contrary that $m = 3$. Let $G' = G - \{u, v, x, y\}$. By the minimality of $G$, $G'$ admits an $(L,1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. Notice that $|L_\pi(y)| \geq 1$, $|L_\pi(v)| \geq 1$, $|L_\pi(u)| \geq 2$ and $|L_\pi(x)| \geq 2$. First, we color $v$ with $a \in L_\pi(v)$ and color $y$ with $b \in L_\pi(y)$. Then color $u$ with $c \in L_\pi(u) \setminus \{a\}$ and $x$ with $d \in L_\pi(x) \setminus \{b\}$. One can easily check that the resulting coloring of $G$ is an $(L,1)^*$-coloring. This contradicts the assumption of $G$.

(F2) Suppose to the contrary that $x$ is a soft $4$-vertex. By definition, $x$ has other two neighbors whose degree are both 3, say $x_1$ and $x_2$. Observe that neither $x_1$ nor $x_2$ is on $b(f)$. Let $G' = G - \{u, v, x, y, x_1, x_2\}$. Obviously, $G'$ admits an $(L,1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. For each $w \in \{v, y, x_1, x_2\}$, we deduce that $|L_\pi(w)| \geq 1$. Moreover, $|L_\pi(u)| \geq 2$. We first color $w$ with $\pi(w) \in L_\pi(w)$ and color $u$ with a color in $L_\pi(u) \setminus \{\pi(v)\}$. If at least one of $x_1$ and $x_2$ has the same color as $\pi(v)$, we can color $x$ with a color different from that of $v$ and $y$. Otherwise, we can color $x$ with a color different from $x_1$ and $y$. Therefore, we achieve an $(L,1)^*$-coloring of $G$, which is a contradiction. $\square$

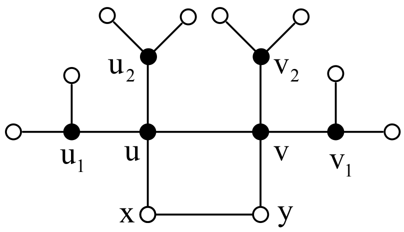


Figure 3: Adjacent soft $4$-vertices $u$ and $v$.

**Lemma 5** *There is no adjacent soft* $4$*-vertices.*

**Proof.** Suppose to the contrary that $u$ and $v$ are adjacent soft $4$-vertices such that $[uxyv]$ is a $4$-face and $u_1, u_2, v_1, v_2$ are $3$-vertices, which is depicted in Figure 3. By Observation 1(b), $u_i$ cannot be coincided with $v_j$, where $i, j \in \{1, 2\}$. Let $G' = G - \{u_1, u_2, v_1, v_2, u, v\}$. For each $i \in \{1, 2\}$, we color $u_i$ and $v_i$ with a color in $L_\pi(u_i)$ and $L_\pi(v_i)$, respectively. If $L(u) \neq \{\pi(x), \pi(u_1), \pi(u_2)\}$, then color $u$ with $a \in L(u) \setminus \{\pi(x), \pi(u_1), \pi(u_2)\}$. It is easy to see that there exists at least one color in $L(v) \setminus \{\pi(y)\}$ which appears at most once on the set $\{u, v_1, v_2\}$. So we may assign such a color to $v$. Now suppose that $L(u) = \{\pi(x), \pi(u_1), \pi(u_2)\}$. By symmetry, we may suppose that $L(v) = \{\pi(y), \pi(v_1), \pi(v_2)\}$. This implies that $\pi(v_1) \neq \pi(v_2)$. Thus, we can first color $u$ with $\pi(u_1)$ and then assign a color in $L(v) \setminus \{\pi(u_1), \pi(y)\}$ to $v$. $\square$

**Lemma 6** *Suppose $v$ is a 5-vertex incident to two 3-faces $f_1 = [vv_1v_2]$ and $f_3 = [vv_3v_4]$. Let $v_5$ be the neighbour of $v$ not belonging to $f_1$ and $f_3$. Then the following holds.*
(C1) *If $f_1$ and $f_3$ are both $(5, 4^-, 4^-)$-faces, then $d(v_5) \geq 4$.*
(C2) *If $f_1$ is a $(5, *, 4)$-face and $f_3$ is a $(5, *, 4^+)$-face, then $d(v_5) \geq 4$.*
(C3) *$f_1$ and $f_3$ cannot be both $(5, *, 4)$-faces.*

**Proof.** In each of following cases, we will show that an $(L, 1)^*$-coloring of $G' \subset G$ can be extended to $G$, which is a contradiction.

(C1) We only need to show that $d(v_5) \neq 3$ since $\delta(G) \geq 3$ by (A1). Suppose that $v_5$ is a 3-vertex. Let $G' = G - \{v, v_1, \cdots, v_5\}$. By the minimality of $G$, $G'$ has an $(L, 1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. It is easy to deduce that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \cdots, 5\}$ and $|L_\pi(v)| \geq 3$. So we first color each $v_i$ with $\pi(v_i) \in L_\pi(v_i)$. Observe that there exists a color $a \in L_\pi(v)$ that appears at most once on the set $\{v_1, v_2, \cdots, v_5\}$. Therefore, we can color $v$ with $a$ to obtain an $(L, 1)^*$-coloring of $G$.

(C2) Suppose that $d(v_2) = 4$, $d(v_5) = 3$ and $v_1$ and $v_3$ are both $\mathcal{S}$-vertices. By definition, we see that $v_i$ is either a 3-vertex or a light 4-vertex, where $i \in \{1, 3\}$. Let $G' = G - \{v, v_1, v_2, v_3, v_5\}$. By the minimality of $G$, $G'$ has an $(L, 1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. The proof is split into two cases in light of the conditions of $v_3$.

- Assume $v_3$ is a 3-vertex. It is easy to calculate that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, 2, 3, 5\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, $\pi$ can be extended to $G$.
- Assume $v_3$ is a light 4-vertex. By definition, let $x_3, y_3$ denote the other two neighbors of $v_3$ not on $b(f_3)$. Recolor $x_3$ and $y_3$ with a color different from its neighbors. Next, we will show how to extend the resulting coloring $\pi'$ to $G$. Denote $L_{\pi'}$ be the induced assignment of $G - G'$. Notice that $|L_{\pi'}(v_i)| \geq 1$ for each $i \in \{1, 2, 5\}$. If $|L_{\pi'}(v_3)| \geq 1$, then by Lemma 2, $\pi'$ can be extended to $G$. Otherwise, we derive that $L(v_3) =$

$\{\pi'(x_3), \pi'(y_3), \pi'(v_4)\}$. First we assign a color in $L_{\pi'}(v_i)$ to each $v_i$, where $i \in \{1, 2, 5\}$. It is easy to see that there is at least one color, say $a$, belonging to $L(v) \setminus \{\pi'(v_4)\}$ that appears at most once on the set $\{v_1, v_2, v_5\}$. We assign such a color $a$ to $v$. Then color $v_3$ with a color in $\{\pi'(x_3), \pi'(y_3)\}$ but different from $a$.

(C3) Suppose that $f_1$ and $f_3$ are both $(5, *, 4)$-faces such that $d(v_2) = d(v_4) = 4$ and $v_1$ and $v_3$ are $\mathcal{S}$-vertices. Let $G' = G - \{v, v_1, \cdots, v_4\}$. Obviously, $G'$ has an $(L, 1)^*$-coloring $\pi$ by the minimality of $G$. Let $L_\pi$ be an induced list assignment of $G - G'$. We assert that $v_i$ satisfies that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \cdots, 4\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, we can extend $\pi$ to the whole graph $G$ successfully. $\square$

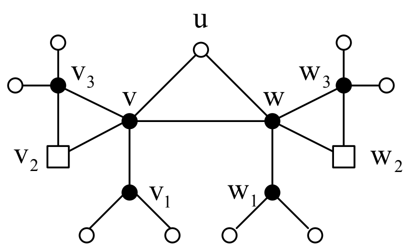


Figure 4: The configuration in Lemma 7.

**Lemma 7** *There is no* $3$*-face incident to two bad* $5$*-vertices.*

**Proof.** Suppose to the contrary that there is a $3$-face $[uvw]$ incident to two bad $5$-vertices $v$ and $w$, depicted in Figure 4. Let $G' = G - \{v, w, v_1, v_2, v_3, w_1, w_2, w_3\}$. By the minimality of $G$, $G'$ has an $(L, 1)^*$-coloring $\pi$. Let $L_\pi$ be an induced list assignment of $G - G'$. Since each $w_i$ has at most two neighbors in $G'$, we deduce that $|L_\pi(w_i)| \geq 1$ for each $i \in \{1, 2, 3\}$. So we first color each $w_i$ with a color $\pi(w_i) \in L_\pi(w_i)$. If $|L_\pi(w)| \geq 1$, namely $L(w) \neq \{\pi(u), \pi(w_1), \pi(w_2), \pi(w_3)\}$, then by Lemma 2 we may easy extend $\pi$ to $G$, since $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, 2, 3\}$. Otherwise, we deduce that there exists a color $a$ in $L(w) \setminus \{\pi(u)\}$ that is the same as $\pi(w_{i^*})$ for some fixed $i^* \in \{1, 2, 3\}$. Color $w$ with $a$ and $v_i$ with a color $\pi(v_i) \in L_\pi(v_i)$ firstly, where $i \in \{1, 2, 3\}$. For our simplicity, denote $V^* = \{v_1, v_2, v_3, w\}$.

First, suppose that there is a color, say $b \in L(v) \setminus \{\pi(u)\}$, appearing at most once on the set $V^*$. We assign such a color $b$ to $v$. If $b \neq a$, the obtained coloring is obvious an $(L, 1)^*$-coloring. Otherwise, assume that $b = a$. Now we erase the color $a$ from $w$. One may check that the resulting coloring, say $\pi'$, satisfies that each of $v, w_1, w_2, w_3$ has at least one possible color in $G - G'$. In other words, $|L_{\pi'}(s)| \geq 1$ for each $s \in \{v, w_1, w_2, w_3\}$. Hence, by Lemma 2, we can easily extend $\pi'$ to $G$.

Now, w.l.o.g., suppose that $L(v) = \{1, 2, 3\}$, $\pi(u) = 1$, $\pi(w) = 2$ and each color in $\{2, 3\}$ appears exactly twice on the set $V^*$. It implies that $\pi(v_1) \in \{2, 3\}$. We apply versions of discussion (i) and (ii) in the proof of Lemma 2. After doing that, one may check that now $v$ is colored with $\pi(v_2)$ and $v_1$ is recolored with a new color, say $\alpha$. There are two cases left to discuss: if $\pi(v_2) = 3$, namely the new color of $v$ is 3, then the obtained coloring is an $(L, 1)^*$-coloring and thus we are done; otherwise, we uncolor $w$. Again, it is easy to see that the resulting coloring, say $\pi''$, satisfies that $|L_{\pi''}(s)| \geq 1$ for each $s \in \{v, w_1, w_2, w_3\}$. Therefore, we can easily extend $\pi''$ to $G$ successfully by Lemma 2. $\square$

## 3.2 Discharging progress

We now apply a discharging procedure to reach a contradiction. Suppose that $u$ is adjacent to a 3-vertex $v$ such that $uv$ is not incident to any 3-faces. We call $v$ a *free* 3-vertex if $t(v) = 0$ and a *pendant* 3-vertex if $t(v) = 1$. For simplicity, we use $\nu_3(u)$ to denote the number of free 3-vertices adjacent to $u$ and $p_3(u)$ to denote the number of pendant 3-vertices of $u$. Suppose that $v$ is a soft 4-vertex such that $f_1 = [vv_1uv_2]$ is a 4-face and $d(v_3) = d(v_4) = 3$. If the opposite face to $f_1$ via $v$, i.e., $f_3$, is of degree at least 5, then we call $v$ a *weak* 4-vertex. We notice that every weak 4-vertex is soft but not vice versa.

For $x \in V(G)$ and $y \in F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from $x$ to $y$. Suppose that $f = [v_1v_2v_3]$ is a 3-face. We use $(d(v_1), d(v_2), d(v_3)) \to (c_1, c_2, c_3)$ to denote $\tau(v_i \to f) = c_i$ for $i = 1, 2, 3$. Our discharging rules are defined as follows:

(R1) Let $f = [v_1v_2v_3]$ be a 3-face. We set

(R1.1) $(3, 4, 5^+) \to (0, 1, 3)$;

(R1.2) $(3, 5^+, 5^+) \to (0, 2, 2)$;

(R1.3)

$$(4, 4, 5^+) \to \begin{cases} (0, 1, 3) & \text{if } v_1 \text{ is a light 4-vertex;} \\ (1, 1, 2) & \text{if neither } v_1 \text{ nor } v_2 \text{ is a light 4-vertex.} \end{cases}$$

(R1.4)

$$(4, 5^+, 5^+) \to \begin{cases} (1, 1, 2) & \text{if } v_2 \text{ is a bad 5-vertex;} \\ (0, 2, 2) & \text{if neither } v_2 \text{ nor } v_3 \text{ is a bad 5-vertex.} \end{cases}$$

(R1.5)

$$(5^+, 5^+, 5^+) \to \begin{cases} (1, \frac{3}{2}, \frac{3}{2}) & \text{if } v_1 \text{ is a bad 5-vertex;} \\ (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) & \text{if none of } v_1, v_2, v_3 \text{ is a bad 5-vertex.} \end{cases}$$

(R2) Suppose that $v$ is a $5^+$-vertex incident to a 4-face $f = [vv_1uv_2]$. Then

(R2.1) $\tau(v \to f) = 1$ if $d(v_1) \geq 4$ and $d(v_2) \geq 4$;

(R2.2) $\tau(v \to f) = \frac{4}{3}$ otherwise.

(R3) Suppose that $v$ is a non-weak 4-vertex incident to a 4-face $f = [vv_1uv_2]$.

(R3.1) Assume $d(v_1) = d(v_2) = 3$. Then

(R3.1.1) $\tau(v \to f) = \frac{4}{3}$ if the opposite face to $f$ via $v$ is of degree 3;

(R3.1.2) $\tau(v \to f) = \frac{2}{3}$ otherwise.

(R3.2) Assume $d(v_1) \geq 4$ and $d(v_2) \geq 4$. Then

(R3.2.1) $\tau(v \to f) = 1$ if at least one of $v_1$ and $v_2$ is a soft 4-vertex;

(R3.2.2) $\tau(v \to f) = \frac{2}{3}$ otherwise.

(R3.3) Assume $d(v_1) = 3$ and $d(v_2) \geq 4$. Then $\tau(v \to f) = \frac{2}{3}$.

(R4) Every $4^+$-vertex sends 1 to each pendant 3-vertex and $\frac{1}{3}$ to each free 3-vertex.

According to (R3), we notice that a weak 4-vertex does not send any charge.

We first consider the faces. Let $f$ be a $k$-face.

**Case** $k = 3$. Initially $\omega(f) = -4$. Let $f = [v_1v_2v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. By (A1), $d(v_1) \geq 3$. If $d(v_1) = 3$, then $d(v_2) \geq 4$ by (A2). Together with (B2), we deduce that $f$ is either a $(3, 4, 5^+)$-face, a $(3, 5^+, 5^+)$-face, a $(4, 4, 5^+)$-face, a $(4, 5^+, 5^+)$-face or a $(5^+, 5^+, 5^+)$-face. It follows from (B3) and Lemma 7 that every possibility is indeed covered by rule (R1). Obviously, $f$ takes charge 4 in total from its incident vertices. Therefore, $\omega^*(f) = -4 + 4 = 0$.

**Case** $k = 4$. Clearly, $w(f) = -2$. Assume that $f = [vxuy]$ is a 4-face. By (A2), there are no adjacent 3-vertices in $G$. It follows that $f$ is incident to at most two 3-vertices. By symmetry, we have to discuss three cases depending on the conditions of these 3-vertices.

- $d(x) = d(y) = 3$. By (F1), we deduce that at least one of $u$ and $v$ is of degree at least 5. Moreover, if one of $u$ and $v$ is a 4-vertex, say $v$, we claim that $v$ cannot be weak by definition and (B1). Hence, $\omega^*(f) \geq -2 + \frac{4}{3} + \frac{2}{3} = 0$ by (R2) and (R3).

- $d(x) = 3$ and $d(y) \geq 4$. Note that $u$ and $v$ are both $4^+$-vertices. Similarly, neither $u$ nor $v$ can be a weak 4-vertex. It follows from (R3.3) and (R2) that each of $u$ and $v$ sends charge at least $\frac{2}{3}$ to $f$. So if one of them is a $5^+$-vertex, say $v$, then by (R2) we have that $\tau(v \to f) = \frac{4}{3}$ and thus $f$ gets $\frac{2}{3} + \frac{4}{3} = 2$ in total from incident vertices of $f$. Otherwise, suppose $d(u) = d(v) = 4$. Now by (F2), $y$ cannot be a soft 4-vertex and thus not weak. Hence, $\omega^*(f) \geq -2 + \frac{2}{3} \times 3 = 0$ by (R3.2).

- $d(x) \geq 4$ and $d(y) \geq 4$. Namely, $f$ is a $(4^+, 4^+, 4^+, 4^+)$-face. If at most one of $u, v, x, y$ is a weak 4-vertex, then $\omega^*(f) \geq -2 + \frac{2}{3} \times 3 = 0$. Otherwise, by Lemma 5, assume that $v$ and $u$ are weak 4-vertices and thus soft. We see that $\tau(x \to f) = \tau(y \to f) = 1$ by (R3.2.1) and (R2.1) which implies that $\omega^*(f) \geq -2 + 1 \times 2 = 0$.

**Case** $k \geq 5$. Then $\omega^*(f) = \omega(f) = 2d(f) - 10 \geq 0$.

Now we consider the vertices. Let $v$ be a $k$-vertex with $k \geq 3$ by (A1). For $v \in V(G)$, we use $m_4(v)$ to denote the number of 4-faces incident to $v$. So by Observation 1 (a) and (b), we derive that $t(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ and $m_4(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. Furthermore, $t(v) + m_4(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ by Observation 1 (c).

**Observation 2** *Suppose $v$ is a $4^+$-vertex which is incident to a 3-face $f$. Then, by (R1), we have the following:*
(a) $\tau(v \to f) \leq 1$ *if* $d(v) = 4$*;*
(b) $\tau(v \to f) \in \{3, 2, \frac{3}{2}, \frac{4}{3}, 1\}$ *if* $d(v) \geq 5$*; moreover, if* $\tau(v \to f) = 3$ *then $f$ is a $(5^+, *, 4)$-face.*

**Case** $k = 3$. Then $\omega(v) = -1$. Clearly, $t(v) \leq 1$. If $t(v) = 1$, then there exists a neighbor of $v$, say $u$, so that $v$ is a pendant 3-vertex of $u$. By (A2), $d(u) \geq 4$. Thus, $\omega^*(v) = -1 + 1 = 0$ by (R4). Otherwise, we obtain that $\omega^*(v) = -1 + \frac{1}{3} \times 3 = 0$ by (R4).

**Case** $k = 4$. Then $\omega(v) = 2$. Note that $t(v) \leq 2$. If $t(v) = 2$, then $m_4(v) = 0$ and $p_3(v) = 0$. So $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by Observation 2 (a). If $t(v) = 0$, then $n_3(v) \leq 2$ by (B1) and $m_4(v) \leq 2$. We need to consider following cases.

- $m_4(v) = 2$. W.l.o.g., assume that $f_1 = [vv_1uv_2]$ and $f_3 = [vv_3wv_4]$ are incident 4-faces. Obviously, $p_3(v) = 0$ by Observation 1 (b). However, $\nu_3(v) \leq 2$ by (B1). By (R3), $v$ sends charge at most 1 to $f_i$, where $i = 1, 3$. If $n_3(v) = 0$, then $\nu_3(v) = 0$ and thus $\omega^*(v) \geq 2 - 1 \times 2 = 0$. If $n_3(v) = 1$, say $v_1$ is a 3-vertex, then $\tau(v \to f_1) \leq \frac{2}{3}$ by (R3.3) and thus $\omega^*(v) \geq 2 - \frac{2}{3} - 1 - \frac{1}{3} = 0$ by (R4). Now suppose that $n_3(v) = 2$. By symmetry, we have two cases depending on the conditions of these two 3-vertices. If $d(v_1) = d(v_2) = 3$, then $\tau(v \to f_1) = \frac{2}{3}$ by (R3.1.2). By (B1), $v_3$ and $v_4$ are both $4^+$-vertices. Moreover, neither $v_3$ nor $v_4$ is a soft 4-vertex according to Lemma 5. So by (R3.2.2), $\tau(v \to f_3) \leq \frac{2}{3}$. Hence $\omega^*(v) \geq 2 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \times 2 = 0$. Otherwise, suppose that $d(v_i) = d(v_j) = 3$, where $i \in \{1, 2\}$ and $j \in \{3, 4\}$. We derive that $\omega^*(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R3.3).

- $m_4(v) = 1$. W.l.o.g, assume that $d(f_1) = 4$. This implies that $d(f_3) \geq 5$. Again, $\tau(v \to f_1) \leq 1$ by (R3). If $n_3(v) \leq 1$ then we have that $\omega^*(v) \geq 2 - 1 - 1 = 0$ by (R4). So in what follows, we assume that $n_3(v)=2$. If $d(v_3)=d(v_4)=3$ then $v$ is a weak 4-vertex, implying that $v$ sends nothing to $f_1$. So $\omega^*(v)\geq 2-1\times 2=0$ by (R4). If $d(v_1)=d(v_2)=3$, then $p_3(v)=0$ by Observation 1 (b). We deduce that $\omega^*(v)\geq 2-\frac{2}{3}-\frac{1}{3}\times 2=\frac{2}{3}$ by (R3.1.2) and (R4). Otherwise, suppose $d(v_i)=d(v_j)=3$, where $i\in\{1,2\}$ and $j\in\{3,4\}$. It follows immediately from (R3.3)and (R4) that $\omega^*(v)\geq 2-\frac{2}{3}-1-\frac{1}{3}=0$.

- $m_4(v)=0$. Obviously, $\omega^*(v)\geq 2-1\times 2=0$ by (R4).

Now, in the following, we consider the case $t(v)=1$. Assume that $f_1$ is a 3-face. By (A1) and (B2), $f_1$ is either a $(4,3,5^+)$-face, a $(4,4,5^+)$-face or a $(4,5^+,5^+)$-face. Observe that $m_4(v)\leq 1$. First assume that $m_4(v)=0$. If $f_1$ is a $(4,3,5^+)$-face, then $p_3(v)\leq 1$ by (B1) and hence $\omega^*(v)\geq 2-1-1=0$ by Observation 2 (a) and (R2). Next suppose that $f_1$ is a $(4,4,5^+)$-face. If $n_3(v)=2$, then $v$ is a light 4-vertex. By (R1.3), we see that $v$ sends nothing to $f_1$ and therefore $\omega^*(v)\geq 2-1\times 2=0$ by (R4). Otherwise, at most one of $v_3,v_4$ is a 3-vertex and hence $\omega^*(v)\geq 2-1-1=0$ by Observation 2 (a) and (R4). Finally, we suppose that $f_1$ is a $(4,5^+,5^+)$-face. If neither $v_1$ nor $v_2$ is a bad 5-vertex, then $v$ sends nothing to $f_1$ by (R1.4) and thus $\omega^*(v)\geq 2-1\times 2=0$ by (R4). Otherwise, one of $v_1$ and $v_2$ is a bad 5-vertex. If follows directly from (C2) that $n_3(v)\leq 1$. Therefore, $\omega^*(v)\geq 2-1-1=0$ by (R2). Now suppose that $m_4(v)=1$. By Observation 1 (c), we may assume that $f_3=[vv_3wv_4]$ is a 4-face. In this case, $p_3(v)=0$. If $d(v_3)=d(v_4)=3$, then $\tau(v\to f_3)=\frac{4}{3}$ by (R3.1.1). It follows from (B1) and (C2) that $f$ is neither a $(4,3,5^+)$-face nor a $(4,5,5^+)$-face such that $v_2$ is a bad 5-vertex. So we deduce that $f_1$ gets nothing from $v$ by (R1.3), which implies that $\omega^*(v)\geq 2-\frac{4}{3}-\frac{1}{3}\times 2=0$. If exactly one of $v_3,v_4$ is a 3-vertex, then $\tau(v\to f_3)\leq\frac{2}{3}$ by (R3,3). Thus, $\omega^*(v)\geq 2-1-\frac{2}{3}-\frac{1}{3}=0$ by Observation 2 (a) and (R4). Otherwise, we suppose that $v_3,v_4$ are both of degree at least 4. In this case, $\nu_3(v)=0$ and hence $\omega^*(v)\geq 2-1-1=0$ by (R3.2) and Observation 2 (a).

**Case** $k=5$. Then $\omega(v)=5$. Also, $t(v)\leq 2$. we have three cases to discuss.

Assume $t(v)=0$. If $m_4(v)=0$, then $\omega^*(v)\geq 5-1\times 5=0$ by (R4). If $m_4(v)=1$, then $p_3(v)\leq 3$. Thus $\omega^*(v)\geq 5-\frac{4}{3}-1\times 3-2\times\frac{1}{3}=0$ by (R2) and (R4). Now suppose that $m_4(v)=2$. By Observation 1 (c), we assert that $p_3(v)\leq 1$. So $\omega^*(v)\geq 5-\frac{4}{3}\times 2-\frac{1}{3}\times 4-1=0$.

Next assume $t(v)=1$, say $f_1$. Then $\tau(v\to f_1)\leq 3$ by Observation 2 (b). Moreover, equality holds iff $f_1$ is a $(5,*,4)$-face. So if $\tau(v\to f_1)=3$ then at most one of $v_3,v_4,v_5$ is a 3-vertex by (B4). Furthermore, $m_4(v)\leq 1$. When $m_4(v)=0$, we deduce that $\omega^*(v)\geq 5-3-1=1$ by (R4). When $m_4(v)=1$, by symmetry, say $f_3$ is a 4-face, we have two cases to discuss: if $p_3(v)=1$, namely, $v_5$ is a 3-vertex, then $\tau(v\to f_3)\leq 1$ by (R2) and neither $v_3$ nor $v_4$ takes charge from $v$. Thus $\omega^*(v)\geq 5-3-1-1=0$; otherwise, $p_3(v)=0$ and we have $\omega^*(v)\geq 5-3-\frac{4}{3}-\frac{1}{3}=\frac{1}{3}$. Now suppose that $\tau(v \to f_1) \leq 2$. By (R2) and (R4), $\omega^*(v) \geq 5 - 2 - 1 \times 3 = 0$ if $m_4(v) = 0$ and $\omega^*(v) \geq 5 - 2 - \frac{4}{3} - 1 - 2 \times \frac{1}{3} = 0$ if $m_4(v) = 1$.

Now assume $t(v) = 2$. By symmetry, assume $f_1$ and $f_3$ are both 3-faces. Observe that $m_4(v) = 0$. For simplicity, denote $\tau(v \to f_1) = \sigma_1$ and $\tau(v \to f_3) = \sigma_2$. Let $\sigma = \max\{\sigma_1, \sigma_2\}$. If $\sigma \leq 2$, then $\omega^*(v) \geq 5 - 2 \times 2 - 1 = 0$ by (R2). Now assume that $\sigma = 3$, i.e., $f_1$ gets charge $3$ from $v$. It means that $f_1$ is a $(5, *, 4)$-face by Observation 2. By (C3), $f_3$ cannot be a $(5, *, 4)$-face. This implies that $\sigma_2 \leq 2$. Moreover, if $v_5$ is a $3$-vertex, then $f_3$ is neither a $(5, *, 4^+)$-face by (C2) nor a $(5, 4, 4)$-face by (C1). It follows from (R1.4) and (R1.5) that $\sigma_2 \leq 1$, since $v$ is a bad $5$-vertex. Thus, $\omega^*(v) \geq 5 - 3 - 1 - 1 = 0$ by (R2). Otherwise, we easily obtain that $\omega^*(v) \geq 5 - 3 - 2 = 0$.

**Case** $k \geq 6$. Notice that $t(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. If $v$ is incident to a $4$-face $f_i$, then by (R2) we inspect $v$ sends a charge at most $\frac{4}{3}$ to $f_i$, while $\frac{1}{3}$ to each of $v_i$ and $v_{i+1}$. So we may consider $v$ as a vertex which sends charge at most $\frac{4}{3} + 2 \times \frac{1}{3} = 2$ to $f_i$. So by (R4) and Observation 2, we have

$$\begin{aligned} \omega^*(v) &\geq 3d(v) - 10 - 3t(v) - 2m_4(v) - (d(v) - 2t(v) - 2m_4(v)) \\ &= 2d(v) - 10 - t(v) \equiv \tau(v) \end{aligned}$$

If $d(v) \geq 7$, then $\tau(v) \geq 2d(v) - 10 - \frac{d(v)}{2} = \frac{3}{2}d(v) - 10 \geq \frac{3}{2} \times 7 - 10 = \frac{1}{2} > 0$. Now suppose that $d(v) = 6$. If $t(v) \leq 2$ then $\tau(v) \geq 2 \times 6 - 10 - 2 = 0$. So, in what follows, assume that $t(v) = 3$ and $d(f_i) = 3$ for $i = 1, 3, 5$. Clearly, $m_4(v) = 0$. Similarly, if there are at most two of 3-faces get charge $3 \times 2$ in total from $v$, then $\omega^*(v) \geq 8 - 2 \times 3 - 2 = 0$. Otherwise, suppose $\tau(v \to f_i) = 3$ for each $i \in \{1, 3, 5\}$. By Observation 2 (b), we assert that $f_i$ is a $(6, *, 4)$-face. Noting that a $(6, *, 4)$-face is also a $(6, 4^-, 4^-)$-face, we may regard $v$ as a $6$-vertex which is incident to two $(6, 4^-, 4^-)$-faces and one $(6, *, 4)$-face. However, it is impossible by (B5).

Therefore, we complete the proof of Theorem 1. $\square$

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