MINIMUM DEGREES AND CODEGREES OF MINIMAL RAMSEY 3-UNIFORM HYPERGRAPHS

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Abstract. A uniform hypergraph H is called k-Ramsey for a hypergraph F, if no matter how one colors the edges of H with k colors, there is always a monochromatic copy of F. We say that H is minimal k-Ramsey for F, if H is k-Ramsey for F but every proper subhypergraph of H is not. Burr, Erdős and Lovasz [S. A. Burr, P. Erdős, and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria 1 (1976), no. 1, 167–190] studied various parameters of minimal Ramsey graphs. In this paper we initiate the study of minimum degrees and codegrees of minimal Ramsey 3-uniform hypergraphs. We show that the smallest minimum vertex degree over all minimal k-Ramsey 3-uniform hypergraphs for $K_t^{(3)}$ is exponential in some polynomial in k and t. We also study the smallest possible minimum codegrees over minimal 2-Ramsey 3-uniform hypergraphs.

1. Introduction and New Results

A graph G is said to be Ramsey for a graph F if no matter how one colors the edges of G with two colors, say red and blue, there is a monochromatic copy of F (we write $G \longrightarrow (F)_2$ for this). A classical result of Ramsey [12] states that for every F there is an integer n such that K_n is Ramsey for F. Moreover, generalizations to more than two colors and to hypergraphs hold as well [12]. We say that G is minimal Ramsey for F if G is Ramsey for F but every proper subgraph of G is not. More generally, we denote by $\mathcal{M}_k(F)$ the set of minimal graphs G with the property that no matter how one colors the edges of G with k colors, there is a monochromatic copy of F in it, and refer to these as minimal k-Ramsey graphs for F. There are many challenging open questions concerning the study of various parameters of minimal k-Ramsey graphs for various F. The most studied ones are the classical (vertex) Ramsey numbers $r_k(F) := \min_{G\in\mathcal{M}_k(F)} v(G)$ and the size Ramsey number $\hat{r}_k(F) := \min_{G\in\mathcal{M}_k(F)} e(G)$, where $v(G)$ is the number of vertices in G and $e(G)$ is its number of edges. To determine the classical Ramsey number $r_2(K_t)$ is a notorously difficult problem and essentially the best known bounds are $2^{(1+o(1))t/2}$ and $2^{(2+o(1))t}$ due to Spencer [14] and Conlon [3].

Burr, Erdős and Lovász [1] were the first to study other possible parameters of the class $\mathcal{M}_2(K_t)$. In particular they determined the minimum degree $s_2(K_t) := \min_{G\in\mathcal{M}_2(K_t)} \delta(G) = (t-1)^2$ which looks surprising given the exponential bound on the minimum degree of K_n with $K_n \longrightarrow (K_t)_2$ and $n = r_2(K_t)$ (it is not difficult to see that such K_n is indeed minimal 2-Ramsey for K_t). Generalizing their results, Fox, Grinshpun, Liebenau, Person and Szabó [7] studied the minimum degree $s_k(K_t) := \min_{G\in\mathcal{M}_k(K_t)} \delta(G)$ for more colors showing a general bound on $s_k(K_t) \le 8(t-1)^6k^3$ and proving quasiquadratic bounds in k on $s_k(K_t)$ for fixed t. Further results concerning minimal Ramsey graphs were studied in [2, 9, 13, 15, 8].

In this paper we initiate the study of minimal Ramsey 3-uniform hypergraphs and provide first bounds on various notions of minimum degrees for minimal Ramsey

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hypergraphs. Generally, an r-uniform hypergraph H is a tuple (V, E) with vertex set V and $E \subseteq \binom{V}{r}$ being its edge set. We define link(v), the *link* of a vertex $v \in V$, to be the edges of H that contain v, minus the vertex v (thus, these form an $(r-1)$-uniform hypergraph). Formally, the edge set of link(v) is $\{e \setminus \{v\} :\ v \in e \in E\}$. The random r-uniform hypergraph $H^{(r)}(n, p)$ is the probability space of all labeled r-uniform hypergraphs on the vertex set $[n]$ where each edge exists with probability p independently of the other edges. In this paper we will be dealing exclusively with 3-uniform hypergraphs, thus the links of their vertices are just the edges of some graph.

Ramsey's theorem holds for r-uniform hypergraphs as well as shown originally by Ramsey himself [12], and we write $G \longrightarrow (F)_k$, if no matter how one colors the edges of the r-uniform hypergraph G, there is a monochromatic copy of F. We denote by $K_t^{(r)}$ the complete r-uniform hypergraph with t vertices, i.e. $K_t^{(3)} = ([t], \binom{[t]}{r})$, and by the hypergraph Ramsey number $r_k(F)$ the smallest n such that $K_n^{(r)} \longrightarrow (F)_k$. While in the graph case the known bounds on $r_2(K_t)$ are only polynomially far apart, already in the case of 3-uniform hypergraphs the bounds on $r_2(K_t^{(r)})$ differ in one exponent: $2^{c_1 t^2} \leq r_2(K_t^{(3)}) \leq 2^{2^{c_2 t}}$ for some absolute positive constants c_1 and c_2. More generally, it holds $t_{r-1}(c_1 t^2) \leq r_2(K_t^{(r)}) \leq t_r(c_2 t)$ for some absolute constants $c_1 = c_1(r), c_2 = c_2(r) > 0$ and where $t_i(x)$ is the tower function defined by $t_1(x) := x$, $t_i(x) := 2^{t_{i-1}(x)}$. For further information on hypergraph Ramsey numbers we refer the reader to the standard book on Ramsey theory [10] and for newer results to the work of Conlon, Fox and Sudakov [4].

Given $\ell \in [r-1]$, we define the degree $\deg(S)$ of an ℓ-set S in an r-uniform hypergraph $H = (V, E)$ as the number of edges that contain S and the minimum ℓ-degree $\delta_\ell(H) := \min_{S \in \binom{V}{\ell}} \deg(S)$. For two vertices u and v we simply write $\deg(u, v)$ for the *codegree* $\deg(\{u, v\})$.

Similar to the graph case we extend verbatim the notion of minimal Ramsey graphs to minimal Ramsey r-uniform hypergraphs $\mathcal{M}_k(F)$ in a natural way. That is, $\mathcal{M}_k(F)$ is the set of all minimal k-Ramsey r-uniform hypergraphs H with $H \longrightarrow (F)_k$. We define

$$s_{k,\ell}(K_t^{(r)}) := \min_{G \in \mathcal{M}_k(K_t^{(r)})} \delta_\ell(G), \tag{1}$$

which extends the introduced graph parameter $s_2(K_t)$. It will be shown actually that $s_{2,2}(K_t^{(3)})$ is zero and thus it makes sense to ask for the second smallest value of the codegrees. This motivates the following parameter $s'_{k,\ell}(K_t^{(r)})$:

$$s'_{k,\ell}(K_t^{(r)}) := \min_{G \in \mathcal{M}_k(K_t^{(r)})} \left(\min \left\{ \deg_G(S) \colon S \in \binom{V(G)}{\ell}, \deg_G(S) > 0 \right\} \right).$$

We prove the following results on the minimum degree and codegree of minimal Ramsey 3-uniform hypergraphs for cliques $K_t^{(3)}$.

Theorem 1. *The following holds for all $t \geq 4$ and $k \geq 2$*

$$2^{\frac{1}{2}kt(1-o(1))} \leq \binom{r_k(K_{t-1})}{2} \leq s_{k,1}(K_t^{(3)}) \leq k^{20kt^4}. \tag{2}$$

For the lower bound see [4].

Theorem 2. *Let $t \geq 4$ be an integer. Then,*

$$s_{2,2}(K_t^{(3)}) = 0 \text{ and } s'_{2,2}(K_t^{(3)}) = (t-2)^2.$$

Observe that with $s'_{2,2}$ we ask for the smallest *positive* codegree, while for $s_{2,2}$ we also allow the codegree to be zero. This in particular means that in *any* minimal

2-Ramsey hypergraph H for $K_t^{(3)}$ we have that a pair of vertices u and v are either not contained in a common edge or have codegree at least $(t-2)^2$.

Methods. The methods we are going to use are generalizations of signal senders introduced first by Burr, Erdős and Lovász in [1], and generalized later by Burr, Nešetřil and Rödl [2] and by Rödl and Siggers [13], that we combine with probabilistic arguments analyzing certain properties of random 3-uniform hypergraphs.

Organization of the paper. In the next section, Section 2, we generalize "almost" Ramsey graphs, i.e. graphs whose edge colorings without a monochromatic copy of some complete graph K_t impose certain color pattern, first introduced by Burr, Erdős and Lovász [1] to hypergraphs. Then we study in Section 3 the vertex degree for minimal k-Ramsey 3-uniform hypergraphs for $K_t^{(3)}$, while in Section 4 we look into the case of codegrees in minimal 2-Ramsey 3-uniform hypergraphs for $K_t^{(3)}$.

2. BEL-Gadgets for 3-uniform hypergraphs

First we show a lemma that asserts the existence of a 3-uniform hypergraph H and two edges f, $e \in E(H)$ with $|f \cap e| = 2$ and $e(H[e \cup f]) = 2$ so that H is not k-Ramsey for $K_t^{(3)}$ with the property that any k-coloring of $E(H)$ without a monochromatic $K_t^{(3)}$ colors the edges e and f differently. We will refer to such hypergraphs that impose certain structure on $K_t^{(3)}$-free colorings as *BEL-gadgets*. Moreover, we refer in the following to a coloring without a monochromatic copy of F as an *F-free coloring*.

Lemma 3. *Let $t \geq 4$ and $k \geq 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$ and two edges $e_{\mathcal{H}}, f_{\mathcal{H}} \in E(\mathcal{H})$ with $|f_{\mathcal{H}} \cap e_{\mathcal{H}}| = 2$ and $e(\mathcal{H}[e_{\mathcal{H}} \cup f_{\mathcal{H}}]) = 2$ such that the following properties hold:*

(1) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$,

(2) for every k-coloring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $c(e_{\mathcal{H}}) \neq c(f_{\mathcal{H}})$.

Proof. Set $m = r_k(K_t^{(3)})$ and define a hypergraph F' on the vertex set $[m]$ as follows: delete from $K_m^{(3)}$ all edges that contain vertices $m-1$ and m. It is easy to see that then $F' \not\longrightarrow (K_t^{(3)})_k$. Indeed, fix a k-coloring of $E(K_{m-1}^{(3)})$ without a monochromatic $K_t^{(3)}$, then extend this coloring to $E(F')$ by coloring each edge (x, y, m) with the color of $(x, y, m-1)$. Since every copy of $K_t^{(3)}$ in F' may contain at most one of the vertices $m-1$ and m, we see $F' \not\longrightarrow (K_t^{(3)})_k$.

Define $F_i := ([m], E(F') \cup \{\{j, m-1, m\} : j \leq i\})$ and set $F := F_\ell$ where ℓ is maximal such that F_ℓ is not k-Ramsey for $K_t^{(3)}$ but $F_{\ell+1}$ is (this is possible since $F_{m-2} = K_m^{(3)}$ is k-Ramsey for $K_t^{(3)}$ by the choice of $m = r_k(K_t^{(3)})$).

For a coloring $\psi \colon E(F) \to [k]$ without a monochromatic copy of $K_t^{(3)}$ we define an *admissible pattern* $(a_1, \ldots, a_k)$, where a_i denotes the number of edges in the color i containing both vertices $m-1$ and m. Moreover, with $\mathcal{P}$ we denote the set of all admissible patterns. In particular, by the choice of ℓ we have that $\mathcal{P} \neq \emptyset$.

Notice that $\sum_{i \in [k]} a_i = \ell$ for every $(a_1, \ldots, a_k) \in \mathcal{P}$, and $a_c \notin \{0, \ell\}$ for every $c \in [k]$. Indeed if, say, there is a pattern $(a_1, \ldots, a_k) \in \mathcal{P}$ with $a_j = 0$ for some $j \in [k]$, then we could take a corresponding k-coloring of the edges of F_ℓ avoiding monochromatic copies of $K_t^{(3)}$ with pattern $(a_1, \ldots, a_k)$, which then we would extend to a k-coloring of $E(F_{\ell+1})$ without a monochromatic copy of $K_t^{(3)}$ just by coloring the edge $\{\ell+1, m-1, m\}$ in color j. Indeed, this new edge cannot participate in a monochromatic copy of $K_t^{(3)}$ in this coloring, as its color is j, while all other edges containing both $m-1$ and m have colors different from j. But this is a contradiction to the definition of ℓ.

Moreover, notice that the following holds: If $\varphi\colon [\ell] \to [k]$ is a coloring of the first ℓ vertices of F such that $(|\varphi^{-1}(1)|, \dots, |\varphi^{-1}(k)|) \in \mathcal{P}$, then there exists a coloring $c\colon E(F) \to [k]$ avoiding monochromatic copies of $K_t^{(3)}$ such that $c(i, m-1, m) = \varphi(i)$ for every $i \in [\ell]$.

Now, let H be an ℓ-uniform hypergraph. We say that a coloring $\psi\colon V(H) \to [k]$ is admissible, if for every edge $e \in E(H)$ we have $(c_1, \dots, c_k) \in \mathcal{P}$ where c_i denotes the number of vertices in e colored i.

Now we proceed analogously to Claim 2 from [1]. We find an ℓ-uniform hypergraph H^* with $\mathrm{girth}(H^*) \geq 3$ (this means that any two distinct edges e and f satisfy $|e \cap f| \leq 1$) and two vertices $x, y \in V(H^*)$ with $\deg_{H^*}(x, y) = 0$ such that there exist admissible colorings for H^* and in every such coloring the color of x differs from the color of y. For completeness we provide this elegant argument here. We start with an ℓ-uniform hypergraph H with $\mathrm{girth}(H) \geq 3$ and chromatic number $\chi(H) \geq k+1$. It was shown that such hypergraphs exist by Erdős and Hajnal in [6].

Then, as every k-coloring of the vertices of H yields a monochromatic edge, while $(\ell, 0, \dots, 0)$,…,$(0, \dots, 0, \ell) \notin \mathcal{P}$, H does not have admissible colorings. Now, we can take a subhypergraph H' of H which is minimal (with respect to the number of edges) for the property of not having admissible k-colorings. For an arbitrary edge $f = \{x_1, \dots, x_\ell\} \in H'$ and arbitrary vertices $y_1, \dots, y_\ell \notin V(H')$, we define a sequence of hypergraphs H_i on $V(H') \cup \{y_1, \dots, y_i\}$ with $H_i = H' - f + f_i$, where $f_i = \{y_1, \dots, y_i, x_{i+1}, \dots, x_\ell\}$. By the definition, $H_0 = H'$ does not have admissible colorings while H_ℓ does, so there is a minimal index $i \in [\ell]$ such that H_{i-1} does not have admissible colorings, but H_i does. We now set $H^* = H_i$ and $x := x_i$, $y := y_i$. It is clear that $\mathrm{girth}(H^*) \geq 3$, $\deg_{H^*}(x, y) = 0$ and that H^* has admissible colorings. Moreover, for any such admissible k-coloring x and y need to have distinct colors as otherwise, by taking an admissible coloring of H_i with x and y colored the same and then identifying x with y would yield an admissible coloring of H_{i-1}, a contradiction.

Finally, we define a 3-uniform hypergraph $\mathcal{H}$ as follows. First we introduce for each $e \in E(H^*)$ a set $V_e := e \cup \{m-1, m\} \cup (\{e\} \times \{\ell+1, \dots, m-2\})$ and then we define a 3-uniform hypergraph F_e which is a copy of $F = F_\ell$ that contains all vertices from e as follows:

$$F_e := \left(V_e, \binom{V_e}{3} \setminus \{\{(e,i), m-1, m\}\colon i = \ell+1, \dots, m-2\} \right).$$

The hypergraph $\mathcal{H}$ is then the union over all F_e's: $\mathcal{H} := \cup_{e \in E(H^*)} F_e$. In other words, we obtain $\mathcal{H}$ by placing F_e, a copy of F, for each edge $e \in E(H^*)$ so that the vertices $\{1, \dots, \ell\}$ of F are identified with e. Further, we set $e_{\mathcal{H}} = \{m-1, m, x\}$ and $f_{\mathcal{H}} = \{m-1, m, y\}$. Before showing that $\mathcal{H}$, $e_{\mathcal{H}}$ and $f_{\mathcal{H}}$ fulfill the requirements (1) and (2), we establish the following claim.

Claim 4. *Any copy K of $K_t^{(3)}$ in $\mathcal{H}$ is contained in F_e for some $e \in E(H^*)$.*

Proof. Assume first $V(K) \setminus (\{m-1, m\} \cup V(H^*)) \neq \emptyset$ holds. Thus K contains a vertex of the form (e, s), whose link is a graph on $m-1$ vertices which must form the set $V_e \setminus \{(e, s)\}$, by construction of $\mathcal{H}$. This, with $\mathcal{H}[V_e] = F_e$, then implies that $K \subseteq F_e$.

From now on we may assume that $V(K) \subseteq V(H^*) \cup \{m-1, m\}$. First we assume that $K \cong K_4^{(3)}$ and $m-1, m \in V(K)$. Thus, the remaining two vertices, call them a and b, must lie in some edge $e \in E(H^*)$ (since $\{m, a, b\}$ is an edge in $\mathcal{H}[V(H^*) \cup \{m-1, m\}]$), which implies $K \subseteq F_e$. Finally, we may assume that $|V(K) \cap V(H^*)| \geq 3$ and setting $S := V(K) \cap V(H^*)$ we have $K[S] \cong K_s^{(3)}$, $s \geq 3$. Since $\mathcal{H}[V(H^*)]$ consists of cliques $K_\ell^{(3)}$ that intersect in at most one vertex as $\operatorname{girth}(H^*) \geq 3$, this implies that S has to be contained in some $e \in E(H^*)$. Again this yields $K \subseteq F_e$. □

Recall that we defined $e_{\mathcal{H}} = \{m-1, m, x\}$ and $f_{\mathcal{H}} = \{m-1, m, y\}$. By construction of $\mathcal{H}$ and since $\deg_{H^*}(x, y) = 0$, it is clear that $\{x, y, m-1\}$ and $\{x, y, m\}$ are nonedges in $\mathcal{H}$. We now prove that this choice of $\mathcal{H}$, $e_{\mathcal{H}}$ and $f_{\mathcal{H}}$ fulfills the requirements (1) and (2) of our lemma:

(1) By construction there exists an admissible coloring $c\colon V(H^*) \to [k]$. Notice that two hypergraphs F_e and F_f for distinct $e, f \in E(H^*)$ have in common both vertices $m-1$ and m and additionally at most one further vertex v (and if so also the edge $\{v, m-1, m\}$), by construction and since $\operatorname{girth}(H^*) \geq 3$. Since $\mathcal{H}$ consists of copies of F that intersect pairwise in at most one edge (containing both vertices $m-1$ and m), we can find colorings of these copies without monochromatic $K_t^{(3)}$ so that these colorings agree on common edges $\{v, m-1, m\}$. Indeed, for every edge $e \in E(H^*)$ we have an admissible color pattern $(d_1, \ldots, d_k) \in \mathcal{P}$ which depends on c. Thus, there exists a coloring $\varphi_e\colon E(F_e) \to [k]$ without monochromatic $K_t^{(3)}$ so that $\varphi_e(\{v, m-1, m\}) = c(v)$ for all $v \in e$.

We need to show that the union of φ_e over all $e \in E(H^*)$ gives us a k-coloring φ of $E(\mathcal{H})$ without monochromatic copies of $K_t^{(3)}$. By Claim 4, any copy of $K_t^{(3)}$ is contained in F_e for some $e \in E(H^*)$. Since $E(F_e)$ does not contain any monochromatic $K_t^{(3)}$ under φ_e, the requirement (1) is verified.

(2) Now, let $c\colon E(\mathcal{H}) \to [k]$ be a coloring on the edge set of $\mathcal{H}$ which avoids monochromatic copies of $K_t^{(3)}$. Define $\varphi : V(H^*) \to [k]$ with $\varphi(v) := c(\{v, m-1, m\})$. Then φ is an admissible coloring of H^* and thus, by the properties of H^* we know that $c(e_{\mathcal{H}}) = \varphi(x) \neq \varphi(y) = c(f_{\mathcal{H}})$. □

We introduce the following definition of a path in hypergraphs. In an r-uniform path (or r-path for short notation) with t edges e_1,…,e_t the vertices of $\cup_{i\in[t]} e_i$ are ordered linearly and the edges are *consecutive* segments with the property that $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in [t-1]$. We will refer to the edges e_1 and e_t as *ends* of such a path. In particular, in our notation the path is a vertex-connected subhypergraph of a so-called tight path on the vertex set $\cup_{i\in[t]} e_i$ (where in a tight path it is $|e_i \cap e_{i+1}| = r - 1$).

Further we say that two edges e and f have distance $\operatorname{dist}_H(e, f) := s$ in H if any r-uniform path in H with ends e and f contains at least s vertices and there exists at least one such path with exactly s vertices. We call a path from e to f with $\operatorname{dist}_H(e, f)$ vertices a shortest path. If no such path exists, we set $\operatorname{dist}_H(e, f) := \infty$.

First we show a lemma that allows us to obtain a "rainbow star".

Lemma 5. *Let $t \geq 4$ and $k \geq 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$, a 2-element set $S \subseteq V(\mathcal{H})$ and edges $e_1, \ldots, e_k \in E(\mathcal{H})$ with $e_i \cap e_j = S$ (for all $i \neq j \in [k]$), $|\cup_{i\in[k]} e_i| = k+2$ and $e(\mathcal{H}[\cup_{i\in[k]} e_i]) = k$ such that the following properties hold:*

(1) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$,

(2) for every k-coloring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $\{c(e_i)\colon i \in [k]\} = [k]$, that is the colors of e_is are all distinct.

Proof. Take $\binom{k}{2}$ vertex-disjoint copies $(\mathcal{H}_{ij})_{1\le i<j\le k}$ of the hypergraph $\mathcal{H}'$ as guaranteed to us by Lemma 3, and let e_{ij} and f_{ij} be the corresponding edges of $\mathcal{H}'$ that satisfy Property (2) of Lemma 3. We start with the hypergraph H on the vertex set $[k+2]$ and with edge set $\{\{i,k+1,k+2\} : i \in [k]\}$, and we set $S := \{k+1,k+2\}$.

We construct the hypergraph $\mathcal{H}$ as follows. For each $i < j \in [k]$ we identify the vertices $k+1$ and $k+2$ (arbitrarily) with the two vertices from $C_{ij} := e_{ij} \cap f_{ij}$ and the only vertex from $e_{ij} \setminus C_{ij}$ is identified with i while the only vertex from $f_{ij} \setminus C_{ij}$ is identified with j. Otherwise the hypergraphs $\mathcal{H}_{ij}$ don't intersect each other in further vertices. We claim that the properties from Lemma 5 are satisfied. Indeed, since $\mathcal{H}_{ij} \not\to \left(K_t^{(3)}\right)_k$ and by the symmetry of the colors, we can assume that there is a $K_t^{(3)}$-free coloring φ_{ij} of $\mathcal{H}_{ij}$ such that $\varphi(e_{ij}) = i$ and $\varphi(f_{ij}) = j$ (and $i < j$). We obtain the coloring φ of $\mathcal{H}$ by coloring the corresponding edges according to appropriate φ_{ij}s. This is possible since the edge $\{i,k+1,k+2\}$ is identified with e_{ij} and $f_{\ell i}$ for $\ell < i < j$, and these are colored with the color i. The coloring φ is $K_t^{(3)}$-free, since each copy of $K_t^{(3)}$ is contained in one of the $\mathcal{H}_{ij}$s. To see Property (2), we use the Property (2) of Lemma 3, which asserts that in any $K_t^{(3)}$-free coloring of $\mathcal{H}$ the edges $\{i,k+1,k+2\}$ and $\{j,k+1,k+2\}$ are colored differently (with $i < j$). $\square$

The next lemma allows us to construct a BEL-gadget that colors two edges the same.

Lemma 6. *Let $t \ge 4$ and $k \ge 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$ and edges e and f with $|e \cap f| = 2$ and $e(\mathcal{H}[e \cup f]) = 2$ such that the following properties hold:*

(1) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$,

(2) for every k-coloring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $c(e) = c(f)$.

Proof. We take two vertex-disjoint copies of $\mathcal{H}_1$ and $\mathcal{H}_2$ as asserted by Lemma 5, along with the corresponding edges $e_{1,1},\dots, e_{1,k}$ for $\mathcal{H}_1$ and $e_{2,1},\dots, e_{2,k}$ for $\mathcal{H}_2$ respectively. Recall that there exist S_1 and S_2 such that $e_{\ell,i} \cap e_{\ell,j} = S_\ell$ for all $i < j \in [k]$ and $\ell \in [2]$. We obtain the hypergraph $\mathcal{H}$ by identifying the edge $e_{1,i}$ with $e_{2,i}$ for all $2 \le i \le k$ such that the vertices from S_1 are identified with those from S_2.

We set $e := e_{1,1}$ and $f := e_{2,1}$ and claim that $\mathcal{H}$ fulfills the requirements. By the symmetry of the colors, we may assume that $e_{\ell,i}$ may be colored with the color i for all $i \in [k]$ and $\ell \in [2]$, and then we may extend the coloring by coloring the (otherwise disjoint) copies $\mathcal{H}_1$ and $\mathcal{H}_2$ separately. Since any copy of $K_t^{(3)}$ is contained fully either in $\mathcal{H}_1$ or in $\mathcal{H}_2$, we see $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$. On the other hand, any $K_t^{(3)}$-free coloring φ of $\mathcal{H}$ is a $K_t^{(3)}$-free coloring of $\mathcal{H}_1$ and $\mathcal{H}_2$, and from the properties from Lemma 5 we have that the edges $e_{\ell,1},\dots, e_{\ell,k}$ are colored differently for each $\ell \in [2]$ and, by the construction, $\varphi(e_{1,i}) = \varphi(e_{2,i})$ for all $2 \le i \le k$. Thus, we also have $\varphi(e_{1,1}) = \varphi(e_{2,1})$. $\square$

Finally, we construct BEL-gadgets with monochromatic edges in every $K_t^{(3)}$-free coloring that are "far" from each other.

Lemma 7. *Let s, $t \ge 4$ and $k \ge 2$ be integers. There exist a 3-uniform hypergraph H and two edges $e, f \in E(H)$ such that the following properties hold:*

(1) $H \not\to \left(K_t^{(3)}\right)_k$,

(2) e and f have distance at least s, and

(3) for every k-coloring φ on $E(H)$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $\varphi(e) = \varphi(f)$.

Proof. First we construct a hypergraph $\mathcal{H}$ which is not k-Ramsey for $K_t^{(3)}$, but contains two edges e and f at distance 5 that are colored the same by any k-coloring of $E(\mathcal{H})$ without monochromatic $K_t^{(3)}$. We apply Lemma 6 twice and obtain 3-uniform hypergraphs $\mathcal{H}_1$ with edges $e_{\mathcal{H}_1} = \{a, b, x_1\}$ and $f_{\mathcal{H}_1} = \{a, b, y_1\}$ and $\mathcal{H}_2$ with edges $e_{\mathcal{H}_2} = \{c, d, x_2\}$ and $f_{\mathcal{H}_2} = \{c, d, y_2\}$ respectively. Furthermore, we may assume $V(\mathcal{H}_1) \cap V(\mathcal{H}_2) = \emptyset$. We define a new hypergraph $\mathcal{H}$ by taking both $\mathcal{H}_1$ and $\mathcal{H}_2$ and identifying y_1 with d, b with c, and a with y_2. Observe that in $\mathcal{H}$ any copy of $K_t^{(3)}$ is completely contained within one of the $\mathcal{H}_i$'s. This implies that $\mathcal{H} \not\to (K_t^{(3)})_k$. Indeed, according to Lemma 3 we can color $\mathcal{H}_1$ and $\mathcal{H}_2$ without monochromatic $K_t^{(3)}$. Moreover, by swapping the colors appropriately if necessary, we may do so that the edges $f_{\mathcal{H}_1} \in E(\mathcal{H}_1)$ and $f_{\mathcal{H}_2} \in E(\mathcal{H}_2)$ receive the same color. This gives us a $K_t^{(3)}$-free coloring of $E(\mathcal{H})$.

Next we use the Property (2) of Lemma 6 which asserts that any $K_t^{(3)}$-free coloring colors the edges $\{a, b, x_1\}$ and $\{a, b, y_1\}$ the same, and the colors of $\{c, d, x_2\}$ and $\{c, d, y_2\}$ are the same as well. Since $\{a, b, y_1\} = \{c, d, y_2\}$ in $\mathcal{H}$, the edges $f := \{c, d, x_2\}$ and $e := \{a, b, x_1\}$ are colored the same through any $K_t^{(3)}$-free coloring of $\mathcal{H}$. We thus arrived at a hypergraph $\mathcal{H}$ that satisfies the following properties:

(a) there are two edges e and f at distance 5,

(b) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$,

(c) for every k-coloring c on $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $c(e) = c(f)$.

Next we proceed iteratively. We take two isomorphic hypergraphs H_1 and H_2, along with edges e_1, f_1 and e_2, f_2 respectively, which satisfy (b) and (c). Assuming that $\mathrm{dist}_{H_1}(e_1, f_1) = d = \mathrm{dist}_{H_2}(e_2, f_2)$ for some $d \geq 5$, we now aim to construct a hypergraph H', along with edges e, f, such that (b) and (c) hold and $\mathrm{dist}_{H'}(e, f) \geq d + 1$. For the construction, we identify the edge f_1 with e_2 such that none of the vertices of e_1 and f_2 are identified, and we set $e = e_1$ and $f = f_2$. This way the properties (b) and (c) are naturally preserved in H'.

Thus, it remains to show that the distance between e_1 and f_2 is at least $d + 1$ in H'. Let $v_1, \ldots, v_\ell$ be the vertices of a shortest path from e_1 to f_2 in H' in the linear order, i.e. $\{v_1, v_2, v_3\} = e_1$ and $\{v_{\ell-2}, v_{\ell-1}, v_\ell\} = f_2$. Let $i \geq 4$ be the smallest index such that $v_i \notin V(H_1)$. If $i < d - 1$, then we have $v_{i-1} \in f_1$ and in case $\{v_{i-3}, v_{i-2}, v_{i-1}\} \notin E(H_1)$ holds then we additionally have $\{v_{i-4}, v_{i-3}, v_{i-2}\} \in E(H_1)$ and $v_{i-2} \in f_1$. In any case we would obtain a 3-path from e_1 to f_1 with at most $d - 1$ vertices which consists of some edges of P contained in $\{v_1, \ldots, v_{i-1}\}$ and of the edge f_1, a contradiction to $\mathrm{dist}_{H_1}(e_1, f_1) = d$. Thus we may assume $i \geq d - 1$. If, additionally, $d > 5$ then it follows, that none of the vertices from f_2 are among $\{v_1, \ldots, v_{i-1}\}$ resulting in $\mathrm{dist}_{H'}(e_1, f_2) \geq d + 1$. If $d = 5$, then since none of the vertices of e_1 and f_2 are identified, $\mathrm{dist}_{H'}(e_1, f_2) \geq 6 > d$. □

Now we are in position to build non-Ramsey hypergraphs which assert more structure in any $K_t^{(3)}$-free coloring.

Theorem 8. *Let $k \geq 2$ and $t \geq 4$ be integers. Let H be a 3-uniform hypergraph with $H \not\to \left(K_t^{(3)}\right)_k$ and let $c\colon E(H) \to [k]$ be a k-coloring which avoids monochromatic* *copies of* $K_t^{(3)}$*. Then, there exists a 3-uniform hypergraph* $\mathcal{H}$ *with the following properties:*

(1) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$*,*
(2) $\mathcal{H}$ *contains* H *as an induced subhypergraph, and*
(3) for every coloring $\varphi\colon E(\mathcal{H}) \to [k]$ *without a monochromatic copy of* $K_t^{(3)}$*, the coloring of* H *under* φ *agrees with the coloring* c*, up to a permutation of the* k *colors.*
(4) If there are two vertices $a, b \in V(H)$ *with* $\deg_H(a,b) = 0$ *then* $\deg_{\mathcal{H}}(a,b) = 0$ *as well.*
(5) If $|V(H)| \geq 4$ *then for every vertex* $x \in V(\mathcal{H}) \setminus V(H)$ *there exists a vertex* $y \in V(H)$ *such that* $\deg_{\mathcal{H}}(x,y) = 0$*.*

Proof. Let a hypergraph H and a $K_t^{(3)}$-free coloring c be given according to the theorem. We take a hypergraph $\mathcal{H}'$ as asserted to us by Lemma 5, along with the edges $e'_1,\dots, e'_k$, such that $V(H) \cap V(\mathcal{H}') = \emptyset$. Moreover, let H' be given according to Lemma 7, along with edges e' and f' of distance at least 7. Then, for every edge $g \in E(H)$, we take a copy H_g of the hypergraph H' on a set of new vertices, along with edges e_g and f_g representing e' and f'. We identify the edge g with e_g and if g is colored i under the coloring c then we identify f_g with e'_i. We denote the obtained hypergraph by $\mathcal{H}$.

We verify the desired properties one by one.

(1) It is easily seen that every copy F of $K_t^{(3)}$ is contained either in H or in $\mathcal{H}'$ or in some H_g with $g \in E(H)$. Indeed, if such a copy contains a vertex $x \in V(H_g) \setminus (e_g \cup f_g)$ for some $g \in E(H)$, then every other vertex $v \in V(F)$ needs to share an edge with x, which by construction needs to be part of H_g. Thus, $V(F) \subseteq V(H_g)$ and $F \subseteq \mathcal{H}[V(H_g)] = H_g$. Otherwise, F contains no such vertices x, and therefore, $V(F) \subseteq V(H) \cup V(\mathcal{H}')$. By construction of $\mathcal{H}$ we know that $\mathrm{dist}_{H_g}(e_g, f_g) \geq 7$ for all $g \in E(H)$ and thus $\deg_{\mathcal{H}[V(H)\cup V(\mathcal{H}')]}(u,v) = 0$ for every $u \in V(H)$ and $v \in V(\mathcal{H}')$, which yields $F \subseteq H$ or $F \subseteq \mathcal{H}'$.

Now, we color $E(H)$ according to c. As $V(H) \cap V(\mathcal{H}') = \emptyset$ we can easily extend c to a $K_t^{(3)}$-free coloring of $E(H) \cup E(\mathcal{H}')$ such that e'_i is colored i for each $i \in [k]$. Here we use that by Lemma 5, the edges $e'_1,\dots, e'_k$ have different colors in any $K_t^{(3)}$-free coloring. Moreover, observe that for every $g \in E(H)$ we then have that e_g and f_g receive the same color.

Next, we can extend further the above coloring to a $K_t^{(3)}$-free coloring of $E(\mathcal{H})$, by Lemma 7 and since the H_gs have only already colored edges from $\{e'_1, \dots, e'_k\}$ in common. Thus, $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$.

(2) H occurs as an induced subhypergraph in $\mathcal{H}$ since $\mathrm{dist}_{H_g}(e_g, f_g) \geq 6$ and thus $e_g \cap f_g = \emptyset$ for all $g \in E(H)$.

(3) Given any $K_t^{(3)}$-free coloring φ of $\mathcal{H}$, it holds by Lemma 5 that $e'_1,\dots, e'_k$ are colored differently. Moreover, by Lemma 7, the edges f_g and e_g are colored the same (for each $g \in e(H)$) in such a way that the ith color class of H under c obtains the color $\varphi(e'_i)$ for each $i \in [k]$.

(4) Suppose that $\deg_H(a,b) = 0$ for some two distinct vertices a, $b \in V(H)$. By construction, any two of the auxiliary hypergraphs (i.e. $\mathcal{H}'$, H, H_gs) overlap only in one edge (if at all). This way it follows that $\deg_{\mathcal{H}}(a,b) = 0$.

(5) Finally, take some $x \in V(\mathcal{H}) \setminus V(H)$. If $x \in V(\mathcal{H}') \setminus \left(\cup_{g\in E(H)} V(H_g)\right)$, then $\deg_{\mathcal{H}}(x,y) = 0$ for all $y \in V(H)$. If $x \in V(H_g)$ for some $g \in E(H)$, then again, by construction of $\mathcal{H}$, we have that $x \notin g \subseteq V(H)$ and therefore every $y \in V(H) \setminus g$ satisfies $\deg_{\mathcal{H}}(x,y) = 0$. □

3. Minimum degrees of minimal Ramsey 3-uniform hypergraphs

Before we prove Theorem 1, we first show the existence of an appropriate BEL-gadget which will be crucial for the upper bound (2) in Theorem 1.

Lemma 9. *Let $t \geq 4$ and $k \geq 2$ be integers. There is a 3-uniform hypergraph H on $n = k^{10kt^4}$ vertices, which can be written as an edge-disjoint union of k 3-uniform hypergraphs $H_1, \dots, H_k$ with the following properties:*

(*a*) *for every $i \in [k]$, H_i contains no copies of $K_t^{(3)}$, and*
(*b*) *for any coloring c of the edges of the complete graph K_n with k colors there exists a color $x \in [k]$ and k sets $S_1, \dots, S_k$ that induce copies of K_{t-1} in color x under the coloring c such that $H_1[S_1] \cong \dots \cong H_k[S_k] \cong K_{t-1}^{(3)}$.*

Before we proceed we state a simple quantitative version of Ramsey's theorem.

Fact 10. *Let $n \geq r_k(\ell)$. Then, in any k-coloring of $E(K_n)$ there are at least*

$$\frac{n^\ell}{k(r_k(\ell))^\ell}$$

monochromatic copies of K_ℓ in the same color.

Proof. Fix an arbitrary red-blue-coloring φ of $E(K_n)$. First observe that we find in *any* subset of $r_k(\ell)$ vertices of K_n a monochromatic K_ℓ. We estimate pairs of subsets of $[n]$ of the form (R, L) with $|R| = r_k(\ell)$, $|L| = \ell$ and $L \subseteq R$ such that all edges from $\binom{L}{2}$ are colored the same. As a lower bound we obtain $\binom{n}{r_k(\ell)}$, while the upper bound is the number of monochromatic copies of K_ℓ under φ times the number of $r_k(\ell)$-sets containing a particular copy (which is $\binom{n-\ell}{r_k(\ell)-\ell}$). This yields that there are at least

$$\binom{n-\ell}{r_k(\ell)-\ell}^{-1}\binom{n}{r_k(\ell)} = \frac{n \cdot \ldots \cdot (n-\ell+1)}{r_k(\ell) \cdot \ldots (r_k(\ell)-\ell+1)} \geq \left(\frac{n}{r_k(\ell)}\right)^\ell$$

monochromatic K_ℓs. Hence the claim follows. □

The rough idea of the proof of Lemma 9 is to take k random hypergraphs of appropriate density on the same vertex set and then show that even after deleting common edges and edges that lie in copies of $K_t^{(3)}$ we are left with k edge-disjoint hypergraphs that satisfy condition (*b*). We now turn to the details.

Proof of Lemma 9. We choose with foresight

$$p := C \cdot n^{\frac{-6}{(t-1)(t-2)}}, \text{ where } C := k^{100k/t} \text{ and } n = k^{10kt^4}. \tag{3}$$

We use the simple upper bound on $r_k(t) \leq k^{kt-2k+1}$ and we define $f(t) := k^{-kt^2}$ so that, with Fact 10, there are at least $f(t) \cdot n^{t-1}$ monochromatic copies of K_{t-1} in one of the colors in any k-coloring of the edges of K_n.

We take k independent random 3-uniform hypergraphs $H_1', \dots, H_k' \sim H^{(3)}(n,p)$, $i \in [k]$, on the vertex set $[n]$, and we observe first that

$$\mathbb{E}(e(H_i' \cap H_j')) = \binom{n}{3}p^2, \quad \mathbb{E}(e(H_i')) = \binom{n}{3}p \quad \text{and}$$
$$\mathbb{E}(\text{number of copies of } K_t^{(3)} \text{ in } H_i') = \binom{n}{t}p^{\binom{t}{3}}$$

for all $i \neq j \in [k]$.

For $i \in [k]$, we denote by E_i' the (random) set of edges in H_i' that either belong to some copy of $K_t^{(3)}$ in H_i' or to the edge set of some hypergraph H_j', $j \in [k] \setminus \{i\}$. We set $H_i := H_i' \setminus E_i'$. Obviously, $H_1, \dots, H_k$ satisfy (*a*). To prove the lemma, it thus remains to show that (b) is satisfied with positive probability. This will be immediate from the following two claims.

Claim 11. *With probability larger than $3/5$, the following holds. Each H'_i contains at most $0.2 \cdot f(t) \cdot n^{t-1} p^{\binom{t-1}{3}}$ copies of $K^{(3)}_{t-1}$ that contain an edge from E'_i.*

Proof. Fix an $i \in [k]$. We first consider the number X of copies of $K^{(3)}_{t-1}$ in H'_i that contain an edge e which is part of some copy of $K^{(3)}_t$ in H'_i. For a pair (T_1, T_2) of subsets of $[n]$ with $|T_1| = t-1$ and $|T_2| = t$ we define the indicator variable $I_{(T_1,T_2)}$ by

$$I_{(T_1,T_2)} := \begin{cases} 1, & \text{if } H'_i[T_1] \cong K^{(3)}_{t-1} \text{ and } H'_i[T_2] \cong K^{(3)}_t \\ 0, & \text{else} \end{cases}$$

and observe that

$$X \leq \sum_{s=3}^{t-1} \sum_{\substack{(T_1,T_2): \\ |T_1 \cap T_2| = s}} I_{(T_1,T_2)}. \tag{4}$$

By the linearity of expectation it follows that

$$\mathbb{E}(X) \leq \sum_{s=3}^{t-1} n^{t-1} \cdot \binom{t-1}{s} \cdot n^{t-s} \cdot p^{\binom{t-1}{3} + \binom{t}{3} - \binom{s}{3}}$$
$$\leq 2^t n^{2t-1} p^{\binom{t-1}{3} + \binom{t}{3}} \sum_{s=3}^{t-1} n^{-s} p^{-\binom{s}{3}}. \tag{5}$$

Each term above is dominated by the sum of its first and last summand. Indeed, let $g(s) := n^{-s} p^{-\binom{s}{3}}$, then for $3 \leq s \leq t-2$, we have

$$\frac{g(3)}{g(s)} = n^{s-3} \cdot p^{\binom{s}{3} - 1} = \left[n p^{\frac{s^2+2}{6}} \right]^{s-3} \geq \left[n p^{\frac{s(s+1)}{6}} \right]^{s-3} \geq \left[n p^{\frac{(t-1)(t-2)}{6}} \right]^{s-3} \geq 1.$$

Thus, we obtain $\mathbb{E}(X) \leq 2^t n^{2t-1} p^{\binom{t-1}{3} + \binom{t}{3}} \cdot t \cdot \Big(g(3) + g(t-1) \Big)$. And we further upper bound $\mathbb{E}(X)$ with (3) by

$$\begin{aligned} \mathbb{E}(X) &\leq t 2^t n^{t-1} p^{\binom{t-1}{3}} \left(n^t p^{\binom{t}{3}} n^{-3} p^{-1} + n^t p^{\binom{t}{3}} n^{-t+1} p^{-\binom{t-1}{3}} \right) \\ &\overset{(3)}{=} t 2^t n^{t-1} p^{\binom{t-1}{3}} \left(C^{\binom{t}{3}} n^{-3} p^{-1} + n^{-2} C^{\binom{t-1}{2}} \right) \\ &\overset{(3)}{\leq} t 2^t n^{t-1} p^{\binom{t-1}{3}} \left(k^{50kt^2/3} + k^{50kt} \right) n^{-2} \\ &\overset{(3)}{\leq} 2^{t + \log_2 t + 1} k^{50kt^2/3} k^{-20kt^4} n^{t-1} p^{\binom{t-1}{3}} \leq \frac{1}{50k} f(t) n^{t-1} p^{\binom{t-1}{3}}. \end{aligned} \tag{6}$$

So, by Markov's inequality, with probability at least $1 - \frac{1}{5k}$ we have,

$$X \leq 0.1 f(t) n^{t-1} p^{\binom{t-1}{3}}.$$

Next, consider the number Y of copies of $K^{(3)}_{t-1}$s in H'_i that contain an edge e from the intersection $E(H'_i) \cap E(H'_j)$ for a fixed $j \neq i$. For a subset $S \in \binom{[n]}{t-1}$ and an edge $e \in \binom{S}{3}$ let

$$I_{(S,e)} := \begin{cases} 1, & \text{if } H'_i[S] \cong K^{(3)}_{t-1} \text{ and } e \in E(H'_j) \\ 0, & \text{else} \end{cases}$$

so that $Y \leq \sum_{(S,e)} I_{(S,e)}$. Then,

$$\mathbb{E}(Y) \leq n^{t-1}\binom{t-1}{3} \cdot p^{\binom{t-1}{3}+1} \overset{(3)}{=} n^{t-1}p^{\binom{t-1}{3}}\binom{t-1}{3}k^{100k/t}k^{-\frac{60kt^4}{(t-1)(t-2)}}$$
$$\leq n^{t-1}p^{\binom{t-1}{3}}t^3k^{25k}k^{-60kt^2} \leq \frac{1}{50k^3}f(t)n^{t-1}p^{\binom{t-1}{3}}.$$

By Markov's inequality, with probability at least $1-\frac{1}{5k^2}$ we then have

$$Y \leq \frac{1}{10k}f(t)n^{t-1}p^{\binom{t-1}{3}}.$$

In particular, with probability at least $3/5$ it holds for all $i \in [k]$ that H_i' contains at most $0.2 \cdot f(t) \cdot n^{t-1}p^{\binom{t-1}{3}}$ copies of $K_{t-1}^{(3)}$ that contain an edge from E_i'. Therefore the claim follows. □

Claim 12. *The following holds with probability at least 2/3. For every coloring $\psi\colon E(K_n) \to [k]$ there is a color x such that for every $i \in [k]$, there are at least $0.5f(t)n^{t-1}p^{\binom{t-1}{3}}$ monochromatic copies F of K_{t-1} in color x with $\binom{V(F)}{3} \subseteq E(H_i')$.*

Proof. Fix an $i \in [k]$. Let $\psi\colon E(K_n) \to [k]$ be an arbitrary coloring. Then there is a color x such that there are at least $f(t)n^{t-1}$ monochromatic copies of K_{t-1} under coloring ψ which all have the same color x (by Fact 10). We fix a family $\mathcal{F} = \{F_1, \dots, F_m\}$ of exactly $m = f(t)n^{t-1}$ such copies (say lexicographically smallest ones). Now, denote with $X_{\mathcal{F},i}$ the number of such $F_j \in \mathcal{F}$ with $\binom{V(F_j)}{3} \subseteq E(H_i')$. For every $F_j \in \mathcal{F}$ let

$$X_{F_j,i} = \begin{cases} 1, & \text{if } \binom{V(F_j)}{3} \subseteq E(H_i') \\ 0, & \text{else} \end{cases}$$

and observe that $X_{\mathcal{F},i} = \sum_{F\in\mathcal{F}} X_{F,i}$. We define $\lambda := \mathbb{E}(X_{\mathcal{F},i}) = f(t)n^{t-1} \cdot p^{\binom{t-1}{3}}$. Observe that by exploiting the choice of p and n in (3) we obtain

$$\lambda = k^{-kt^2}n^{t-1}C^{\binom{t-1}{3}}n^{-t+3} = k^{-kt^2}k^{50k(t-1)(t-2)(t-3)/(3t)}n^2. \tag{7}$$

Let

$$\overline{\Delta}_i := \sum_{\substack{F,F'\in\mathcal{F} \\ \binom{V(F)}{3}\cap\binom{V(F')}{3}\neq\emptyset}} \mathbb{E}(X_{F,i}X_{F',i}).$$

Next we estimate $\overline{\Delta}_i$ as follows (since each $X_{F,i}$ counts a copy of the complete 3-uniform hypergraph on the vertex set $V(F)$, we can classify pairs of these copies according to the number s of common vertices):

$$\overline{\Delta}_i \leq |\mathcal{F}|\sum_{s=3}^{t-1}\binom{t-1}{s}n^{t-1-s}p^{2\binom{t-1}{3}-\binom{s}{3}} \leq f(t) \cdot n^{2t-2}p^{2\binom{t-1}{3}}2^t\sum_{s=3}^{t-1}n^{-s}p^{-\binom{s}{3}},$$

and thus exactly as in the previous claim, Claim 11, we estimate the sum by $t\left(n^{-3}p^{-1} + n^{-t+1}p^{-\binom{t-1}{3}}\right)$, which leads to the upper bound

$$\overline{\Delta}_i \leq t2^t\lambda\left(n^{t-1}p^{\binom{t-1}{3}}n^{-3}p^{-1} + n^{t-1}p^{\binom{t-1}{3}}n^{-t+1}p^{-\binom{t-1}{3}}\right) =$$
$$t2^t\lambda\left(C^{\binom{t-1}{3}}(pn)^{-1} + 1\right) \overset{(3)}{=} 2^{t+\log_2 t}\lambda\left(k^{\frac{100k}{t}\left[\binom{t-1}{3}-1\right]}k^{-10kt^4+\frac{60kt^4}{(t-1)(t-2)}} + 1\right) \leq 2^{2t}\lambda. \tag{8}$$

Now with Janson's inequality (see e.g. Theorem 2.14 in [11]) we obtain

$$\mathbb{P}(X_{\mathcal{F},i} \leq 0.5\lambda) \leq \exp(-\lambda^2/(8\overline{\Delta}_i)) \overset{(8)}{\leq} \exp(-2^{-2t-3}\lambda)$$
$$\overset{(7)}{\leq} \exp(-2^{-2t-3}k^{-kt^2+50k(t-1)(t-2)(t-3)/(3t)}n^2) \leq$$
$$\exp(-2^{-2t-3}k^{-kt^2+50kt^2/32}n^2) \leq \exp(-k^{-2t-3+9t^2/8}n^2) \leq \exp(-k^7n^2).$$

This tells us that for the color x with probability at least $1 - k\exp(-k^7n^2)$ all graphs H_i', $i \in [k]$, contain at least $0.5 \cdot f(t) \cdot n^{t-1}p^{\binom{t-1}{3}}$ copies F of K_{t-1} in color x and with $\binom{V(F)}{3} \subseteq E(H_i')$. Since there are $k^{\binom{n}{2}}$ different colorings of $E(K_n)$, we may apply the union bound to see that the probability that there is a coloring $\psi\colon E(K_n) \to \{\text{red}, \text{blue}\}$ not satisfying the claim is at most $k^{\binom{n}{2}} \cdot k\exp(-k^7n^2) < 1/3$. □

With positive probability the Claims 11 and 12 hold. So fix H_1', …, H_k' that satisfy the assertions of these claims. Recall that $H_i = H_i' \setminus E_i'$ and we only need to verify *(b)* as H_1,…, H_k obviously satisfy *(a)*. Let $\psi\colon E(K_n) \to [k]$ be an arbitrary coloring. Claim 12 asserts that there is a color x such that for every $i \in [k]$, there are at least $0.5 \cdot f(t) \cdot n^{t-1}p^{\binom{t-1}{3}}$ monochromatic copies F of K_{t-1} in color x and such that $\binom{V(F)}{3} \subseteq E(H_i')$. By Claim 11, for each $i \in [k]$, at most $0.2 \cdot f(t) \cdot n^{t-1}p^{\binom{t-1}{3}}$ of these copies satisfy $\binom{V(F)}{3} \not\subseteq E(H_i)$, and thus condition *(b)* is satisfied. □

3.1. **Proof of Theorem 1.** *A lower bound on* $s_{k,1}(K_t^{(3)})$. The proof of the lower bound is easy. In fact, it follows from the bound on the Ramsey number $r_k(K_t) \geq k^{(1+o(1))t/2}$ and is as follows. Take a minimal k-Ramsey hypergraph $\mathcal{H}$ for $K_t^{(3)}$ such that $\delta(\mathcal{H}) = s_{k,1}(K_t^{(3)})$ and let $v \in V(\mathcal{H})$ be a vertex of minimum degree. By minimality of $\mathcal{H}$, we have $\mathcal{H} \setminus \{v\} \not\longrightarrow (K_t^{(3)})_k$ and fix an edge coloring φ that certifies this. Since $\mathcal{H} \longrightarrow (K_t^{(3)})_k$ it follows that the link graph $\text{link}_{\mathcal{H}}(v)$ is Ramsey: $\text{link}_{\mathcal{H}}(v) \longrightarrow (K_{t-1})_k$. Therefore: $s_{k,1}(K_t^{(3)}) = \deg(v) \geq \hat{r}_k(K_{t-1}) = \binom{r_k(K_{t-1})}{2} \geq k^{(1+o(1))t}$, where $\hat{r}_k(K_\ell)$ is the *size-Ramsey number* for K_ℓ and it was shown by Erdős, Faudree, Rousseau and Schelp [5] that $\hat{r}_k(K_\ell) = \binom{r_k(K_\ell)}{2}$.

An upper bound on $s_{k,1}(K_t^{(3)})$. Let H be the 3-uniform hypergraph as asserted by Lemma 9 along with the hypergraphs H_1, …, H_k that satisfy the conditions *(a)* and *(b)*. We fix the following $K_t^{(3)}$-free k-coloring c of $E(H)$: we color all edges from H_i with color $i \in [k]$. Let further $\mathcal{H}'$ be the hypergraph as guaranteed by Theorem 8 for given H and c. We define the hypergraph $\mathcal{H}$ by adding to $\mathcal{H}'$ a new vertex v whose link is $\text{link}_{\mathcal{H}}(v) := \binom{V(H)}{2}$. So $\deg_{\mathcal{H}}(v) = \binom{n}{2} < k^{20kt^4}$ as asserted by Lemma 9. In the following we argue that $\mathcal{H}' \not\longrightarrow (K_t^{(3)})_k$ but $\mathcal{H} \longrightarrow (K_t^{(3)})_k$. It then follows immediately that every Ramsey subhypergraph of $\mathcal{H}$ (in particular minimal Ramsey subhypergraph of $\mathcal{H}$) for $K_t^{(3)}$ needs to contain the vertex v, whose degree is less than k^{20kt^4}. Thus, once these two properties are proven, the upper bound follows.

In fact, $\mathcal{H}' \not\longrightarrow (K_t^{(3)})_k$ is asserted by Theorem 8. So, we only need to focus on showing that $\mathcal{H} \longrightarrow (K_t^{(3)})_k$. For contradiction, suppose that there is a coloring $\varphi\colon E(\mathcal{H}) \to [k]$ without monochromatic copies of $K_t^{(3)}$. We then know by the Property (3) from Theorem 8 that $E(H_1)$, …, $E(H_k)$ are all colored monochromatically, but in different colors. W.l.o.g. we may assume that, for each $i \in [k]$, H_i is colored with the color i. Now, we define a coloring $\psi\colon \binom{V(H)}{2} \to [k]$ with $\psi(\{u_1, u_2\}) = \varphi(\{u_1, u_2, v\})$. Then, according to Lemma 9 there is a color x and the sets $S_1, \ldots, S_k \in \binom{V(H)}{t-1}$ such that $\binom{S_1}{2}, \ldots, \binom{S_k}{k}$ are monochromatic under ψ in color x, while for every $i \in [k]$ we have that $H[S_i] \cong K_{t-1}^{(3)}$ is colored i. But this implies immediately that we found a monochromatic clique $\mathcal{H}[S_x \cup \{v\}] \cong K_t^{(3)}$ in color x. A contradiction. $\square$

4. Minimum codegrees of minimal Ramsey 3-uniform hypergraphs

In this section we prove Theorem 2 by showing that $s_{2,2}(K_t^{(3)}) = 0$ and that $s'_{2,2}(K_t^{(3)}) = (t-2)^2$. Our proof strategy is similar to that of [1, 7]: for the lower bound we rather provide an adhoc argument, while for the upper bound we employ the BEL-gadgets, Theorem 8, combined with a natural construction that we "plant" via a BEL-gadget (which is an almost Ramsey hypergraph).

Proof of Theorem 2.
Lower bound argument for $s'_{2,2}$. We first prove that $s'_{2,2}(K_t^{(3)}) \geq (t-2)^2$. Take a minimal 2-Ramsey hypergraph H for $K_t^{(3)}$. Fix any two vertices u and $v \in V(H)$ with $\deg_H(u,v) > 0$. We aim to show that $\deg_H(u,v) \geq (t-2)^2$. So, assume the opposite, i.e. $\deg_H(u,v) < (t-2)^2$.

Let H' be the subhypergraph obtained from H by deleting all edges containing both vertices u and v. Since H is Ramsey-minimal, $H' \not\to \left(K_t^{(3)}\right)_2$. Thus, there is a coloring c with red and blue of $E(H')$ which does not create a monochromatic copy of $K_t^{(3)}$. Define $N(u,v) := \{w \in V(H) : \{u,v,w\} \in E(H)\}$, thus $\deg_H(u,v) = |N(u,v)|$. Take a longest sequence $B_1,\ldots,B_k$ of vertex disjoint sets of size $t-2$ in $N(u,v)$, such that both $B_i \cup \{u\}$ and $B_i \cup \{v\}$ span only blue edges under the coloring c in H. By assumption on the codegree $\deg_H(u,v)$, we know that $k < t-2$.

Next we can extend the coloring c as follows. For each edge $e = \{u,v,w\} \in E(H)$ with $w \in \bigcup B_i$ we set $c(e) = \text{red}$, while for all other edges $e = \{u,v,w\} \in E(H)$ we set $c(e) = \text{blue}$. We claim that under this coloring there is no monochromatic copy of $K_t^{(3)}$ in H. Indeed, if there were a monochromatic subgraph F isomorphic to $K_t^{(3)}$, then necessarily $u, v \in V(F)$ (since $E(H')$ were colored without monochromatic $K_t^{(3)}$). If F is red, then by construction F can have at most one vertex from each of the sets B_i and no vertex from $N(u,v) \setminus \bigcup B_i$, so $|V(F)| < t$, a contradiction. If F is blue, then it cannot contain vertices from $\bigcup B_i$, and therefore $V(F) \subseteq (N(u,v) \setminus \bigcup B_i) \cup \{u,v\}$. But then, we could extend the sequence of B_is by the set $V(F) \setminus \{u,v\}$, in contradiction to its maximality. So, under the assumption $\deg_H(u,v) < (t-2)^2$ we conclude that $H \not\to (K_t^{(3)})_2$, a contradiction. Thus, we need to have $\deg_H(u,v) \geq (t-2)^2$ for every $u, v \in V$ with $\deg_H(u,v) > 0$. Therefore, $s'_{2,2}(K_t^{(3)}) \geq (t-2)^2$.
Upper bound argument for $s'_{2,2}$. First we provide a hypergraph H with a prescribed coloring of $E(H)$ without a monochromatic $K_t^{(3)}$. We set $V(H) := [(t-2)^2] \cup \{a,b\}$ and we further partition the vertices of $[(t-2)^2]$ into $(t-2)$ equal-sized sets $V_1,\ldots,V_{t-2}$. Next we choose the edges for H as follows.

$$
\begin{aligned}
E(H) := \bigcup_{i}^{t-2} \binom{V_i}{3} &\cup \left\{ e \cup \{w\} \colon e \in \binom{V_i}{2} \text{ for some } i \in [t-2], w \in \{a,b\} \right\} \\
&\cup \left\{ f \colon f \in \binom{[(t-2)^2]}{3}, |f \cap V_i| \leq 1 \ \forall i \in [t-2] \right\} \qquad (9) \\
&\cup \left\{ e \cup \{w\} \colon e \in \binom{[(t-2)^2]}{2}, |e \cap V_i| \leq 1 \ \forall i \in [t-2], w \in \{a,b\} \right\}.
\end{aligned}
$$

Thus, H is obtained from the clique $K^{(3)}_{(t-2)^2+2}$ on the vertex set $\bigcup V_i \cup \{a,b\}$, where we delete all edges that contain both a and b and moreover we delete all edges that cross exactly two different V_is and contain neither a nor b. Next we provide a red-blue-coloring c of the edges of H as follows: the edges contained in $V_i \cup \{a\}$ and in $V_i \cup \{b\}$ for $i \in [t-2]$ are colored *blue*, while the other edges of H are colored *red* – thus the edges in the first line of (9) are colored blue, while the edges defined in the second and third line of (9) are colored red. It is immediate that such a coloring does not yield a monochromatic copy of $K_t^{(3)}$. Indeed, a blue copy of $K_s^{(3)}$ cannot use vertices from different sets V_i and, since $\deg_H(a,b)=0$, it also cannot contain both vertices a,b, which gives $s \le t-1$. Similarly, a red copy of $K_s^{(3)}$ can use at most one vertex from each V_i and, as $\deg_H(a,b)=0$, it also cannot contain both vertices a,b, which again gives $s \le t-1$.

Applying Theorem 8 to the colored hypergraph H for this coloring c, we obtain a 3-uniform hypergraph $\mathcal{H}$ which contains H as an induced hypergraph, which is not 2-Ramsey for $K_t^{(3)}$ and such that any red-blue $K_t^{(3)}$-free coloring φ of $E(\mathcal{H})$ agrees on $E(H)$ with the coloring c up to permutation of the two colors. Also, Theorem 8 asserts that $\deg_{\mathcal{H}}(a,b)=0$. Next we define $\mathcal{H}'$ by adding to $\mathcal{H}$ all $(t-2)^2$ edges $\{a,b,u\}$ where $u \in [(t-2)^2]$.

Let us see why $\mathcal{H}' \longrightarrow (K_t^{(3)})_2$. Fix any coloring φ of $E(\mathcal{H}')$ and assume that no copy of $K_t^{(3)}$ is monochromatic in $\mathcal{H}'$ under φ. Since $\mathcal{H} \subseteq \mathcal{H}'$, it follows that the color pattern c as described above (up to permutation) is enforced in H. Assume w.l.o.g. that $E(H)$ is colored according to c. Then if there is a set V_i such that all edges $\{v,a,b\}$ are colored blue for all $v \in V_i$ this would yield a blue copy of $K_t^{(3)}$. So, assume that for every V_i there is at least one edge $\{v_i,a,b\}$ which is colored red for some $v_i \in V_i$. Then $\{a,b,v_1,\dots,v_{t-2}\}$ forms a red clique $K_t^{(3)}$. Thus, in any case, we find a monochromatic copy of $K_t^{(3)}$, i.e. $\mathcal{H} \longrightarrow (K_t^{(3)})_2$. Moreover, since $\mathcal{H}$ is not 2-Ramsey for $K_t^{(3)}$, any minimal 2-Ramsey subhypergraph of $\mathcal{H}'$ must contain edges that contain both a and b. This shows $s'_{2,2}(K_t^{(3)}) \le (t-2)^2$.

In fact, notice that by the previous discussion of the lower bound on $s'_{2,2}$, any such minimal 2-Ramsey subhypergraph of $\mathcal{H}'$ must contain all the $(t-2)^2$ edges that contain both a and b. This will be important in the following proof.

Showing $s_{2,2}(K_t^{(3)})=0$. This looks surprising at the first sight since taking $K_n^{(3)}$ with $n=r_2(K_t^{(3)})$ and then deleting all edges that contain two distinguished vertices gives a non-Ramsey hypergraph (which suggests $s_{2,2}(K_t^{(3)})>0$). However this is not the case and it will follow from the above construction of the hypergraph $\mathcal{H}'$.

As argued above, *any* minimal Ramsey subhypergraph of $\mathcal{H}'$ for $K_t^{(3)}$ has to contain *all* $(t-2)^2$ edges that contain a and b. Thus, any such minimal hypergraph $\mathcal{H}''$ contains all vertices of H. Next we argue that $\mathcal{H}''[V(H)] \not\longrightarrow (K_t^{(3)})_2$. Indeed, by construction of $\mathcal{H}'$, we observe that $\mathcal{H}'[V(H)] \supseteq \mathcal{H}''[V(H)]$ contains exactly $(t-2)+(t-2)^{t-2}$ copies of $K_t^{(3)}$, namely exactly $(t-2)$ ones that are induced on $V_i \cup \{a,b\}$ for some $i \in [t-2]$, and $(t-2)^{t-2}$ ones that contain one vertex from each of the V_is and additionally a and b. There are no further copies of $K_t^{(3)}$ since $H[\bigcup V_i]$ contains only copies of $K_{t-2}^{(3)}$ which either cross all V_is or are equal to some $H[V_i]$. It is now easy to see that $\mathcal{H}'[V(H)] \not\longrightarrow (K_t^{(3)})_2$ as follows. We can color the edges of $\mathcal{H}''[V(H)]$ uniformly at random with colors red and blue. Then, the expected number of monochromatic copies of $K_t^{(3)}$ is $[(t-2)+(t-2)^{t-2}]\cdot 2^{1-\binom{t}{3}} < 1$, as $t \ge 4$, i.e. there exists a 2-coloring which avoids monochromatic copies of $K_t^{(3)}$.

Thus, $\mathcal{H}''$ has to contain at least one further vertex $x \notin V(H)$. Then, since $|V(H)| = (t-2)^2+2 \ge 6$, it follows by Property (5) of Theorem 8 that there exists a vertex $y \in V(H)$ such that $0 = \deg_{\mathcal{H}'}(x, y) \geq \deg_{\mathcal{H}''}(x, y)$. Therefore, $s_{2,2}(K_t^{(3)}) = 0$. $\square$

5. CONCLUDING REMARKS

In this paper we studied the smallest minimum degree and codegree of minimal Ramsey 3-uniform hypergraphs for complete hypergraphs $K_t^{(3)}$, $t \geq 4$. In particular we showed that the smallest minimum degree $s_{2,1}(K_t^{(3)})$ of minimal 2-Ramsey 3-uniform hypergraph lies between 2^t and 2^{40t^4}. It would be interesting to determine the right order of the exponent. We leave the study of minimal Ramsey r-uniform hypergraphs for $r \geq 4$ for future work.

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