Large unavoidable subtournaments

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Abstract

Let D_k denote the tournament on $3k$ vertices consisting of three disjoint vertex classes V_1, V_2 and V_3 of size k, each oriented as a transitive subtournament, and with edges directed from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. Fox and Sudakov proved that given a natural number k and $\epsilon > 0$ there is $n_0(k, \epsilon)$ such that every tournament of order $n \geq n_0(k, \epsilon)$ which is ϵ-far from being transitive contains D_k as a subtournament. Their proof showed that $n_0(k, \epsilon) \leq \epsilon^{-O(k/\epsilon^2)}$ and they conjectured that this could be reduced to $n_0(k, \epsilon) \leq \epsilon^{-O(k)}$. Here we prove this conjecture.

1 Introduction

Ramsey theory refers to a large and active branch of combinatorics mainly concerned with understanding which properties of a structure are preserved in dense substructures or upon finite partition. It is often introduced with the phrase 'complete disorder is impossible', attributed to Motzkin, and part of the subject's growth can be attributed to the surprising variety of contexts in which this philosophy can be applied (for a small sample, see [1], [2], [11], [13]).

A central result in the area is Ramsey's theorem [14], which says that given any natural number k, there is an integer N such that every 2-colouring of the edges of the complete graph K_N contains a monochromatic copy of K_k. An important problem in the area is to estimate the smallest value of N for which the theorem holds, denoted $R(k)$. It is known that $2^{(1/2+o(1))k} \leq R(k) \leq 4^{(1+o(1))k}$ (see [6], [16], [8], [4]).

Clearly it is not possible to guarantee the existence of non-monochromatic cliques in general 2-colourings of K_N. Bollobás raised the question of which 2-coloured subgraphs occur in 2-colourings of K_N where each colour appears on at least ϵ proportion of the edges. Let $\mathcal{F}_k$ denote the collection of 2-coloured graphs of order $2k$, in which one colour appears as either a clique of order k or two disjoint cliques of order k. Bollobás asked whether, given a natural k and $\epsilon > 0$ there is $M = M(k, \epsilon)$ with the following property: in every 2-colouring of the edges of K_M containing both colours on at least ϵ proportion of the edges, some element of $\mathcal{F}_k$ appears as a coloured subgraph. Cutler and Montágh [5] answered this question in the affirmative and proved that it is possible to take $M(k, \epsilon) \leq 4^{k/\epsilon}$. Fox and Sudakov [9] subsequently improved this bound to show that $M(k, \epsilon) \leq \epsilon^{-ck}$, for some constant

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$c > 0$. As shown in [9], this bound is tight up to the value of the constant c in the exponent, which can be seen by taking a random 2-colouring of a graph on $\epsilon^{-(k-1)/2}$ vertices with appropriate densities.

Here we will be concerned with an analogous question for tournaments. A *tournament* is a directed graph obtained by assigning a direction to the edges of a complete graph. A tournament is said to be *transitive* if it is possible to order the vertices of the tournament so that all of its edges point in the same direction. Let $T(k)$ denote the smallest integer such that every tournament on $T(k)$ vertices contains a transitive subtournament on k vertices. A classic result due to Erdős and Moser [7] shows that $T(k)$ is finite for all k and gives that $2^{(k-1)/2} \leq T(k) \leq 2^{k-1}$ (in fact, as pointed out by the referee, the upper bound here is attributed to Stearns in [7]).

As in the case of 2-colouring graphs, it is natural to ask which subtournaments must occur in every large tournament which is 'not too similar' to a transitive tournament. An n-vertex tournament T is *ϵ-far from being transitive* if in any ordering of the vertices of T, the direction of at least ϵn^2 edges of T must be switched in order to obtain a transitive tournament. In [9], Fox and Sudakov asked the following question: given $\epsilon > 0$, which subtournaments must an n-vertex tournament which is ϵ-far from being transitive contain?

For any natural number k, let D_k denote the tournament on $3k$ vertices consisting of three disjoint vertex classes V_1, V_2 and V_3 of size k, each oriented as a transitive subtournament, and with all edges directed from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. Taking $T = D_{n/3}$ we obtain an n-vertex tournament which is $\frac{1}{9}$-far from being transitive and whose only subtournaments are contained in D_k for some k. Thus, subtournaments of D_k are the only candidates for unavoidable tournaments which occur in large tournaments that are ϵ-far from transitive for small ϵ. In [9], Fox and Sudakov proved that subtournaments necessarily appear in large tournaments which are ϵ-far from being transitive.

Theorem 1 (Fox–Sudakov)**.** *Given $\epsilon > 0$ and a natural number k, there is $n_0(k, \epsilon)$ such that if T is a tournament on $n \geq n_0(k, \epsilon)$ vertices which is ϵ-far from being transitive, then T contains D_k as a subtournament. Furthermore $n_0(k, \epsilon) \leq \epsilon^{-ck/\epsilon^2}$, for some absolute constant $c > 0$.*

The authors in [9] conjectured that this bound can be further reduced to $n_0(k, \epsilon) \leq \epsilon^{-Ck}$ for some absolute constant $C > 0$. This order of growth agrees with high probability with a random tournament obtained by directing edges backwards independently with probability $\approx \epsilon$. Here we prove this conjecture.

Theorem 2. *There is a constant $C > 0$ such that for $\epsilon > 0$ and any natural number k we have $n_0(k, \epsilon) \leq \epsilon^{-Ck}$.*

Before beginning on the proof let us mention two other results related to Theorems 1 and 2. A tournament T is said to be *c-colourable* if it is possible to partition $V(T)$ into c subsets, each of which is a transitive subtournament. The *chromatic number* $\chi(T)$ of a tournament T equals the smallest value of c such that T is c-colourable. A tournament H is said to be a *hero* if every H-free tournament has bounded chromatic number. The definition of a hero was introduced in by Berger, Choromanski, Chudnovksy, Fox, Loebl, Scott, Seymour and Thomassé in [3] and their main result gave an explicit description of heroes. This notion was recently extended by Shapira and Yuster [15]. A tournament H is said to be *c-unavoidable* if for every $\epsilon > 0$ and $n \geq n_0(\epsilon, H)$, every n-vertex tournament T that is ϵ-far from satisfying $\chi(T) \leq c$ contains a copy of H. A tournament H is said to be *unavoidable* if it is c_H-unavoidable for some constant c_H. Clearly a tournament is 1-chromatic if and only if it is transitive. Thus from Theorem 1 and the discussion preceding it, 1-unavoidable tournaments are precisely those tournaments which appear as subtournaments of D_k for some k. In [15] this result was extended to show that a tournament H is unavoidable iff it is a transitive blowup of a hero (see [3] and [15] for the precise definitions).

Notation: Given a tournament T, we write $V(T)$ to denote its vertex set and $E(T)$ to denote the directed edge set of T. Given $v \in V(T)$ and a set $S \subset V(T)$, let $d^-_S(v) := |\{u \in S : \overrightarrow{uv} \in E(T)\}|$ and $d^+_S(v) := |\{u \in S : \overrightarrow{vu} \in E(T)\}|$. We will also write $T[S]$ to denote the induced subtournament of T on vertex set S. Given $B \subset E(T)$, we write $d^-_B(v) = |\{u \in V(T) : \overrightarrow{uv} \in B\}|$ and $d^+_B(v) = |\{u \in V(T) : \overrightarrow{vu} \in B\}|$. For an ordering $v_1, \ldots, v_{|T|}$ of $V(T)$ and $1 \leq i < j \leq |T|$, let $[v_i, v_j] := \{v_i, v_{i+1}, \ldots, v_j\}$. Lastly, all log functions will be to the base 2.

2 Finding many long backwards edges in T

In [9], Theorem 1 was deduced from two results of independent interest. The first result showed that any tournament which is ϵ-far from being transitive must contain many directed triangles.

Theorem 3 (Theorem 1.3 in [9])**.** *Any n-vertex tournament T which is ϵ-far from being transitive contains at least $c\epsilon^2 n^3$ directed triangles, where $c > 0$ is an absolute constant.*

As pointed out in [9], this bound is also best possible in general, as can be seen from the following tournament. Let T be given by taking k copies of $D_{n/3k}$, say on disjoint vertex sets $V_1, \ldots, V_k$ with all edges between V_i and V_j directed forward, for $i < j$. As at least $(n/3k)^2$ edges from each copy of $D_{n/3k}$ must be reoriented in order to obtain a transitive tournament, T is $k(1/3k)^2 = 1/9k$ far from being transitive, but contains only $k.(n/3k)^3 = n^3/27k^2$ directed triangles. Taking $\epsilon = 1/9k$, we see that the growth rate here agrees with that given by Theorem 3 up to constants.

Our first improvement in the bound for $n_0(k, \epsilon)$ comes from showing that any tournament which is ϵ-far from being transitive must either contain many more directed triangles than the number given in Theorem 3 or contain a slightly smaller subtournament which is 2ϵ-far from being transitive. This density increment argument will allow one of the factors of ϵ to be removed from the exponent in the bound on $n_0(k, \epsilon)$ in Theorem 1.

Given an ordering $v_1, \ldots, v_{|T|}$ of the vertices of a tournament T, edges of the form $\overleftarrow{v_i v_j}$ with $i < j$ are called *backwards edges*. We will often list the vertices of tournaments in an order which minimizes the number of backwards edges. Such orderings are said to be *optimal*. The following proposition gives some simple but useful properties of optimal orderings.

Proposition 4. *Suppose that T is a tournament on n vertices and let $v_1, \ldots, v_n$ be an optimal ordering of $V(T)$. Then the following hold:*

1. *For every $i, j \in [n]$ with $i < j$ we have*
   - $d^+_{[v_{i+1}, v_j]}(v_i) \geq (j - i)/2$;
   - $d^-_{[v_i, v_{j-1}]}(v_j) \geq (j - i)/2$.

2. *If $T[v_{i+1}, v_j] := T[\{v_{i+1}, \dots, v_j\}]$ has $\delta(j-i)^2$ backwards edges in this ordering then the subtournament $T[v_{i+1}, v_j]$ is δ-far from being transitive.*

Proof. If we had $d^+_{[v_{i+1},v_j]}(v_i) < (j-i)/2$, then the ordering $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, v_i, v_{j+1}, \dots, v_n$ would decrease the number of backwards edges of T. A similar switch works if $d^-_{[v_i,v_{j-1}]}(v_j) < (j-i)/2$. Lastly, if $v_{k_1}, \dots, v_{k_{j-i}}$ was an ordering of $[v_{i+1}, v_j]$ with fewer than $\delta(j-i)^2$ backwards edges, then the order of $V(T)$ given by $v_1, \dots, v_i, v_{k_1}, \dots, v_{k_{j-i}}, v_{j+1}, \dots, v_n$ would have less backwards edges than $v_1, \dots, v_n$. $\square$

Given an ordering $v_1, \dots, v_n$ of $V(T)$ with a backwards edge $\overleftarrow{v_iv_j}$ $(i < j)$, the edge $\overleftarrow{v_iv_j} \in B$ is said to have *length* $j - i$.

Lemma 5. *Suppose that T is a tournament on n vertices which is ϵ-far from being transitive and let $v_1, \dots, v_n$ be an optimal ordering of $V(T)$. Let B denote the collection of backwards edges in this ordering. Then one of the following holds:*

1. *The subset B' of B consisting of those edges of length at least $n/16$ satisfies $|B'| \geq |B|/4$;*

2. *T contains a subtournament on at least $n/8$ vertices which is 2ϵ-far from being transitive.*

Proof. We can assume that T itself is not 2ϵ-far from being transitive, as otherwise 2. above would trivially hold. Thus $\epsilon n^2 \leq |B| < 2\epsilon n^2$. Let us assume that $|B'| < |B|/4$, i.e. 1. fails. Note that this gives $n \geq 16$. We wish to show that there exists $S \subset V(T)$ with $|S| \geq n/8$ such that $T[S]$ is 2ϵ-far from being transitive. To prove this, by part 2 of Proposition 4, it suffices to find an interval $v_{i+1}, \dots, v_j$ with $j - i \geq n/8$ containing at least $2\epsilon(j-i)^2$ edges from B.

If either $T_{first} = T[v_1, \dots, v_{n/8}]$ or $T_{last} = T[v_{7n/8+1}, \dots, v_n]$ have at least $2\epsilon(n/8)^2 = \epsilon n^2/32$ backwards edges then we done. Otherwise, let E denote the subset of B consisting of those backwards edges not in B' and not in T_{first} or T_{last}. From the above bounds

$$|E| > |B| - |B'| - 2\frac{\epsilon n^2}{32} > \frac{3|B|}{4} - \frac{\epsilon n^2}{16} \geq \frac{\epsilon n^2}{2}. \tag{1}$$

Now given $i \in [0, 7n/8]$, let T_i denote the subtournament of T which is given by $T_i = T[\{v_{i+1}, \dots, v_{i+n/8}\}]$. Choose $i \in [0, 7n/8]$ uniformly at random and let E_i denote the random variable which counts the number of edges of E which lie in T_i. As each element $e \in E$ has length at most $n/16$, with at least one endpoint in $\{v_{n/8}, \dots, v_{7n/8}\}$, there are at least $n/16$ choices of i with $e \in T_i$. As $n \geq 8$, this gives

$$\mathbb{P}(e \in T_i) \geq \frac{n/16}{7n/8+1} \geq \frac{1}{16}.$$

By linearity of expectation, combined with (1) this gives

$$\mathbb{E}(E_i) = \sum_{e \in E} \mathbb{P}(e \in T_i) \geq \frac{|E|}{16} \geq \frac{\epsilon n^2}{32} = 2\epsilon(\frac{n}{8})^2.$$

Fix a value of i such that E_i is at least as large as its expectation. Then as T_i has $n/8$ vertices and at least $2\epsilon(n/8)^2$ backwards edges. By Proposition 4, T_i is 2ϵ-far from being transitive, as required. $\square$

3 Finding many directed triangles in T

Our second lemma will show that in a tournament with few backwards edges, many of which have large length, there is a large subset of backwards edges which all lie in many directed triangles.

Lemma 6. *Let T be an n-vertex tournament with an optimal ordering $v_1, \dots, v_n$ and let B denote the set of backwards edges in this ordering, $|B| = \alpha n^2$. Suppose that the subset $B' \subset B$ of backwards edges with length at least $n/16$ satisfies $|B'| \geq \alpha n^2/4$. Then, provided that $\alpha \leq 2^{-16}$, there exists $B'' \subset B'$ satisfying $|B''| \geq |B'|/2$ with the property that each edge of B'' lies in at least $n/64$ directed triangles in T.*

Proof. Given B' as in the statement of the lemma, let $B'' \subset B'$ be the set

$$B'' := \{\overleftarrow{v_i v_j} \in B' : \text{ either } d^-_{[v_{i+1}, v_j]}(v_i) \leq 4\alpha^{1/2} n \text{ or } d^+_{[v_i, v_{j-1}]}(v_j) \leq 4\alpha^{1/2} n\}$$

We first claim that $|B''| \geq |B'|/2$. To see this let $S_- = \{v_i \in V(T) : d^-_B(v_i) \geq 4\alpha^{1/2} n\}$ and let $S_+ = \{v_i \in V(T) : d^+_B(v_i) \geq 4\alpha^{1/2} n\}$. Using that

$$4\alpha^{1/2} n |S_-| \leq \sum_{i \in S_-} d^-_B(v_i) \leq \sum_{i \in [n]} d^-_B(v_i) = |B|,$$

gives $|S_-| \leq |B|/4\alpha^{1/2} n = \alpha^{1/2} n/4$. Similarly $|S_+| \leq \alpha^{1/2} n/4$. But all edges $\overleftarrow{v_i v_j} \in B' \setminus B''$ have $v_i \in S_-$ and $v_j \in S_+$. This gives

$$|B' \setminus B''| \leq |S_-||S_+| \leq (\alpha^{1/2} n/4)^2 = \alpha n^2/16.$$

But then $|B''| \geq |B'| - \alpha n^2/16 \geq |B'|/2$, as claimed.

Now recall that by the definition of B', for every $\overleftarrow{v_i v_j} \in B''$ we have $j - i \geq n/16$. Also, by Proposition 4 part 1 we have $d^+_{[v_{i+1}, v_j]}(v_i) \geq (j-i)/2$ and $d^-_{[v_i, v_{j-1}]}(v_j) \geq (j-i)/2$. Furthermore, as $\overleftarrow{v_i v_j} \in B''$ we must have either

$$d^+_{[v_{i+1}, v_j]}(v_i) \geq (j-i) - 4\alpha^{1/2} n \geq 3(j-i)/4$$

or

$$d^-_{[v_i, v_{j-1}]}(v_j) \geq (j-i) - 4\alpha^{1/2} n \geq 3(j-i)/4.$$

The inequalities hold here since $\alpha \leq 2^{-16}$ and $j - i \geq n/16$ gives $(j-i)/4 \geq n/2^6 \geq 4\alpha^{1/2} n$. Thus for every edge $\overleftarrow{v_i v_j} \in B''$ there are at least $(j-i)/4 \geq n/2^6$ vertices $v_k \in \{v_{i+1}, \dots, v_{j-1}\}$ such that $\overrightarrow{v_i v_k}$ and $\overrightarrow{v_k v_j}$ are edges. But this gives that every edge of B'' lies in at least $n/2^6$ directed triangles, as claimed. $\square$

4 Finding a copy of D_k in T

The second half of our argument is based on another result from [9]. Here the authors proved that the following holds:

Theorem 7 (Theorem 3.5 in [9])**.** *Any n-vertex tournament with at least δn^3 directed triangles contains D_k as a subtournament provided that $n \geq \delta^{-4k/\delta}$.*

By combining Lemma 5 and Lemma 6 with Theorem 7 it is already possible to improve the bound $n_0(k,\epsilon)$, to show that $n_0(k,\epsilon) \leq \epsilon^{-ck/\epsilon}$ for some fixed constant $c > 0$. To remove the additional ϵ term from the exponent, we need to modify Theorem 7.

The next lemma shows that if many directed triangles in Theorem 7 occur in a very unbalanced manner, meaning that each of these triangles contain an edge from a small set, the lower bound on n in Theorem 7 can be reduced. Note that this is exactly the situation given by Lemma 6.

Lemma 8. *Let T be an n-vertex tournament and let E be a set of βn^2 edges in T. Suppose that each edge of E occurs in at least γn directed triangles in T. Then T contains D_k as a subtournament provided $n \geq \beta^{-100k/\gamma}$.*

The proof modifies the proof of Theorem 7 in [9], but as the details are somewhat technical, we have included the proof in full. We will use the following formulation of the dependent random choice method (see [10]).

Lemma 9. *Let $G = (A, B, E)$ be a bipartite graph with $|A| = |B| = n$ and αn^2 edges. Given $d, l \in \mathbb{N}$, there exists a set $A' \subset A$ with $|A'| \geq \alpha^l n - 1$ such that every d-set in A' has at least $n^{1-d/l}$ common neighbours in B.*

We will also use of the following bound for the Zarankiewicz problem, due to Kövari, Sós and Turán (see [17], [12]). Here it was shown that any bipartite graph $G = (A, B, E)$, with $|A| = m$, $|B| = n$, which does not contain $K_{s,t}$ as a subgraph, with s vertices in A and t in B satisfies

$$e(G) \leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m. \tag{2}$$

Proof of Lemma 8. To begin, pick a random equipartition of $V(T)$ into three sets V_1, V_2 and V_3, each with size $n/3$. For each edge $e \in E$, let $Q_e^{(i)}$ denote the number of vertices $v \in V_i$ which form a directed triangle with e in T. Let E_{good} denote the collection of (random) edges $e = \overrightarrow{xy} \in E$ with $x \in V_1$ to $y \in V_2$ and $Q_e^{(3)} \geq \gamma n/3$. For all $e \in E$, we have

$$\begin{aligned}\mathbb{P}(e \in E_{good}) = \mathbb{P}(e \in \overrightarrow{V_1V_2} \text{ and } Q_e^{(3)} \geq \gamma n/3) &= \mathbb{P}(Q_e^{(3)} \geq \gamma n/3 | e \in \overrightarrow{V_1V_2}) \times \mathbb{P}(e \in \overrightarrow{V_1V_2}) \\ &\geq \frac{1}{3} \times \frac{|V_1||V_2|}{n(n-1)} \geq \frac{1}{27}.\end{aligned} \tag{3}$$

To see the inequality here, note that as $|V_3| \geq |V_1 \setminus \{x\}|, |V_2 \setminus \{y\}|$ we have $\mathbb{P}(Q_e^{(3)} \geq \gamma n/3 | e \in \overrightarrow{V_1V_2}) \geq \mathbb{P}(Q_e^{(i)} \geq \gamma n/3 | e \in \overrightarrow{V_1V_2})$ for $i \in \{1, 2\}$. As $e \in E$ we also have $\sum_{i=1}^3 Q_e^{(i)} \geq \gamma n$ and so

$$\begin{aligned}3\mathbb{P}(Q_e^{(3)} \geq \gamma n/3 | e \in \overrightarrow{V_1V_2}) &\geq \sum_{i=1}^{3} \mathbb{P}(Q_e^{(i)} \geq \gamma n/3 | e \in \overrightarrow{V_1V_2}) \\ &\geq \mathbb{P}(Q_e^{(i)} \geq \gamma n/3 \text{ for some } i | e \in \overrightarrow{V_1V_2}) = 1.\end{aligned}$$

By (3) we have $\mathbb{E}(|E_{good}|) \geq |E|/27 \geq \beta n^2/27$. Fix a partition with $|E_{good}|$ at least this big.

Now take H to denote the bipartite graph between sets V_1 and V_2 whose edge set is E_{good}. From the previous paragraph $|e(H)| \geq \beta n^2/27 = \frac{\beta}{3}(\frac{n}{3})^2$. Applying Lemma 9 to H with $d = 3k/\gamma$ and $l = 4d$ we can find a set in $W_1 \subset V_1$ with $|W_1| \geq (\beta/3)^l|V_1| - 1 \geq n^{1/2}$ such that every d-set in W_1 has at least $(n/3)^{1-d/l} = (n/3)^{3/4} \geq n^{1/2}$ common neighbours in V_2. The inequality on $|W_1|$ here holds since

$$(\beta/3)^l|V_1| - 1 \geq \beta^{3l}\frac{n}{3} - 1 \geq 2\beta^{4l}n - 1 \geq 2n^{1/2} - 1 \geq n^{1/2}.$$

using that $1/3 \geq \beta^2$ and $\beta^l \leq 1/6$ (since $\beta \leq 1/2$, $l \geq 4$) and $n \geq \beta^{-100k/\gamma} \geq \beta^{-8l}$.

Now by applying the Erdős–Moser theorem to W_1, we find a transitive subtournament T_1 on vertex set $S_1 \subset W_1$ with $|S_1| \geq \log|W_1| \geq \log n^{1/2} \geq d$. Letting $N_H[S_1] \subset V_2$ denote the common neighbourhood of S_1 in H, by choice of W_1 we have $|N_H[S_1]| \geq n^{1/2}$. Again apply the Erdős–Moser theorem to $N_H[S_1]$, we find $S_2 \subset N_H(S_1)$ with $|S_2| \geq \log|N_H[S_1]| = \log n^{1/2} \geq d$ vertices. By the construction of H, this gives that all edges of T between S_1 and S_2 are directed from S_1 to S_2.

For the next section of the argument, fix a matching of size d within this bipartite directed subgraph $T[S_1, S_2]$, say with edges $\{e_1, \ldots, e_d\}$. As each edge $e_i \in E_{good}$, we have $Q^{(3)}_{e_i} \geq \gamma n/3$ for all $i \in [d]$. Now consider the bipartite graph G on vertex set $A = \{e_1, \ldots, e_d\}$ and V_3 in which $e_i \in A$ is joined to $v \in V_3$ if together the vertices of e_i and v form a directed triangle in T. As $Q_{e_i} \geq \gamma n/3$ for all $i \in [d]$, we see $e(G) \geq d\gamma n/3 = kn$.

We now claim that in G there exists $A' \subset A$ and $V_3' \subset V_3$ with $|A'| \geq k$ and $|V_3'| \geq n^{1/2}$ such that $G[A', V_3']$ is complete. Indeed, by (2), if G does not contain a complete bipartite subgraph G' with k vertices in A and $n^{1/2}$ vertices in V_3, then the number of edges in G satisfies

$$\begin{aligned} e(G) &< (n^{1/2} - 1)^{1/k}(d - k + 1)(n/3)^{1-1/k} + (k-1)n/3 \\ &< (dn^{-1/2k} + k/3)n \leq 5kn/6 < e(G). \end{aligned}$$

To see the second last inequality, note that $n^{1/2k} \geq \beta^{-12/\gamma} \geq 2^{12/\gamma} \geq e^{6/\gamma} \geq 6/\gamma$, as $\beta \leq 1/2$ and $e^x \geq x$ for all x. This gives $dn^{-1/2k} \leq d\gamma/6 = k/2$. This contradiction shows that must exist some set of k edges $\{e_{i_1}, \ldots, e_{i_k}\} \subset A$ which is completely joined to a set $W_3 \subset V_3$ of size at least $n^{1/2}$. To complete the proof of the lemma, apply the Erdős–Moser theorem a final time to W_3 to find a transitive subtournament of size $\log n^{1/2} > d > k$ on vertex set S_3. For $i = 1, 2$, let U_i denote the sets $U_i \subset V_i$ which occur in the edges $\{e_{i_1}, \ldots, e_{i_k}\}$. Also let $U_3 \subset S_3$ be a set with $|U_3| = k$.

We claim that $T[U_1 \cup U_2 \cup U_3]$ forms a subtournament isomorphic to D_k. Indeed, $|U_i| = k$ for all $i \in [3]$ and $T[U_i]$ is transitive since $U_i \subset S_i$. Also, all edges in T between U_1 and U_2 are directed from U_1 to U_2, since $U_i \subset S_i$. Lastly, from the definition of H, each $u \in U_3$ forms a directed triangle in T with $e_{i_j} \in \overrightarrow{U_1U_2}$ for all $j \in [k]$ giving that all edges of T are directed from U_2 to U_3 and from U_3 to U_1. $\square$

We can now complete the proof of Theorem 2.

Proof of Theorem 2. Take $c \geq 1$ to be a constant such that Theorem 1 holds and set $C = 2^{33}c$. We will show that an n-vertex tournament T which is ϵ-far from being transitive contains D_k as a subtournament, provided $n \geq \epsilon^{-Ck}$.

To begin, choose $i \in \mathbb{N} \cup \{0\}$ as large as possible so that T contains a subtournament T' satisfying $|T'| \geq |T|/8^i$ and such that T' is $(2^i\epsilon)$-far from being transitive. Let $|T'| = t \geq n/8^i$ and list the vertices of T' in an optimal ordering $v_1, \dots, v_t$. Letting B denote the backwards edges of T' and $|B| = \alpha t^2$, we have $\alpha \geq 2^i \epsilon$. In particular, since $\alpha \leq 1$ we have $1/2^i \geq \epsilon$. Now by the choice of i, the conclusion of Lemma 5 part 2 fails for T'. Lemma 5 therefore guarantees that the subset B' of B consisting of edges of length at least $t/16$ satisfies $|B'| \geq |B|/4 = \alpha t^2/4$.

We first consider the case when $\alpha > 2^{-16}$. Here we apply Theorem 1 to T' taking advantage of the fact that α is quite large. Indeed, as T' is α-far from being transitive, by Theorem 1 we find that T' contains D_k as a subtournament, provided $t \geq \alpha^{-ck/\alpha^2}$. This holds as

$$t \geq n/8^i \geq n\epsilon^3 \geq \epsilon^{-Ck+3} \geq \epsilon^{-Ck/2} \geq \alpha^{-Ck/2} \geq \alpha^{-2^{32}ck} \geq \alpha^{-ck/\alpha^2}.$$

Here we used that $1/2^i \geq \epsilon$, that $C \geq 6$ and $k \geq 1$ and that $\alpha \geq \epsilon$.

Now we deal with the case when $\alpha \leq 2^{-16}$. We can apply Lemma 6 to T' taking B and B' as given above, to find a subset $B'' \subset B'$, satisfying $|B''| \geq |B'|/2 \geq (\alpha/8)t^2$ with the property that each edge of B'' lies in at least $t/64$ directed triangles in T'. We now apply Lemma 8 to T' taking $E = B''$, $\beta = \alpha/8$ and $\gamma = 1/64$. This shows that T' contains a copy of D_k, provided that $|T'| = t \geq \beta^{-100k/\gamma} = \beta^{-6400k}$. To see that this holds, first note that $t \geq n/8^i \geq \epsilon^{-Ck}/8^i \geq \epsilon^{-Ck+3} \geq \epsilon^{-Ck/2}$ as $C \geq 6$. Using $\beta \geq 2^i\epsilon/8 \geq \epsilon/8 \geq \epsilon^4$ (since $1/2 \geq \epsilon$) gives $t \geq \epsilon^{-Ck/2} \geq \beta^{-Ck/8} \geq \beta^{-2^{30}ck} \geq \beta^{-6400k}$, as required. □

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References

[1] P. Agarwal, J. Pach, **Combinatorial geometry**, Wiley-Interscience, 1995.

[2] N. Alon and J. Spencer, **The probabilistic method**, 3rd ed., John Wiley and Sons (2008).

[3] E. Berger, K. Choromanski, M. Chudnovksy, J. Fox, M. Loebl, A. Scott, P. Seymour and S. Thomassé, Tournaments and colouring, *J. Combin. Theory Ser. B* **103** (2013), 1-20.

[4] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **170** (2009), 941-960.

[5] J. Cutler and B. Montagh, Unavoidable subgraphs of colored graphs, *Discrete Math.*, **308** (2008), 4396-4413.

[6] P. Erdős, Some remarks on the theory of graphs, *Bulletin of the Amer. Math. Soc.*, **53** (1947), 292-294.

[7] Erdős, L. Moser, On the representation of directed graphs as unions of orderings, *Publ. Math. Inst. Hungar. Acad. Sci.*, **9** (1964), 125-132.

[8] P. Erdős, G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463-470.

[9] J. Fox, B. Sudakov: Unavoidable patterns, *J. Combin. Theory Ser. A* **115** (2008), 1561-1569.

[10] J. Fox, B. Sudakov: Dependent random choice, *Random Structures Algorithms* **38**(1-2) (2011), 68-99.

[11] H. Furstenberg, **Recurrence in ergodic theory and combinatorial number theory**, Princeton University Press, Princeton, NJ, 1981.

[12] T. Kövari, V. T. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50-57.

[13] R.L. Graham, B.L. Rothschild, J.H. Spencer, **Ramsey theory**, 2nd ed., John Wiley & Sons, 1990.

[14] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), 264-286.

[15] A. Shapira and R. Yuster, Unavoidable tournaments, *J. Combin. Theory Ser. B*, **116** (2016), 191-207.

[16] J. Spencer, Ramsey's theorem - A new lower bound , *J. Combin. Theory Ser. A*, **18**1 (1975), 108-115.

[17] K. Zarankiewicz, Problem p 101, *Colloq. Math.* **2** (1951), 301.