

ON PATH DECOMPOSITIONS OF $2k$ -REGULAR GRAPHS

FÁBIO BOTLER¹ AND ANDREA JIMÉNEZ²

¹*Instituto de Matemática e Estatística, Universidade de São Paulo*

²*CIMFAV, Facultad de Ingeniería, Universidad de Valparaíso*

ABSTRACT. Tibor Gallai conjectured that the edge set of every connected graph G on n vertices can be partitioned into $\lceil n/2 \rceil$ paths. Let \mathcal{G}_k be the class of all $2k$ -regular graphs of girth at least $2k - 2$ that admit a pair of disjoint perfect matchings. In this work, we show that Gallai's conjecture holds in \mathcal{G}_k , for every $k \geq 3$. Further, we prove that for every graph G in \mathcal{G}_k on n vertices, there exists a partition of its edge set into $n/2$ paths of lengths in $\{2k - 1, 2k, 2k + 1\}$.

Keywords. Path decomposition, length-constrained, $2k$ -regular graphs, perfect matchings.

1. INTRODUCTION

A *decomposition* of a graph is a set of subgraphs that partition its edge set. If all subgraphs are paths, then it is called a *path decomposition*. In 1966, Erdős (see [10]) raised the question: *given $n > 0$, what is the minimum λ such that every connected graph G on n vertices admits a decomposition into λ paths?* Gallai conjectured that λ is at most $\lfloor (n + 1)/2 \rfloor$. Despite the best efforts [5, 8, 10, 11], the conjecture of Gallai remains widely open. Over the years, Gallai's conjecture has been explored on particular classes of graphs. In this paper, we focus on regular graphs. A seminal result of Lovász [10] shows that all odd regular graphs satisfy the conjecture of Gallai. In contrast, for even regular graphs not much is known. Indeed, in addition to classical results for complete graphs, it is known that the conjecture holds on 4-regular graphs [6]. We make progress within this direction. Let $k \geq 3$ an integer and \mathcal{G}_k be the class of all $2k$ -regular graphs of girth at least $2k - 2$ that admit a pair of disjoint perfect matchings. As usual, the *girth* of a graph is the length of a shortest cycle. We establish the following theorem.

We thank the partial support of FAPESP (2013/03447-6) and CNPq (477203/2012-4), Brazil. The first author thanks partial support FAPESP Projects (Proc. 2014/01460-8 and 2011/08033-0) and CNPq Project (456792/2014-7), Brazil. The second author thanks the partial support of CONICYT/FONDECYT/POSTDOCTORADO 3150673 and of Nucleo Milenio Información y Coordinación en Redes ICM/FIC RC130003, Chile.

Email: ¹fbotler@ime.usp.br, ²andrea.jimenez@uv.cl.

Theorem 1. *Every $G \in \mathcal{G}_k$ on n vertices has a decomposition into $n/2$ paths.*

A closely related problem to the conjecture of Gallai that has drew great interest [2, 7, 9, 12, 13] is as follows: given a family of paths \mathcal{H} , is there a decomposition \mathcal{D} of G such that each graph in \mathcal{D} is isomorphic to a graph in \mathcal{H} ? We give a step forward towards obtaining constrained path decompositions of even regular graphs. Actually, Theorem 1 follows from a stronger statement. In this work, the *length* of a graph refers to its number of edges. The main contribution of this paper is the following statement.

Theorem 2. *For every $G \in \mathcal{G}_k$ there exists a decomposition \mathcal{D} into paths of lengths in $\{2k - 1, 2k, 2k + 1\}$. Further, in \mathcal{D} , the number of paths of length $2k - 1$ is equal to the number of paths of length $2k + 1$.*

In the same vein, Botler et al. (Theorem 4.2 of [2] and Theorem 3.2 of [1]) guaranteed existence of length-constrained path decompositions for $(2k - 1)$ -regular graphs.

Theorem 3 (Botler et al. [1, 2]). *Let $k \geq 3$ and G be a $2k - 1$ -regular graph of girth at least $2k - 2$ that admits a perfect matching. Then G has a decomposition into paths of length $2k - 1$.*

This result is the outset of the proof of Theorem 2.

1.1. Organization of the paper. The proof of Theorem 2 consists of two main steps. In the first step, developed in Section 2, we prove that one can decompose graphs in \mathcal{G}_k into paths of lengths in $\{2k - 1, 2k, 2k + 1\}$ and cycles of length $2k$. In the second step, developed in Section 3, we show how to turn the decompositions obtained in the first step into the desired ones. The last section contains the proof of a technical lemma (Lemma 2.6) on which our results are based on.

2. DECOMPOSING INTO PATHS AND CYCLES OF LENGTHS IN $\{2k - 1, 2k, 2k + 1\}$

Let $N \subset \mathbb{N}$. We refer to a decomposition into paths and cycles as an N -*decomposition* if all its paths and cycles have their lengths in N . The following definition introduces a type of decomposition that is fundamental to prove the existence of $\{2k - 1, 2k, 2k + 1\}$ -decompositions into paths only.

Definition 2.1. Let G be a graph and $k \geq 3$. Let v be a vertex of G and \mathcal{D} be a $\{2k - 1, 2k, 2k + 1\}$ -decomposition of G . We say that \mathcal{D} is k -*balanced* (or simply *balanced*) at v if the number of paths of \mathcal{D} of length at most $2k$ with end vertex v is at least the number of paths of \mathcal{D} of length at least $2k + 1$ with end vertex v . Further, we say that \mathcal{D} is a k -*balanced decomposition* of G if it is k -balanced at every vertex of G and the number of paths of length $2k - 1$ is equal to the number of paths of length $2k + 1$.

Recall that \mathcal{G}_k denotes the class of all $2k$ -regular graphs of girth at least $2k - 2$ that admit a pair of disjoint perfect matchings. The following is the main result of this section.

Theorem 4. *For every graph in \mathcal{G}_k there is a k -balanced decomposition \mathcal{D} . Moreover, the cycles in \mathcal{D} have length $2k$.*

Throughout this section, $G \in \mathcal{G}_k$ is a graph on n vertices and M denotes one of its two disjoint perfect matchings. The starting point of the proof of Theorem 4 is the following. By removing all edges of M from G , we obtain a $(2k - 1)$ -regular graph G' with girth at least $2k - 2$ provided with a perfect matching. Due to Theorem 3, there exists a decomposition \mathcal{P} of G' into paths of length $2k - 1$. The rest of the proof of Theorem 4 relies on our ability to extend \mathcal{P} to a k -balanced decomposition of G . Indeed, Lemma 2.6 characterizes, and shows a way to overcome, the obstructions to such extension. Before we present the proof of Theorem 4, we formulate this key lemma.

2.1. A toolbox. A sequence of vertices and edges $W := v_1e_1v_2 \cdots v_te_tv_{t+1}$ is called a *trail* if $e_i = v_iv_{i+1} \in E(W)$ for each $i \in \{1, \dots, t\}$, and $e_i \neq e_j$ if $i \neq j$. If $v_1 = v_{t+1}$, then W is a *closed trail*. A closed trail W is called an (M, \mathcal{P}) -*alternating r -closed trail* (or simply an *alternating closed trail*) if W is the union of r distinct paths P_1, \dots, P_r of \mathcal{P} and r distinct matching edges e_1, \dots, e_r of M , for some $r \geq 1$, such that, for each $i \in \mathbb{Z}_r$ the matching edge $e_i = uv$ is such that u is an end vertex of P_i and v is an end vertex of P_{i+1} . We write W as $P_0e_0 \cdots P_{r-1}e_{r-1}P_0$. Note that a 1-closed trail is a cycle of length $2k$. A subsequence of W is called an (M, \mathcal{P}) -*alternating trail* (or simply an *alternating trail*).

The following definition presents the structure of the main obstruction.

Definition 2.2. Let $G \in \mathcal{G}_3$. We say that a subgraph of G is *exceptional* if it is isomorphic to an (M, \mathcal{P}) -alternating trail $PeP'e'$ where $P = x_0 \cdots x_5$, $P' = y_0 \cdots y_5$, and $e = x_5y_0$ such that $e' = y_5y_2$, $y_0 = x_1$, $y_4 = x_5$ and $y_2 = x_3$ (see Figure 1).

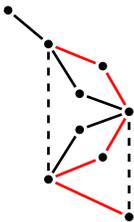


FIGURE 1. Exceptional graph: P is the black path, P' is the red path and the dashed edges represent the matching edges.

The notation $s(j, l)$ stands for the (M, \mathcal{P}) -alternating trail $P_je_j \cdots P_{j+l}e_{j+l}P_{j+l+1}$, which embraces the matching edges e_j, \dots, e_{j+l} . In the case that $P_j = P_{j+l+1}$, we say that $s(j, l)$ is cyclic. If $s(j, l)$ is non-cyclic, we refer to the alternating trails

$$s[j, l] = e_{j-1}s(j, l), \quad s(j, l] = s(j, l)e_{j+l+1}, \quad s[j, l] = e_{j-1}s(j, l)e_{j+l+1},$$

and to $s(j, l)$ itself, as *extensions* of $s(j, l)$.

Observation 2.3. *Because graphs in \mathcal{G}_3 are triangle-free, if the (M, \mathcal{P}) -alternating trail $s(j, l)$ is cyclic and contains an exceptional subgraph, then $l \geq 2$.*

The following definition captures the notion of balanced decompositions in extensions.

Definition 2.4 (Quasi-balanced decompositions). Let X be an extension of $s(j, l)$. We refer to a $\{2k - 1, 2k, 2k + 1\}$ -decomposition \mathcal{D} of X as *quasi- k -balanced* (or simply, quasi-balanced) if \mathcal{D} is k -balanced at all vertices of X , except possibly, at the vertices of odd degree of X incident to an edge of $\{e_{j-1}, e_{j+l+1}\}$. In addition, if \mathcal{D} is not k -balanced at $v \in e$ for $e \in \{e_{j-1}, e_{j+l+1}\}$, then $e \in X$.

For a set S , the notation $|S|$ stands for the cardinality of S . The next observation follows.

Observation 2.5. *Let X_1, \dots, X_l be a partition of the edges of an (M, \mathcal{P}) -alternating trail s (not necessarily closed) into extensions and $\mathcal{D}_1, \dots, \mathcal{D}_l$ be a set of quasi- k -balanced decompositions of X_1, \dots, X_l , respectively. Then, $\mathcal{D} = \cup_{i=1, \dots, l} \mathcal{D}_i$ is a quasi- k -balanced decomposition of s . Further, if for each $i \in \{1, \dots, l\}$, $|\mathcal{D}_i|$ equals the number of paths of \mathcal{P} in X_i and s is an alternating closed trail, then \mathcal{D} is a balanced decomposition of s .*

The toolbox of this paper is a result (Lemma 2.6) which claims that each alternating trail containing 2 paths from \mathcal{P} has a quasi- k -balanced decomposition into 2 paths, unless it contains an exceptional subgraph (in particular, $k = 3$). Moreover, Lemma 2.6 shows that exceptional subgraphs are disjoint and are not subsequent, which allows us to overcome them. The proof of Theorem 4 relies on this result; specifically, it is used to prove Claim 2.11.

Lemma 2.6. *Let $e_{i-1}P_i \cdots P_{i+2}e_{i+2}$ be an (M, \mathcal{P}) -alternating trail and X be an extension of $P_i e_i P_{i+1}$. The following two statements hold.*

- (i) *If X does not contain an exceptional subgraph, then X has a quasi- k -balanced decomposition into 2 paths.*
- (ii) *If X is exceptional (in particular, $k = 3$), then $e_{i-1} \neq e_{i+1}$. If, in addition, $X = P_i e_i P_{i+1} e_{i+1}$ (analogously for $e_{i-1} P_i e_i P_{i+1}$), then neither $e_i P_{i+1} e_{i+1} P_{i+2}$, nor $P_{i+1} e_{i+1} P_{i+2} e_{i+2}$ is exceptional.*

In the following we present the proof of Theorem 4. The proof of Lemma 2.6 is in Section 4.

2.2. Proof of Theorem 4. Recall that $G \in \mathcal{G}_k$ is a $2k$ -regular graph on n vertices of girth at least $2k - 2$ that admits a pair of disjoint perfect matchings, M denotes one of its two disjoint perfect matchings and the $(2k - 1)$ -regular graph $G' = G - M$ has a decomposition \mathcal{P} into paths of length $2k - 1$.

Since each vertex of G' is the end vertex of exactly one path in \mathcal{P} , it follows that the edge set of G admits a partition \mathcal{W} into (M, \mathcal{P}) -alternating closed trails. Each alternating 1-closed trail in \mathcal{W} is a cycle of length $2k$. Hence, to conclude the result of Theorem 4, it suffices to prove that for each alternating r -closed trail with $r \geq 2$ there exists a k -balanced decomposition into r paths; this is the statement of our next lemma. Thus, the validity of Theorem 4 follows from Lemma 2.7.

Lemma 2.7. *Let $r \geq 2$ and W be an (M, \mathcal{P}) -alternating r -closed trail. Then, W admits a k -balanced decomposition.*

Before we show Lemma 2.7, we present two useful definitions.

Definition 2.8 (malicious paths, nice paths and trapped edges). Let PeP' be a path-edge-path (M, \mathcal{P}) -alternating trail. Suppose $e = xx'$, where x is an end vertex of P and x' is an end vertex of P' . If x' is an internal vertex of P , then we say that P is *malicious* for e ; otherwise, we say P is *nice* for e . If P and P' are malicious for e , we say that edge e is *trapped* (in the sequence PeP').

Definition 2.9 (trapped sequences, nice and malicious extensions). Let $r \geq 2$ and W be the (M, \mathcal{P}) -alternating r -closed trail $P_0e_0 \cdots P_{r-1}e_{r-1}P_0$. If all matching edges $e_j, e_{j+1}, \dots, e_{j+l}$ are trapped in $s(j, l)$, then we say that $s(j, l)$ is a *trapped sequence*. Moreover, we say that an extension X of $s(j, l)$ is *nice*, if the following holds: if $e_{j-1} \in X$ (resp. $e_{j+l+1} \in X$) and $e_{j-1} \neq e_{j+l+1}$, then P_j (resp. P_{j+l+1}) is nice for e_{j-1} (resp. e_{j+l+1}). Otherwise, it is called *malicious*. In particular, $s(j, l)$ is nice as an extension of itself.

Note that an extension X of $s(j, l)$ is malicious only if at least one of the following holds.

- $e_{j-1} \in X$, $e_{j-1} \neq e_{j+l+1}$, and P_j is malicious for e_{j-1} ; or
- $e_{j+l+1} \in X$, $e_{j-1} \neq e_{j+l+1}$, and P_{j+l+1} is malicious for e_{j+l+1} .

We are ready to prove Lemma 2.7.

Proof of Lemma 2.7. Let $W = P_0e_0 \cdots P_{r-1}e_{r-1}P_0$ be an alternating r -closed trail, for some $r \geq 2$. Recall that we want to prove that W admits a k -balanced decomposition. Note that the property that the number of paths of length $2k - 1$ equals the number of paths of length $2k + 1$ of the decomposition is equivalent to the one that the decomposition consists of r paths (Observation 2.3).

Let us first suppose that for every $i \in \mathbb{Z}_r$, the matching edge e_i is not trapped in $P_i e_i P_{i+1}$. Since $r \geq 2$, for each $i \in \mathbb{Z}_r$, we have $P_i e_i$ or $e_i P_{i+1}$ is a path. Moreover, if $e_{i-1} P_i$ and $P_i e_i$ are paths, then $e_{i-1} P_i e_i$ is a path (because e_0, \dots, e_{r-1} are elements of a matching of G) and it has length $2k + 1$. Therefore, it is possible to decompose W into r paths whose lengths are in $\{2k - 1, 2k, 2k + 1\}$ such that each path of length $2k + 1$ has both end edges in the perfect matching M . Hence, each vertex of W is the end vertex of at most one path of length $2k + 1$, thus, the decomposition is k -balanced.

We now assume that W contains trapped edges. We study first the case that W is not a cyclic trapped sequence, and later the remaining case.

Case that W is not a cyclic trapped sequence. We say that the matching edges e_{i-1}, e_i are the *neighbouring edges* of P_i in W , for all $i \in \mathbb{Z}_r$. Let \mathcal{S} be the set of all maximal trapped sequences in W , $\mathcal{P}' \subset \mathcal{P}$ be the set of all paths that do not belong to any trapped sequence in W , and W' be the subgraph of W consisting of the union of all the paths in \mathcal{P}' and all matching edges in M satisfying that for each of them there exists a path in \mathcal{P}' that is nice. Note that each maximal trapped sequence starts and ends with paths in \mathcal{P} and that $P \in \mathcal{P}'$ if and only if either P is nice for one of its two neighbouring edges, or there are paths in \mathcal{P} that are nice for its neighbouring edges (and thus, such paths are also in \mathcal{P}). Hence, W' is a disjoint union of extensions. Trivially, it follows that there exists a $\{2k-1, 2k, 2k+1\}$ -decomposition $\mathcal{P}(W')$ of W' into $|\mathcal{P}'|$ paths such that $\mathcal{P}(W')$ is quasi- k -balanced at each extension of W' .

Let $W^* = W - E(W')$ be the graph obtained from W by deleting the edge set $E(W')$ and all (if any) isolated vertices. Recall that $s(j, l)$ denotes $P_j e_j \cdots P_{j+l} e_{j+l} P_{j+l+1}$. We refer to the paths P_j and P_{j+l+1} as the *end paths* of $s(j, l)$, to P_j as its *initial path* and to P_{j+l+1} as its *final path*. Observe that W^* is the disjoint union of all the maximal trapped sequences in \mathcal{S} and a set $M^* \subset M$ (possibly empty). Let us understand the behavior of the edges in M^* arise. If $e \in M^*$, then there is no path in \mathcal{P}' that is nice for e . Thus, e is a neighbouring edge of an end path P_e of a maximal trapped sequence s_e of \mathcal{S} ; otherwise, there are paths P, P' in \mathcal{P}' such that PeP' is a trapped sequence, a contradiction. We now prove that we can choose s_e and P_e so that P_e is nice for e . We have already shown that there exists a maximal trapped sequence $s(j, l) \in \mathcal{S}$, such that $e = e_{j-1}$, or $e = e_{j+l+1}$. Suppose, without loss of generality, that $e = e_{j-1}$. Suppose that the result does not hold with such sequence; that is, if $e = e_{j-1}$ then P_j is malicious for e . Due to the maximality of $s(j, l)$, the path P_{j-1} is nice for e . Hence, P_{j-1} is neither an end path of $s(j, l)$, nor a path in \mathcal{P}' . Therefore, P_{j-1} is the final path of a maximal trapped sequence $s \in \mathcal{S}$, obviously distinct of $s(j, l)$, and the result follows by taking $s_e = s$ and $P_e = P_{j-1}$. Consequently, we can claim the following.

Claim 2.10. *W^* can be decomposed into nice extensions of maximal trapped sequences.*

Let us assume that the following claim holds:

Claim 2.11. *For every nice extension X of a trapped sequence $s(j, l)$, there exists a quasi- k -balanced decomposition D of X into $l+2$ paths.*

Then, according to Observation 2.5, Lemma 2.7 follows from the fact that W is the edge disjoint union of W' and W^* , the existence of a quasi- k -balanced decomposition for each extension of W' and, the validity of Claims 2.10 and 2.11.

In what follows, we prove Claim 2.11. The proof of Claim 2.11 relies on the technical result, namely Lemma 2.6, described in Subsection 2.1.

Proof of Claim 2.11. Let us proceed by induction on l (clearly, $l \geq 0$). For the case that $l = 0$, we have that $s(j, 0)$ is a path-edge-path alternating trail. Let X be a nice extension of $s(j, 0)$. By Lemma 2.6(i), if $k > 3$, then X has a quasi- k -balanced decomposition into 2 paths. Suppose $k = 3$. If $e_{j-1} \neq e_{j+1}$, then X does not contain exceptional subgraphs (see Figure 1), because if e_{j-1} (resp. e_{j+1}) is in X , then P_j (resp. P_{j+1}) is nice for e_{j-1} (resp. e_{j+1}). Thus, using Lemma 2.6(i) the result follows. If $e_{j-1} = e_{j+1}$, then by Observation 2.3, X does not contain an exceptional subgraph and the result holds, again, by Lemma 2.6(i).

We now suppose $l \geq 1$. Let $eP_j e_j P_{j+1} e_{j+1}$ be the beginning of the nice extension $X(j, l)$, where e may exist or not. If $e \in X(j, l)$, we can suppose without loss of generality that P_j is nice for e ; otherwise, namely, if $e \in X(j, l)$ and P_j is malicious for e , by definition of nice extensions, we have that $e_{j+l+1} \in X(j, l)$ and P_{j+l+1} is nice for e_{j+l+1} , and thus we can use the inverse sequence of $X(j, l)$, instead of $X(j, l)$ itself. In particular, in the case that $k = 3$, we have that $eP_j e_j P_{j+1}$ is not exceptional.

If $k = 3$ and $P_j e_j P_{j+1} e_{j+1}$ is not exceptional, then by Lemma 2.6(i), we have that $eP_j e_j P_{j+1} e_{j+1}$ has a quasi-3-balanced decomposition into two paths. For $k > 3$, we have that $eP_j e_j P_{j+1} e_{j+1}$ is an extension of $P_j e_j P_{j+1}$ and thus Lemma 2.6(i) claims that there exists a quasi- k -balanced decomposition of $eP_j e_j P_{j+1} e_{j+1}$ into two paths. For $k \geq 3$, let $X := X(j, l) - eP_j e_j P_{j+1} e_{j+1}$; the nice extension of the trapped sequence $s(j+2, l-2)$. If $l = 1$, then X is simply a path of length $2k-1$ or $2k$. If $l \geq 2$, by the induction hypothesis, X has a quasi- k -balanced decomposition into l paths. Thus, by Observation 2.5, there exists a quasi- k -balanced decomposition of $X(j, l)$ into $l+2$ paths.

We now study the case that $k = 3$ and $P_j e_j P_{j+1} e_{j+1}$ is exceptional. Using Lemma 2.6(ii), we have that $e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$ does not contain an exceptional subgraph, and therefore, by Lemma 2.6(i), it has a quasi-3-balanced decomposition into 2 paths. Since P_j is nice for e (by previous assumption), we have that eP_j is a path of length 6 (if $e \in X$) or simply a path of length 5. It follows, by Observation 2.5, that $eP_j e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$ can be decomposed into three paths that form a quasi-3-balanced decomposition. Analogously to the case above, (but taking $X = X(j, l) - eP_j e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$) we prove again that $X(j, l)$ has a quasi-3-balanced decomposition into $l+2$ paths. \square

Case that W is a cyclic trapped sequence. Let $s := s(0, r-1)$. We show that s has a quasi- k -balanced decomposition. For the particular case that $r-1 = 1$, the result follows by Lemma 2.6(i) and, in the case that $k = 3$, in addition by Observation 2.3. We now suppose that $r-1 \geq 2$. Assume first that for $k = 3$ and some $j \in \{0, \dots, r-1\}$,

we have that $P_j e_j P_{j+1} e_{j+1}$, $e_{j-1} P_j e_j P_{j+1}$ are not exceptionals; without loss of generality $j = 0$. Therefore, if $k = 3$ and $P_0 e_0 P_1 e_1$, $e_{r-1} P_0 e_0 P_1$ are not exceptionals (analogously if $k > 3$), then by Lemma 2.6 we can obtain a quasi- k -balanced decomposition of $e_{r-1} P_0 e_0 P_1 e_1$ into two paths. In addition, we use Claim 2.11 to obtain a quasi- k -balanced decomposition of $s(j+2, l-3) = s - e_{r-1} P_0 e_0 P_1 e_1$ and thus, by Observation 2.5, we have that s has a k -balanced decomposition. We suppose now that $P_j e_j P_{j+1} e_{j+1}$ (analogously, for $e_{j+1} P_j e_j P_{j+1}$) is exceptional for $k = 3$. Then by Observation 2.3, $P_j \neq P_{j+2}$ and by Lemma 2.6(ii), $e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$ does not contain an exceptional subgraph, thus, again by Lemma 2.6(i), $e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$ can be decomposed into two paths that form a quasi-3-balanced decomposition. As before, due to Claim 2.11 we have that $s - e_j P_{j+1} e_{j+1} P_{j+2} e_{j+2}$ admits a quasi-3-balanced decomposition, and thus, by Observation 2.5, we have that s has a 3-balanced decomposition.

3. PATH-CYCLES INTO LENGTH-CONSTRAINED PATHS

In this section, we prove Theorem 2. Namely, we show that for every graph in \mathcal{G}_k there exists a $\{2k-1, 2k, 2k+1\}$ -decomposition into paths such that the number of paths of length $2k-1$ equals the number of paths of length $2k+1$. From now on, the length of a graph G is denoted by $l(G)$. The following lemma is useful in the proof of the upcoming results. In what follows, let $C = y_0 \cdots y_{2k-1}$ and $P = v_0 \cdots v_\ell$, and take the indices of the vertices of C modulo $2k$.

Lemma 3.1. *Let C be a cycle of length $2k$ and let P be a path with its end vertices in $V(C)$ such that $E(C) \cap E(P) = \emptyset$. Assume that $H := C \cup P$ has girth at least $2k-2$. Then, $l(P) \geq k-2$, and if $y_0 = v_0$, then $v_\ell = y_k$. Moreover, if P contains a vertex of C as an internal vertex, then $l(P) \geq 2k-3$.*

Proof. Let $x, y \in V(C)$ be the end vertices of P . Since $l(C) = 2k$, there is a path P' in C with end vertices x and y , satisfies $l(P') \leq k$. Trivially, $P \cup P'$ contains a cycle and due to the assumption on the girth of H , we have $l(P) + l(P') \geq 2k-2$. The first part of the result follows. Let us now suppose that $z \neq x, y$ is a vertex of C and an internal vertex of P . Given distinct vertices $u, v, w \in V(C)$, let $C_w(u, v)$ be the path in C with end vertices u and v that avoids w . Consider the cycles $C_1 = P(x, z) \cup C_y(x, z)$, $C_2 = P(z, y) \cup C_x(z, y)$, and $C_3 = P \cup C_z(x, y)$. Since H has girth at least $2k-2$, we have $l(C_i) \geq 2k-2$, for each $i = 1, 2, 3$. Note that every edge of P is contained in exactly two cycles of $\{C_1, C_2, C_3\}$ and each edge of C is contained in exactly one of these cycles. Then, we have $2k + 2l(P) = l(C_1) + l(C_2) + l(C_3) \geq 6k-6$. The lemma follows. \square

In what follows we prove four lemmas that are used as the initial step in the proof of our main theorem.

Lemma 3.2. *Let $k \geq 3$. Let C be a cycle of length $2k$ and P be a path of length $\ell \in \{2k-1, 2k, 2k+1\}$ such that $E(C) \cap E(P) = \emptyset$, $V(C) \cap V(P) \neq \emptyset$ and no end vertex of P is in $V(C)$. If $H := C \cup P$ has girth at least $2k-2$, then H can be decomposed into two paths whose lengths are in $\{2k, \ell\}$ so that each path contains exactly one end vertex of P as an end-vertex.*

Proof. In what follows, the indices of the vertices in C are in \mathbb{Z}_{2k} . We denote by $C(y_i, y_j)$ (resp. $P(v_i, v_j)$) the path $y_i \cdots y_j$ (resp. $v_i \cdots v_j$). Let x_0, \dots, x_r be the vertices of $V(C) \cap V(P)$ in the order that they appear when reading P starting from v_0 . Suppose without loss of generality that $x_0 = y_0$.

Suppose that $r = 0$; namely, $|V(C) \cap V(P)| = 1$. Let t be the length of $P(x_0, v_\ell)$ and without loss of generality suppose that $t \leq \lfloor \ell/2 \rfloor \leq k$. Since no end vertex of P is in $V(C)$, we have $1 \leq t \leq k$. It implies that the paths $P_1 = P(v_0, x_0) \cup C(y_0, y_t)$ and $P_2 = C(y_t, y_0) \cup P(x_0, v_\ell)$ have lengths ℓ and $2k$, respectively. Moreover, each of P_1 and P_2 contains an end vertex of P . Thus, they form the desired decomposition of H .

From now on, we can assume that $r > 0$, i.e., $|V(C) \cap V(P)| > 1$. We claim that $r \leq 3$ and if $r = 3$, then $k \leq 4$.

Claim 3.3. *$r \leq 3$ and if $r = 3$, then $k \leq 4$.*

Proof. Assume the opposite for the first conclusion, i.e. $r \geq 4$ (resp. $r = 3$ for the second conclusion). Then, there are distinct x_0, x_1, x_2, x_3, x_4 in $V(C) \cap V(P)$ (resp. x_0, x_1, x_2, x_3 in $V(C) \cap V(P)$). Due to the assumption on the length of P and to that no end vertex of P is in $V(C)$, $P(x_0, x_4)$ (resp. $P(x_0, x_3)$) has length at most $2k-1$. For $r \geq 4$, using Lemma 3.1, we obtain that the length of $P(x_0, x_4)$ is at least $2(2k-3)$ and thus, $k < 3$, a contradiction. For $r = 3$, again using Lemma 3.1, we obtain that the length of $P(x_0, x_4)$ is at least $3k-5$ and thus, $k \leq 4$, as desired. \square

We divide the rest of the proof depending on $r \in \{1, 2, 3\}$.

Case 1: $r = 1$. Let l_1, l_2, l_3 be the lengths of $P(v_0, x_0), P(x_0, x_1), P(x_1, v_\ell)$, respectively, and let c_1, c_2 be the lengths of $C(x_0, x_1)$ and $C(x_1, x_0)$, respectively. We suppose without loss of generality that $l_1 \leq l_3$ and $c_2 \leq c_1$. Note that $c_1 \geq k$. By Lemma 3.1, we have $l_2 \geq k-2$. Thus $l_1 + l_3 \leq k+3$ and $1 \leq l_1 \leq (k+3)/2$. Since $k \geq 3$, we have $l_1 \leq k$. Moreover, $l_1 = k$ if and only if $k = 3$. If $l_1 < k$, we have $l_1 < c_1$. Let $P_1 = C(y_0, y_{l_1}) \cup P(x_0, v_\ell)$, and $P_2 = P(v_0, x_0) \cup C(y_{l_1}, y_0)$. Note that the only vertex in common between $C(y_0, y_{l_1})$ and P is $x_0 = y_0$. Therefore, P_1 is a path of length $l_1 + l_2 + l_3 = \ell$. Also, the only vertex in common between C and $P(v_0, x_0)$ is x_0 , and since $l_1 \geq 1$, we have that $C(y_{l_1}, x_0)$ is not a cycle. Therefore P_2 is a path of length $2k$.

Now, suppose $l_1 = k = 3$. Since $l_1 \leq l_3$, we have $l_3 = 3$, and $l_2 = 1$. Since $l_2 = 1$, by Lemma 3.1, we have $c_1 = c_2 = 3$. Let $P_1 = C(y_0, y_2) \cup P(x_0, v_\ell)$, and $P_2 = P(v_0, x_0) \cup$

$C(y_2, y_0)$. Clearly, $l(P_1) = 6$ and $l(P_2) = 7$ and each of P_1 and P_2 contains one of the end vertices of P .

Case 2: $r = 2$. Let $x_0 = y_0$, $x_1 = y_i$, and $x_2 = y_j$. We suppose without loss of generality that $0 < i < j$. Let l_1, l_2, l_3, l_4 be the lengths of $P(v_0, x_0), P(x_0, x_1), P(x_1, x_2), P(x_2, v_\ell)$, and let c_1, c_2, c_3 be the lengths of $C(y_0, y_i), C(y_i, y_j), C(y_j, y_0)$, respectively. If $c_1 > l_1$, then we make $P_1 = C(y_0, y_{l_1}) \cup P(x_0, v_\ell)$, and $P_2 = P(v_0, x_0) \cup C(y_{l_1}, y_0)$. Clearly, $l(P_1) = \ell$ and $l(P_2) = 2k$. Thus, we can suppose $c_1 \leq l_1$. Consider the cycles $P(x_0, x_1) \cup C(y_0, y_i)$ of length $l_2 + c_1$, and $P(x_1, x_2) \cup C(y_i, y_j)$ of length $l_3 + c_2$. Since the girth of H is at least $2k - 2$, we have $l_2 + c_1 \geq 2k - 2$ and $l_3 + c_2 \geq 2k - 2$, from which we obtain $c_1 + c_2 + l_2 + l_3 \geq 4k - 4$. Moreover, since P is a path of length at most $2k + 1$, we have $l_1 + l_2 + l_3 + l_4 \leq 2k + 1$. Thus, subtracting this inequality from the previous one, we obtain $c_1 + c_2 - l_1 - l_4 \geq 2k - 5$. Since $k \geq 3$, we have $c_1 + c_2 - l_1 - l_4 > 0$, which implies $c_2 - l_4 > l_1 - c_1$. Since $l_1 \geq c_1$, we have $c_2 > l_4$. Thus, the paths $C(y_{k-l_4}, y_0) \cup P(v_0, x_2)$, and $P(x_2, v_\ell) \cup C(y_j, y_{k-l_4})$ of lengths ℓ , and $2k$, respectively form the desired decomposition.

Case 3: $r = 3$. Let l_1, \dots, l_5 be the lengths of $P(v_0, x_0), P(x_0, x_1), P(x_1, x_2), P(x_2, x_3), P(x_3, v_\ell)$, respectively. Due to Claim 3.3, we can suppose $k \leq 4$. First, suppose $k = 4$. We state that the following hold

$$l_1 = l_5 = 1, l_2 = l_4 = 2, l_3 = 3.$$

By Lemma 3.1, we have $l_2, l_3, l_4 \geq 2$, $l_2 + l_3 \geq 5$ and $l_3 + l_4 \geq 5$. Therefore, we have $l_2 + l_3 + l_4 \geq 7$. Since $l_1 \geq 1$, $l_5 \geq 1$ and $\ell \leq 9$, we have $l_1 = l_5 = 1$, and $l_2 + l_3 + l_4 = 7$. Moreover, $\ell = 9$. Note that if $l_3 \leq 2$, then $l_2 = 3$ and $l_3 = 3$, implying $l_2 + l_3 + l_4 \geq 8$, a contradiction. Therefore, we have $l_3 \geq 3$ and $l_2 = l_3 = 2$.

Now, since $l_2 = 2$, we have that $x_1 = y_4$. Since $l_3 = 3$, and $x_2 \neq x_0$, we have $x_2 \in \{y_1, y_7\}$. By symmetry, we can suppose $x_2 = y_1$. Since $l_4 = 2$, we have that $x_3 = y_5$. Thus, y_7 is not a vertex of P . Let $P_1 = (P \setminus x_0v_0) \cup x_0y_7$ and $P_2 = (C \setminus x_0y_7) \cup x_0v_0$. Clearly P_1 and P_2 are two paths whose lengths are in $\{2k, \ell\}$. Moreover, each of P_1 and P_2 contains one of the end vertices of P .

We now suppose $k = 3$. By Lemma 3.1, we have $l_2 + l_3 \geq 3$. Therefore, $l_2 + l_3 + l_4 \geq 4$ and thus, at least one of $l_1 = 1$, $l_5 = 1$ holds. Suppose, without loss of generality, that $l_1 = 1$. We claim that $x_1 \notin \{y_1, y_5\}$. In fact if $x_1 \in \{y_1, y_5\}$, then $l_2 \geq 3$, otherwise $P(x_0, x_1) \cup y_0x_0$ would be a cycle with length smaller than 4. By Lemma 3.1, we have $l_3 + l_4 \geq 3$, hence $l_2 + l_3 + l_4 \geq 6$ and $\ell \geq 8$. Therefore, $x_1 \notin \{y_1, y_5\}$. On the other hand, if a vertex y in $\{y_1, y_5\}$ is not a vertex of P , then $P_1 = (P \setminus x_0v_0) \cup x_0y$ and $P_2 = (C \setminus x_0y) \cup x_0v_0$ decompose H into paths of lengths in $\{6, \ell\}$. Thus, we may suppose $y_1, y_5 \in V(P)$. Since $r = 3$, we have $\{y_1, y_5\} = \{x_2, x_3\}$. Since $P(x_2, x_3) \cup C(y_5, y_1)$ induce a cycle in H , we have that $l_4 \geq 2$. By Lemma 3.1, we have $l_2 + l_3 \geq 3$. Since $l_2 + l_3 + l_4 \leq 5$, we have $l_2 + l_3 = 3$, $l_4 = 2$, and $l_1 = l_5 = 1$. Let v be the neighbor of x_3 in $P(x_2, x_3)$.

Since $l_4 = 2$, we have that v is not a vertex of C . Let $P_1 = (P \setminus \{v_0x_0, vx_3\}) \cup x_0x_3$ and $P_2 = (C \setminus x_0x_3) \cup \{vx_3, v_0x_0\}$. Clearly, P_1 is a path of length 6, and P_2 is a path of length $\ell = 7$. Moreover, each of P_1 and P_2 contains one of the end vertices of P . \square

We now extend Lemma 3.2.

Lemma 3.4. *Let C be a cycle of length $2k$ and P be a path of length $2k$ such that $E(C) \cap E(P) = \emptyset$ and $V(C) \cap V(P) \neq \emptyset$. If $H := C \cup P$ has girth $2k - 2$, then H can be decomposed into two paths whose lengths are in $\{2k - 1, 2k, 2k + 1\}$.*

Proof. The case that no end vertex of P is in $V(C)$ holds due to Lemma 3.2. Suppose now that exactly one end vertex of P , say z , is in $V(C)$. Let P' be the path that consists of the union of P , a new vertex z' and a new edge zz' . As P' has length $2k + 1$, by Lemma 3.2, the graph $C \cup P'$ has a decomposition into two paths of lengths $2k$ and $2k + 1$. Since the edge zz' is an end edge of one of these paths, by removing it from such path we obtain a decomposition of H into two paths of length $2k - 1$ and $2k + 1$, or into two paths, each of length $2k$.

We now assume that both end vertices of P are in $V(C)$. Moreover, we can assume that $y_1, y_{2k-1} \in \{x_0, \dots, x_r\}$. To see this, we suppose, without loss of generality, that $y_1 \notin \{x_0, \dots, x_r\}$. Hence $C \setminus x_0y_1$ and $P \cup x_0y_1$ are paths that decompose H , whose lengths are $2k - 1$ and $2k + 1$, respectively.

The next claim helps to complete the proof of the lemma.

Claim 3.5. *If $x \in \{y_1, y_{2k-1}\}$ and the neighbor x' of x in the path $P(x_0, x)$ is not in $V(C)$, then H admits a decomposition into two paths, each of length $2k$.*

Proof. The paths $P(x', x_0) + x_0x + P(x, v_\ell)$ and $C - x_0x + xx'$ form a $2k$ -decomposition of H . \square

Recall that for every i in $\{0, \dots, r - 1\}$, the path $P(x_i, x_{i+1})$ has length at least $k - 2$. Thus, if $k \geq 4$, the path $P(x_i, x_{i+1})$ has length at least 2. Therefore, the neighbor x' of x_{i+1} in the path $P(x_i, x_{i+1})$ (and also in the path $P(x_0, x_{i+1})$) is not a vertex of C , and by Claim 3.5, we can decompose H into two paths of length $2k$.

In consequence, we can assume $k = 3$. Let us assume that H does not have a $\{5, 6, 7\}$ -decomposition into paths. If $y_1 = x_i$, then by Claim 3.5, $P(x_{i-1}, x_i)$ has length 1 and thus, since H is triangle-free $x_{i-1} = y_4$. Analogously, if $y_5 = x_j$, then $x_{j-1} = y_2$. Suppose without loss of generality that $i < j$. We can write

$$P = P(x_0, x_{i-1}) \cup P(x_{i-1}, x_i) \cup P(x_i, x_{j-1}) \cup P(x_{j-1}, x_j) \cup P(x_j, v_\ell).$$

We have $x_{i-1} = y_4$, thus $P(x_0, x_{i-1})$ has length at least 2, because H is triangle-free. In addition, we have that $P(x_i, x_{j-1})$ has length at least 3, because $x_i = y_1$ and $x_{j-1} = y_2$. In consequence, P has length at least 7, a contradiction. \square

Lemma 3.6. *Let C be a cycle of length $2k$ and P be a path of length $2k - 1$ such that $E(C) \cap E(P) = \emptyset$ and $V(C) \cap V(P) \neq \emptyset$. If $H := C \cup P$ has girth $2k - 2$, then H can be decomposed into two paths whose lengths are in $\{2k - 1, 2k\}$.*

Proof. Let z_1, z_2 be the two end-vertices of P . First, we add two new vertices z'_1, z'_2 and two new edges $z_1z'_1, z_2z'_2$ to P . By Lemma 3.2, we obtain a decomposition of this new graph into two paths, one of length $2k$ and the other of length $2k + 1$ such that each path contains one of the new vertices z'_1, z'_2 . By removing the edges $z_1z'_1$ and $z_2z'_2$ from them, we obtain two paths of lengths $2k - 1$ and $2k$. \square

Finally, we consider a lemma that helps decomposing the union of 2 cycles into paths.

Lemma 3.7. *Let C, C' be cycles of length $2k$ such that $E(C) \cap E(C') = \emptyset$ and $V(C) \cap V(C') \neq \emptyset$. If $H := C \cup C'$ has girth at least $2k - 2$, then H can be decomposed into two paths of length $2k$.*

Proof. Let $C = y_0 \cdots y_{2k-1}$ and $C' = z_0 \cdots z_{2k-1}$. Without loss of generality $y_0 = z_0$. First, we claim that y_1 or y_{2k-1} is not in $V(C) \cap V(C')$ and z_1 or z_{2k-1} is not in $V(C) \cap V(C')$. Suppose z_{2k-1} is a vertex of C . Let P be a path in C of length at most k and end vertices y_0, z_{2k-1} . Note that $P \cup y_0z_{2k-1}$ is a cycle of length c at most $k + 1$ in H . Since the girth of H is at least $2k - 2$, we have

$$k + 1 \geq c \geq 2k - 2$$

Therefore, $k = 3$. Now, since the girth of H is at least 4, and C has length 6, if z_i is a vertex of C , for $i \in \{1, 5\}$, then $z_i = y_3$. Therefore $z_1 = z_5 = y_3$, and y_0z_1, y_0z_5 is a cycle of length 2, a contradiction.

Now, suppose without loss of generality, that y_1 and z_1 are not in $V(C) \cap V(C')$, then the graphs $C - y_0y_1 + y_0z_1$ and $C' - y_0z_1 + y_0y_1$ are paths of length $2k$. \square

In order to prove Theorem 2, we consider the following definition.

Definition 3.8. Let G be a graph in \mathcal{G}_k , and let \mathcal{L} be a $\{2k - 1, 2k, 2k + 1\}$ -decomposition of G with cycles of length $2k$ such that the number of paths of length $2k - 1$ equals the number of paths of length $2k + 1$. We say that \mathcal{L} is *complete* if the following two conditions hold.

- (i) the cycles in \mathcal{L} are vertex-disjoint; and
- (ii) if v is a vertex of a cycle in \mathcal{L} , then \mathcal{L} is k -balanced at v .

Proof of Theorem 2. Let $G \in \mathcal{G}_k$. Due to Theorem 4, there exists a k -balanced decomposition \mathcal{L} of G with each cycle of length $2k$. Actually, because of Lemma 3.7, we can also assume that the cycles in \mathcal{L} are vertex-disjoint. In consequence, \mathcal{L} is a complete decomposition of G . Among all complete decompositions of G , we consider one, say \mathcal{D} ,

that minimizes the number of cycles; note that \mathcal{D} is not necessarily k -balanced. If \mathcal{D} has no cycles, then \mathcal{D} is the desired decomposition. Therefore, we assume that \mathcal{D} has at least one cycle. Let us first show the following statement.

Claim 3.9. *For every cycle C in \mathcal{D} there are at least two paths in \mathcal{D} of length $2k$ such that each of them has exactly one end vertex in $V(C)$.*

Proof. Let \mathcal{P} be the set of paths in \mathcal{D} that have vertices in common with C . Since the degree of each vertex in G is greater than 5, and any two cycles of \mathcal{D} are vertex-disjoint, we have $\mathcal{P} \neq \emptyset$; indeed, for each vertex of C there are at least two paths in \mathcal{D} containing such vertex.

We claim that every path in \mathcal{P} has an end vertex in $V(C)$. In fact, suppose that there is a path P in \mathcal{P} of length $\ell \in \{2k - 1, 2k, 2k + 1\}$ that has no end vertices in $V(C)$. By Lemma 3.2 we can decompose $C \cup P$ into paths P_1, P_2 of lengths $2k, \ell$, respectively. Moreover, each path P_1, P_2 contains an end vertex of P . Hence, we have $\mathcal{D}' = (\mathcal{D} \setminus \{C, P\}) \cup \{P_1, P_2\}$ is a decomposition of G into paths of lengths in $\{2k - 1, 2k, 2k + 1\}$ and cycles of length $2k$, and the number of cycles in \mathcal{D}' is strictly smaller than the number of cycles in \mathcal{D} . If, in addition, \mathcal{D}' is complete, then we have a contradiction to the minimality of \mathcal{D} . Indeed, since \mathcal{D} satisfies (i), \mathcal{D}' also does. Note that, since \mathcal{D} satisfies (ii) and \mathcal{D}' is trivially balanced at the common end vertex of P_1, P_2 , the only vertices where \mathcal{D}' may contradict property (ii) are the end vertices of P ; let v_1, v_2 be the end vertices of P which are end vertices of P_1, P_2 , respectively. Suppose \mathcal{D} is balanced at v_1, v_2 . If the lengths of P_1, P_2 are in $\{2k - 1, 2k\}$, then \mathcal{D}' is balanced at v_1, v_2 . If the length of P_2 is $2k + 1$, then the length of P is $2k + 1$ as well and thus, the number of paths of length $2k + 1$ ending at v_2 in \mathcal{D}' is the same as the number of paths of length $2k + 1$ ending at v_2 in \mathcal{D} . Hence, \mathcal{D}' satisfies (ii).

We now claim that \mathcal{P} does not contain paths of length $2k - 1$. In fact, if there is a P of length $2k - 1$ in \mathcal{P} , due to Lemma 3.6, $C \cup P$ can be decomposed into paths P_1 and P_2 of lengths $2k - 1$ and $2k$. As \mathcal{D} is complete, $(\mathcal{D} \setminus \{C, P\}) \cup \{P_1, P_2\}$ is complete as well, and it has less cycles than \mathcal{D} , a contradiction.

Suppose that there is a path P of length $2k$ in \mathcal{P} such that both end vertices are in $V(C)$. By Lemma 3.4, we can decompose $C \cup P$ into paths P_1 and P_2 of length in $\{2k - 1, 2k, 2k + 1\}$. Since both end vertices of P are in $V(C)$, because of (i), they are not vertices of a cycle in the decomposition $(\mathcal{D} \setminus \{C, P\}) \cup \{P_1, P_2\}$, and thus, it is complete; a contradiction to the minimality of the number of cycles in \mathcal{D} . Therefore, every path of length $2k$ in \mathcal{P} has exactly one end vertex that is a vertex of C .

We conclude that if P is an element of \mathcal{P} , then one of the following happens: either P has length $2k + 1$ and at least one end vertex of P is in $V(C)$, or P has length $2k$ and exactly one end vertex of P is in $V(C)$. Therefore, there exists a vertex v in $V(C)$ that is an end vertex of a path in \mathcal{P} . Since the degree of v is even (recall it is $2k$) and \mathcal{D} is

balanced at v , there is a positive even number of paths in \mathcal{P} ending at v . Furthermore, at most half of them have length $2k + 1$; that is, at least half of them have length $2k$. Thus, if there are at least four paths, the result follows. Therefore, we suppose that exactly two paths P, P' end at v . If the length of each of these paths is $2k$, the claim holds. If not, one of them has length $2k$ and the other one has length $2k + 1$. Since the degree of v is at least 6, there is a path P^* that contains v as an internal vertex and $P^* \notin \{P, P'\}$. If P^* has length $2k$, the claim follows. If P^* has length $2k + 1$, then there is an end vertex u of P^* in $V(C)$ and $u \neq v$. As \mathcal{D} is balanced at u , there is a path P'' of length $2k$ with u as an end vertex. Since P'' has exactly one end vertex in $V(C)$, we have $P'' \notin \{P, P'\}$ and the result follows. \square

Let P be a path of length $2k$ in \mathcal{D} . This path exists due the assumption of the existence of cycles and Claim 3.9. We observe that the number N of cycles in \mathcal{D} that contain an end vertex of P is in $\{0, 2\}$. In fact, since cycles in \mathcal{D} are vertex-disjoint, we have $N \leq 2$ and if $N = 1$, then, by Lemma 3.4, we can obtain a complete decomposition of G with less cycles than \mathcal{D} , a contradiction to the minimality of \mathcal{D} .

Now, consider the auxiliary graph K with the set of cycles in \mathcal{D} as vertex set of K , and such that two vertices C_i, C_j form an edge if and only if there exists a path of length $2k$ in \mathcal{D} with an end vertex in C_i and an end vertex in C_j . Because of the previous observation and by Claim 3.9, the minimum degree of K is 2 and thus, K contains a cycle $C_0C_1 \cdots C_{t-1}C_0$. For each $i \in \mathbb{Z}_t$, let P_i be a path in \mathcal{D} such that each cycle C_i, C_{i+1} contains an end vertex of P_i . By Lemma 3.4, for each $i \in \mathbb{Z}_t$, there exists a decomposition of $C_i \cup P_i$ into two paths of lengths in $\{2k - 1, 2k, 2k + 1\}$ and thus, we can obtain a decomposition of $\bigcup_{i \in \mathbb{Z}_t} (C_i \cup P_i)$ into $2t$ paths of lengths in $\{2k - 1, 2k, 2k + 1\}$ such that the number of paths of length $2k - 1$ equals the number of paths of length $2k + 1$. This yields a complete decomposition of G with t cycles less than \mathcal{D} , a contradiction to the minimality of \mathcal{D} . \square

4. PROOF OF LEMMA 2.6

We recall Lemma 2.6. In the following, $G \in \mathcal{G}_k$, M is a perfect matching of G and \mathcal{P} is a $(2k - 1)$ -decomposition of $G - M$ into paths.

Lemma (Lemma 2.6). Let $e_{i-1}P_i \cdots P_{i+2}e_{i+2}$ be an (M, \mathcal{P}) -alternating trail and X be an extension of $P_i e_i P_{i+1}$. The following two statements hold.

- (i) If X does not contain an exceptional sequence, then X has a quasi- k -balanced decomposition into 2 paths.
- (ii) If X is an exceptional sequence (in particular, $k = 3$), then $e_{i-1} \neq e_{i+1}$. If, in addition, $X = P_i e_i P_{i+1} e_{i+1}$ (analogously for $e_{i-1} P_i e_i P_{i+1}$), then neither $e_i P_{i+1} e_{i+1} P_{i+2}$, nor $P_{i+1} e_{i+1} P_{i+2} e_{i+2}$ is exceptional.

Let P be a path and x, y be vertices of P . In what follows, the notation $P(x, y)$ stands for the subpath of P with end vertices x and y ; we refer to it as a *segment* of P ; if $x = y$, then $P(x, y)$ is simply a vertex of P .

Recall that for a trapped sequence PeP' , it holds that $e \subset V(P) \cap V(P')$. The following claim follows due to the girth condition on $2k - 2$ and since P, P' do not have common end vertices.

Claim 4.1. *Let PeP' be a trapped sequence, $e := xx'$ and suppose that $|V(P) \cap V(P')| > 2$. If $k > 3$, then $|V(P) \cap V(P')| = 3$, the length of $P(x, x')$ and of $P'(x, x')$ is $2k - 2$ and the middle vertex of $P(x, x')$ corresponds to the middle vertex of $P'(x, x')$. If $k = 3$, then PeP' is isomorphic to one of the graphs described in Figure 2.*

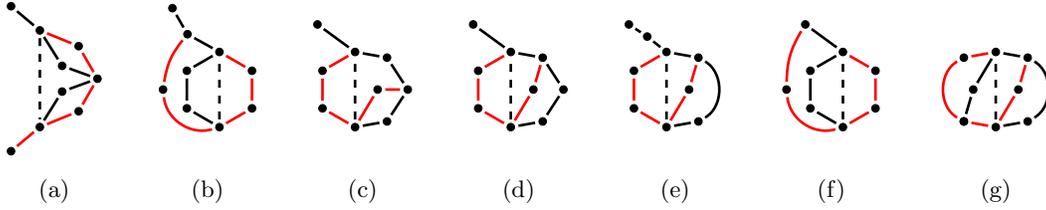


FIGURE 2. Dashed edges depict matching edges. One path is depicted in black, the other one in red. Figures (a), (b), (c), (d) and (e) correspond to $|V(P) \cap V(P')| = 3$ and (f), (g) to $|V(P) \cap V(P')| = 4$.

Proof of Lemma 2.6(i). We first set some notation. We write $X = e_{i-1}P_i e_i P_{i+1} e_{i+1}$ as $e^*PeP'e'$ (e^*, e' might not exist). Let $e^* := zz'$, $e := xx'$ and $e' := yy'$. Further, in the segment $P(x, x')$, let \tilde{z} and \hat{z} be the neighbors of x and x' , respectively; and, in the segment, $P'(x, x')$ let \tilde{y} and \hat{y} be the neighbors of x and x' , respectively. Thus, X is the trail $zz' \cdots x\tilde{z} \cdots \hat{z}x'x\hat{y} \cdots \tilde{y}x' \cdots y'y$ and P, P' are the segments of X given by $z' \cdots x\tilde{z} \cdots \hat{z}x'$, $x\hat{y} \cdots \tilde{y}x' \cdots y'$, respectively (see Figure 3). Note that the segments $P(z', x), P'(y', x')$ are paths of length 1 (edges) or 2, and the segments $P(\tilde{z}, \hat{z}), P'(\tilde{y}, \hat{y})$ are paths of length $2k - 3$ or $2k - 4$.

We split the proof into three cases: (1) $e^* \neq e', z \notin V(P)$ and $y \notin V(P')$, (2) $e^* \neq e'$ and $z \in V(P)$, and (3) $e^* = e'$. Note that if $e^* \neq e'$, then $|\{z, z', y, y'\}| = 4$. In particular $z \neq y'$ and $y \neq z'$.

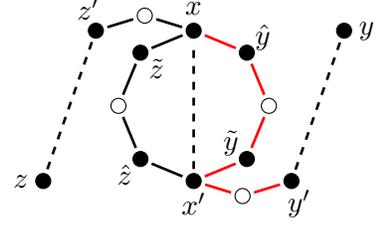
CASE 1 $e^* \neq e', z \notin V(P)$ and $y \notin V(P')$.

In other words, P is nice for e^* and P' is nice for e' .

We first assume that $y' \neq \tilde{z}$. By the symmetry of X , this situation also covers the case that $z' \neq \tilde{y}$. If $y \neq \tilde{z}$, then due to Claim 4.1 the following subgraphs:

$$P_1 := (P - x\tilde{z}) \cup e \cup e^* \quad \text{and} \quad P_2 := P' \cup x\tilde{z} \cup e'$$

FIGURE 3. P and P' are represented by black and red lines, respectively. Dashed edges depict matching edges. For $k = 3$, unfilled vertices may exist or not.



form a path decomposition of X . Since the length of $(P - x\tilde{z}) \cup e$ is $2k - 1$, the path P_1 has its length in $\{2k - 1, 2k\}$, depending on whether e^* exists. Analogously, we obtain that the path P_2 has its length in $\{2k, 2k + 1\}$. Moreover, if $l(P_2) = 2k + 1$, then e' belongs to X and there is only one vertex y at which $\{P_1, P_2\}$ is not balanced. However, $y \in e'$ has odd degree in X . Hence, $\{P_1, P_2\}$ is a quasi- k -balanced decomposition of X . In the forthcoming analysis we use the same argument to guarantee that a path decomposition is quasi- k -balanced, and thus, we might omit details. Assume now that $y = \tilde{z}$. Therefore $k = 3$, otherwise there is a contradiction to the girth condition on X . Due to Claim 4.1, we have $y' \notin V(P)$. Note that if $P_1 \cup e'$ has length 7, then e^* belongs to X . Thus, $\{P_1 \cup e', P_2 - e'\}$ is a quasi-balanced path decomposition of X .

Secondly, we suppose that $y' = \tilde{z}$ and $z' = \tilde{y}$. Then, according to Claim 4.1, $k = 3$ and PeP' is isomorphic to the graph depicted in (g) of Figure 2. Since X is triangle-free, $z \notin V(P')$ and $y \notin V(P)$. Let z'' (resp. y'') denote the neighbor of z' (resp. y') in P (resp. P'). Therefore, the subgraph $P(z'', x') \cup x'z' \cup e^*$ and its complement with respect to X form a quasi-balanced decomposition of X . We clarify that along this work, the complement is edge-wise. That is, K' is the complement of K with respect to X if K' is obtained from X by removing all edges of K and all isolated vertices of $X - E(K)$; by abuse of notation, $K' = X - K$.

CASE 2 $e^* \neq e'$ and $z \in V(P)$

In this case $k = 3$. Otherwise, there is a contradiction to the condition on the girth of X . Recall that the vertex sets of e^* , e and e' are pairwise disjoint. Trivially, if $z \in V(P)$, then z is an internal vertex of $P(x, x')$. Since the segment $P(x, x')$ has length at least 3, there exists internal vertex of $P(x, x')$, say n_z , such that n_z is a neighbor of z . Observe that n_z is determined by the location of z , except in the particular case that $P(x, x')$ has length 4 and z is the middle vertex of $P(x, x')$, in which case $n_z \in \{\tilde{z}, \hat{z}\}$.

We use the following subgraphs to split the proof into subcases:

$$H_1 := (P(x, x') - zn_z) \cup P'(x, x') \text{ and } H_2 := X - H_1.$$

Note that H_1 and H_2 decompose the edge set of X into two trails. Moreover, due to that the lengths of $P(x, x')$ and $P'(x, x')$ are in $\{3, 4\}$, and $l(H_1) = l(P(x, x')) + l(P'(x, x')) - 1$, we have $l(H_1) \in \{5, 6, 7\}$.

If H_1 and H_2 are paths and H_2 has length in $\{5, 6, 7\}$, then $\{H_1, H_2\}$ is a quasi-balanced decomposition of X ; observe that if H_1 (resp. H_2) has length 7, then the decomposition $\{H_1, H_2\}$ is balanced at all vertices of X , except at z (resp. y), and $z \in e^*$ (resp. $y \in e'$) and e^* (resp. e') belongs to X . Note that if $e' \in X$ and $l(H_1) = 5$, then $l(H_2) = 8$. We assume that $\{H_1, H_2\}$ is not a $\{5, 6, 7\}$ -decomposition of X into paths. Thus, one of the following situations holds:

- H_1 is not a path.
- H_2 is not a path.
- the length of H_1 or H_2 is not in $\{5, 6, 7\}$.

If H_1 is not a path, then due to Claim 4.1, PeP' is isomorphic to the graph depicted in (a) of Figure 2. Thus, $l(H_1) = 7$ and $l(H_2) \in \{5, 6\}$ subject to the existence of e' . Moreover, $P(x, x')$ and $P'(x, x')$ have length 4 and $z \in \{\hat{z}, \tilde{z}, z^*\}$, where z^* is the middle vertex of $P(x, x')$. If $z = z^*$, then X is exceptional, a contradiction (see Figure 1). If $z = \tilde{z}$, then $\{x, z', \tilde{z}\}$ induces a triangle. Thus, we assume that $z = \hat{z}$ and thus, $n_z = z^*$.

If $y \neq \hat{y}$ (resp. $y = \hat{y}$), then $H'_1 = H_1 - n_z \hat{y}$ and $H'_2 = H_2 \cup n_z \hat{y}$ (resp. $H'_1 = H_1 - n_z \tilde{z}$ and $H'_2 = H_2 \cup n_z \tilde{z}$) form a quasi-balanced decomposition of X . This is because $l(H'_1) \in \{5, 6\}$, and if $l(H'_2) = 7$, then $\{H'_1, H'_2\}$ is balanced at all vertices of X , but at $y \in e'$ and e' belongs to X . Therefore, from now on, we can assume that H_1 is a path.

Let us discuss the second scenario; namely, H_2 is not a path. We first assume that the cycles in H_2 arise from the intersection of the paths P and P' . By definition of H_2 , the segments of P and P' in H_2 are $P(z', x)$, $P'(y', x')$, and zn_z . Recall that z'' (resp. y'') denotes the neighbor of z' (resp. y') in P (resp. P'). We have $y' = z''$, $z' = y''$, or $y' = n_z$. Further, due to Claim 4.1, exactly one of these equalities holds. Note that $z'' \neq y''$; otherwise, X would have a triangle on $\{x, x', y''\}$. Suppose $y' = z''$. Clearly, this case is possible only if $z'' \neq x$ and $y'' \neq x'$. Due to Claim 4.1, $P(x, x')$ and $P'(x, x')$ have length 3. Since X is triangle-free, $y \notin \{\hat{y}, \tilde{z}\}$. Let us consider the following decomposition of X :

$$H'_1 := e^* \cup P(z', x) \cup e \cup P'(x', \hat{y}) \text{ and } H'_2 := X - H'_1.$$

If $y \neq \hat{z}$, then $\{H'_1, H'_2\}$ is a quasi-balanced decomposition of X . If $y = \hat{z}$, then $z \neq \hat{z}$, otherwise z is incident to two matching edges, which implies $n_z = \hat{z}$ and $z = \tilde{z}$. Thus, $(H'_1 - e^* - z'z'') \cup e' \cup n_z z$ and its complement with respect to X form a quasi-balanced decomposition.

We now suppose that $y' = n_z$. Then, $y' \neq \hat{z}$. Thus, $n_z = \tilde{z}$ or $P(x, x')$ has length 4. We claim that $z = \hat{z}$. In fact, if $z = \tilde{z}$, then $P(x, x')$ has length 4, and $z'' = x$ and $\{x, z', \tilde{z}\}$ induces a triangle in X . If $z = z^*$, then the paths $H_1 - z\hat{z} + zn_z$ and $H_2 - zn_z + z\hat{z}$ form a quasi-balanced decomposition of X .

If $y \notin \{\tilde{y}, \hat{y}\}$, then the path $e^* \cup P(z', n_z) \cup P'(n_z, \tilde{y})$ and its complement form a quasi-balanced decomposition of X . Let e_x denote the edge incident to x in the segment $P(x, z')$.

If $y \in \{\tilde{y}, \hat{y}\}$, then $H'_1 = e_x \cup P'(x, y) \cup e' \cup P'(y', x') \cup zx'$ and $H'_2 = X - H'_1$ form a $\{5, 6, 7\}$ -decomposition of X into paths. Moreover, if H'_1 (resp. H'_2) has length 7, then $\{H'_1, H'_2\}$ is balanced at all vertices of X but at y (resp. at z). Thus, $\{H'_1, H'_2\}$ is a quasi-balanced decomposition of X .

We now need to study the case that $z' = y''$. According to Claim 4.1, PeP' is isomorphic to the graph depicted in (b) or (f) of Figure 2. Thus, the segments $P(x, x')$ and $P'(x, x')$ have length 3, and we have $z = \tilde{z}$ and $y \in \{n_z, \tilde{y}, \hat{y}\}$; note that, if $z = \hat{z}$, then $\{y'', x', \hat{z}\}$ induces a triangle in X . Moreover, $y' \neq \hat{z}$; otherwise X creates a triangle or a multiple edge. If $y \neq n_z$, then $H'_1 := e^* \cup P(z', x) \cup P'(x, x') \cup x'\hat{z}$ and $H'_2 = X - H'_1$ is a $\{5, 6, 7\}$ -decomposition of X into paths with $l(H'_1) = 7$ and $l(H'_2) \in \{5, 6\}$. As $\{H'_1, H'_2\}$ is balanced at all vertices of X except at z , the result follows. Finally, if $y = n_z = \hat{z}$, then the paths

$$H'_1 := P(\hat{z}, z') \cup z'x' \cup x'\tilde{y} \text{ and } H'_2 = X - H'_1$$

form the desired decomposition of X ; observe that $l(H'_1) = 6$, $l(H'_2) = 7$ and H'_2 ends at z .

We now move to the case that the cycles in H_2 do not arise from the intersection of the paths P and P' . Hence, we necessarily have $y \in \{z'', n_z\}$. If $y = z''$, then $e^* \cup P(z', x) \cup e \cup P'(x', \hat{y})$ and $X - H'_1$ form a quasi-balanced decomposition of X . If $y = n_z$, then $zn_z \cup e^* \cup P(z', x) \cup e \cup P'(x', y')$ and $X - H'_1$ form a quasi-balanced decomposition of X . This completes the analysis for the case that H_2 is not a path.

Finally, we study the case that H_1 and H_2 are paths, but the length of at least one of them is not in $\{5, 6, 7\}$. Note that, by definition, $5 \leq l(H_1) \leq 7$. On the other hand, $5 \leq l(H_2) \leq 8$. The case $l(H_2) = 8$ arises whenever e' exists and $P(z', x)$, $P'(y', x')$ have length 2. In this case, if $y \neq \tilde{z}$, then the path $(H_1 - x\tilde{z}) \cup P(z', x)$ and its complement form a quasi-balanced decomposition of X . If $y = \tilde{z}$, then the paths $H_1 \cup e^*$, of length 6, and $H_2 - e^*$, of length 7, form a quasi-balanced decomposition of X . The lemma follows.

CASE 3 $e^* = e'$

Again, in this case $k = 3$. Otherwise, there is a contradiction to the girth condition on X . Since all vertices of X have even degree in X , every $\{5, 6, 7\}$ -decomposition of X into paths is balanced at every vertex of X . If PeP' is not isomorphic to (a) of Figure 2, $z' \neq \tilde{y}$, and $y' \neq \tilde{z}$, then, by Claim 4.1, we have that $P(\tilde{z}, x') \cup e \cup P'(x, \tilde{y})$ and its complement form a $\{5, 6, 7\}$ -decomposition of X into paths. Suppose that PeP' is isomorphic to (a) of Figure 2 and let u be the vertex in $V(P) \cap V(P') - \{x, x'\}$. Then, by Claim 4.1, $z' \neq \tilde{y}$ and $y' \neq \tilde{z}$, and thus the paths $\tilde{z}u \cup P'(u, x) \cup e \cup x'\hat{z}$ and its complement form a $\{5, 6, 7\}$ -decomposition of X .

Finally, suppose that at least one of the following equalities $z' = \tilde{y}$, $y' = \tilde{z}$, occurs. Therefore, $z'' \neq x$ and $y'' \neq x'$; since otherwise $\{x, x', \tilde{y}\}$ or $\{x', y', \tilde{y}\}$ induces a triangle in X . Without loss of generality assume that $z' = \tilde{y}$ holds. Then, $x\hat{y} \cup P(x, x') \cup x'\tilde{y} \cup \tilde{y}z''$ and its complement with respect to X form the desired decomposition of X . \square

We say that a graph is an (k, l) -lollipop if it can be obtained from the edge disjoint union of a path of length k , say P , and a cycle of length l , say C , satisfying that $V(P) \cap V(C)$ is an end vertex of P .

Proof of Lemma 2.6(ii). We consider the names of the vertices of P, P', e, e', e^* as in the proof of Lemma 2.6(i). Without loss of generality we assume that $X := PeP'e'$. If $e^* = e'$, then P' is not a path, a contradiction. Thus, $e^* \neq e'$. Observe that a necessary condition on $eP'e'\tilde{P}$ (and $P'e'\tilde{P}\tilde{e}$) to be exceptional is that $P'e'$ is a $(1, 5)$ -lollipop, but $P'e'$ is a $(2, 4)$ -lollipop. \square

REFERENCES

- [1] F. Botler, G. O. Mota, M. T. I. Oshiro, and Y. Wakabayashi. Path decompositions of regular graphs with prescribed girth. *Electronic Notes in Discrete Mathematics*, 2015.
- [2] F. Botler, G. O. Mota, and Y. Wakabayashi. Decompositions of triangle-free 5-regular graphs into paths of length five. In *The 9th International colloquium on graph theory and combinatorics*, 2014.
- [3] F. Botler and Y. Wakabayashi. A constrained path decomposition of cubic graphs and the path number of cacti. In *The Seventh European Conference on Combinatorics, Graph Theory and Applications, CRM Series*, 16(1):615–616, 2013.
- [4] A. Donald. An upper bound for the path number of a graph. *Journal of Graph Theory*, 4(2):189–201, 1980.
- [5] G. Fan. Path decompositions and Gallai’s conjecture. *Journal of Combinatorial Theory Series B*, 93(2):117–125, 2005.
- [6] O. Favaron and M. Kouider. Path partitions and cycle partitions of Eulerian graphs of maximum degree 4. *Studia Scientiarum Mathematicarum Hungarica*, 23(1):237–244, 1988.
- [7] O. Favaron, F. Genest, and M. Kouider. Regular path decompositions of odd regular graphs. *Journal of Graph Theory*, 63(2):114–128, 2010.
- [8] A. Jiménez and Y. Wakabayashi. On path-cycle decompositions of triangle-free graphs. In *The 9th International colloquium on graph theory and combinatorics*, 2014. ArXiv 1402.3741.
- [9] A. Kotzig. Aus der Theorie der endlichen regulären Graphen dritten und vierten Grades. *Časopis pro pěstování matematiky a fysiky*, 82:76–92, 1957.
- [10] L. Lovász. On covering of graphs. *Studia Scientiarum Mathematicarum Hungarica*, 1:237–238, 1966.
- [11] L. Pyber. Covering the edges of a connected graph by paths. *Journal of Combinatorial Theory Series B*, 66(1):152–159, 1996.
- [12] C. Thomassen. Decomposing graphs into paths of fixed length. *Combinatorica*, 33(1):97–123, 2013.
- [13] M. Zhai and C. Lü. Path decomposition of graphs with given path length. *Acta Mathematicae Applicatae Sinica, English Series*, 22(4):633–638, 2006.