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Some links between identifying codes and separating, dominating and total dominating sets in graphs. [1]

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Abstract

In the search for a dynamic programming-based algorithm derived from the modular decomposition of graphs, we analyze the behavior of the identifying code number under disjoint union and join operations. This study lead us to investigate the behavior of new parameters related to separating, dominating and total dominating sets under the same operations. The obtained results and the modular decomposition of graphs easily result in a dynamic programming-based algorithm to calculate the identifying code number (and the related parameters) of a graph from the parameter values of its modular subgraphs. In particular, we obtain closed formulas for the parameters on spider and quasi-spider graphs which allow us to derive a simple and easy-to-implement linear time algorithm to obtain the identifying code number (and the related parameters) of P_4-tidy graphs.

Keywords: identifying code, modular decomposition, P_4-tidy graph.

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1 Preliminaries, definitions and notations

We consider finite and simple graphs G, where $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. For every terminology and notation on graphs not defined in this paper, we follow [7].

Given a graph G and $R \subseteq V(G)$, $\overline{G}$ denotes its complementary graph and $G[R]$ the *subgraph of G induced by R*. For any $v \in V(G)$, $N(v)$ is the *open neighborhood of* v and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood of* v. If $N[v] = V(G)$ we say that v is a *universal vertex* of G. Two different vertices u, v are called *true twins* if $N[u] = N[v]$. A true twin free graph is called *identifiable*.

A subset D of $V(G)$ is a *dominating set* (resp. *total dominating set*) of G if, for each $v \in V(G)$, $D \cap N[v] \neq \emptyset$ (resp. $D \cap N(v) \neq \emptyset$). A *separating set* of G is a subset C of vertices of G such that for each pair $u, v \in V(G)$, $u \neq v$, $N[u] \cap C \neq N[v] \cap C$, or equivalently $(N[u] \triangle N[v]) \cap C \neq \emptyset$, where $\triangle$ is the symmetric difference between sets. An *identifying code* of G is a dominating and separating set of G. It can be easily seen that a graph admits an identifying code if and only if it is true twin free.

The concept of identifying code of graphs was introduced in 1998 [6] and has been applied to several problems which can be modeled as finding a subset of vertices of a graph such that each vertex in the graph can be uniquely determined by its (nonempty) neighbourhood within the code C. As usual, the *identifying code number* of G is the smallest size of an identifying code of G and is denoted by $\gamma_{Id}(G)$. There are some few families of graphs for which we have explicit formulas for their identifying code numbers. In particular, paths, cycles and complete r-ary trees. From a computational point of view, it is known that the problem of finding the identifying code number of a graph (IdCod) is NP-hard for many classes of graphs as planar, interval, split and bipartite graphs. On the other hand, from Courcelle et al.'s results relating to *Monadic Second Order Logic* [2], IdCod is linear time solvable for bounded clique-width graphs and their line graphs. More details and references can be found in [3].

Clearly, if G (resp. $\overline{G}$) is not connected, there exist two graphs G_1 and G_2 such that $G = G_1 + G_2$ (resp. $G = G_1 \vee G_2$), where $G_1 + G_2$ (resp. $G_1 \vee G_2$) denotes the *disjoint union* (resp. *join*) of graphs G_1 and G_2. If G and $\overline{G}$ are both connected, we say that G is a *modular graph*. Given a family of graphs $\mathcal{F}$, we denote by $M(\mathcal{F})$ the family of modular graphs in $\mathcal{F}$.

We say that a subset $C \subseteq V(G)$ satisfies the property D (resp. $\triangle$) if C is a dominating set (resp. separating set) of G. Besides, we say that C satisfies the property $\overline{D}$ if C is a total dominating set of $\overline{G}$, i.e. $C \setminus N[v] \neq \emptyset$ for each $v \in V(G)$. If $\mathcal{P} = \{\triangle, D, \overline{D}\}$, for every $R \subseteq \mathcal{P}$ we define $\mathcal{C}_R(G) = \{C \subseteq V(G) : C$ satisfies property Q, $\forall Q \in R\}$ and $\gamma_R(G) = \min\{|C| : C \in \mathcal{C}_R(G)\}$ if $\mathcal{C}_R(G) \neq \emptyset$ and $\gamma_R(G) = \infty$ otherwise.

It is straightforward to see that $\gamma_{\triangle D}(G) = \gamma_{Id}(G)$ and the parameters $\gamma_D(G)$ and $\gamma_{\overline{D}}(\overline{G})$ are the well known *domination number* and *total domination number* of G, respectively (for a survey on domination see, for example, [5]).

2 The parameters under disjoint union and join graphs

For the case of disjoint union of graphs, it is not hard to see that $\gamma_{Id}(G_1 + G_2) = \gamma_{Id}(G_1) + \gamma_{Id}(G_2)$ for every pair of graphs G_1 and G_2. However, the next theorem shows the links among the parameters presented in the previous section. These parameters are involved in the expression of the identifying code number for join of graphs.

If $|V(G_1)| = |V(G_2)| = 1$, it is clear that $\gamma_{Id}(G_1 \vee G_2) = \infty$. For the remaining cases we have:

Theorem 2.1 *Let G_1 and G_2 be two identifiable graphs. Then:*

(i) *If $|V(G_1)| = 1$ and $|V(G_2)| \geq 2$, then $\gamma_{Id}(G_1 \vee G_2) = \gamma_{D\triangle\overline{D}}(G_2)$.*

(ii) *If $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then*

$$\gamma_{Id}(G_1 \vee G_2) = \min\{\gamma_{\triangle}(G_1) + \gamma_{\triangle\overline{D}}(G_2), \gamma_{\triangle}(G_2) + \gamma_{\triangle\overline{D}}(G_1)\}.$$

Let $\mathcal{P}_{\triangle} = \{Q \subseteq \mathcal{P} : \triangle \in Q\}$. Theorem 2.1 shows that, in order to derive an algorithm based on the modular decomposition of graphs (see [4]) to compute γ_{Id}, we need to know the behavior of all the parameters γ_R with $R \in \mathcal{P}_{\triangle}$ under disjoint union and join operations on graphs. The results concerning to $\gamma_{\triangle D}$ are already presented in Theorem 2.1. For the remaining parameters, if both graphs G_1 and G_2 have one vertex, we obtain that $\gamma_{\triangle}(G_1 + G_2) = 1$, $\gamma_{\triangle\overline{D}}(G_1 + G_2) = \gamma_{\triangle D\overline{D}}(G_1 + G_2) = 2$ and $\gamma_R(G_1 \vee G_2) = \infty$, for all $R \in \mathcal{P}_{\triangle}$. In general we have the following results.

Theorem 2.2 *Let G be a graph and G_1 and G_2 be two graphs such that $G = G_1 + G_2$. Then:*

(i) *If $|V(G_1)| = 1$ and $|V(G_2)| \geq 2$, then $\gamma_{\triangle}(G) = \gamma_{\triangle D}(G_2)$, $\gamma_{\triangle D\overline{D}}(G) = 1 + \gamma_{\triangle D}(G_2)$ and $\gamma_{\triangle\overline{D}}(G) = \min\{1 + \gamma_{\triangle}(G_2), \gamma_{\triangle D\overline{D}}(G_2)\}$.*

(ii) *If $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then $\gamma_{\triangle}(G) = \gamma_{\triangle\overline{D}}(G) = \min\{\gamma_{\triangle}(G_1) + \gamma_{\triangle D}(G_2), \gamma_{\triangle D}(G_1) + \gamma_{\triangle}(G_2)\}$ and $\gamma_{\triangle D\overline{D}}(G) = \gamma_{\triangle D}(G_1) + \gamma_{\triangle D}(G_2)$.*

Theorem 2.3 *Let G be a graph and G_1 and G_2 two graphs such that $G = G_1 \vee G_2$. Then,*

(i) *If $|V(G_1)| = 1$ and $|V(G_2)| \geq 2$, then $\gamma_{\triangle}(G) = \gamma_{\triangle\overline{D}}(G_2)$, $\gamma_{\triangle D}(G) = \gamma_{\triangle D\overline{D}}(G_2)$ and $\gamma_{\triangle\overline{D}}(G) = \gamma_{\triangle D\overline{D}}(G) = \infty$.*

(ii) *If $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then $\gamma_{\triangle}(G) = \gamma_{\triangle D}(G) = \min\{\gamma_{\triangle}(G_1) + \gamma_{\triangle\overline{D}}(G_2), \gamma_{\triangle}(G_2) + \gamma_{\triangle\overline{D}}(G_1)\}$ and $\gamma_{\triangle\overline{D}}(G) = \gamma_{\triangle D\overline{D}}(G) = \gamma_{\triangle\overline{D}}(G_1) + \gamma_{\triangle\overline{D}}(G_2)$.*

From the previous theorems and the modular decomposition of graphs we can state:

Theorem 2.4 *Let $\mathcal{F}$ be a graph class. If, for all $R \subseteq \mathcal{P}_{\triangle}$ and $G \in M(\mathcal{F})$, $\gamma_R(G)$ can be computed in polynomial (resp. linear) time then, for every $R \subseteq \mathcal{P}_{\triangle}$ and $G \in \mathcal{F}$, $\gamma_R(G)$ can be computed in polynomial (resp. linear) time .*

Then, we can easily derive a linear time dynamic programming-based algorithm to obtain the parameter values (including the identifying code number) of a graph from the parameter values on its modular subgraphs. These results allow us to reduce the study of the identifying code number (and the associated parameters) to modular graphs.

3 Spider and quasi-spider graphs.

Spider graphs are modular graphs which play an important role in the characterization of many *few P_4's* families of graphs [1]. A graph is a *spider graph* if its vertex set can be partitioned into three sets S, C and H (H possible empty) where S is a stable set, C is a complete set, $|S| = |C| = r \geq 2$, all vertices in H are adjacent to all vertices in C, no vertex of H is adjacent to some vertex in S. Moreover, if $S = \{s_1, \ldots, s_r\}$ and $C = \{c_1, \ldots, c_r\}$ one of the following conditions must hold:

(i) *thin spider*: s_i is adjacent to c_j if and only if $i = j$.

(ii) *thick spider*: s_i is adjacent to c_j if and only if $i \neq j$.

The size of C (and S) is called the *weight* of G and the set H in the partition is called *the head* of the spider. A spider graph G with partition S, C, H will be denoted $G = (S, C, H)$.

Given two graphs G and H and $v \in V(G)$, the graph obtained by *replacing v by H in G* is the graph whose vertex set is $(V(G) \setminus \{v\}) \cup V(H)$ and whose edges either belong to $E(G - \{v\}) \cup E(H)$ or connect any vertex in $V(H)$ with any vertex in $N(v)$.

If $G = (S, C, H)$ is a thin (resp. thick) spider graph, the graph obtained by replacing one vertex $v \in S \cup C$ by S_2 or K_2 is called *thin (resp. thick) quasi-spider graph.* We denote $(S \hookleftarrow S_2, C, H)$ and $(S, C \hookleftarrow S_2, H)$ the quasi-spider graph obtaining from a spider graph with partition (S, C, H) by replacing one vertex in S by S_2 and one vertex in C by S_2, respectively. The weight of a quasi-spider graph is the weight of the original spider graph.

It is known that the partition for spider and quasi-spider graphs is unique and its recognition as well as its partition can be performed in linear time (see [4]).

Observe that quasi-spider graphs obtained by replacing a vertex by K_2 are not identifiable and then $\gamma_R(G) = \infty$ for all $R \in \mathcal{P}_\triangle$. Then, we specify the values of parameters $\gamma_R(G)$ with $R \in \mathcal{P}_\triangle$ for every identifiable spider and quasi-spider graph.

For spider graphs with empty head we have the following:

Theorem 3.1 *Let $G = (S, C, \emptyset)$ be a spider graph.*

(i) *If $r = 2$, $\gamma_\triangle(P_4) = \gamma_{\triangle D}(P_4) = 3$ and $\gamma_{\triangle \overline{D}}(P_4) = \gamma_{\triangle D\overline{D}}(P_4) = 4$.*

(ii) *If $r \geq 3$ and G is a thin spider, then $\gamma_\triangle(G) = \gamma_{\triangle D}(G) = \gamma_{\triangle \overline{D}}(G) = \gamma_{\triangle D\overline{D}}(G) = r + 1$.*

(iii) *If $r \geq 3$ and G is a thick spider, then $\gamma_\triangle(G) = r - 1$ and $\gamma_{\triangle D}(G) = \gamma_{\triangle \overline{D}}(G) = \gamma_{\triangle D\overline{D}}(G) = r$.*

We obtain similar formulas for the parameters on the remaining spiders and quasi-spiders. For lack of space, we summarize these results in the next two theorems.

Theorem 3.2 *Let G be a spider or a quasi-spider graph with weight r and empty head. Then, for all $R \in \mathcal{P}_\triangle$, $\gamma_R(G)$ is a linear function on its weight, i.e.*

$$\gamma_R(G) = \alpha(G, R)r + \beta(G, R)$$

where the coefficients $\alpha(G, R)$ and $\beta(G, R)$ can be computed in linear time from G.

Theorem 3.3 *Let G be a spider or a quasi-spider graph with non-empty head H, weight r and $R \in \mathcal{P}_\triangle$. Then,*

$$\gamma_R(G) = r + f_G^R(H)$$

where f_G^R can be determined in linear time from G and R and $f_G^R(H)$ can be computed in constant time from the values $\gamma_T(G[H])$, with $T \in \mathcal{P}_\triangle$.

From previous theorems we can conclude the following:

Corollary 3.4 *Let $\mathcal{F}$ be a graph class such that, for all $R \in \mathcal{P}_\triangle$, the parameter γ_R can be obtained in polynomial (linear) time. Then, for all $R \in \mathcal{P}_\triangle$, the parameter γ_R can be obtained in polynomial (linear) time on the family of spider and quasi-spider graphs for which the subgraph induced by their heads belongs to $\mathcal{F}$.*

Moreover, as we have mentioned before, spider graphs play an important role in the characterization of many few P_4's families of graphs. In particular, for *P_4-tidy graphs*, a super class of cographs and P_4-sparse graphs, defined by Rusu et al. (see [4]). Modular P_4-tidy graphs are spider graphs and quasi-spider graphs with P_4-tidy heads and the graphs C_5, P_5 and $\bar{P}_5$. Then, the dynamic-programming based algorithm derived from Theorem 2.4 allow us to solve IdCod in linear time with a simple and easy-to-implement algorithm.

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