Squares of Low Clique Number

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Abstract

The Square Root problem is that of deciding whether a given graph admits a square root. This problem is only known to be NP-complete for chordal graphs and polynomial-time solvable for non-trivial minor-closed graph classes and a very limited number of other graph classes. By researching boundedness of the treewidth of a graph, we prove that Square Root is polynomial-time solvable on various graph classes of low clique number that are not minor-closed.

Keywords: graph classes, square roots, squares, treewidth

The *square* $G = H^2$ of a graph $H = (V_G, E_G)$ is the graph with vertex set V_H, such that any two distinct vertices $u, v \in V_H$ are adjacent in G if and only if u and v are of distance at most 2 in H. A graph H is a *square root* of G if $G = H^2$. There exist graphs with no square root, graphs with a unique square root as well as graphs with many square roots. The corresponding

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recognition problem, which asks whether a given graph admits a square root, is called the Square Root problem and is known to be NP-complete [9]. As such, it is natural to restrict the input to special graph classes in order to obtain polynomial-time results. For many graph classes the complexity of Square Root is still unknown. For instance, Milanic and Schaudt [8] posed the complexity of Square Root restricted to split graphs and cographs as open problems. In Table 1 we survey the known results (note that the row for planar graphs could be absorbed by the row above of it). We explain this table in more detail below. In this paper we aim to identify new classes of squares of bounded treewidth. Our motivation for this question stems from the following result (obtained via applying Courcelle's meta-theorem).

Lemma 1 **([2])** *The* Square Root *problem can be solved in time $O(f(t)n)$ for n-vertex graphs of treewidth at most t.*

The unreferenced results in Table 1 correspond to our new results. The last column of this table indicates whether the squares of the graph class have bounded treewidth, where an $*$ means that these squares have bounded treewidth after some appropriate edge reduction (see [3] for further details). Note that the seven graph classes in the bottom seven rows not only have bounded treewidth but also have bounded clique number. We also observe that Nestoridis and Thilikos [10] proved that Square Root is polynomial-time solvable for non-trivial minor-closed graph classes by showing boundedness of carving width instead of treewidth. However, it is possible, by using the graph minor decomposition of Robertson and Seymour, to show that squares of graphs from such classes have in fact bounded treewidth as well.

We sketch the proof of one of our results from Table 1, namely the proof for 3-degenerate graphs (that is, graphs for which every subgraph has a vertex of degree at most 3.) We need one known and one new lemma (proof omitted).

Lemma 2 **([1])** *For any fixed constant k, it is possible to decide in linear time whether the treewidth of a graph is at most k.*

Lemma 3 *Let H be a square root of a graph G. Let T be the bipartite graph with $V_T = \mathcal{C} \cup \mathcal{B}$, where partition classes $\mathcal{C}$ and $\mathcal{B}$ are the set of cut vertices and blocks of H, respectively, such that $u \in \mathcal{C}$ and $Q \in \mathcal{B}$ are adjacent if and only if Q contains u. For $u \in \mathcal{C}$, let X_u consist of u and all neighbours of u in H. For $Q \in \mathcal{B}$, let $X_Q = V_Q$. Then (T, X) is a tree decomposition of G.*

We call the tree decomposition (T, X) the *H-tree decomposition* of G. We also need the following lemma.

Lemma 4 *If G is a 3-degenerate graph with a square root, then* $\mathbf{tw}(G) \leq 3$.

graph class	complexity	square bounded tw
trivially perfect graphs [8]	polynomial	no
threshold graphs [8]	polynomial	no
chordal graphs [4]	NP-complete	no
line graphs [7]	polynomial	no
non-trivial and minor-closed [10]	linear	yes
planar graphs [6]	linear	yes
K_4-free graphs	linear	yes
(K_r, P_t)-free graphs	linear	yes
3-degenerate graphs	linear	yes
graphs of maximum degree ≤ 5	linear	yes
graphs of maximum degree ≤ 6 [2]	polynomial	yes*
graphs of maximum average degree $< \frac{46}{11}$ [3]	polynomial	yes*

Table 1
The known results for Square Root restricted to some special graph class.

Proof. Without loss of generality we may assume that G is connected and has at least one edge. Let H be a square root of G; let $\mathcal{C}$ be the set of cut vertices of H and let $\mathcal{B}$ be the set of blocks of H. We construct the H-tree decomposition (T, X) of G (cf. Lemma 3) and show that (T, X) has width at most 3.

We start with two useful observations. If $v \in V_H$, then $N_H[v]$ is a clique in G. Because G is 3-degenerate, this means that $\Delta(H) \leq 3$. For the same reason H contains no cycles of length at least 5 as a subgraph, since a square of a cycle of length at least 5 has minimum degree 4.

We claim that X_Q has size at most 4 for every $Q \in \mathcal{B}$. In order to see this let Q be a block of H and let $u \in V_Q$. Suppose that Q has a vertex v at distance at least 3 from u. Since Q is 2-connected, Q has two internally vertex disjoint paths that join u and v. Therefore, Q (and thus H) contain a cycle of length at least 6 which, as we saw, is not possible. We find that each vertex $v \in V_Q$ is at distance at most 2 from u. Hence, u is adjacent to all other vertices of Q in G. Similarly, any two vertices in Q are of distance at most 2 from each other. Hence, Q is a clique in G. As G is 3-degenerate, this means that Q is a clique in G of size at most 4. Consequently, X_Q, has size at most 4. As $\Delta(H) \leq 3$, X_u has size at most 4 for every cut vertex u of H.□

We can now prove the following result.

Theorem 1 Square Root *can be solved in* $O(n)$ *time for* 3*-degenerate graphs.*

Proof. Let G be an 3-degenerate graph on n vertices. By Lemma 2 we can check in $O(n)$ time whether $\mathbf{tw}(G) \leq 3$. If $\mathbf{tw}(G) > 3$, then G has no square root by Lemma 4. Otherwise, apply Lemma 1. □

We cannot claim any upper bound for the treewidth of 4-degenerate graphs with a square root: take the square of a wall of arbitrarily large treewidth in which each edge is subdivided three times. We pose determining the complexity of the Square Root problem for 4-degenerate graphs as an open problem.

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