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First steps in combinatorial optimization on graphons: Matchings

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Abstract

Much of discrete optimization concerns problems whose underlying structures are graphs. Here, we translate the theory around the maximum matching problem to the setting of graphons. We study continuity properties of the thus defined matching ratio, limit versions of matching polytopes and vertex cover polytopes, and deduce a version of the LP duality for the problem of maximum fractional matching in the graphon setting.

Keywords: graphon, graph limits, matching, combinatorial optimization

1 Introduction

The study of matchings is central both in graph theory and in theoretical computer science. It has three sides: structural, polyhedral, and algorithmic. The structural part of the theory includes results such as the Gallai–Edmonds matching theorem. The study of polyhedral aspects — which include the geometry of the matching polytope, the vertex cover polytope — is much motivated by linear programming. Finally, algorithmic questions include, e.g., the study of fast algorithms for finding the maximum matching, or are motivated by theory related to property testing and parameter estimation. These three sides are very much intertwined.

We initiate and study concepts related to matchings in the setting of graphons. Graphons are analytic object which capture properties of large graphs. They were introduced in [1,6] as limit representation of large dense graphs. Since then they have played a key role in extremal graph theory, theory of random graphs, and other parts of mathematics. While this announcement is rather dry due to space constraints, we believe that it will convince the reader that many interesting combinatorial phenomena about matchings extend in a nontrivial way to the graphon setting. Also, note that while here we treat the topic *per se*, in an accompanying paper [3] we use (an

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extended version of) the theory to prove a strengthening of a tiling theorem of Komlós [5]. We expect further similar applications of our graphon approach to extremal graph theory in the future.

We shall assume that Ω is an atomless Borel probability space.

2 Matchings in graphons

Let us give our definition of matchings in graphons. In the graphon world, there is no distinction between integral and fractional matchings. This is why our definition is actually inspired by the notion of fractional matchings in finite graphs. To see the analogy, let us recall that for a finite graph G, a function $f : V(G)^2 \to \mathbb{R}$ represents a *fractional matching* if

(a) $f \geq 0$,

(b) if $f(x, y) > 0$ then $xy \in E(G)$, and

(c) for every $x \in V(G)$, we have $\sum_y f(x, y) + \sum_y f(y, x) \leq 1$.

Note that usually fractional matchings are represented using symmetric functions. This is however only a matter of notation. [6]

Definition 2.1 *Suppose that* $W : \Omega^2 \to [0, 1]$ *is a graphon. We say that a function* $\mathfrak{m} \in \mathcal{L}^1(\Omega^2)$ *is a* matching *in* W *if*

*(*a*)* $\mathfrak{m} \geq 0$ *almost everywhere,*

*(*b*) for the supports we have* $\operatorname{supp} \mathfrak{m} \subset \operatorname{supp} W$ *up to a null-set, and*

*(*c*) for almost every* $x \in \Omega$*, we have* $\int_y \mathfrak{m}(x, y) + \int_y \mathfrak{m}(y, x) \leq 1$.

As said already, even though Definition 2.1 is inspired by fractional matchings in finite graphs, the resulting graphon concept is referred to as "matchings". Note also that in Definition 2.1 the values of W are immaterial, only the support of W matters. To hint for the reason to this end, observe that by the (non-spanning) variant of the Blow-up lemma, a regular pair contains an almost perfect matching regardless of its density, provided that the density is non-zero. The Blow-up lemma is also the reason why there in the graphon

[6] The current choice for these functions being not-necessarily symmetric is adopted from [4]. In [4], we have worked out a more general concept of F*-tilings*, which is a collection of vertex-disjoint copies of a fixed graph F. When we take $F = K_2$, we get the notion of matchings. If F is on the vertex set $V(F) = [k]$ then a fractional F-tiling in a graph G is a function $f : V(G)^k \to \mathbb{R}$ satisfying (1) $f \geq 0$, (2) if $f(x_1, \ldots x_k) > 0$ then for each $ij \in E(F)$ we have that $x_i x_j \in E(G)$, and (3) for every $x \in V(G)$, we have $\sum_{i=1}^k \sum_{\mathbf{y} \in V(G)^k, \mathbf{y}_i = x} f(\mathbf{y}) \leq 1$. Observe that if F is not a complete graph then fractional F-tilings are generally not symmetric under permuting their coordinates, and no symmetrization may be possible.

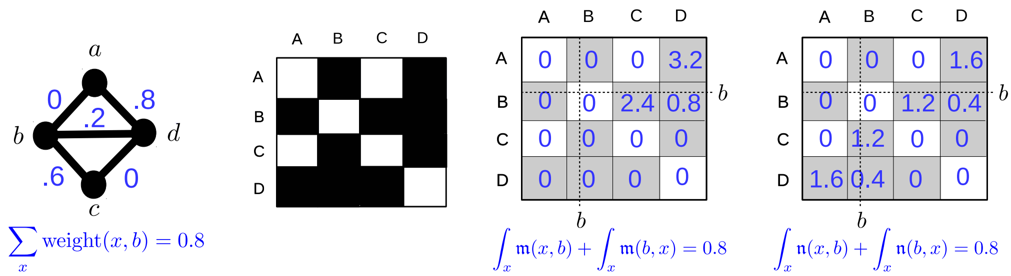

Fig. 1. A graph with a sample fractional matching on the left. Two sample matchings in its graphon representation are shown on the two figures on the right. The "antisymmetric" matching $\mathfrak{m}$ and the "symmetric" matching $\mathfrak{n}$. The formulas below the figures show the correspondence between the total weight at the vertex b, and the corresponding integrals in the representations of the fractional matchings.

setting, each fractional matching can be also viewed as an integral matching (the other direction is of course obvious even in finite graphs). Indeed, suppose that a graph G is close to a given graphon W in the cut-norm, and $\mathfrak{m}$ is a matching in W. Take a cluster graph R of G and observe that R is also close to W. Now, we shall have tools which allow us to transfer $\mathfrak{m}$ to a proportional fractional matching in R. After splitting the clusters according to this fractional matching, we may apply the Blow-up lemma and find an appropriate integral matching in G. Figure 1 shows how to translate a fractional matching on a finite graph G to a matching on a graphon that represents G.

Given a matching $\mathfrak{m}$ in a graphon W we define its *size*, $\|\mathfrak{m}\| = \int_x \int_y \mathfrak{m}(x,y)$. We write $\mathrm{MATCH}(W) \subset \mathcal{L}^1(\Omega^2)$ for the set of all matchings in W. Following the *-on* word ending already used for graph*ons* and permut*ons*, we call $\mathrm{MATCH}(W)$ the *matching polyton* of W. Last, we define the *matching ratio* $\mathsf{match}(W)$ of a graphon W as the supremum of sizes of its matchings.

There are two connected areas of questions regarding matchings in graphons. One of them concerns their continuity properties. That is, we have a sequence of graphs (or graphons) converging to a graphon W in the cut-distance, and we want to know how their matching ratios/polytons relate to that of W. The other area lies in investigating the properties of a single matching polyton. The latter is a direct counterpart to investigating the properties of the fractional matching polytope.

Given a finite graph G, we write $\mathsf{match}(G)$ for the matching number of G and $\mathsf{fmatch}(G)$ for the fractional matching number of G. If G has n vertices then we have $0 \leq \mathsf{match}(G) \leq \mathsf{fmatch}(G) \leq \frac{n}{2}$. It is easy to show that the normalized fractional matching number of G equals to the matching ratio of its graphon representation.

The bad news is that the matching ratio is not continuous with respect to the cut-distance. For example, take Z_n to be constant $\frac{1}{n}$. We have $\mathsf{match}(Z_n) = \frac{1}{2}$, but the sequence $(Z_n)_n$ converges to the zero graphon O for which we have $\mathsf{match}(O) = 0$. Another example is to take Z_n^* to be a graphon representation of a perfect matching of order $2n$ (i.e., the n-th graph has $2n$ vertices and n edges that form a perfect matching). Again, we have $\mathsf{match}(Z_n^*) = \frac{1}{2}$, but the sequence $(Z_n^*)_n$ converges to the zero graphon O. However, the matching ratio is lower-semicontinuous. We state this result in two versions: for sequences of graphs and for sequences of graphons.

Theorem 2.2 *Suppose that $(G_n)_n$ is a sequence of graphs of growing orders converging to a graphon $W : \Omega^2 \to [0,1]$ in the cut-distance. Then we have that $\liminf_n \frac{\mathsf{fmatch}(G_n)}{v(G_n)} \geq \liminf_n \frac{\mathsf{match}(G_n)}{v(G_n)} \geq \mathsf{match}(W)$.*

Theorem 2.3 *Suppose that $(W_n)_n$ is a sequence of graphons $W_n : \Omega^2 \to [0,1]$ converging to a graphon $W : \Omega^2 \to [0,1]$ in the cut-distance. Then we have that $\liminf_n \mathsf{match}(W_n) \geq \mathsf{match}(W)$.*

Lower–semicontinuity is the more important half of continuity for purposes of extremal graph theory. This is because in extremal graph theory typically we want to prove *lower* bounds on the matching number of some graph class $\mathcal{C}$ (say, in an asymptotic sense). We can then proceed by contradiction: Should this conjectured lower bound be violated, we can take a sequence of graphs that serve as counterexamples. These graphs have at least one graphon W as an accumulation point. By Theorem 2.2, W has a small matching ratio. At the same time, W inherited properties of $\mathcal{C}$, such as the edge density. This can then lead to contradiction, depending on a particular problem. For example, apart from the already advertised work on Komlós's Theorem, in [2, Section 3.6] we use this to prove a stability version of the Erdős–Gallai Theorem.

Our next theorem tells that the matching polyton of a limit is smaller than the "limit" of matching polytons of graphons converging to that limit.

Theorem 2.4 *Suppose that $(W_n)_n$ is a sequence of graphons, $W_n : \Omega^2 \to [0,1]$, converging to a graphon $W : \Omega^2 \to [0,1]$ in the cut-norm. Then for every $\mathfrak{m} \in \mathrm{MATCH}(W)$ there exists a sequence $(\mathfrak{m}_n \in \mathrm{MATCH}(W_n))_n$ which converges to $\mathfrak{m}$ in the cut-norm. In particular, $(\mathfrak{m}_n)_n$ converges weakly to $\mathfrak{m}$.*

3 Vertex covers

We proceed now with the definition of fractional vertex covers of a graphon. Recall that given a finite graph G, a function $c : V(G) \to [0, 1]$ is a fractional vertex cover if for each edge xy of G we have $c(x) + c(y) \geq 1$. If G is a finite graph then we write $\mathsf{fcov}(G)$ for the size of the minimum fractional vertex cover of G. The graphon counterpart of these concepts is as follows.

Definition 3.1 *Suppose that $W : \Omega^2 \to [0, 1]$ is a graphon. We say that a function $\mathfrak{c} \in \mathcal{L}^\infty(\Omega)$ is a* fractional vertex cover *of W if $0 \leq \mathfrak{c} \leq 1$ and the set $\mathrm{supp}\, W \setminus \{(x, y) : \mathfrak{c}(x) + \mathfrak{c}(y) \geq 1\}$ has measure 0.*

Given a fractional vertex cover $\mathfrak{c}$ of a graphon W we define its *size*, $\|\mathfrak{c}\| = \int_x \mathfrak{c}(x)$. We write $\mathrm{FCOV}(W) \subset \mathcal{L}^\infty(\Omega)$ for the set of all fractional vertex covers which we call the *fractional vertex cover polyton*. The fractional vertex cover ratio $\mathsf{fcov}(W)$ of a graphon W is defined as the infimum of sizes of its fractional vertex covers.

As with the concept of matchings, we shall investigate continuity properties of these notions and the structure of a fractional vertex cover polyton of a single graphon. However, the most important result which connects fractional vertex cover ratio and (fractional) matching ratio is the LP uality.

3.1 LP duality

Recall that for a finite graph G, the LP duality asserts that $\mathsf{fmatch}(G) = \mathsf{fcov}(G)$. The graphon version has exactly the same form.

Theorem 3.2 *Suppose that W is a graphon. Then $\mathsf{match}(W) = \mathsf{fcov}(W)$.*

Let us remark that all the versions of the LP duality we have found in the literature were formulated in terms of matrices and vectors in finite-dimensional vector spaces. Thus, Theorem 3.2 seems to be the first instance of an "analytic LP duality" in which the usual "$k \times \ell$-matrix" becomes a measurable function with domain $\Omega \times \Lambda$ for two measurable spaces Ω and Λ. It is interesting to study such an analytic LP duality in general.

3.2 Continuity properties

Combining Theorem 2.3 and Theorem 3.2 we immediately get that if a sequence of graphons W_n converges to a graphon W in the cut-distance then $\liminf_n \mathsf{fcov}(W_n) \geq \mathsf{fcov}(W)$. Our next theorem is more descriptive.

Theorem 3.3 *Suppose that $(W_n)_n$ is a sequence of graphons $W_n : \Omega^2 \to [0, 1]$*

converging to a graphon W in the cut-norm. Suppose that for each n, $\mathfrak{c}_n$ is a fractional vertex cover of W_n. Then any accumulation point of the sequence $(\mathfrak{c}_n)_n$ in the weak topology on $\mathcal{L}^\infty(\Omega)$ is a fractional vertex cover of W.*

3.3 Structure of fractional vertex cover polyton

One of the basic facts of polyhedral combinatorics stated that the vertices of the fractional vertex cover polytope $\mathrm{FCOV}(G)$ of a graph G are all half-integral, and they are integral if and only if G is bipartite. Here, we formulate a counterpart of this statement for graphons. To this end, we first define a counterpart of vertices of a polytope. Suppose that $\mathcal{L}$ is a vector space, and suppose that $X \subset \mathcal{L}$ is a convex set. Recall that a point $x \in X$ is called an *extreme point of* X if the only pair $x', x'' \in X$ for which $x = \frac{1}{2}(x' + x'')$ is the pair $x' = x$, $x'' = x$. The analogy between vertices of polytopes in a finite-dimensional vector space and extreme points of a convex compact set $X \subset \mathcal{L}^\infty(\Omega)$ is provided by the Krein–Milman Theorem which asserts that X equals to the closure of the convex hull of its extreme points. The Krein–Milman Theorem is indeed relevant in the current setting as $\mathrm{FCOV}(W)$ is a convex compact set in the weak* topology on $\mathcal{L}^\infty(\Omega)$.

A fractional vertex cover $\mathfrak{c} \in \mathrm{FCOV}(W)$ is *half-integral* if $\mathfrak{c}(x) \in \left\{0, \frac{1}{2}, 1\right\}$ for almost every $x \in \Omega$. *Integral* vertex covers are defined analogously. Thus, our results are as follows.

Theorem 3.4 *Let W be a graphon. Then all the extreme points of* $\mathrm{FCOV}(W)$ *are half-integral. Further, they are integral if and only if W is bipartite.*

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