# ADDITIVE BASES AND FLOWS IN GRAPHS 

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#### Abstract

It was conjectured by Jaeger, Linial, Payan, and Tarsi in 1992 that for any prime number $p$, there is a constant $c$ such that for any $n$, the union (with repetition) of the vectors of any family of $c$ linear bases of $\mathbb{Z}_{p}^{n}$ forms an additive basis of $\mathbb{Z}_{p}^{n}$ (i.e. any element of $\mathbb{Z}_{p}^{n}$ can be expressed as the sum of a subset of these vectors). In this note, we prove this conjecture when each vector contains at most two non-zero entries. As an application, we prove several results on flows in highly edge-connected graphs, extending known results. For instance, assume that $p \geqslant 3$ is a prime number and $\vec{G}$ is a directed, highly edge-connected graph in which each arc is given a list of two distinct values in $\mathbb{Z}_{p}$. Then $\vec{G}$ has a $\mathbb{Z}_{p}$-flow in which each arc is assigned a value of its own list.


## 1. Introduction

Graphs considered in this paper may have multiple edges but no loops. An additive basis $B$ of a vector space $F$ is a multiset of elements from $F$ such that for all $\beta \in F$, there is a subset of $B$ which sums to $\beta$. Let $\mathbb{Z}_{p}^{n}$ be the $n$-dimensional linear space over the prime field $\mathbb{Z}_{p}$. The following result is a simple consequence of the Cauchy-Davenport Theorem [5] (see also [2]).

Theorem 1 ([5]). For any prime $p$, any multiset of $p-1$ non-zero elements of $\mathbb{Z}_{p}$ forms an additive basis of $\mathbb{Z}_{p}$.

This result can be rephrased as: for $n=1$, any family of $p-1$ linear bases of $\mathbb{Z}_{p}^{n}$ forms an additive basis of $\mathbb{Z}_{p}^{n}$. A natural question is whether this can be extended to all integers $n$. Given a collection of sets $X_{1}, \ldots, X_{k}$, we denote by $\biguplus_{i=1}^{k} X_{i}$ the union with repetitions of $X_{1}, \ldots, X_{k}$. Jaeger, Linial, Payan and Tarsi [12] conjectured the

[^0]following, a generalization of important results regarding nowhere-zero flows in graphs.

Conjecture 2 ([12]). For every prime number p, there is a constant $c(p)$ such that for any $t \geqslant c(p)$ linear bases $B_{1}, \ldots, B_{t}$ of $\mathbb{Z}_{p}^{n}$, the union $\biguplus_{s=1}^{t} B_{s}$ forms an additive basis of $\mathbb{Z}_{p}^{n}$.

Alon, Linial and Meshulam [1] proved a weaker version of Conjecture 2 , that the union of any $p\lceil\log n\rceil$ linear bases of $\mathbb{Z}_{p}^{n}$ contains an additive basis of $\mathbb{Z}_{p}^{n}$ (note that their bound depends on $n$ ). The support of a vector $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}_{p}^{n}$ is the set of indices $i$ such that $x_{i} \neq 0$. The shadow of a vector $x$ is the (unordered) multiset of non-zero entries of $x$. Note that sizes of the support and of the shadow of a vector are equal. In this note, we prove that Conjecture 2 holds if the support of each vector has size at most two.

Theorem 3. Let $p \geqslant 3$ be a prime number. For some integer $\ell \geqslant 1$, consider $t \geqslant 8 \ell(3 p-4)+p-2$ linear bases $B_{1}, \ldots, B_{t}$ of $\mathbb{Z}_{p}^{n}$, such that the support of each vector has size at most 2 , and at most $\ell$ different shadows of size 2 appear among the vectors of $\mathcal{B}=\biguplus_{s=1}^{t} B_{s}$. Then $\mathcal{B}$ forms an additive basis of $\mathbb{Z}_{p}^{n}$.

Theorem 3 will be proved in Section 3 using a result of Lovász, Thomassen, Wu and Zhang [15] (Theorem 6 below) on flows in highly edge-connected graphs. It was mentioned to us by one of the referees that Lai and Li [14] established the equivalence between Theorem 6 and Theorem 3 in the special case where all the shadows are equal to $\{-1,+1\}(\bmod p)$.

The number of possibilities for an (unordered) multiset of $\mathbb{Z}_{p} \backslash\{0\}$ of size 2 is $\binom{p-1}{2}+p-1=\binom{p}{2}$. As a consequence, Theorem 3 has the following immediate corollary.

Corollary 4. Let $p \geqslant 3$ be a prime number. For any $t \geqslant 8\binom{p}{2}(3 p-$ $4)+p-2$ linear bases $B_{1}, \ldots, B_{t}$ of $\mathbb{Z}_{p}^{n}$ such that the support of each vector has size at most $\mathcal{Z}, \biguplus_{s=1}^{t} B_{s}$ forms an additive basis of $\mathbb{Z}_{p}^{n}$.

Another interesting consequence of Theorem 3 concerns the linear subspace $\left(\mathbb{Z}_{p}^{n}\right)_{0}$ of vectors of $\mathbb{Z}_{p}^{n}$ whose entries sum to $0(\bmod p)$.

Corollary 5. Let $p \geqslant 3$ be a prime number. For any $t \geqslant 4(p-1)(3 p-$ $4)+p-2$ linear bases $B_{1}, \ldots, B_{t}$ of $\left(\mathbb{Z}_{p}^{n}\right)_{0}$ such that the support of each vector has size at most 2, $\biguplus_{s=1}^{t} B_{s}$ forms an additive basis of $\left(\mathbb{Z}_{p}^{n}\right)_{0}$.

Proof. Note that for any $1 \leqslant s \leqslant t$, the linear basis $B_{s}$ consists of $n-1$ vectors, each of which has a support of size 2 , and the two elements of the shadow sum to $0(\bmod p)$. In particular, at most $\frac{p-1}{2}$ different shadows appear among the vectors of the linear bases $B_{1}, \ldots, B_{t}$. It is convenient to view each $B_{s}$ as a matrix in which the elements of the basis are column vectors. For each $1 \leqslant s \leqslant t$, let $B_{s}^{\prime}$ be obtained from $B_{s}$ by deleting the last row. It is easy to see that $B_{s}^{\prime}$ is a linear basis of $\mathbb{Z}_{p}^{n-1}$. Moreover, at most $\frac{p-1}{2}$ different shadows of size 2 appear among the vectors of the linear bases $B_{1}^{\prime}, \ldots, B_{t}^{\prime}$ (note that the removal of the last row may have created vectors with shadows of size 1). In particular, it follows from Theorem 3 that for any vector $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in\left(\mathbb{Z}_{p}^{n}\right)_{0}$, the vector $\left(\beta_{1}, \ldots, \beta_{n-1}\right) \in \mathbb{Z}_{p}^{n-1}$ can be written as a sum of a subset of elements of $\biguplus_{s=1}^{t} B_{s}^{\prime}$. Clearly, the corresponding subset of elements of $\biguplus_{s=1}^{t} B_{s}$ sums to $\beta$. This concludes the proof of Corollary 5 .

In the next section, we explore some consequences of Corollary 5.

## 2. Orientations and flows in graphs

Let $G=(V, E)$ be a non-oriented graph. An orientation $\vec{G}=(V, \vec{E})$ of $G$ is obtained by giving each edge of $E$ a direction. For each edge $e \in E$, we denote the corresponding arc of $\vec{E}$ by $\vec{e}$, and vice versa. For a vertex $v \in V$, we denote by $\delta_{\vec{G}}^{+}(v)$ the set of $\operatorname{arcs}$ of $\vec{E}$ leaving $v$, and by $\delta_{\vec{G}}^{-}(v)$ the set of arcs of $\vec{E}$ entering $v$.

For an integer $k \geqslant 2$, a mapping $\beta: V \rightarrow \mathbb{Z}_{k}$ is said to be a $\mathbb{Z}_{k}{ }^{-}$ boundary of $G$ if $\sum_{v \in V} \beta(v) \equiv 0(\bmod k)$. Given a $\mathbb{Z}_{k}$-boundary $\beta$ of $G$, an orientation $\vec{G}$ of $G$ is a $\beta$-orientation if $d_{\vec{G}}^{+}(v)-d_{\vec{G}}^{-}(v) \equiv \beta(v)$ $(\bmod k)$ for every $v \in V$, where $d_{\vec{G}}^{+}(v)$ and $d_{\vec{G}}^{-}(v)$ stand for the outdegree and the in-degree of $v$ in $\vec{G}$.

The following major result was obtained by Lovász, Thomassen, Wu and Zhang [15]:

Theorem 6. [15] For any $k \geqslant 1$, any $6 k$-edge-connected graph $G$, and any $\mathbb{Z}_{2 k+1}$-boundary $\beta$ of $G$, the graph $G$ has a $\beta$-orientation.

A natural question is whether a weighted counterpart of Theorem 6 exists. Given a graph $G=(V, E)$, a $\mathbb{Z}_{k}$-boundary $\beta$ of $G$ and a mapping $f: E \rightarrow \mathbb{Z}_{k}$, an orientation $\vec{G}$ of $G$ is called an $f$-weighted $\beta$-orientation if $\partial f(v) \equiv \beta(v)(\bmod k)$ for every $v$, where $\partial f(v)=\sum_{\vec{e} \in \delta_{\vec{G}}^{+}(v)} f(e)-$
$\sum_{\vec{e} \in \delta_{\vec{G}}^{-}(v)} f(e)$. Note that if $f(e) \equiv 1(\bmod k)$ for every edge $e$, an $f$-weighted $\beta$-orientation is precisely a $\beta$-orientation.

An immediate observation is that if we wish to have a general result of the form of Theorem 6 for weighted orientations, it is necessary to assume that $2 k+1$ is a prime number. For instance, take $G$ to consist of two vertices $u, v$ with an arbitrary number of edges between $u$ and $v$, consider a non-trivial divisor $p$ of $2 k+1$, and ask for a $\mathbf{p}$-weighted $\mathbb{Z}_{2 k+1^{-}}$ orientation $\vec{G}$ of $G$ (here, $\mathbf{p}$ denotes the function that maps each edge to $p(\bmod 2 k+1))$. Note that for any orientation, $\partial \mathbf{p}(v)$ is in the subgroup of $\mathbb{Z}_{2 k+1}$ generated by $p$, and this subgroup does not contain $1,-1$ $(\bmod 2 k+1)$. In particular, there is no $\mathbf{p}$-weighted $\mathbb{Z}_{2 k+1}$-orientation of $G$ with boundary $\beta$ satisfying $\beta(u) \equiv-\beta(v) \equiv 1(\bmod 2 k+1)$.

In Section 4, we will prove that Corollary 5 easily implies a weighted counterpart of Theorem 6 as in the following theorem, but with a stronger requirement on the edge-connectivity. Theorem 7 itself will be deduced directly from Theorem 6.

Theorem 7. Let $p \geqslant 3$ be a prime number and let $G=(V, E)$ be a $(6 p-8)(p-1)$-edge-connected graph. For any mapping $f: E \rightarrow \mathbb{Z}_{p} \backslash\{0\}$ and any $\mathbb{Z}_{p}$-boundary $\beta$, G has an $f$-weighted $\beta$-orientation.

Theorem 7 turns out to be equivalent to the following seemingly more general result. Assume that we are given a directed graph $\vec{G}=(V, \vec{E})$ and a $\mathbb{Z}_{p}$-boundary $\beta$. A $\mathbb{Z}_{p}$-flow with boundary $\beta$ in $\vec{G}$ is a mapping $f: \vec{E} \rightarrow \mathbb{Z}_{p}$ such that $\partial f(v) \equiv \beta(v)(\bmod p)$ for every $v$. In other words, $f$ is a $\mathbb{Z}_{p}$-flow with boundary $\beta$ in $\vec{G}=(V, \vec{E})$ if and only if $\vec{G}$ is an $f$-weighted $\beta$-orientation of its underlying non-oriented graph $G=(V, E)$, where $f$ is extended from $\vec{E}$ to $E$ in the natural way (i.e. for each $e \in E, f(e):=f(\vec{e}))$.

In the remainder of the paper we will say that a directed graph $\vec{G}$ is $t$-edge-connected if its underlying non-oriented graph, denoted by $G$, is $t$-edge-connected.

Theorem 8. Let $p \geqslant 3$ be a prime number and let $\vec{G}=(V, \vec{E})$ be a directed $(6 p-8)(p-1)$-edge-connected graph. For any arc $\vec{e} \in \vec{E}$, let $L(\vec{e})$ be a pair of distinct elements of $\mathbb{Z}_{p}$. Then for every $\mathbb{Z}_{p}$-boundary $\beta, \vec{G}$ has a $\mathbb{Z}_{p}$-flow $f$ with boundary $\beta$ such that for any $\vec{e} \in \vec{E}, f(\vec{e}) \in$ $L(\vec{e})$.

This result can been seen as a choosability version of Theorem 6 (the reader is referred to [6] for choosability versions of some classical results on flows). To see that Theorem 8 implies Theorem 7, simply fix an arbitrary orientation of $G$ and set $L(\vec{e})=\{f(e),-f(e)\}$ for each $\operatorname{arc} \vec{e}$. We now prove that Theorem 7 implies Theorem 8 . We actually prove a slightly stronger statement (holding in $\mathbb{Z}_{2 k+1}$ for any integer $k \geqslant 1$ ).

Lemma 9. Let $k \geqslant 1$ be an integer, and let $\vec{G}=(V, \vec{E})$ be a directed graph such that the underlying non-oriented graph $G$ has an $f$-weighted $\beta$-orientation for any mapping $f: E \rightarrow \mathbb{Z}_{2 k+1} \backslash\{0\}$ and any $\mathbb{Z}_{2 k+1}$ boundary $\beta$. For every arc $\vec{e} \in \vec{E}$, let $L(\vec{e})$ be a pair of distinct elements of $\mathbb{Z}_{2 k+1}$. Then for every $\mathbb{Z}_{2 k+1}$-boundary $\beta, \vec{G}$ has a $\mathbb{Z}_{2 k+1}$-flow $g$ with boundary $\beta$ such that $g(\vec{e}) \in L(\vec{e})$ for every $\vec{e}$.

Proof. Let $\beta$ be a $\mathbb{Z}_{2 k+1}$-boundary of $\vec{G}$. Consider a single $\operatorname{arc} \vec{e}=(u, v)$ of $\vec{G}$. Choosing one of the two values of $L(\vec{e})$, say $a$ or $b$, will either add $a$ to $\partial g(u)$ and subtract $a$ from $\partial g(v)$, or add $b$ to $\partial g(u)$ and subtract $b$ from $\partial g(v)$. Note that 2 and $2 k+1$ are relatively prime, so the element $2^{-1}$ is well-defined in $\mathbb{Z}_{2 k+1}$. If we now add $2^{-1}(a+b)$ to $\beta(v)$ and subtract $2^{-1}(a+b)$ from $\beta(u)$, the earlier choice is equivalent to choosing between the two following options: adding $2^{-1}(a-b)$ to $\partial g(u)$ and subtracting $2^{-1}(a-b)$ from $\partial g(v)$, or adding $2^{-1}(b-a)$ to $\partial g(u)$ and subtracting $2^{-1}(b-a)$ from $\partial g(v)$. This is equivalent to choosing an orientation for an edge of weight $2^{-1}(a-b)$. It follows that finding a $\mathbb{Z}_{2 k+1}$-flow $g$ with boundary $\beta$ such that for any $\vec{e} \in \vec{E}, g(\vec{e}) \in L(\vec{e})$ is equivalent to finding an $f$-weighted $\beta^{\prime}$-orientation for some other $\mathbb{Z}_{2 k+1}$-boundary $\beta^{\prime}$ of $G$, where the weight $f(e)$ of each edge $e$ is $2^{-1}$ times the difference between the two elements of $L(\vec{e})$.

We now consider the case where $L(\vec{e})=\{0,1\}$ for every arc $\vec{e} \in \vec{E}$. Let $f_{2^{-1}}: \vec{E} \rightarrow \mathbb{Z}_{2 k+1}$ denote the function that maps each arc $\vec{e}$ to $2^{-1}$ $(\bmod 2 k+1)$. The same argument as in the proof of Lemma 9 implies that if $G$ has an $f_{2^{-1}}$-weighted $\beta$-orientation for every $\mathbb{Z}_{2 k+1}$-boundary $\beta$, then for every $\mathbb{Z}_{2 k+1}$-boundary $\beta$, the digraph $\vec{G}$ has a $\mathbb{Z}_{2 k+1}$-flow $f$ with boundary $\beta$ such that $f(\vec{e}) \in L(\vec{e})$ for every $\vec{e}$.

The following is a simple corollary of Theorem 6.
Corollary 10. Let $\ell \geqslant 1$ be an odd integer and let $k \geqslant 1$ be relatively prime with $\ell$. Let $G=(V, E)$ be a $(3 \ell-3)$-edge-connected graph, and let $\mathbf{k}: E \rightarrow \mathbb{Z}_{\ell}$ be the mapping that assigns $k(\bmod \ell)$ to each edge $e \in E$. Then for any $\mathbb{Z}_{\ell}$-boundary $\beta$, $G$ has a $\mathbf{k}$-weighted $\beta$-orientation.

Proof. Observe that $\beta^{\prime}=k^{-1} \cdot \beta$ is a $\mathbb{Z}_{\ell^{-}}$-boundary $\left(k^{-1}\right.$ is well defined in $\left.\mathbb{Z}_{\ell}\right)$. It follows from Theorem 6 that $G$ has a $\beta^{\prime}$-orientation. Note that this corresponds to a $\mathbf{k}$-weighted $\beta$-orientation of $G$, as desired.

As a consequence, the following is an equivalent version of Theorem 6 (see also [12, 14]).

Theorem 11. Let $k \geqslant 1$ be an integer and let $\vec{G}=(V, \vec{E})$ be a directed $6 k$-edge-connected graph. Then for every $\mathbb{Z}_{2 k+1}$-boundary $\beta, \vec{G}$ has a $\mathbb{Z}_{2 k+1}$-flow $f$ with boundary $\beta$ such that $f(\vec{E}) \in\{0,1\}(\bmod 2 k+1)$.

This version of Theorem 6 will allow us to derive interesting results on antisymmetric flows in directed highly edge-connected graphs. Given an abelian group $(B,+)$, a $B$-flow in $\vec{G}$ is a mapping $f: \vec{E} \rightarrow B$ such that $\partial f(v)=0$ for every vertex $v$, where all operations are performed in $B$. A $B$-flow $f$ in $\vec{G}=(V, \vec{E})$ is a nowhere-zero $B$-flow (or a $B$-NZF) if $0 \notin f(\vec{E})$, i.e. each arc of $\vec{G}$ is assigned a non-zero element of $B$. If no two arcs receive inverse elements of $B$, then $f$ is an antisymmetric $B$-flow (or a $B$-ASF).

Since $0=-0$, a $B$-ASF is also a $B$-NZF. It was conjectured by Tutte that every directed 2-edge-connected graph has a $\mathbb{Z}_{5}$-NZF [21], and that every directed 4 -edge-connected graph has a $\mathbb{Z}_{3}$-NZF (see [18] and [3]). Antisymmetric flows were introduced by Nešetřil and Raspaud in [16]. A natural obstruction for the existence of an antisymmetric flow in a directed graph $\vec{G}$ is the presence of directed 2-edge-cut in $\vec{G}$. Nešetřil and Raspaud asked whether any directed graph without directed 2-edge-cut has a $B$-ASF, for some $B$. This was proved by DeVos, Johnson, and Seymour in [7], who showed that any directed graph without directed 2-edge-cut has a $\mathbb{Z}_{2}^{8} \times \mathbb{Z}_{3}^{17}$-ASF. It was later proved by DeVos, Nešetřil, and Raspaud [8], that the group could be replaced by $\mathbb{Z}_{2}^{6} \times \mathbb{Z}_{3}^{9}$. The best known result is due to Dvorrák, Kaiser, Král', and Sereni [10], who showed that any directed graph without directed 2-edge-cut has a $\mathbb{Z}_{2}^{3} \times \mathbb{Z}_{3}^{9}$-ASF (this group has 157464 elements).

Adding a stronger condition on the edge-connectivity allows to prove stronger results on the size of the group $B$. It was proved by DeVos, Nešetřil, and Raspaud [8], that every directed 4-edge-connected graph has a $\mathbb{Z}_{2}^{2} \times \mathbb{Z}_{3}^{4}$-ASF, that every directed 5-edge-connected graph has a $\mathbb{Z}_{3}^{5}$-ASF, and that every directed 6 -edge-connected graph has a $\mathbb{Z}_{2} \times \mathbb{Z}_{3}^{2}$ ASF.

In [11], Jaeger conjectured the following weaker version of Tutte's 3 -flow conjecture: there is a constant $k$ such that every $k$-edgeconnected graph has a $\mathbb{Z}_{3}$-NZF. This conjecture was recently solved by Thomassen [19], who proved that every 8-edge-connected graph has a $\mathbb{Z}_{3}$-NZF, and was improved by Lovász, Thomassen, Wu and Zhang [15], that every 6-edge-connected graph has a $\mathbb{Z}_{3}$-NZF (this is a simple consequence of Theorem 6).

The natural antisymmetric variant of Jaeger's weak 3-flow conjecture would be the following: there is a constant $k$ such that every directed $k$-edge-connected graph has a $\mathbb{Z}_{5}$-ASF.

Note that the size of the group would be best possible, since in $\mathbb{Z}_{2}$ and $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ every element is its own inverse, while a $\mathbb{Z}_{3}$-ASF or a $\mathbb{Z}_{4^{-}}$ ASF has to assign the same value to all the arcs (and this is impossible in the digraph on two vertices $u, v$ with exactly $k$ arcs directed from $u$ to $v$, for any integer $k \equiv 1(\bmod 12))$.

Our final result is the following.
Theorem 12. For any $k \geqslant 2$, every directed $\left\lceil\frac{6 k}{k-1}\right\rceil$-edge-connected graph has a $\mathbb{Z}_{2 k+1}$-ASF.

Proof. Let $k \geqslant 2$, and let $\vec{G}$ be a directed $\left\lceil\frac{6 k}{k-1}\right\rceil$-edge-connected graph. Let $\vec{H}$ be the directed graph obtained from $\vec{G}$ by replacing every arc $\vec{e}$ by $k-1$ arcs with the same tail and head as $\vec{e}$, and let $H$ be the non-oriented graph underlying $\vec{H}$. Let $\beta(v)=d_{\vec{G}}^{-}(v)-d_{\vec{G}}^{+}(v)$ for every $v$. Since $\vec{G}$ is $\left\lceil\frac{6 k}{k-1}\right\rceil$-edge-connected, $H$ is $6 k$-edge-connected and by Theorem $11, \vec{H}$ has a $\mathbb{Z}_{2 k+1}$-flow $f$ with boundary $\beta$ with flow values in the set $\{0,1\}(\bmod 2 k+1)$. For any $\operatorname{arc} \vec{e}$ of $\vec{G}$, let $g(\vec{e})$ be the sum of the values of the flow $f$ on the $t$ arcs corresponding to $\vec{e}$ in $\vec{H}$. Then $g$ is a $\mathbb{Z}_{2 k+1}$-flow with boundary $\beta$ in $\vec{G}$, with flow values in the set $\{0,1, \ldots, k-1\}(\bmod 2 k+1)$. Now, set $g^{\prime}(\vec{e})=g(\vec{e})+1$ for every arc $\vec{e}$. Hence every $\vec{e}$ is assigned a value in $\{1, \ldots, k\}(\bmod 2 k+1)$, and $\partial g^{\prime}(v) \equiv \partial g(v)+d_{\vec{G}}^{+}(v)-d_{\vec{G}}^{-}(v) \equiv \beta^{\prime}(v)+d_{\vec{G}}^{+}(v)-d_{\vec{G}}^{-}(v) \equiv 0(\bmod 2 k+1)$ for every $v$. Thus $g^{\prime}$ is a $\mathbb{Z}_{2 k+1}$ flow of $\vec{G}$ with flow values in the set $\{1, \ldots, k\}(\bmod 2 k+1)$, and thus a $\mathbb{Z}_{2 k+1}$-ASF in $\vec{G}$, as desired. This concludes the proof of Theorem 12.

As a corollary, we directly obtain:

## Corollary 13.

(i) Every directed 7-edge-connected graph has a $\mathbb{Z}_{15}$-ASF.
(ii) Every directed 8-edge-connected graph has a $\mathbb{Z}_{9}$-ASF.
(iii) Every directed 9-edge-connected graph has a $\mathbb{Z}_{7}$-ASF.
(iv) Every directed 12-edge-connected graph has a $\mathbb{Z}_{5}$-ASF.

By duality, using the results of Nešetřil and Raspaud [16], Corollary 13 (which, again, can be seen as an antisymmetric analogue of the statement of Jaeger's conjecture) directly implies that every orientation of a planar graph of girth (length of a shortest cycle) at least 12 has a homomorphism to an oriented graph on at most 5 vertices. This was proved by Borodin, Ivanova and Kostochka in 2007 [4], and it is not known whether the same holds for planar graphs of girth at least 11. On the other hand, it was proved by Nešetřil, Raspaud and Sopena [17] that there are orientations of some planar graphs of girth at least 7 that have no homomorphism to an oriented graph of at most 5 vertices. By duality again, this implies that there are directed 7-edge-connected graphs with no $\mathbb{Z}_{5}$-ASF. We conjecture the following:

Conjecture 14. Every directed 8-edge-connected graph has a $\mathbb{Z}_{5}$-ASF.
It was conjectured by Lai [13] that for every $k \geqslant 1$, every $(4 k+1)$ -edge-connected graph $G$ has a $\beta$-orientation for every $\mathbb{Z}_{2 k+1}$-boundary $\beta$ of $G$. If true, this conjecture would directly imply (using the same proof as that of Theorem 12) that for any $k \geqslant 2$, every directed $\left\lceil\frac{4 k+1}{k-1}\right\rceil-$ edge-connected graph has a $\mathbb{Z}_{2 k+1}$-ASF. In particular, this would show that directed 5-edge-connected graph have a $\mathbb{Z}_{13}$-ASF, directed 6 -edgeconnected graph have a $\mathbb{Z}_{9}$-ASF, directed 7 -edge-connected graph have a $\mathbb{Z}_{7}$-ASF, and directed 9-edge-connected graph have a $\mathbb{Z}_{5}$-ASF. The bound on directed 5 -edge-connected graph would also directly imply, using the proof of the main result of [10], that directed graphs with no directed 2-edge-cut have a $\mathbb{Z}_{2}^{2} \times \mathbb{Z}_{3}^{4} \times \mathbb{Z}_{13}$-ASF.

## 3. Proof of Theorem 3

We first recall the following (weak form of a) classical result by Mader (see [9], Theorem 1.4.3):

Lemma 15. Given an integer $k \geqslant 1$, if $G=(V, E)$ is a graph with average degree at least $4 k$, then there is a subset $X$ of $V$ such that $|X|>1$ and $G[X]$ is $(k+1)$-edge-connected.

We will also need the following result of Thomassen [20], which is a simple consequence of Theorem 6 .

Theorem 16 ([20]). Let $k \geqslant 3$ be an odd integer, $G=\left(V_{1}, V_{2}, E\right)$ be a bipartite graph, and $f: V_{1} \cup V_{2} \rightarrow \mathbb{Z}_{k}$ be a mapping satisfying $\sum_{v \in V_{1}} f(v) \equiv \sum_{v \in V_{2}} f(v)(\bmod k)$. If $G$ is $(3 k-3)$-edge-connected, then $G$ has a spanning subgraph $H$ such that for any $v \in V, d_{H}(v) \equiv$ $f(v)(\bmod k)$.

Let $G$ be a graph, and let $X$ and $Y$ be two disjoint subsets of vertices of $G$. The set of edges of $G$ with one endpoint in $X$ and the other in $Y$ is denoted by $E(X, Y)$.

We are now ready to prove Theorem 3.
Proof of Theorem 3. We proceed by induction on $n$. For $n=1$, this is a direct consequence of Theorem 1, so suppose that $n \geqslant 2$. Each basis $B_{s}$ can be considered as an $n \times n$ matrix where each column is a vector with support of size at most 2 . Let $\mathcal{B}=\biguplus_{i=1}^{t} B_{i}$.

For $1 \leqslant i \leqslant n$, a vector is called an $i$-vector if its support is the singleton $\{i\}$ (in other words, the $i$-th entry is non-zero and all the other entries are zero). Suppose that for some $1 \leqslant i \leqslant n, \mathcal{B}$ contains at least $p-1 i$-vectors. Let $\mathcal{C}$ be the set of $i$-vectors of $\mathcal{B}$. Clearly, each basis contains at most one $i$-vector. For every $B_{s}$, let $B_{s}^{\prime}$ be the matrix obtained from $B_{s}$ by removing its $i$-vector (if any) and the $i^{\text {th }}$ row. Clearly $B_{s}^{\prime}$ is or contains a basis of $\mathbb{Z}_{p}^{n-1}$. By induction hypothesis, $\biguplus_{s=1}^{t} B_{s}^{\prime}$ forms an additive basis of $\mathbb{Z}_{p}^{n-1}$. In other words, for any vector $\beta=\left(\beta_{1}, \ldots, \beta_{i}, \ldots, \beta_{n}\right) \in \mathbb{Z}_{p}^{n}$, there is a subset $Y_{1}$ of $\mathcal{B} \backslash \mathcal{C}$ which sums to $\left(\beta_{1}, \ldots, \hat{\beta}_{i}, . ., \beta_{n}\right)$ for some $\hat{\beta}_{i}$. Since $|\mathcal{C}| \geqslant p-1$, it follows from Theorem 1 that there is a subset $Y_{2}$ of $\mathcal{C}$ which sums to $\left(0, \ldots, \beta_{i}-\hat{\beta}_{i}, . ., 0\right)$. Hence $Y_{1} \cup Y_{2}$ sums to $\beta$.

Thus we can suppose that there are at most $p-2 i$-vectors for every $i$. Then there are at least $8 \ell(3 p-4) n$ vectors with a support of size 2 in $\mathcal{B}$. Since there are at most $\ell$ distinct shadows of size 2 in $\mathcal{B}$, there are at least $8(3 p-4) n$ vectors with the same (unordered) shadow of size 2 , say $\left\{a_{1}, a_{2}\right\}$ (recall that shadows are multisets, so $a_{1}$ and $a_{2}$ might coincide).

Let $G$ be the graph (recall that graphs in this paper are allowed to have multiple edges) with vertex set $V=\left\{v_{1}, \ldots, v_{n}\right\}$ and edge set $E$, where edges $v_{i} v_{j}$ are in one-to-one correspondence with vectors of $\mathcal{B}$ with support $\{i, j\}$ and shadow $\left\{a_{1}, a_{2}\right\}$. Then $G$ contains at least $8(3 p-4) n$ edges.

We now consider a random partition of $V$ into 2 sets $V_{1}, V_{2}$ (by assigning each vertex of $V$ uniformly at random to one of the sets $V_{k}$,
$k=1,2)$. Let $e=v_{i} v_{j}$ be some edge of $G$. Recall that $e$ corresponds to some vector with only two non-zero entries, say without loss of generality $a_{1}$ at $i^{\text {th }}$ index and $a_{2}$ at $j^{\text {th }}$ index. The probability that $v_{i}$ is assigned to $V_{1}$ and $v_{j}$ is assigned to $V_{2}$ is at least $\frac{1}{4}$. As a consequence, there is a partition of $V$ into 2 sets $V_{1}, V_{2}$ and a subset $E^{\prime} \subseteq E\left(V_{1}, V_{2}\right)$ of at least $8(3 p-4) n / 4=2(3 p-4) n$ edges such that for every $e \in E^{\prime}$, the vector of $\mathcal{B}$ corresponding with $e$ has entry $a_{1}$ (resp. $a_{2}$ ) at the index associated to the endpoint of $e$ in $V_{1}$ (resp. $V_{2}$ ).

Since the graph $G^{\prime}=\left(V, E^{\prime}\right)$ has average degree at least $4(3 p-4)$, it follows from Lemma 15 that there is a set $X \subseteq V$ of at least 2 vertices, such that $G^{\prime}[X]$ is $(3 p-3)$-edge-connected. Set $H=G^{\prime}[X]$ and $F$ the edge set of $H$. Note that $H$ is bipartite with bipartition $X_{1}=X \cap V_{1}$ and $X_{2}=X \cap V_{2}$.

For each integer $1 \leqslant s \leqslant t$, let $B_{s}^{*}$ be the matrix obtained from $B_{s}$ by doing the following: for each vertex $v_{i}$ in $X_{1}$ (resp. $X_{2}$ ), we multiply all the elements of the $i^{\text {th }}$ row of $B_{s}$ by $a_{1}^{-1}$ (resp. $-a_{2}^{-1}$ ), noting that all the operations are performed in $\mathbb{Z}_{p}$. Let $\mathcal{B}^{*}=\biguplus_{s=1}^{t} B_{s}^{*}$. Note that each vector of $\mathcal{B}^{*}$ corresponding to some edge $e \in F$ has shadow $\{1,-1\}$ (1 is the entry indexed by the endpoint of $e$ in $X_{1}$ and -1 is the entry indexed by the endpoint of $e$ in $X_{2}$ ). It is easy to verify the following.

- Each $B_{s}^{*}$ is a linear basis of $\mathbb{Z}_{p}^{n}$.
- $\mathcal{B}$ is an additive basis if and only if $\mathcal{B}^{*}$ is an additive basis.

Hence it suffices to prove that $\mathcal{B}^{*}$ is an additive basis.
Without loss of generality, suppose that $X=\left\{v_{m}, \ldots, v_{n}\right\}$ for some $m \leqslant n-1$. By contracting $k$ rows of a matrix, we mean deleting these $k$ rows and adding a new row consisting of the sum of the $k$ rows. For each $1 \leqslant s \leqslant t$, let $B_{s}^{\prime}$ be the matrix of $m$ rows obtained from $B_{s}^{*}$ by contracting all $m^{\text {th }},(m+1)^{t h}, \ldots, n^{\text {th }}$ rows. Note that the operation of contracting $k$ rows decreases the rank of the matrix by at most $k-1$ (since it is the same as replacing one of the rows by the sum of the $k$ rows, which preserves the rank, and then deleting the $k-1$ other rows). Let $\mathcal{B}^{\prime}=\biguplus_{s=1}^{t} B_{s}^{\prime}$. Since each $B_{s}^{*}$ is a linear basis of $\mathbb{Z}_{p}^{n}$, each $B_{s}^{\prime}$ has rank at least $m$ and therefore contains a basis of $\mathbb{Z}_{p}^{m}$. Hence, by induction hypothesis, $\mathcal{B}^{\prime} \backslash \mathcal{B}_{0}^{\prime}$ is an additive basis of $\mathbb{Z}_{p}^{m}$, where $\mathcal{B}_{0}^{\prime}$ is the set of all columns with empty support in $\mathcal{B}^{\prime}$. For every $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{Z}_{p}^{n}$, let $\beta^{\prime}=\left(\beta_{1}, \ldots, \beta_{m-1}, \sum_{i=m}^{n} \beta_{i}\right) \in \mathbb{Z}_{p}^{m}$. Then there is a subset $Y^{\prime}$ of $\mathcal{B}^{\prime} \backslash \mathcal{B}_{0}^{\prime}$ which sums to $\beta^{\prime}$. Let $Y^{*}$ and $\mathcal{B}_{0}^{*}$ be the subsets of $\mathcal{B}^{*}$ corresponding to $Y^{\prime}$ and $\mathcal{B}_{0}^{\prime}$, respectively. Then $Y^{*}$
sums to some $\hat{\beta}=\left(\beta_{1}, \ldots, \beta_{m-1}, \hat{\beta}_{m}, \ldots, \hat{\beta}_{n}\right)$, where $\sum_{i=m}^{n} \hat{\beta}_{i} \equiv \sum_{i=m}^{n} \beta_{i}$ $(\bmod p)$.

Recall that for each edge $e \in F$, the corresponding vector in $\mathcal{B}^{*}$ has precisely two non-zero entries, $(1,-1)$, each with index in $X$. Hence the vector corresponding to each $e \in F$ in $\mathcal{B}^{\prime}$ has empty support. Thus the set of vectors in $\mathcal{B}^{*}$ corresponding to the edge set $F$ is a subset of $\mathcal{B}_{0}^{*}$, which is disjoint from $Y$.

For each $v_{i} \in X_{1}$, let $\beta_{X}\left(v_{i}\right)=\beta_{i}-\hat{\beta}_{i}$, and for each $v_{i} \in X_{2}$, let $\beta_{X}\left(v_{i}\right)=\hat{\beta}_{i}-\beta_{i}$. Since $\sum_{i=m}^{n} \hat{\beta}_{i} \equiv \sum_{i=m}^{n} \beta_{i}(\bmod p)$, we have $\sum_{v_{i} \in X \cap V_{1}} \beta_{X}\left(v_{i}\right)=\sum_{v_{i} \in X \cap V_{2}} \beta_{X}\left(v_{i}\right)$. Since $H$ is ( $3 p-3$ )-edgeconnected, it follows from Theorem 16 that there is a subset $F^{\prime} \subseteq F$ such that, in the graph $\left(X, F^{\prime}\right)$, each vertex $v_{i} \in X_{1}$ has degree $\beta_{i}-\hat{\beta}_{i}$ $(\bmod p)$ and each vertex $v_{i} \in X_{2}$ has degree $\hat{\beta}_{i}-\beta_{i}(\bmod p)$. Therefore, $F^{\prime}$ corresponds to a subset $Z^{*}$ of vectors of $\mathcal{B}_{0}^{*}$, summing to $\left(0, \ldots, 0, \beta_{m}-\hat{\beta}_{m}, \ldots, \beta_{n}-\hat{\beta}_{n}\right)$. Then $Y^{*} \cup Z^{*}$ sums to $\beta$. It follows that $\mathcal{B}^{*}$ is an additive basis of $\mathbb{Z}_{p}^{n}$, and so is $\mathcal{B}$. This completes the proof.

## 4. Two proofs of (versions of) Theorem 7

We now give two proofs of (versions of) Theorem 7. The first one is a direct application of Corollary 5, but requires a stronger assumption on the edge-connectivity of $G\left(24 p^{2}-54 p+28\right.$ instead of $6 p^{2}-14 p+8$ for the second proof).

First proof of Theorem 7. We fix some arbitrary orientation $\vec{G}=(V, \vec{E})$ of $G$ and denote the vertices of $G$ by $v_{1}, \ldots, v_{n}$. The number of edges of $G$ is denoted by $m$. For each arc $\vec{e}=\left(v_{i}, v_{j}\right)$ of $\vec{G}$, we associate $\vec{e}$ to a vector $x_{e} \in\left(\mathbb{Z}_{p}^{n}\right)_{0}$ in which the $i^{t h}$-entry is equal to $f(e)(\bmod p)$, the $j^{\text {th }}$-entry is equal to $-f(e)(\bmod p)$ and all the remaining entries are equal to $0(\bmod p)$.

Let us consider the following statements.
(a) For each $\mathbb{Z}_{p}$-boundary $\beta$, there is an $f$-weighted $\beta$-orientation of $G$.
(b) For each $\mathbb{Z}_{p}$-boundary $\beta$ there is a vector $\left(a_{e}\right)_{e \in E} \in\{-1,1\}^{m}$, such that $\sum_{e \in E} a_{e} x_{e} \equiv \beta(\bmod p)$.
(c) For each $\mathbb{Z}_{p}$-boundary $\beta$ there is a vector $\left(a_{e}\right)_{e \in E} \in\{0,1\}^{m}$ such that $\sum_{e \in E} 2 a_{e} x_{e} \equiv \beta(\bmod p)$.

Clearly, a is equivalent to b . We now claim that b is equivalent to c. To see this, simply do the following for each $\operatorname{arc} \vec{e}=\left(v_{i}, v_{j}\right)$ of
$\vec{G}$ : add $f(e)$ to the $j^{\text {th }}$-entry of $x_{e}$ and to $\beta\left(v_{j}\right)$, and subtract $f(e)$ from the $i^{t h}$-entry of $x_{e}$ and from $\beta\left(v_{i}\right)$. To deduce c from Corollary 5 , what is left is to show that $\left\{a_{e}: e \in E\right\}$ can be decomposed into sufficiently many linear bases of $\left(\mathbb{Z}_{p}^{n}\right)_{0}$. This follows from the fact that $G$ is $(8(p-1)(3 p-4)+2 p-4)$-edge-connected (and therefore contains $4(p-1)(3 p-4)+p-2$ edge-disjoint spanning trees) and that the set of vectors $a_{e}$ corresponding to the edges of a spanning tree of $G$ forms a linear basis of $\left(\mathbb{Z}_{p}^{n}\right)_{0}$ (see [12]).

A second proof consists in mimicking the proof of Theorem 3 (it turns out to give a better bound for the edge-connectivity of $G$ ).

Second proof of Theorem 7. As before, all values and operations are considered modulo $p$. We can assume without loss of generality that $f(E) \in\left\{1,2, \ldots, \frac{p-1}{2}\right\}$, since otherwise we can replace the value $f(e)$ of an edge $e$ by $-f(e)$, without changing the problem.

We prove the result by induction on the number of vertices of $G$. The result is trivial if $G$ contains only one vertex, so assume that $G$ has at least two vertices.

For any $1 \leqslant i \leqslant k$, let $E_{i}$ be the set of edges $e \in E$ with $f(e)=i$, and let $G_{i}=\left(V, E_{i}\right)$. Since $G$ is $(6 p-8)(p-1)$-edge-connected, $G$ has minimum degree at least $(6 p-8)(p-1)$ and then average degree at least $(6 p-8)(p-1)$. As a consequence, there exists $i$ such that $G_{i}$ has average degree at least $12 p-16$. By Lemma 15 , since $\frac{12 p-16}{4}+1=3 p-3$, $G_{i}$ has an induced subgraph $H=(X, F)$ with at least two vertices such that $H$ is $(3 p-3)$-edge-connected. Let $G / X$ be the graph obtained from $G$ by contracting $X$ into a single vertex $x$ (and removing possible loops). Since $H$ contains more than one vertex, $G / X$ has less vertices than $G$ (note that possibly, $X=V$ and in this case $G / X$ consists of the single vertex $x)$. Since $G$ is $(6 p-8)(p-1)$-edge-connected, $G / X$ is also $(6 p-8)(p-1)$-edge-connected. Hence by the induction hypothesis it has an $f$-weighted $\beta$-orientation, where we consider the restriction of $f$ to the edge-set of $G / X$, and we define $\beta(x)=\beta(X)$. Note that this orientation corresponds to an orientation of all the edges of $G$ with at most one endpoint in $X$.

We now orient arbitrarily the edges of $G[X]$ not in $F$ (the edge-set of $H$ ), and update the values of the $\mathbb{Z}_{p}$-boundary $\beta$ accordingly (i.e. for each $v \in X$, we subtract from $\beta(v)$ the contribution of the arcs that were already oriented). It is easy to see that as the original $\beta$ was a boundary, the new $\beta$ is indeed a boundary. Finally, since all the edges of $H$ have the same weight, and since $H$ is (3p-3)-edge-connected,
it follows from Corollary 10 that $H$ has an $f$-weighted $\beta$-orientation (with respect to the updated boundary $\beta$ ). The orientations combine into an $f$-weighted $\beta$-orientation of $G$, as desired.

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