Characteristic polynomials of production matrices for geometric graphs

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Abstract

An $n \times n$ production matrix for a class of geometric graphs has the property that the numbers of these geometric graphs on up to n vertices can be read off from the powers of the matrix. Recently, we obtained such production matrices for non-crossing geometric graphs on point sets in convex position [6]. In this note, we determine the characteristic polynomials of these matrices. Then, the Cayley-Hamilton theorem implies relations among the numbers of geometric graphs with different numbers of vertices. Further, relations between characteristic polynomials of production matrices for geometric graphs and Fibonacci numbers are revealed.

1 Introduction

A *geometric graph* on a point set $\mathcal{S}$ is a graph with vertex set $\mathcal{S}$ whose edges are straight-line segments with endpoints in $\mathcal{S}$. It is called *non-crossing* if no two edges intersect except at common endpoints. Here, we consider non-crossing geometric graphs on sets $\mathcal{S}$ of n points in convex position for the following graph classes: triangulations, matchings, spanning trees, forests, spanning paths, and all geometric graphs on n vertices. The numbers of these graphs are well known, see for instance the work of Flajolet and Noy [4]. Recently, in [6], we counted such geometric graphs by using an $n \times n$ matrix A_n, called *production matrix*, associated to the graph class. The numbers of these graphs on a certain number of vertices are then given by (a column of) powers of A_n. In order to derive a production matrix, first the graphs on $i \leq n$ vertices are partitioned according to the degree of a specified root vertex. Each part is counted in the elements of an n-element integer vector $\vec{v}^i$, and hence the sum of the elements gives the number of geometric graphs on i vertices. The *production matrix* A_n is such that $\vec{v}^{i+1} = A_n \vec{v}^i = A_n^{i+1-c} \vec{v}^c$, when starting with a vector $\vec{v}^c$ for a constant number of vertices, which will usually be $(1, 0, \ldots, 0)^\top$. To find the matrix A_n,

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$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$
(a) Triangulations	(b) Matchings	(c) Spanning trees
$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$
(d) Forests	(e) Geometric graphs	(f) Paths

Figure 1: Production matrices for six different graph classes, for $n = 6$. Matrix (f) is for paths on at most $n/2$ points.

the graphs are implicitly arranged in a tree structure (called *generating tree*), s.t., for each graph on i vertices and with root degree j, the number of its descendants on $i+1$ vertices with root degree ℓ (for each ℓ) is known. Generating trees are the basis of the *ECO method* [1], and have been used to obtain matrix representations for combinatorial objects [3, 8]. Here we omit how the production matrices for geometric graphs are obtained, and only refer to works by Hurtado and Noy [7] for a generating tree of triangulations, and by Hernando et al. [5] for a generating tree of spanning trees; and also to [6]. Figure 1 shows the obtained production matrices for the studied graph classes, for $n = 6$. The matrix for triangulations was well-known, see e.g. [3, 8]. All matrices except for the matrix for paths are upper Hessenberg matrices. The different structure of the production matrix for paths, which is formed by four blocks, is due to a necessary distinction between paths with root vertex begin an endpoint of a path or an interior point. Also, for paths, the degree of the root vertex is defined in a different way, based on visibility (we omit the definition here). For all other graph classes, the degree is the number of edges incident to the root vertex.

The aim of the present work is to analyze these matrices by finding their characteristic polynomials. Previously, these polynomials were only known for triangulations [2], matchings, and spanning trees [6]. Here we determine the characteristic polynomial of the production matrices for geometric graphs, for forests, and for paths, solving the problem which was left open in [6]. An application of the characteristic polynomial of a production matrix follows from the Cayley-Hamilton theorem, which then implies a relation among the numbers of graphs with given root vertex degree. For example, via the production matrix for triangulations and $t_n(\lambda)$ one obtains the relation $\sum_{j=0}^{\lceil n/2 \rceil} \binom{n-j+1}{j}(-1)^{n+j} C_{n-j} = 0$

(a) triangulations	$t_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k} (-1)^{n+k} \lambda^{n-k}$
(b) matchings	$m_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \sum_{j=0}^{n-k} \binom{n-k+1}{k} \binom{n-k}{j} (-1)^{n+k} \lambda^j$
(c) spanning trees	$s_n(\lambda) = \sum_{k=0}^{n} \binom{2k+2}{n-k} \lambda^k (-1)^k$
(d) forests	$f_n(\lambda) = \sum_{k=0}^{n} \lambda^k (-1)^k \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell}{2\ell+j-n} \binom{2\ell+j-n}{k+\ell+j-n}$
(e) geometric graphs	$g_n(\lambda) = 2^{n-1} + \sum_{k=1}^{n} \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-j}{k} 2^{n-k} (-1)^k \lambda^k$
(f) paths	$p_n(\lambda) = \sum_{k=0}^{n/2-1} \binom{n/2-1}{k} (-1)^{n/2+k+1} \lambda^k + \sum_{k=0}^{n/2} 2 \frac{n-k}{n} \binom{n/2}{k} (-1)^{n/2+k} \lambda^{n/2+k}$

Figure 2: The characteristic polynomials of $n \times n$ production matrices for several graph classes.[2]

for the Catalan numbers C_i; we only refer to [6] for the full example.

Figure 2 shows the characteristic polynomials of the production matrices for the six studied graph classes. We give an outline of the proofs for geometric graphs and for forests in the following two sections.

2 Geometric graphs

Theorem 2.1 *The characteristic polynomial $g_n(\lambda)$ of the $n \times n$ production matrix G_n of geometric graphs is*

$$g_n(\lambda) = 2^{n-1} + \sum_{k=1}^{n} \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-j}{k} 2^{n-k} (-1)^k \lambda^k.$$

The proof is by induction on n, using the following two lemmas. First, we develop the determinant of $G_n - \lambda I_n$, where I_n is the $n \times n$ identity matrix, to obtain a recurrence equation for $g_n(\lambda)$, and then to make induction work, we need to show a binomial identity.

Lemma 2.2 *The characteristic polynomial $g_n(\lambda)$ of the matrix G_n satisfies the recurrence relation*

$$g_n(\lambda) = (2 - \lambda) g_{n-1}(\lambda) - 2\lambda g_{n-2}(\lambda). \tag{1}$$

Lemma 2.3 *For $\ell \geq 2$ and $n \in \mathbb{N}$,*

$$\sum_{j=0}^{\ell-2} \left(\binom{\ell-2}{j} \binom{n-2-j}{\ell-1} - \binom{\ell-2}{j-1} \binom{n-1-j}{\ell-1} \right) = \binom{n-\ell}{\ell-1}.$$

[2]The polynomials are also defined for $\lambda = 0$ by setting the indeterminate form $0^0 := 1$.

3 Forests

Theorem 3.1 *The characteristic polynomial $f_n(\lambda)$ of the $n \times n$ production matrix F_n of forests is*

$$f_n(\lambda) = \sum_{k=0}^{n} \lambda^k (-1)^k \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell}{2\ell + j - n} \binom{2\ell + j - n}{k + \ell + j - n}.$$

Lemma 3.2 *The characteristic polynomial $f_n(\lambda)$ of the matrix F_n satisfies the recurrence relation*

$$f_n(\lambda) = (1 - \lambda) f_{n-1}(\lambda) + (1 - 2\lambda) f_{n-2}(\lambda) - \lambda f_{n-3}(\lambda). \tag{2}$$

To prove Theorem 3.1, we apply Riordan arrays as described for instance in [8]. Consider the infinite matrix

$$M = \begin{pmatrix} 1-\lambda & 1-2\lambda & -\lambda & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\vec{w}^0$ be the vector $(1, 0, 0, \dots)^\top$, and let $\vec{w}^i = (f_i(\lambda), f_{i-1}(\lambda), f_{i-2}(\lambda), \dots, f_1(\lambda), 1, 0, 0, \dots)^\top$. Then $\vec{w}^{i+1} = M \cdot \vec{w}^i$. Note that $\vec{w}^n$ is the first column of M^n. Using the notation from [8], we have that the Z-sequence is $\{1 - \lambda, 1 - 2\lambda, -\lambda, 0, \dots\}$ and the A-sequence is $\{1, 0, \dots\}$; it follows that $h(t) = 1$ and

$$d(t) = \frac{d_0}{1 - t \cdot Z(th(t))} = \frac{1}{1 - t(1 - \lambda + (1 - 2\lambda)t - \lambda t^2)} =$$

$$= \sum_{k=0}^{\infty} t^k (t+1)^k (1 - \lambda - \lambda t)^k;$$

the characteristic polynomial $f_n(\lambda)$ is then the coefficient of t^n in this expression.

3.1 Relation to Fibonacci numbers

Denote with $Fib(n)$ the n-th Fibonacci number, where $Fib(1) = Fib(2) = 1$, and $Fib(n) = Fib(n-1) + Fib(n-2)$ for $n > 2$.

Corollary 3.3 *The determinant of F_n is the Fibonacci number $Fib(n+1)$.*

Since the determinant of F_n is $f_n(0)$, Corrollary 3.3 follows immediately from Equation (2) and $f_0(0) = f_1(0) = 1$. Another family of matrices whose determinants are Fibonacci numbers can be found for instance in [9]. We remark that when we substitute $\lambda = 0$ in the formula of Theorem 3.1 we also obtain, after simplification, the following well known identity.

Corollary 3.4 *The Fibonacci numbers satisfy the equation*

$$Fib(n+1) = \sum_{k=0}^{n} \binom{k}{n-k}.$$

Also, the characteristic polynomial $f_n(\lambda)$ can be expressed recursively using Fibonacci numbers.

Corollary 3.5

$$f_n(\lambda) + \lambda f_{n-1}(\lambda) = Fib(n+1) - \sum_{k=2}^{n} Fib(k+2)\lambda f_{n-k}(\lambda).$$

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