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Co-Bipartite Neighborhood Edge Elimination Orderings ⋆

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Abstract

In SODA 2001, Raghavan and Spinrad introduced robust algorithms as a way to solve hard combinatorial graph problems in polynomial time even when the input graph falls slightly outside a graph class for which a polynomial-time algorithm exists. As a leading example, the Maximum Clique problem on unit disk graphs (intersection graphs of unit disks in the plane) was shown to have a robust, polynomial-time algorithm by proving that such graphs admit a co-bipartite neighborhood edge elimination ordering (CNEEO). This begs the question whether other graph classes also admit a CNEEO.

In this paper, we answer this question positively, and identify many graph classes that admit a CNEEO, including several graph classes for which no polynomial-time recognition algorithm exists (unless P=NP). As a consequence, we obtain robust, polynomial-time algorithms for Maximum Clique on all identified graph classes.

We also prove some negative results, and identify graph classes that do not admit a CNEEO. This implies an almost-perfect dichotomy for subclasses of perfect graphs.

Keywords: maximum clique, polynomial-time algorithm, robust algorithm, graph classes, edge ordering

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1 Introduction

Many hard combinatorial graph problems can be solved in polynomial time only if the input graph belongs to a structured graph class (unless P=NP), such as planar graphs, perfect graphs, or geometric intersection graphs. Typically, these efficient algorithms rely on deep structural properties of the graph class or on a geometric or structural representation of the graph. Therefore, we first need to run an algorithm that recognizes whether the input graph actually belongs to the graph class or finds a suitable representation. This step, however, puts the approach at a distinct disadvantage, because recognizing whether the input graph belongs to the graph class could be NP-hard or could only have a very complicated polynomial-time algorithm.

To alleviate this disadvantage, Raghavan and Spinrad [16,17] introduced the notion of robust algorithms. A *robust algorithm* for a problem Π on a graph class $\mathcal{C}$ must always solve Π if the input graph belongs to $\mathcal{C}$. Otherwise, the algorithm either solves Π or (correctly) returns that the input graph does not belong to $\mathcal{C}$. An additional advantage is that the algorithm might still work if the input graph does not belong to the class $\mathcal{C}$.

As the main example of this concept, Raghavan and Spinrad showed that Maximum Clique has a polynomial-time, robust algorithm on unit disk graphs. Recall that a graph is a unit disk graph if it has a representation: a set of equal-sized disks in the plane where each vertex can be made to correspond to a disk and there is an edge between two vertices if and only if the corresponding disks intersect. Maximum Clique can be solved in polynomial time on unit disk graphs if a representation is given [5]. However, finding a representation is NP-hard [3] and famously not known to be in NP [15]. To give a robust algorithm for Maximum Clique on unit disk graphs, and circumvent the challenge of finding a representation, Raghavan and Spinrad introduced the notion of a co-bipartite neighborhood edge elimination ordering.

Definition 1.1 Let $\sigma = e_1, e_2, \ldots, e_m$ be an ordering of all m edges of an undirected graph G. Let $G_\sigma[i]$ be the subgraph of G with edge set $\{e_i, e_{i+1}, \ldots, e_m\}$. Then σ is a *co-bipartite neighborhood edge elimination ordering (CNEEO)* if, for each $e_i = u_i v_i$, the common neighborhood of u_i and v_i in $G_\sigma[i]$ (i.e.

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$N_{G_\sigma[i]}[u_i] \cap N_{G_\sigma[i]}[v_i]$) induces a co-bipartite graph in G.

Theorem 1.2 (Raghavan & Spinrad [16,17]) *If each graph in a class $\mathcal{C}$ admits a CNEEO, then there is a robust, polynomial-time algorithm for* Maximum Clique *on $\mathcal{C}$.*

Theorem 1.3 (Raghavan & Spinrad [16,17]) *Unit disk graphs admit a CNEEO.*

The success of robust algorithms for Maximum Clique on unit disk graphs makes it natural to ask whether Maximum Clique also has polynomial-time, robust algorithms on other graph classes. In order to make progress on this question, Theorem 1.2 suggests we should find further graph classes that admit a CNEEO. However, we are not aware of any work in this direction.

Our Results

In this paper, we identify many graph classes that also admit a CNEEO. This includes several graph classes for which the problem of recognizing the graph class is NP-hard. As a consequence of our results and Theorem 1.2, there are polynomial-time, robust algorithms for Maximum Clique on all identified graph classes. We highlight our main results.

First, we consider intersection graphs of axis-parallel squares of arbitrary size. This class is NP-hard to recognize [12] and a polynomial-time algorithm for Maximum Clique is known when a representation is given [8].

Theorem 1.4 *Intersection graphs of squares admit a CNEEO.*

We also consider intersection graphs of increasing semi-squares. These are semi-squares (the lower part of a square that is cut at the diagonal from top-left to bottom-right) whose y-coordinate of their base increases with their size. This class generalizes the class of c-max-tolerance graphs [13], which have a representation where each vertex v corresponds to an interval I_v on the real line and a tolerance $t_v = c \cdot |I_v|$, and there is an edge between u, v if and only if $|I_u \cap I_v| \geq \max\{t_u, t_v\}$. Cliques in c-max-tolerance graphs have applications in bioinformatics [10,13]. We are not aware of any result on the recognition of c-max-tolerance graphs.

Theorem 1.5 *Intersection graphs of increasing semi-squares admit a CNEEO. Hence, for any $c \in [0, 1]$, c-max-tolerance graphs admit a CNEEO.*

Second, we consider (Π_A, Π_B)-graphs for graph properties Π_A, Π_B, which admit a partition (A, B) of their vertex set such that $A \in \Pi_A, B \in \Pi_B$. Well-known examples of (Π_A, Π_B)-graphs include 2-subcolorable graphs (both A and B should induce a cluster graph: a disjoint union of cliques) and monopolar graphs (A induces a cluster graph, B an independent set). Monopolar graphs have applications in bioinformatics [4]. Both aforementioned classes are NP-hard to recognize [1,7]. Let Π^* denote the graph property 'is a chordal graph'; (Π^*, Π^*)-graphs are NP-hard to recognize [7].

Theorem 1.6 *The class of* (Π^*, Π^*)*-graphs admits a CNEEO.*

This result covers 2-subcolorable, monopolar graphs, and $(2, 2)$-colorable graphs. A graph is (p, q)*-colorable* if its vertex set can be partitioned into $p+q$ parts of which p each induce a clique and q each induce an independent set.

Third, we consider graph classes in which the neighborhood of a vertex is co-bipartite. For any integer k, k-simplicial graphs G admit an ordering $v_1, \ldots, v_n$ of $V(G)$ such that, for each vertex v_i, the subset of neighbors of v_i contained in $\{v_j \mid j > i\}$ can be partitioned into k sets $S_1, \ldots, S_k$ that each induce a clique [9]. It is easy to see that Maximum Clique can be solved in polynomial time if $k = 2$, but is NP-hard if $k = 3$ (by reducing from Independent Set on 2-subdivisions of general graphs).

Theorem 1.7 2*-simplicial graphs admit a CNEEO.*

Fourth, we consider several broadly studied subclasses of perfect graphs, as well as planar graphs. This demonstrates the versatility of CNEEOs, even though Maximum Clique is well known to be solvable in polynomial time on all of these classes, as is the recognition problem [6].

Theorem 1.8 *Chordal graphs, proper tolerance graphs,* 2*-threshold graphs, and bipartite graphs all admit a CNEEO.*

Theorem 1.9 5*-degenerate graphs admit a CNEEO. So do planar graphs.*

Finally, we consider several graph classes that do not admit a CNEEO.

Theorem 1.10 *The complete union of any three graphs that are not co-bipartite does not admit a CNEEO. In particular, for any* $i, j, k \geq 3$, $K_{i,j,k}$ *does not admit a CNEEO, and for any odd* $i, j, k \geq 5$*, the complete union of* $\overline{C_i}$, $\overline{C_j}$, $\overline{C_k}$ *does not admit a CNEEO.*

Corollary 1.11 *The following subclasses of perfect graphs do not admit a CNEEO: cographs, co-chordal graphs, and Gallai graphs. The following graph classes also do not admit a CNEEO: claw-free graphs,* 6*-degenerate graphs,* 3*-subcolorable graphs, and* (p, q)*-colorable graphs for* $p \geq 3$ *or* $q \geq 3$.

This result combined with Theorem 1.8 implies an almost-perfect dichotomy for subclasses of perfect graphs.

Discussion

This paper establishes CNEEOs as a powerful method to identify robust and polynomial-time algorithms for Maximum Clique on graph classes. This makes CNEEOs among the first approaches to try when determining the computational complexity of Maximum Clique on a graph class.

We remark that another useful approach to obtaining a polynomial-time algorithm for Maximum Clique on many graph classes is to bound the number of maximal cliques of the graph. Since all maximal cliques in a graph can be enumerated with $O(nm)$ or $O(n^{\omega})$ time delay [18,14], where ω denotes the matrix multiplication constant, such a bound immediately implies a polynomial-time algorithm for Maximum Clique. There are several trade-offs between using this approach and the CNEEO-based approach. We refer to the full version of the paper for a detailed comparison and discussion.

We now pose several open questions. One of the more famous questions surrounding Maximum Clique is its complexity on disk graphs: intersection graphs of arbitrary disks in the plane (see e.g. [2]). Given the work in this paper and Theorem 1.3, the following question is reasonable: do disk graphs admit a CNEEO? Natural approaches of selecting a next edge in the ordering, such as smallest intersection area or closest/furthest centers of the disks corresponding to the endpoint of the edge, all seem to fail.

Another interesting question is whether intersection graphs of arbitrary rectangles in the plane admit a CNEEO. Note that Maximum Clique can be solved in polynomial time when a representation is given [8], but finding a representation is NP-hard [19,11].

Finally, we ask whether intersection graphs of general semi-squares admit a CNEEO, which would generalize Theorem 1.5. These graphs have a polynomial number of maximal cliques and are NP-hard to recognize [10], but CNEEO's be faster than the maximal-clique-enumeration approach.

2 Proof Sketches

Complete proofs of all results are available in the full version of the paper, which is to appear on the arXiv.

Proof sketch of Theorem 1.5. The order is determined iteratively as follows. Let $E' \subseteq E(G)$ be the set of edges not yet chosen and let $G' =$

$(V(G), E')$. Let u be a vertex with degree at least 1 in G' for which S_u is smallest. Let v be a vertex with $uv \in E'$ for which the cathetus of S_v is rightmost among all neighbors of u in G'. Then $e = uv$ is the next edge. $\square$

Proof sketch of Theorem 1.6. Chordal graphs admit a CNEEO by using a perfect elimination ordering and repeatedly picking all edges incident on the next simplicial vertex. For a (Π^*, Π^*)-graph G with partition (A, B) of $V(G)$ such that $G[A], G[B] \in \Pi^*$, we consider the vertices of A ordered by a perfect elimination order of $G[A]$. For the next $a \in A$, we consider the edges of a to B in the order determined by a perfect elimination order of $G[B]$. This ensures a co-bipartite common neighborhood. After dealing with all edges between A and B, a chordal graph remains, which admits a CNEEO as mentioned. $\square$

Proof of Theorem 1.9. The order is determined iteratively as follows. Let $E' \subseteq E(G)$ be the set of edges not yet chosen and let $G' = (V(G), E')$. Let $v \in V(G)$ have degree at most 5 in G'. If, in G', v has a neighbor w that is adjacent to at most two neighbors of v, then pick vw as the next edge; as v and w have at most two common neighbors in G', this must induce a co-bipartite graph in G. If v does not have such a neighbor w, then let X denote the set of non-edges in the subgraph of G' induced by the neighborhood of v in G'. If $|X| \geq 3$, then the pigeonhole principle and the fact that v has at most 5 neighbors in G' implies we are in the preceding case. If v has at most 4 neighbors in G', then $|X| = 0$ or we are in the preceding case. It follows that v has a neighbor u in G' that is not incident on a non-edge of X. Pick uv as the next edge. The common neighborhood of u and v in G' induces either a clique, a diamond, or a C_4 in G', and thus in G. $\square$

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