

# NEAR-PERFECT CLIQUE-FACTORS IN SPARSE PSEUDORANDOM GRAPHS

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**ABSTRACT.** We prove that, for any  $t \geq 3$ , there exists a constant  $c = c(t) > 0$  such that any  $d$ -regular  $n$ -vertex graph with the second largest eigenvalue in absolute value  $\lambda$  satisfying  $\lambda \leq cd^{t-1}/n^{t-2}$  contains vertex-disjoint copies of  $K_t$  covering all but at most  $n^{1-1/(8t^4)}$  vertices. This provides further support for the conjecture of Krivelevich, Sudakov and Szábo [*Triangle factors in sparse pseudo-random graphs*, *Combinatorica* **24** (2004), pp. 403–426] that  $(n, d, \lambda)$ -graphs with  $n \in 3\mathbb{N}$  and  $\lambda \leq cd^2/n$  for a suitably small absolute constant  $c > 0$  contain triangle-factors.

## 1. INTRODUCTION

The study of conditions under which certain spanning or almost spanning structures are forced in random or pseudorandom graphs is one of the central topics in extremal graph theory and in random graphs.

An  $(n, d, \lambda)$ -graph is an  $n$ -vertex  $d$ -regular graph whose second largest eigenvalue in absolute value is at most  $\lambda$ . Graphs with  $\lambda \ll d$  are considered to be *pseudorandom*, i.e., they behave in certain respects as random graphs do; for example, the edge count between ‘not too small’ vertex subsets is close to what one sees in random graphs of the same density. As usual, let  $e(A, B) = e_G(A, B)$  denote the number of pairs  $(a, b) \in A \times B$  so that  $ab$  is an edge of  $G$  (note that edges in  $A \cap B$  are counted twice). The following result makes what we discussed above precise.

**Theorem 1.1** (Expander mixing lemma [3]). *If  $G$  is an  $(n, d, \lambda)$ -graph and  $A, B \subseteq V(G)$ , then*

$$\left| e(A, B) - \frac{d}{n}|A||B| \right| < \lambda \sqrt{|A||B|}. \quad (1)$$

As starting points to the extensive literature on pseudorandom graphs, the reader is referred to, e.g., [17], [9] or [7, Chapter 9].

It is an interesting problem to understand optimal or asymptotically optimal conditions on the parameter  $\lambda$  in terms of  $d$  and  $n$  that force an  $(n, d, \lambda)$ -graph to possess a desired property. To demonstrate the optimality of a condition, one needs to show the existence of an  $(n, d, \lambda)$ -graph that certifies that the condition is indeed optimal.

Unfortunately, there are very few examples certifying optimality. A celebrated example is due to Alon, who showed [4] that there are  $(n, d, \lambda)$ -graphs that are  $K_3$ -free and yet satisfy  $\lambda = cd^2/n$  for some absolute constant  $c > 0$ . This is in contrast with the fact that, as it follows easily from the expander mixing lemma above, for, say,  $\lambda \leq 0.1d^2/n$ , any  $(n, d, \lambda)$ -graph contains a triangle (in fact, every vertex lies in a triangle). It turns out that  $(n, d, \lambda)$ -graphs with  $\lambda = \Theta(d^2/n)$

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must satisfy  $d = \Omega(n^{2/3})$ . The construction of Alon [4] provides an example of the essentially sparsest possible  $K_3$ -free  $(n, d, \lambda)$ -graph with  $d = \Theta(n^{2/3})$  and  $\lambda = \Theta(n^{1/3})$ . The other known example is a generalization of this construction by Alon and Kahale [6] (see also [17, Section 3]) to graphs without odd cycles of length at most  $2\ell + 1$ .

Our focus here is on spanning or almost spanning structures in  $(n, d, \lambda)$ -graphs. One of the simplest spanning structures is that of a perfect matching. Alon, Krivelevich and Sudakov (see [17]) proved that  $(n, d, \lambda)$ -graphs with  $\lambda \leq d - 2$  and  $n$  even contain perfect matchings. *Factors* generalize perfect matchings: for a graph  $F$ , an  $F$ -factor in a graph  $G$  is a collection of vertex-disjoint copies of  $F$  in  $G$  whose vertex sets cover  $V(G)$  (this requires that  $v(G) := |V(G)|$  should be divisible by  $v(F)$ ). Motivated by the study of spanning structures in graphs, Krivelevich, Sudakov and Szabó [18] proved that  $(n, d, \lambda)$ -graphs with  $\lambda = o(d^3/(n^2 \log n))$  contain a triangle-factor if  $3 \mid n$ .

A *fractional triangle-factor* in a graph  $G = (V, E)$  is a non-negative weight function  $f$  on the set  $\mathcal{K}_3(G)$  of all triangles  $T$  of  $G$ , such that, for every  $v \in V$ , we have  $\sum_{T: v \in V(T)} f(T) = 1$ . Krivelevich, Sudakov and Szabó further proved [18] that  $(n, d, \lambda)$ -graphs with  $\lambda \leq 0.1d^2/n$  admit a fractional triangle-factor. Moreover, they conjectured the following.

**Conjecture 1.2** (Conjecture 7.1 in [18]). *There exists an absolute constant  $c > 0$  such that if  $\lambda \leq cd^2/n$ , then every  $(n, d, \lambda)$ -graph  $G$  on  $n \in 3\mathbb{N}$  vertices has a triangle-factor.*

The  $t^{\text{th}}$  power  $H^t$  of a graph  $H$  is the graph on the vertex set  $V(H)$  where  $uv$  ( $u \neq v$ ) is an edge if there is a  $u$ - $v$ -path of length at most  $t$  in  $H$ . Since the  $(t-1)^{\text{st}}$  power of a Hamilton cycle contains a  $K_t$ -factor if  $t \mid n$ , powers of Hamilton cycles are also of interest when investigating clique-factors.

Allen, Böttcher, Hàn and two of the authors [2] proved that, if  $\lambda = o(d^{3t/2}n^{1-3t/2})$  and  $t \geq 3$ , then any  $(n, d, \lambda)$ -graph contains the  $t^{\text{th}}$  power of the Hamilton cycle (and thus a  $K_{t+1}$ -factor if  $(t+1) \mid n$ ). In the case  $t = 2$ , it was further proved in [2] that the condition  $\lambda = o(d^{5/2}/n^{3/2})$  suffices to guarantee squares of Hamilton cycles, and thus  $K_3$ -factors, improving over the aforementioned result of Krivelevich, Sudakov and Szabó. For very recent progress, due to Nenadov [20], see Section 7 below.

The construction of Alon of  $K_3$ -free  $(n, d, \lambda)$ -graphs shows that the condition on  $\lambda$  in Conjecture 1.2 cannot be weakened. The result from [18] on the existence of fractional triangle-factors supports Conjecture 1.2. As a further evidence in support of that conjecture we prove here the following result.

**Theorem 1.3** (Main result<sup>1,2</sup>). *For any  $t \geq 3$  there is  $n_0 > 0$  for which the following holds. Every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$  and  $\lambda \leq (1/(50t4^{t-2}))d^{t-1}/n^{t-2}$  contains vertex-disjoint copies of  $K_t$  covering all but at most  $n^{1-1/(8t^4)}$  vertices of  $G$ .*

We remark that, under the condition  $\lambda \leq cd^{t-1}/n^{t-2}$  for some appropriate  $c = c(t) > 0$ , Krivelevich, Sudakov and Szabó [18] proved that any  $(n, d, \lambda)$ -graph contains a fractional  $K_t$ -factor.

A naïve approach to proving Theorem 1.3 is to pick cliques  $K_t$  one after another, each vertex-disjoint from the previous ones, by appealing to the pseudorandomness of  $G$  via the expander

<sup>1</sup>This result appears in an extended abstract in the Proceedings of Discrete Mathematics Days 2018 (Sevilla) [11].

<sup>2</sup>Theorem 1.3 answers a question raised by Nenadov (see of [20, Concluding remarks]).

mixing lemma. However, even for triangles, if  $\lambda = cd^2/n$ , then all what one gets this way is that  $G$  has  $(1-c)n/3$  vertex-disjoint triangles: one can see that a set of  $cn$  vertices in  $G$  induces a graph of average degree roughly  $cd$ , but the condition on  $\lambda$  and the expander mixing lemma do not guarantee that sets of size roughly  $cd$  contain an edge, and hence we do not know whether  $cn$  vertices necessarily span a triangle. Thus our naïve greedy approach will get stuck leaving  $cn$  vertices uncovered. What our result establishes is that, even for some absolute constant  $c > 0$ , we can cover all but  $o(n)$  vertices of  $G$  by vertex-disjoint copies of  $K_3$ . Moreover,  $o(n)$  can be taken to be of the form  $n^{1-\varepsilon}$  for some  $\varepsilon > 0$ . We have restricted ourselves to triangles in this paragraph, but a similar reasoning applies to general cliques  $K_t$  as well.

Now let  $p = d/n$  and suppose  $G = (V, E)$  is an  $(n, d, \lambda)$ -graph with  $\lambda \leq cd^2/n$ . Inequality (1) implies that

$$\left| \frac{e(A, B)}{|A||B|} - p \right| < \frac{cp^2n}{\sqrt{|A||B|}} \leq c^{1/2} \quad (2)$$

for all  $A, B \subseteq V$  with  $|A|, |B| \geq c^{1/2}n$ . Let us now focus on the case in which  $d$  is linear in  $n$ , that is,  $p = d/n$  is a constant independent of  $n$ . The powerful blow-up lemma of Komlós, Sárközy and Szemerédi [14] implies that, if  $c$  is small enough in comparison with  $p$  and  $1/t$ , then *any* graph  $G = (V, E)$  on  $n$  vertices with minimum degree at least  $pn$  that satisfies (2) contains a  $K_t$ -factor as long as  $t \mid n$ . Thus, Conjecture 1.2 holds for dense graphs.

We remark that the blow-up lemmas for sparse graphs developed recently by Allen, Böttcher, Hàn and two of the authors [1] provide bounds on  $\lambda$  to establish the existence of  $K_t$ -factors, but those bounds are worse than those from [2] discussed above.

Throughout the paper floor and ceiling signs are omitted for the sake of readability. For graph theory terminology and notation we refer the reader to Bollobás [8].

This paper is organized as follows. In Section 2 we collect some of the necessary tools and prove auxiliary results. In Section 3 we provide an overview of the proof of Theorem 1.3, which splits into two cases (the ‘dense’ and ‘sparse’ cases). We deal with these two cases in Sections 4 and 5, separately, and then prove Theorem 1.3 in Section 6. Finally we give some concluding remarks in Section 7.

## 2. TOOLS AND AUXILIARY RESULTS

**2.1. Probabilistic techniques.** We shall use the following theorem of Kostochka and Rödl [15] (see also Rödl [21] and Alon and Spencer [7, Theorem 4.7.1]), which asserts the existence of an almost perfect matching in ‘pseudorandom’ hypergraphs.

**Theorem 2.1.** *Let integers  $t \geq 3$  and  $k \geq 8$  and real numbers  $\delta'$  and  $\gamma$  with  $0 < \delta', \gamma < 1$  be fixed. Then there exists  $D_0$  such that the following holds for  $D \geq D_0$ . Let  $H$  be a  $t$ -uniform hypergraph on a set  $V$  of  $n$  vertices such that*

- (1) *for all vertices  $v \in V$ , we have  $D - k\sqrt{D \log D} \leq \deg_H(v) \leq D$  and*
- (2) *for any two distinct vertices  $u$  and  $v \in V$ , we have  $\deg_H(u, v) \leq C < D^{1-\gamma}$ .*

*Then  $H$  contains a matching covering all but  $O(n(C/D)^{(1-\delta')/(t-1)})$  vertices.*

We shall use the following concentration results.

**Theorem 2.2** (Chernoff bounds [13, Corollary 2.4 and Theorems 2.8 and 2.10]). *Suppose  $X$  is a sum of a collection of independent Bernoulli random variables. Then, for  $\delta \in (0, 3/2)$ , we*

have

$$\mathbb{P}(X > (1 + \delta)\mathbb{E}X) < e^{-\delta^2\mathbb{E}X/3} \quad \text{and} \quad \mathbb{P}(X < (1 - \delta)\mathbb{E}X) < e^{-\delta^2\mathbb{E}X/2}.$$

Moreover, for any  $t \geq 6\mathbb{E}X$ , we have

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq e^{-t}.$$

For a graph  $G = (V, E)$  we denote by  $G_p$  the spanning random subgraph of  $G$  in which each edge from  $E$  is included with probability  $p$ , independently of all other edges.

**Theorem 2.3** (Janson's inequality [13, Theorem 2.14]). *Let  $p \in (0, 1)$  be given and consider a family  $\{H_i\}_{i \in \mathcal{I}}$  of subgraphs of a graph  $G$ . For each  $i \in \mathcal{I}$ , let  $X_i$  denote the indicator random variable for the event that  $H_i \subseteq G_p$ . Write  $H_i \sim H_j$  for each ordered pair  $(i, j) \in \mathcal{I} \times \mathcal{I}$  such that  $E(H_i) \cap E(H_j) \neq \emptyset$ . Let  $X = \sum_{i \in \mathcal{I}} X_i$ . Then  $\mathbb{E}[X] = \sum_{i \in \mathcal{I}} p^{e(H_i)}$ . Furthermore, let*

$$\Delta = \sum_{H_i \sim H_j} \mathbb{E}[X_i X_j] = \sum_{H_i \sim H_j} p^{e(H_i) + e(H_j) - e(H_i \cap H_j)}. \quad (3)$$

Then, for any  $0 < \gamma < 1$ , we have

$$\mathbb{P}[X \leq (1 - \gamma)\mathbb{E}[X]] \leq \exp\left(-\frac{\gamma^2 \mathbb{E}[X]^2}{2\Delta}\right). \quad (4)$$

**2.2. Linear programming techniques.** We shall consider *weighted graphs*  $(G, w)$  where  $G = (V, E)$  is a graph and  $w: E \rightarrow [0, 1]$  is a function on its edges. If  $w \equiv 1$ , we identify  $(G, w)$  with  $G$ . For every vertex  $v \in V$ , we define its *weighted degree*  $\deg_w(v)$  to be  $\sum_{u \in N(v)} w(uv)$ .

Let  $\mathcal{K}_t(G)$  denote the set of all copies of  $K_t$  in  $G$ . A function  $f: \mathcal{K}_t(G) \rightarrow [0, 1]$  is called a *fractional  $K_t$ -factor* if  $\sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) = 1$  for every  $v \in V$ , and for every edge  $uv \in E(G)$  one has

$$\sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u, v\}} f(T) \leq w(uv). \quad (5)$$

A fractional  $K_t$ -factor in a weighted graph  $(G, w)$  thus ‘respects’ the weight function  $w$ . We remark that the definition of a fractional  $K_t$ -factor in a graph generalizes the definition of a fractional triangle-factor from the introduction in a straightforward way, and it is itself generalized by the above one for weighted graphs, since condition (5) is satisfied in the case of fractional  $K_t$ -factors in unweighted graphs because of our convention that  $w \equiv 1$  in that case.

We shall use the duality theorem of linear programming; see e.g. [19]. The maximum weight of a fractional  $K_t$ -matching in  $(G, w)$  is given by the following linear programme:

$$\begin{aligned} \max \quad & \sum_{T \in \mathcal{K}_t(G)} f(T) \\ \text{subject to} \quad & \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) \leq 1 \quad (\forall v) \\ & \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u, v\}} f(T) \leq w(uv) \quad (\forall uv \in E(G)) \\ & f(T) \geq 0 \quad (\forall T \in \mathcal{K}_t(G)) \end{aligned} \quad (6)$$

The dual of (6) is the following linear programme:

$$\begin{aligned}
\min \quad & \sum_{v \in V} g(v) + \sum_{uv \in E(G)} h(uv)w(uv) \\
\sum_{v \in V(T)} g(v) + \sum_{uv \in E(T)} h(uv) & \geq 1 \quad (\forall T \in \mathcal{K}_t(G)) \\
g(v) & \geq 0 \quad (\forall v) \\
h(uv) & \geq 0 \quad (\forall uv \in E(G))
\end{aligned} \tag{7}$$

Both linear programmes above are clearly feasible, and therefore the duality theorem tells us that both admit optimal solutions and that, moreover, these optimal values are equal. Given optimal solutions  $f$  and  $(g, h)$ , the complementary slackness conditions tell us that if  $g(v) > 0$  then the corresponding inequality  $\sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) \leq 1$  in the primal linear programme (6) holds with equality, i.e.,  $\sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) = 1$ .

Let  $t^*(G, w)$  be the optimum value of (6) and (7). For a collection  $\mathcal{F}$  of vertex-disjoint copies of  $K_t$  from  $\mathcal{K}_t(G)$ , let  $w(\mathcal{F}) := \sum_{T \in \mathcal{F}} \min_{uv \in E(T)} w(uv)$ . Furthermore, let  $t(G, w)$  be the maximum possible value of  $w(\mathcal{F})$  for such a collection  $\mathcal{F}$ . Write  $|g|$  for  $\sum_{v \in V} g(v)$  and  $|h|$  for  $\sum_{uv \in E(G)} h(uv)w(uv)$  (see the objective function in (7)).

The following proposition collects some useful properties of linear programmes (6) and (7). A variant for unweighted uniform hypergraphs and fractional matchings was first stated and proved by Krivelevich in [16, Proposition 2].

**Proposition 2.4.** *Let  $t \geq 3$  be given and let  $(G, w)$  be a weighted graph. Suppose  $G = (V, E)$ . Then the following hold.*

- (1)  $t^*(G, w) \geq t(G, w)$ .
- (2)  $t^*(G, w) \leq |V|/t$ . Furthermore, if  $t^*(G, w) = |V|/t$ , then  $(G, w)$  has a fractional  $K_t$ -factor.
- (3) If  $g: V \rightarrow \mathbb{R}_{\geq 0}$  and  $h: E \rightarrow \mathbb{R}_{\geq 0}$  form a feasible solution to (7), then for every subset  $U \subseteq V$  the functions  $g' := g \upharpoonright_U$  and  $h' := h \upharpoonright_{E \cap \binom{U}{2}}$  form a feasible solution to (7) with  $G[U]$  in place of  $G$ ; in particular we have  $|g'| + |h'| \geq t^*(G[U], w)$ .
- (4) If  $g: V \rightarrow \mathbb{R}_{\geq 0}$  and  $h: E \rightarrow \mathbb{R}_{\geq 0}$  form an optimal solution to (7), then  $t^*(G, w) \geq |V_1|/t$ , where  $V_1 := \{v \in V : g(v) > 0\}$ .

*Proof.* Let  $\mathcal{F}$  be a collection of vertex-disjoint copies of  $K_t$ . Let  $f(T) := \min_{uv \in E(T)} w(uv)$  for every  $T \in \mathcal{F}$  and  $f(T) := 0$  for every  $T \in \mathcal{K}_t(G) \setminus \mathcal{F}$ . Clearly  $f$  satisfies (6), whence (1) follows.

From  $\sum_v \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) \leq |V|$  and  $\sum_v \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) = t \sum_{T \in \mathcal{K}_t(G)} f(T)$ , assertion (2) follows immediately.

It is also clear that restricting a feasible solution of (7) to a subset  $U \subseteq V$ , we obtain functions for which the relevant inequalities hold. Since the value of a feasible solution of the dual (7) is always at least the value of a feasible solution of the primal programme (6) (if both are feasible, which is the case here), we obtain  $|g'| + |h'| \geq t^*(G[U], w)$ . Thus, assertion (3) follows.

From the complementary slackness conditions we know that, whenever  $g(v) > 0$ , we have  $\sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) = 1$ . This implies that

$$t^*(G, w) = \sum_{T \in \mathcal{K}_t(G)} f(T) = \frac{1}{t} \sum_{v \in V} \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) \geq \frac{1}{t} \sum_{v \in V_1} 1 = \frac{|V_1|}{t},$$

which proves (4). □

**2.3. Applications of linear programming to weighted graphs.** In this section we provide an auxiliary lemma that will help us verify later that certain weighted subgraphs of  $(n, d, \lambda)$ -graphs possess fractional  $K_t$ -factors. The main result of this subsection, Corollary 2.8, generalizes results from Krivelevich, Sudakov and Szabó [18, Section 5] to weighted graphs and fractional  $K_t$ -factors. Our proofs follow their proof strategy.

We first state a simple fact. Given  $\alpha \geq 0$  and a weighted graph  $(G, w)$ , we call an edge  $uv \in E(G)$   $\alpha$ -rich if  $w(uv) \geq 1 - \alpha$ . Similarly, we call a copy  $T$  of  $K_t$  in  $G$   $\alpha$ -rich if  $w(T) \geq 1 - \alpha$ , where  $w(T) := \min_{uv \in E(T)} w(uv)$ .

**Fact 2.5.** *Let  $(G, w)$  be a weighted  $d$ -regular graph. If  $\sum_{u \in N(v)} w(uv) \geq d(1 - \alpha^2)$  for a vertex  $v$ , then  $v$  is incident to at least  $(1 - \alpha)d$  many  $\alpha$ -rich edges.*

*Proof.* Let  $m$  be the number of non- $\alpha$ -rich edges incident to a vertex  $v$ . Then we have  $\sum_{u \in N(v)} w(uv) \leq d - m\alpha$ . If  $\sum_{u \in N(v)} w(uv) \geq d(1 - \alpha^2)$ , then  $d\alpha^2 \geq m\alpha$  and thus  $m \leq \alpha d$ .  $\square$

Let a weighted graph  $(G, w)$  with  $G = (V, E)$  be given. In what follows, we shall consider the spanning subgraph  $H = (V, F)$  of  $G$ , with  $F \subseteq E$  the set of  $\alpha$ -rich edges of  $(G, w)$ , where  $\alpha$  will be chosen suitably.

Let  $t \geq 3$  and  $0 < D, D' < n$  be given. An  $n$ -vertex graph  $G$  has property  $\mathcal{P}(t, D, D', n)$  if the following holds. For every subset  $U \subseteq V(G)$  of cardinality  $|U| \geq n - D$  and for every subset  $U_0 \subseteq U$  with  $|U_0| = D/t$ , there exists a family  $\mathcal{T}_0$  of at least  $D'/(t-1)$  copies of  $K_t$  in  $G[U]$  with the following properties:

- (1)  $V(T) \subseteq U$  for every  $T \in \mathcal{T}_0$ ,
- (2)  $|V(T) \cap U_0| = 1$  for every  $T \in \mathcal{T}_0$ ,
- (3)  $V(T) \cap V(T') \subseteq U_0$  for any distinct  $T$  and  $T' \in \mathcal{T}_0$ .

The copies of  $K_t$  specified above thus have the property that they are all edge-disjoint and intersect pairwise in at most one vertex, which must be from  $U_0$ .

**Lemma 2.6.** *Let  $(G, w)$  be a weighted graph and let  $\alpha \in [0, 1/(20t))$  be given. Suppose  $H$  is the spanning subgraph of  $G$  formed by the  $\alpha$ -rich edges in  $(G, w)$ . Suppose  $H$  is such that*

- (1) *every  $0.11n/t$  vertices span a copy of  $K_t$  in  $H$  and*
- (2)  *$H$  has property  $\mathcal{P}(t, D, 0.2n, n)$  for some  $D \leq n/2$ .*

*Then  $t^*(G[U], w) \geq t^*(H[U], w) \geq (|U| - D/t)/t$  for every  $U \subseteq V = V(G)$  with  $|U| \geq n - D$ .*

*Proof.* Let  $U$  as in the statement of the lemma be given. Let  $g$  and  $h$  form an optimal solution of the linear programme (7) applied to  $H[U]$  instead of  $G$ . Let  $U' = \{u \in U : g(u) = 0\}$ . If  $|U'| \leq D/t$  then we have by Proposition 2.4 (4) that  $t^*(H[U], w) \geq |U \setminus U'|/t \geq (|U| - D/t)/t$  and we are done.

We now assume that  $|U'| > D/t$  and derive a contradiction. First we fix a subset  $U_0 \subseteq U'$  with  $|U_0| = D/t$  and then we consider a family  $\mathcal{T}_0$  of cardinality  $0.2n/(t-1)$  as given by property  $\mathcal{P}$ . Let  $W := (\cup_{T \in \mathcal{T}_0} V(T)) \setminus U_0$ . We have  $|W| = 0.2n$  and, moreover,

$$\begin{aligned} \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_0} \sum_{uv \in E(T)} h(uv)w(uv) &\geq \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_0} \sum_{uv \in E(T)} h(uv)(1 - \alpha) \\ &\geq (1 - \alpha) \left( \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_0} \sum_{uv \in E(T)} h(uv) \right) = (1 - \alpha) \sum_{T \in \mathcal{T}_0} \left( \sum_{v \in W \cap V(T)} g(v) + \sum_{uv \in E(T)} h(uv) \right) \end{aligned}$$

$$\geq (1 - \alpha)|W|/(t - 1)$$

because  $g$  and  $h$  form an optimal solution of (7),  $g \upharpoonright_{U_0} \equiv 0$  and  $\sum_{v:v \in V(T)} g(v) + \sum_{V(T) \supset \{u,v\}} h(uv) \geq 1$  for every  $T \in \mathcal{K}_t(H[U])$ .

Since every  $0.11n/t$  vertices of  $H$  span a copy of  $K_t$ , we find at least  $(|U \setminus W| - 0.11n/t)/t$  vertex-disjoint copies of  $K_t$  in  $H[U \setminus W]$ . Proposition 2.4 (1) then implies that

$$t^*(H[U \setminus W], w) \geq t(H[U \setminus W], w) \geq (1 - \alpha) \frac{|U \setminus W| - 0.11n/t}{t}.$$

By Proposition 2.4 (3),  $g \upharpoonright_{U \setminus W}$  and  $h \upharpoonright_{E(G) \cap \binom{U \setminus W}{2}}$  form a feasible solution to the linear programme (7) applied to  $H[U \setminus W]$  instead of  $H$  and, moreover,

$$|g \upharpoonright_{U \setminus W}| + |h \upharpoonright_{E(H) \cap \binom{U \setminus W}{2}}| \geq t^*(H[U \setminus W], w) \geq (1 - \alpha) \frac{|U \setminus W| - 0.11n/t}{t}.$$

Thus we have

$$\begin{aligned} t^*(H[U], w) &\geq \left( \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_0} \sum_{uv \in E(T)} h(uv)w(uv) \right) + |g \upharpoonright_{U \setminus W}| + |h \upharpoonright_{E(H) \cap \binom{U \setminus W}{2}}| \\ &\geq (1 - \alpha) \left( \frac{|W|}{t - 1} + \frac{|U| - |W| - 0.11n/t}{t} \right) \\ &\geq (1 - \alpha) \frac{|U| + |W|/t - 0.11n/t}{t} \\ &= (1 - \alpha) \frac{|U| + 0.09n/t}{t} = \frac{|U|}{t} + \frac{0.09n}{t^2} - \alpha \frac{1.1n}{t} > \frac{|U|}{t}, \end{aligned}$$

which contradicts Proposition 2.4 (2).  $\square$

We denote by  $\mathcal{T}_v$  a family of copies of  $K_t$  in  $G$  with  $v \in T$  for every  $T \in \mathcal{T}_v$  and with  $V(T) \cap V(T') = \{v\}$  for all distinct  $T$  and  $T'$  in  $\mathcal{T}_v$ . The next lemma establishes a sufficient condition for the existence of a fractional  $K_t$ -factor in a weighted graph.

**Lemma 2.7.** *Let  $\alpha \geq 0$  be a real number and let  $t, n$  and  $D \geq 3$  be integers with  $\alpha < 1/t^2$  and  $D \leq n/2$ . Suppose  $(G, w)$  is an  $n$ -vertex weighted graph such that (i) for every  $v \in V(G)$ , there exists a family  $\mathcal{T}_v$  of at least  $D/(t - 1)$   $\alpha$ -rich copies of  $K_t$  and (ii) for every  $U \subseteq V$  of size  $|U| \geq n - D$ , one has  $t^*(G[U], w) \geq (|U| - D/t)/t$ . Then  $(G, w)$  contains a fractional  $K_t$ -factor.*

*Proof.* Let  $g: V = V(G) \rightarrow \mathbb{R}_{\geq 0}$  and  $h: E = E(G) \rightarrow \mathbb{R}_{\geq 0}$  form an optimal solution to (7). If  $g(v) > 0$  for each  $v \in V$ , we obtain, by Proposition 2.4 (4), that  $t^*(G, w) \geq |V|/t$ . By Proposition 2.4 (2), we deduce that  $t^*(G, w) = |V|/t$  and conclude that  $(G, w)$  has a fractional  $K_t$ -factor.

Assume now for a contradiction that there is a vertex  $x$  with  $g(x) = 0$ . Let  $\mathcal{T}_x$  be a family of  $D/(t - 1)$   $\alpha$ -rich copies of  $K_t$ . Let  $W := (\bigcup_{T \in \mathcal{T}_x} V(T)) \setminus \{x\}$  and observe that  $|W| = D$ . Since the copies of  $K_t$  from  $\mathcal{T}_x$  pairwise intersect only at  $x$  and they are  $\alpha$ -rich, it follows that

$$\begin{aligned} \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_x} \sum_{uv \in E(T)} h(uv)w(uv) &\geq \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_x} \sum_{uv \in E(T)} h(uv)(1 - \alpha) \\ &\geq (1 - \alpha) \left( \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_x} \sum_{uv \in E(T)} h(uv) \right) = (1 - \alpha) \sum_{T \in \mathcal{T}_x} \left( \sum_{v \in W \cap V(T)} g(v) + \sum_{uv \in E(T)} h(uv) \right) \\ &\geq (1 - \alpha)D/(t - 1). \end{aligned}$$



Proposition 2.4 (3) yields that  $g_1 := g \upharpoonright_{V \setminus W}$  and  $h_1 := h \upharpoonright_{E \cap \binom{V \setminus W}{2}}$  is a feasible solution to (7) with  $G$  replaced by  $G[V \setminus W]$ . Therefore we have  $|g_1| + |h_1| \geq t^*(G[V \setminus W], w)$ . By assumption (ii), we get

$$|g_1| + |h_1| \geq t^*(G[V \setminus W], w) \geq \frac{|V \setminus W| - D/t}{t} = \frac{n - D - D/t}{t}.$$

Together we obtain

$$\begin{aligned} t^*(G, w) &= |g| + |h| \geq \sum_{v \in W} g(v) + \sum_{T \in \mathcal{T}_x} \sum_{uv \in E(T)} h(uv)w(uv) + |g_1| + |h_1| \\ &\geq \frac{(1 - \alpha)D}{t - 1} + \frac{n - D - D/t}{t} > \frac{n}{t}. \end{aligned}$$

This contradicts  $t^*(G, w) \leq n/t$  (see Proposition 2.4 (2)). Thus, this case never happens; i.e.,  $g(v) > 0$  for all  $v \in V$ . We conclude that  $(G, w)$  contains a fractional  $K_t$ -factor.  $\square$

The following corollary of Lemmas 2.6 and 2.7 is our main tool for finding fractional  $K_t$ -factors.

**Corollary 2.8.** *Let  $\alpha \geq 0$  be a real number and let  $t, n$  and  $D \geq 3$  be integers with  $\alpha < 1/(7t^2)$  and  $D \leq n/2$ . Suppose  $(G, w)$  is an  $n$ -vertex weighted graph and suppose  $H$  is the spanning subgraph of  $G$  formed by the  $\alpha$ -rich edges of  $(G, w)$ . Suppose*

- (1) *for every  $v \in V(G)$  there exists a family  $\mathcal{T}_v$  of at least  $D/(t - 1)$   $\alpha$ -rich copies of  $K_t$ ;*
- (2) *every  $0.11n/t$  vertices span a copy of  $K_t$  in  $H$ ;*
- (3)  *$H$  has property  $\mathcal{P}(t, D, 0.2n, n)$  for some  $D \leq n/2$ .*

*Then  $(G, w)$  contains a fractional  $K_t$ -factor.*

**2.4. Further useful properties of  $(n, d, \lambda)$ -graphs.** We will use the following auxiliary results.

**Proposition 2.9** (Proposition 2.3 in [18]). *Let  $G$  be an  $(n, d, \lambda)$ -graph with  $d \leq n/2$ . Then  $\lambda \geq \sqrt{d/2}$ .*

**Fact 2.10.** *Let  $G$  be an  $(n, d, \lambda)$ -graph with  $d \leq n/2$ . Suppose  $\lambda \leq d^{t-1}/n^{t-2}$  for some  $t \geq 3$ . Then  $d \geq n^{1-1/(2t-3)}/2$ .*

*Proof.* Proposition 2.9 tells us that  $\lambda \geq \sqrt{d/2}$ . Thus  $\lambda \leq d^{t-1}/n^{t-2}$  implies that  $d^{2t-3} \geq n^{2t-4}/2$ , whence  $d \geq n^{1-1/(2t-3)}/2^{1/(2t-3)}$  follows.  $\square$

Given two graphs  $G$  and  $G'$  on the same vertex set  $V$ , let  $G \setminus G' = (V, E(G) \setminus E(G'))$ . The following proposition gives a rough estimate for the number of copies of  $K_{t-1}$  in induced subgraphs of  $(n, d, \lambda)$ -graphs, even after removing a small number of edges incident to each vertex.

**Proposition 2.11.** *For any integer  $t \geq 3$ , there exists  $n_0$  such that every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$  satisfies the following. Suppose  $\lambda(4n/d)^{t-2} \leq m \leq d$ . Let  $G'$  be a graph on  $V(G)$  with maximum degree  $(d/(4n))^{t-2}m$ . Then, for any  $2 \leq i \leq t-1$  and any set  $U$  of at least  $(d/(4n))^{t-i-1}m$  vertices of  $G$ , the number of copies of  $K_i$  in  $(G \setminus G')[U]$  is at least  $2^{-i^2}i!^{-1}|U|^i(d/n)^{\binom{i}{2}}$  and at most  $2^{i^2}i!^{-1}|U|^i(d/n)^{\binom{i}{2}}$ .*



*Proof.* Let  $m_i = (d/(4n))^{t-i-1}m$  for  $2 \leq i \leq t-1$ . Hence  $\lambda \leq m(d/(4n))^{t-2} = m_i(d/(4n))^{i-1}$ . Suppose  $|U| \geq m_i$ . We prove by induction on  $i$  that the stated estimates hold. Note that, since  $i \geq 2$ , we have  $\Delta(G') \leq (d/(4n))^{t-2}m \leq dm_i/(4n) \leq d|U|/(4n)$ . Let first  $i = 2$ . Theorem 1.1 implies that  $|2e_G(U) - (d/n)|U|^2| \leq \lambda|U|$ . Since  $\lambda \leq |U|(d/(4n))$ , we have  $(3/8)(d/n)|U|^2 \leq e_G(U) \leq (5/8)(d/n)|U|^2$ . Hence the number of edges in  $(G \setminus G')[U]$  is at most  $(5/8)(d/n)|U|^2$  and at least  $(3/8)(d/n)|U|^2 - |U| \cdot d|U|/(4n) = (1/8)(d/n)|U|^2$ , which verifies our claim for  $i = 2$ . Now suppose  $3 \leq i \leq t-1$  and that the estimates hold for smaller values of  $i$ . Note that, in particular, we have  $t \geq 4$ . Let  $X_1$  be the set of vertices  $v \in U$  such that  $\deg(v, U) \geq 2d|U|/n$ . By the definition of  $X_1$  and Theorem 1.1, we have

$$\frac{2d}{n}|U||X_1| \leq e(U, X_1) \leq \frac{d}{n}|U||X_1| + \lambda\sqrt{|U||X_1|}.$$

Together with  $\lambda \leq m_i(d/(4n))^{i-1} \leq |U|(d/(4n))^{i-1}$  and  $i \geq 3$ , this implies that

$$|X_1| \leq \left(\frac{\lambda n}{d}\right)^2 \frac{1}{|U|} \leq \frac{|U|}{16} \left(\frac{d}{4n}\right)^{2i-4} \leq \frac{|U|}{16} \left(\frac{d}{4n}\right)^{i-1}.$$

Note that  $2d|U|/n \geq 2dm_i/n \geq m_{i-1}$ . By the inductive hypothesis, the number of copies of  $K_i$  in  $(G \setminus G')[U]$  is at most

$$\frac{1}{i} \left( |X_1| \cdot \frac{2^{(i-1)^2}}{(i-1)!} |U|^{i-1} \left(\frac{d}{n}\right)^{\binom{i-1}{2}} + |U| \cdot \frac{2^{(i-1)^2}}{(i-1)!} \left(\frac{2d|U|}{n}\right)^{i-1} \left(\frac{d}{n}\right)^{\binom{i-1}{2}} \right) \leq \frac{2^{i^2}}{i!} |U|^i \left(\frac{d}{n}\right)^{\binom{i}{2}},$$

where the  $1/i$  factor avoids counting  $i$  times each copy of  $K_i$ .

Similarly, let  $X_2$  be the set of vertices  $v \in U$  such that  $\deg(v, U) \leq d|U|/(2n)$ . By the definition of  $X_2$  and Theorem 1.1, we have

$$\frac{d}{n}|U||X_2| - \lambda\sqrt{|U||X_2|} \leq e(U, X_2) \leq \frac{d}{2n}|U||X_2|.$$

This implies that  $|X_2| \leq (2\lambda n/d)^2/|U| \leq |U|/4$ . Note that  $d|U|/(4n) \geq dm_i/(4n) = m_{i-1}$ . By the inductive hypothesis, the number of copies of  $K_i$  in  $(G \setminus G')[U]$  is at least

$$\frac{1}{i} \cdot (|U| - |X_2|) \frac{1}{2^{(i-1)^2}(i-1)!} \left(\frac{d|U|}{2n} - \frac{d|U|}{4n}\right)^{i-1} \left(\frac{d}{n}\right)^{\binom{i-1}{2}} \geq \frac{1}{2^{i^2}i!} |U|^i \left(\frac{d}{n}\right)^{\binom{i}{2}}$$

(the  $1/i$  factor takes care of the fact that each copy of  $K_i$  is counted at most  $i$  times).  $\square$

We remark that a better estimate on the number of copies of  $K_i$  in  $m$ -vertex subgraphs of  $(n, d, \lambda)$ -graphs  $G$ , of the form  $(1 + o(1))\binom{m}{i}(d/n)^{\binom{i}{2}}$ , was obtained by Alon (see [17, Theorem 4.10]). Furthermore, a Turán-type result was proven by Sudakov, Szabó and Vu [22] for the containment of a copy of  $K_i$  in dense enough subgraphs of  $(n, d, \lambda)$ -graphs. However, both results require a stronger condition on  $\lambda$  than the one we shall have available in our applications of Proposition 2.11 (see Sections 4 and 5).

### 3. PROOF OUTLINE

In the following we provide a proof overview in the case of triangles, since the general case is similar. Our arguments combine tools from linear programming with probabilistic techniques. In fact, they can be seen as a synthesis of some methods in Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [5] and in Krivelevich, Sudakov and Szabó [18].

Let an  $(n, d, \lambda)$ -graph  $G$  with  $\lambda \leq cd^2/n$  be given. From the expander mixing lemma, Theorem 1.1, it follows that every vertex of  $G$  lies in  $\frac{1}{2}(d^3/n \pm \lambda d) = (d^3/2n)(1 \pm c)$  triangles. The naïve greedy approach outlined in the introduction (see the discussion soon after Theorem 1.3) does not guarantee a collection of  $(1 - o(1))n/3$  vertex-disjoint triangles. Another attempt would be to apply some theorem that would tell us that the 3-uniform hypergraph  $\mathcal{K}_3(G)$  of the triangles in  $G$  contains an almost perfect matching. A theorem of Pippenger (see [10]) would do if we knew that  $\mathcal{K}_3(G)$  is pseudorandom enough (roughly speaking, one needs that  $\mathcal{K}_3(G)$  should be approximately  $\ell$ -regular for some  $\ell \rightarrow \infty$  and that pairs of vertices of  $\mathcal{K}_3(G)$  should be contained in  $o(\ell)$  triples of  $\mathcal{K}_3(G)$  (i.e., the ‘codegrees’ should be small)). However, for  $c$  an absolute constant, this property of  $\mathcal{K}_3(G)$  cannot be deduced.

We circumvent the fact that  $\mathcal{K}_3(G)$  is not necessarily pseudorandom enough by finding a subhypergraph  $H$  of  $\mathcal{K}_3(G)$  in which the ‘deviation’ of the number of triangles at any vertex is ‘smoothed out’ (thus  $H$  will be almost  $\ell$ -regular). This can be done if  $G$  has  $\ell = n^{\Theta(1)}$  fractional  $K_3$ -factors  $f_1, \dots, f_\ell$  such that  $\sum_{i=1}^\ell f_i(T) \leq 1$  for each  $T \in \mathcal{K}_3(G)$  and, for any edge  $e \in E(G)$ , the sum of the weights on the triangles containing  $e$  across  $f_1, \dots, f_\ell$  is at most  $\ell^{1-\gamma}$  for some  $\gamma \in (0, 1)$ . This latter condition helps us force small codegrees.

Indeed, with these fractional  $K_3$ -factors, we can select  $H \subseteq \mathcal{K}_3(G)$  at random, by including each  $T \in \mathcal{K}_3(G)$  in  $H$  independently with probability  $\sum_{i=1}^\ell f_i(T)$ . Then Chernoff’s inequality guarantees that  $H$  satisfies, with high probability, the assumptions of Theorem 2.1, which is a packing result from [15] strengthening the aforementioned result of Pippenger. Such a ‘randomization’ strategy has previously been successfully employed in [5] in the context of perfect matchings in hypergraphs.

Thus, it suffices to find such fractional  $K_3$ -factors  $f_1, \dots, f_\ell$ . In fact, we find such  $f_i$  with the property that, for any  $e \in E(G)$ , we have  $\sum_{e \in E(T)} \sum_{i=1}^\ell f_i(T) \leq 1$  (hence  $\sum_{i=1}^\ell f_i(T) \leq 1$  for each  $T \in \mathcal{K}_3(G)$  is automatically true).

Theorem 1.3 is vacuously true for  $d = o(n^{2/3})$  when  $t = 3$  (owing to Fact 2.10). We thus suppose  $d = \Omega(n^{2/3})$ . We consider two cases. We pick any  $\beta \in (0, 1/3)$  independent of  $n$ . Our first approach (Theorem 4.1) works as long as  $d$  is not too small, say,  $d \geq n^{(2/3)+\beta}$ . In contrast, the second approach (Theorem 5.1) works as long as  $d$  is not too large, say,  $d \leq n^{1-\beta}$ .

In the first approach, we consider edge-weighted graphs and we repeatedly ‘remove’ fractional  $K_3$ -factors from  $G$  (removing from edges  $e$  the weights of the triangles  $T$  with  $e \subseteq V(T)$ ). This is done by Theorem 4.1 below, in which we show that we can repeatedly apply Corollary 2.8 in the remaining weighted graph  $n^\beta$  times.

When  $d$  is close to  $n^{2/3}$ , our approach above fails because we cannot execute it sufficiently many times. To circumvent this, we randomly split  $E(G)$  into  $\ell = n^{\Omega(1)}$  sets  $E_1, \dots, E_\ell$ , with each subgraph  $G_i := (V, E_i)$  distributed as a random subgraph  $G_p$  of  $G$ , where each edge is included in  $G_p$  with probability  $p = 1/\ell$ , independently of all the other edges. Then we show that with high probability each  $G_i$  satisfies the assumptions of Corollary 2.8, and thus contains a fractional  $K_3$ -factor  $f_i$ . This second approach works only for  $d \leq n^{1-o(1)}$ , which makes both approaches necessary.

#### 4. FRACTIONAL $K_t$ -FACTORS: THE DENSE CASE

In this section we prove the following theorem.

**Theorem 4.1.** *For any integer  $t \geq 3$  and  $\beta \in (0, 1/(2t-3)]$ , there exists  $n_0$  such that every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$ ,  $d \geq n^{1-1/(2t-3)+\beta}$  and  $\lambda \leq (1/(20t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$  contains  $\ell = n^\beta$  fractional  $K_t$ -factors  $f_1, \dots, f_\ell$  such that*

$$\sum_{i=1}^{\ell} \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f_i(T) \leq 1 \text{ for every edge } uv \in E(G). \quad (8)$$

**4.1. Proof idea.** Our main idea for proving Theorem 4.1 is to view the  $(n, d, \lambda)$ -graph  $G = (V, E)$  as a graph equipped with the weight function  $w: E \rightarrow [0, 1]$ . Once we manage to find a fractional  $K_t$ -factor  $f$  in  $(G, w)$  (by Corollary 2.8), we update the weight function  $w$  as follows:  $w(uv) := w(uv) - \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f(T)$ , which remains non-negative by condition (5) from Section 2.2. Moreover, the weighted degree of every vertex decreases by *exactly*  $t-1$ , since  $\sum_{u \in N(v)} \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f(T) = (t-1) \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f(T) = t-1$ . Therefore, if in our graph  $(G, w)$  all weighted degrees were the same, then, after updating  $w$ , the weighted degrees stay the same. To prove Theorem 4.1, it suffices to show that we can iterate this procedure  $\ell = n^\beta$  times.

**4.2. Cliques in weighted subgraphs of  $(n, d, \lambda)$ -graphs.** In the next two propositions, we use Proposition 2.11 to derive the assumptions of Corollary 2.8. Recall that, given a graph  $G$  and  $v \in V(G)$ , we denote by  $\mathcal{T}_v$  a family of copies of  $K_t$  in  $G$  with  $v \in T$  for every  $T \in \mathcal{T}_v$  and with  $V(T) \cap V(T') = \{v\}$  for all distinct  $T$  and  $T'$  in  $\mathcal{T}_v$ .

**Proposition 4.2.** *For any integer  $t \geq 3$ , there exists  $n_0$  such that every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$  and  $\lambda \leq (1/(20t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$  satisfies the following. Let  $G'$  be a graph on  $V(G)$  with maximum degree at most  $(d/(4n))^{t-2}d/(20t)$ . Then for every  $v \in V(G)$  there exists a family  $\mathcal{T}_v$  of at least  $d/(2t-2)$  copies of  $K_t$  in  $G \setminus G'$ .*

*Proof.* Fix a vertex  $v \in V(G)$  and let  $U = N_{G \setminus G'}(v)$ . Then  $|U| \geq (1 - d/(80tn))d \geq 0.9d$ . Let  $\mathcal{T}_v$  be a collection of copies of  $K_t$  in  $G \setminus G'$  such that  $|\mathcal{T}_v| < d/(2t-2)$ . Then  $|U \setminus \bigcup_{T \in \mathcal{T}_v} V(T)| \geq 0.9d - d/2 \geq d/(20t)$ . By Proposition 2.11,  $G[U \setminus \bigcup_{T \in \mathcal{T}_v} V(T)]$  contains a copy of  $K_{t-1}$  in  $G \setminus G'$ , which, together with  $v$ , gives a copy of  $K_t$  in  $G \setminus G'$ . The proposition follows.  $\square$

The following proposition establishes Property  $\mathcal{P}(t, D, D', n)$  from Section 2.3 in subgraphs of  $(n, d, \lambda)$ -graphs for certain values of the parameters.

**Proposition 4.3.** *For any integer  $t \geq 3$ , there exists  $n_0$  such that every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$  and  $\lambda \leq (1/(20t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$  satisfies the following. Let  $G'$  be a graph on  $V(G)$  with maximum degree at most  $(d/(4n))^{t-2}d/(20t)$ . Then every  $0.11n/t$  vertices of  $G \setminus G'$  span a copy of  $K_t$  in  $G \setminus G'$ . Moreover,  $G \setminus G'$  has property  $\mathcal{P}(t, d/2, 0.2n, n)$ .*

*Proof.* We first prove that  $G \setminus G'$  has property  $\mathcal{P}(t, d/2, 0.2n, n)$ . Set  $c := 1/(20t \cdot 4^{t-2})$  so that  $\lambda \leq cd^{t-1}/n^{t-2}$ . Let  $\mathcal{T}_0$  be a family of copies of  $K_t$  in  $G \setminus G'$  of maximum cardinality satisfying properties (1)–(3) in the definition of  $\mathcal{P}(t, d/2, 0.2n, n)$ . If  $|\mathcal{T}_0| < 0.2n/(t-1)$ , then let  $W := (\bigcup_{T \in \mathcal{T}_0} V(T)) \setminus U_0$ . It follows that  $|W| = (t-1)|\mathcal{T}_0| < 0.2n$ . Because of the maximality of  $\mathcal{T}_0$ , there are no copies of  $K_t$  in  $G \setminus G'$  with one vertex from  $U_0$  and the other  $t-1$  from  $U \setminus (U_0 \cup W)$ . Note that  $|U \setminus (U_0 \cup W)| \geq n - d/2 - d/(2t) - 0.2n \geq n/(10t)$ . Take any subset  $U' \subseteq U \setminus (U_0 \cup W)$  of cardinality  $n/(10t)$  and note that, by Theorem 1.1, we have

$$e_G(U_0, U') \geq \frac{d}{n} \cdot \frac{n}{10t} \cdot \frac{d}{2t} - \lambda \sqrt{\frac{n}{10t} \cdot \frac{d}{2t}} \geq \frac{d^2}{20t^2} - c \frac{d^{t-1}}{n^{t-3}} \sqrt{\frac{d}{20t^2n}} \geq \frac{d^2}{30t^2}.$$

Thus, there is a vertex  $v \in U_0$  of degree at least  $d/(15t)$  into  $U'$  in  $G$  and hence  $v$  is connected to at least  $d/(15t) - \Delta(G') \geq d/(15t) - d/(80t) \geq d/(20t)$  vertices in  $U'$  via edges in  $G \setminus G'$ . Let  $R := N_{G \setminus G'}(v) \cap U'$ . We have  $|R| \geq d/(20t)$ . By Proposition 2.11,  $(G \setminus G')[R]$  contains a copy of  $K_{t-1}$ , which, together with  $v$ , gives a copy of  $K_t$  in  $G \setminus G'$ . This contradicts the maximality of  $\mathcal{T}_0$ . This shows that  $|\mathcal{T}_0| \geq 0.2n/(t-1)$ , as required.

It remains to prove that every set  $U$  of  $0.11n/(t-1)$  vertices of  $G \setminus G'$  spans a copy of  $K_t$  in  $G \setminus G'$ . This can be done in a similar way by showing that Theorem 1.1 implies that the average degree in  $(G \setminus G')[U]$  is at least  $7d/(80t)$  and then finding a copy of  $K_{t-1}$  in the neighborhood of a vertex of maximum degree in  $(G \setminus G')[U]$ . We omit the details.  $\square$

We now are ready to prove Theorem 4.1.

**4.3. Proof of Theorem 4.1.** We set  $\alpha := (d/(4n))^{t-2}/(20t)$  and choose  $n_0$  sufficiently large. We start with the graph  $G$  and at the beginning we set all edge weights to one, i.e.,  $w(e) := 1$  for all  $e \in E(G)$ . We shall iteratively apply Corollary 2.8 to find  $\ell = n^\beta$  fractional  $K_t$ -factors  $f_1, \dots, f_\ell$ . In doing so we will iteratively update the weights of the edges in  $G$ . By Fact 2.5 and Propositions 4.2 and 4.3, the assumptions of Corollary 2.8 will be satisfied after each iteration, and we shall be able to find a fractional  $K_t$ -factor in the weighted graph at hand. Recall that the ‘weighted degree’  $\deg_w(v)$  of a vertex  $v \in V = V(G)$  is defined to be  $\sum_{u \in N(v)} w(uv)$ . We observe that this degree is exactly  $d$  at the beginning (when  $w \equiv 1$ ). Then we update the edge weights for all  $uv \in E(G)$  in the  $i$ th iteration as follows:

$$w(uv) := w(uv) - \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f_i(T). \quad (9)$$

Observe that the weighted degree of *every* vertex decreased by  $t-1$ , since

$$\sum_{u \in N(v)} \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f_i(T) = (t-1) \sum_{T \in \mathcal{K}_t(G): V(T) \ni v} f_i(T) = t-1,$$

because  $f_i$  is a fractional  $K_t$ -factor. Now suppose  $\sum_{u \in N(v)} w(uv) \geq (1 - \alpha^2)d$  for any  $v \in V$  throughout the process. Let  $G'$  be the graph on  $V$  consisting of edges of  $G$  that are not  $\alpha$ -rich. By Fact 2.5,  $\Delta(G') \leq \alpha d$ . Thus, by Propositions 4.2 and 4.3, we can apply Corollary 2.8 with  $D = d/2$  to  $(G, w)$  iteratively, updating the weights

$$\frac{d\alpha^2}{t-1} = \frac{d}{400t^2(t-1)} \left( \frac{d}{4n} \right)^{2t-4} = \frac{d^{2t-3}}{400t^2(t-1)(4n)^{2t-4}} \geq \frac{n^{(2t-3)\beta}}{400t^2(t-1)4^{2t-4}} \geq n^\beta = \ell$$

times. Because of our update rule (9), condition (8) does hold for the  $f_i$  ( $1 \leq i \leq \ell$ ) that we have obtained.  $\square$

## 5. FRACTIONAL $K_t$ -FACTORS: THE SPARSE CASE

In this section we prove the following theorem.

**Theorem 5.1.** *For any integer  $t \geq 3$  and  $\delta \in (0, 1/(2t-3))$ , there exists  $n_0$  such that every  $(n, d, \lambda)$ -graph  $G$  with  $n \geq n_0$ ,  $d \leq n^{1-\delta}$  and  $\lambda \leq (1/(50t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$  contains  $\ell = n^{\delta/(4t^2)}$  fractional  $K_t$ -factors  $f_1, \dots, f_\ell$  such that*

$$\sum_{i=1}^{\ell} \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u,v\}} f_i(T) \leq 1 \text{ for every edge } uv \in E(G). \quad (10)$$

**5.1. Proof overview.** Our proof strategy this time will be to *partition*  $G$  randomly into  $\ell = n^{\delta/(4t^2)}$  edge-disjoint subgraphs  $G_i$ , and then to show that, with high probability, each such random subgraph  $G_i$  satisfies the assumptions of Corollary 2.8 and thus contains a fractional  $K_t$ -factor  $f_i$ . Because the host graphs  $G_i$  of such  $f_i$  are edge-disjoint, condition (10) will hold because, for any  $T \in \mathcal{K}_t(G)$ , there is at most one  $i \in [\ell]$  such that  $f_i(T) > 0$ . Note that, in this section, when we use results from Sections 2.2 and 2.3, we always use them on standard graphs, i.e., with  $w \equiv 1$ . In particular, here, every edge will be  $\alpha$ -rich for any  $\alpha \geq 0$ . For the sake of definiteness, we always take  $\alpha = 0$  in this section.

**5.2. Probabilistic lemmas.** Recall that, for a graph  $G = (V, E)$ , the random subgraph  $G_p$  of  $G$  is a spanning subgraph of  $G$  in which each edge from  $E$  is included with probability  $p$ , independently of all other edges. In this subsection, we show that, for suitable  $(n, d, \lambda)$ -graphs  $G$ , with high probability  $G_p$  satisfies the assumptions of Corollary 2.8 and thus contains a fractional  $K_t$ -factor (Theorem 5.5 below). We first show that, with high probability,  $G_p$  satisfies (1) in Corollary 2.8, with  $D = p^{\binom{t}{2}} d/4$ .

**Proposition 5.2.** *For any  $t \geq 3$  and  $\delta \in (0, 1/(2t-3))$  there exists  $n_0$  such that the following holds. Suppose  $G$  is an  $(n, d, \lambda)$ -graph with  $n \geq n_0$ ,  $d \leq n^{1-\delta}$  and  $\lambda \leq (1/(20t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$ . Let  $p = d^{-\eta}$  for some  $\eta \in (0, 1/t^2)$ . Then, with probability at least  $1 - n \exp(-\sqrt{d})$ , for any vertex  $v \in V(G_p)$  there exists a family  $\mathcal{T}_v$  in  $G_p$  with  $|\mathcal{T}_v| \geq p^{\binom{t}{2}} d/(4t-4)$ .*

*Proof.* For any  $v \in V(G_p) = V(G)$ , by Proposition 4.2, there is a family  $\mathcal{T}'_v$  in  $G$  with  $|\mathcal{T}'_v| = d/(2t-2)$ . Let  $X$  be the number of cliques  $K_t$  from  $\mathcal{T}'_v$  in  $G_p$ . Since the cliques in  $\mathcal{T}_v$  are edge-disjoint,  $X \sim \text{Bin}(d/(2t-2), p^{\binom{t}{2}})$ . By Chernoff's inequality (Theorem 2.2),  $\mathbb{P}[X < \mathbb{E}X/2] < e^{-\mathbb{E}X/12} = \exp(-p^{\binom{t}{2}} d/(24t)) \leq \exp(-d^{1-\binom{t}{2}\eta}/(24t)) \leq \exp(-\sqrt{d})$ . The union bound over all vertices yields that, with probability at least  $1 - n \exp(-\sqrt{d})$ , every vertex  $v$  lies in some family  $\mathcal{T}_v$  in  $G_p$  of at least  $p^{\binom{t}{2}} d/(4t-4)$  copies of  $K_t$ .  $\square$

We now state and prove a technical lemma that will be required to show that  $G_p$  is very likely to have property  $\mathcal{P}(t, D, D', n)$  for certain values of  $D$  and  $D'$ . The proof of this lemma is based on Janson's inequality (Theorem 2.3). However, to have 'weak enough dependence' when applying inequality (4), we shall have to employ an additional trick.

**Lemma 5.3.** *For any  $t \geq 3$  and  $\delta \in (0, 1/(2t-3))$  there exists  $n_0$  such that the following holds. Suppose  $G$  is an  $(n, d, \lambda)$ -graph with  $n \geq n_0$ ,  $d \leq n^{1-\delta}$  and  $\lambda \leq (1/(50t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$ . Let  $p = d^{-\eta}$  for some  $\eta \in (0, \delta/(2t^2)]$ . Then, with probability  $1 - 2^{-n}$ , for every pair of disjoint subsets  $U_0$  and  $U'$  of  $V(G)$  with  $|U_0| = D = p^{\binom{t}{2}} d/(4t)$  and  $|U'| = n/(10t)$ , there is a clique  $K$  on  $t$  vertices in  $G_p$  with  $|V(K) \cap U_0| = 1$  and  $|V(K) \cap U'| = t-1$ .*

*Proof.* Let  $c := 1/(50t \cdot 4^{t-2})$ . We shall use Proposition 2.11 to show that there are sufficiently many cliques  $K_t$  in  $G$  with one vertex from  $U_0$  and  $t-1$  from  $U'$ , so that after keeping each edge at random, the probability that none of the cliques survives is very small. Taking the union bound over all possible choices of  $U_0$  and  $U'$  will then finish the proof. To estimate the survival probability of some clique, we shall apply Janson's inequality, Theorem 2.3. However, some special care needs to be taken, as otherwise the parameter  $\Delta$  in (3) may turn out to be too large. Therefore, we shall restrict our attention to certain 'nice' cliques between  $U_0$  and  $U'$ .

For a given set  $U_0$  we say that a vertex  $u$  from  $U'$  is *bad with respect to  $U_0$*  if  $\deg_{U_0}(u) \geq 2Dd/n$ , and otherwise we say it is *good*. Let  $B$  be the set of bad vertices from  $U'$  with respect to  $U_0$ . Then  $e(B, U_0) \geq 2|B|Dd/n$ , whereas Theorem 1.1 asserts that

$$e(B, U_0) \leq |B|Dd/n + \lambda\sqrt{|B|D} \leq |B|Dd/n + cd^{t-1}\sqrt{|B|D}/n^{t-2}.$$

From this we infer that  $|B|Dd/n \leq cd^{t-1}\sqrt{|B|D}/n^{t-2}$ , and therefore

$$|B| \leq c^2 d^{2t-4} / (Dn^{2t-6}) \leq (4t)c^2 d^{2t-5+\binom{t}{2}\eta} / n^{2t-6} < n/(30t),$$

by the choice of  $\eta < \delta/\binom{t}{2}$  and  $c$ . Thus, the number of good vertices in  $U'$  is  $|U' \setminus B| \geq n/(15t)$ .

Next we estimate the number of edges  $e_G(U_0, U' \setminus B)$  between  $U_0$  and  $U' \setminus B$  in  $G$  (we omit the subscript  $G$  whenever it is clear from the context). Note that  $Dn/d^2 \geq n^\delta p^{(t)}(2t) / (4t) \geq n^\delta n^{-\eta t^2} / (4t) \geq 1$ , because  $\eta \leq \delta/(2t^2)$ . By  $|U' \setminus B| \geq n/(15t)$  and Theorem 1.1, we have

$$e(U_0, U' \setminus B) \geq \frac{d}{n} \frac{Dn}{15t} - \lambda \sqrt{\frac{Dn}{15t}} \geq \frac{dD}{15t} - \frac{Dd}{60t} \sqrt{\frac{d^2}{15tDn}} \geq \frac{Dd}{20t},$$

where we used that  $\lambda \leq cd^{t-1}/n^{t-2} \leq cd^2/n < d^2/(60tn)$  and  $d^2/(Dn) \leq 1$ . Thus, since  $\Delta(G) \leq d$ , we have at least  $D/(40t)$  vertices in  $U_0$  of degree at least  $d/(40t)$  into  $U' \setminus B$  in  $G$ .

Given a vertex  $u \in U_0$  with  $\deg_{U' \setminus B}(u) \geq d/(40t)$ , we call a vertex  $v \in N(u) \cap (U' \setminus B)$  *expensive with respect to  $u$*  if  $\deg_{N(u) \cap (U' \setminus B)}(v) \geq 4d^2/n$ , and otherwise we call it *inexpensive*. Let  $R$  be the set of expensive vertices with respect to  $u$ . Then  $e(R, N(u) \cap (U' \setminus B)) \geq 4d^2|R|/n$ , whereas Theorem 1.1 asserts that

$$e(R, N(u) \cap (U' \setminus B)) \leq \deg_{U' \setminus B}(u) |R| \frac{d}{n} + c \frac{d^{t-1}}{n^{t-2}} \sqrt{\deg_{U' \setminus B}(u) |R|} \leq \frac{d^2 |R|}{n} + \frac{cd^2}{n} \sqrt{d |R|},$$

where we used that  $\deg_{U' \setminus B}(u) \leq d$ . Thus we have  $d^2 |R|/n \leq cd^2 \sqrt{d |R|}/n$  and therefore  $|R| \leq c^2 d$ .

We now introduce the notion of ‘nice cliques’ in  $G$ . Let  $K$  be a copy of  $K_t$  in  $G$  with  $K \cap U_0 = \{u\}$  and  $|K \cap (U' \setminus B)| = t-1$ . We call  $K$  *nice* if  $\deg_{U' \setminus B}(u) \geq d/(40t)$ , and any other vertex  $v \in V(K) \setminus \{u\}$  is inexpensive with respect to  $u$ . Since there are at least  $d/(50t)$  such inexpensive vertices in  $N(u) \cap (U' \setminus B)$ , we can deduce the following lower bound on the number of nice cliques in  $G$ . For  $u \in U_0$  with  $\deg_{U' \setminus B}(u) \geq d/(40t)$ , let  $R_u$  be the set of inexpensive vertices with respect to  $u$  and notice that  $|R_u| \geq d/(40t) - c^2 d \geq d/(50t)$ . Let  $\mathcal{C}$  be the family of nice cliques in  $G$ . Then we have

$$|\mathcal{C}| = \sum |\mathcal{K}_{t-1}(R_u)| \stackrel{\text{Proposition 2.11}}{\geq} \frac{D}{40t} \frac{(d/(50t))^{t-1}}{2^{(t-1)^2} (t-1)!} (d/n)^{\binom{t-1}{2}} \geq \frac{Dd^{t-1}}{2^{(t-1)^2} (50t)^t (t-1)!} (d/n)^{\binom{t-1}{2}},$$

where the sum is over all  $u \in U_0$  such that  $\deg_{U' \setminus B}(u) \geq d/(40t)$ .

We aim to employ next Janson’s inequality to show that at least one of these nice cliques survives in  $G_p$  with ‘sufficiently high’ probability. Let  $X$  be the number of nice cliques that are contained in  $G_p$ . By the above bound on  $|\mathcal{C}|$ ,

$$\mathbb{E}X = p^{\binom{t}{2}} |\mathcal{C}| \geq \frac{p^{\binom{t}{2}} Dd^{t-1}}{2^{(t-1)^2} (50t)^t (t-1)!} (d/n)^{\binom{t-1}{2}} \geq \frac{p^{t(t-1)} d^t}{2^{t^2} (50t)^t t!} (d/n)^{\binom{t-1}{2}}. \quad (11)$$

It remains to estimate the parameter  $\Delta$  in Janson’s inequality.

Let a clique  $K$  from  $\mathcal{C}$  be given. First we estimate the number of nice cliques  $K'$  with  $K \cap K' \cap U_0 = \emptyset$  and  $|K \cap K' \cap U'| \geq 2$ . Note that  $|K' \cap U_0| = 1$  and denote this only vertex in  $K' \cap U_0$  by  $v$ . We need to choose at least two vertices to lie in the intersection  $K \cap K' \cap U'$ , and



these have to be connected to  $v$ . Since every vertex from  $K \cap U'$  is good, we have at most  $2Dd/n$  choices for  $v$ . Moreover, we need to specify further  $t-3$  vertices from  $U'$  to belong to the copy of  $K'$  and these have to form a  $(t-3)$ -clique in  $N(v)$ . For  $t \geq 5$ , since  $|N(v)| = d \geq (d/4n)^2 d$ , by Proposition 2.11 with  $m = d$  and  $i = t-3$ , there are at most

$$\binom{t-1}{2} \cdot \frac{2Dd}{n} \cdot 2^{(t-3)^2} d^{t-3} (d/n)^{\binom{t-3}{2}} \leq (d^{t-1}/n)(d/n)^{\binom{t-3}{2}} \quad (12)$$

potential nice cliques  $K'$ . Note that the estimates in (12) also hold for  $t = 3$  and 4.

Next we estimate the number of nice cliques  $K'$  with  $K \cap K' \cap U_0 \neq \emptyset$  and  $|K \cap K' \cap U'| \geq 1$  for our fixed clique  $K$  from  $\mathcal{C}$ . We have  $t-1$  choices for a common vertex  $x$  from  $K \cap K' \cap U'$ . Since  $x$  is inexpensive, its common neighborhood with the vertex  $y$  from  $K \cap K' \cap U_0$  is at most  $4d^2/n$ . It remains to estimate the number of possible extensions of  $x$  and  $y$  to a clique of size  $t$ . For this we would like to count the number of copies of  $K_{t-2}$  on a set of size  $4d^2/n$ . For  $t \geq 4$ , since  $4d^2/n \geq d^2/n$ , by Proposition 2.11 with  $m = d$  and  $i = t-2$ , there are at most

$$(t-1) \cdot 2^{(t-2)^2} (4d^2/n)^{t-2} (d/n)^{\binom{t-2}{2}} \leq (d^{t-1}/n)(d/n)^{\binom{t-2}{2}} \quad (13)$$

potential nice cliques  $K'$ . Note that the estimates in (13) also hold for  $t = 3$ .

By (12) and (13), we get  $\Delta \leq 2p^{\binom{t}{2}} |\mathcal{C}| (d^{t-1}/n)(d/n)^{\binom{t-3}{2}} = 2\mathbb{E}X (d^{t-1}/n)(d/n)^{\binom{t-3}{2}}$ . Then Janson's inequality yields

$$\begin{aligned} \mathbb{P}[X = 0] &\leq \exp\left(-\frac{\mathbb{E}[X]^2}{2\Delta}\right) \leq \exp\left(-\frac{\mathbb{E}[X]}{4(d^{t-1}/n)(d/n)^{\binom{t-3}{2}}}\right) \\ &\stackrel{(11)}{\leq} \exp\left(-\frac{p^{t(t-1)} d^t (d/n)^{\binom{t-1}{2}}}{2^{t^2+2} (50t)^t t! \cdot (d^{t-1}/n) \cdot (d/n)^{\binom{t-3}{2}}}\right) = \exp\left(-\frac{p^{t(t-1)}}{2^{t^2+2} (50t)^t t!} n^2 (d/n)^{2t-4}\right). \end{aligned}$$

Note that  $p = d^{-\eta} \geq n^{-\eta} \geq n^{-\delta/(2t^2)}$ . Together with  $d \geq n^{1-1/(2t-3)}/2$  (Fact 2.10) and  $\delta < 1/(2t-3)$ , we get

$$\mathbb{P}[X = 0] \leq \exp\left(-\frac{p^{t(t-1)}}{2^{t^2+2t-2} (50t)^t t!} n^{2-\frac{2t-4}{2t-3}}\right) \leq \exp\left(-n^{1+1/(4t-6)}\right).$$

Taking the union bound over all choices for  $U_0$  and  $U'$  (of which there are at most  $4^n$ ), we obtain the desired claim with probability  $1 - 4^n \exp(-n^{1+1/(4t-6)}) \geq 1 - 2^{-n}$  for  $n$  large enough.  $\square$

Now we show that it is very likely that  $G_p$  satisfies (2) and (3) in Corollary 2.8 with  $D = p^{\binom{t}{2}} d/4$  and  $\alpha = 0$ .

**Proposition 5.4.** *For any  $t \geq 3$  and  $\delta \in (0, 1/(2t-3))$  there exists  $n_0$  such that the following holds. Suppose  $G$  is an  $(n, d, \lambda)$ -graph with  $n \geq n_0$ ,  $d \leq n^{1-\delta}$  and  $\lambda \leq (1/(50t \cdot 4^{t-2})) d^{t-1}/n^{t-2}$ . Let  $p = d^{-\eta}$  for some  $\eta \in (0, \delta/(2t^2)]$ . Then, with probability  $1 - 2^{-n}$ , every  $0.11n/t$  vertices of  $G_p$  span a copy of  $K_t$  and  $G_p$  has property  $\mathcal{P}(t, p^{\binom{t}{2}} d/4, 0.2n, n)$ .*

*Proof.* Let  $t$  and  $\delta$  as in the statement be fixed. Let  $n_0 = n_0(t, \delta)$  be given by Lemma 5.3. Then for any  $\eta \in (0, \delta/(2t^2)]$  the conclusion in Lemma 5.3 holds with probability at least  $1 - 2^{-n}$ . Write  $D = p^{\binom{t}{2}} d/4$ .

Let  $\mathcal{T}_0$  be a family of  $t$ -cliques in  $G_p$  of maximum cardinality satisfying properties (1)–(3) in the definition of  $\mathcal{P}(t, p^{\binom{t}{2}} d/4, 0.2n, n)$  (see Section 2.3). If  $|\mathcal{T}_0| < 0.2n/(t-1)$ , then let  $W := (\cup_{K \in \mathcal{T}_0} V(K)) \setminus U_0$ . It follows that  $|W| = (t-1)|\mathcal{T}_0| < 0.2n$ . Because of the maximality of  $\mathcal{T}_0$  there are no cliques in  $G_p$  with one vertex from  $U_0$  and the other  $t-1$  vertices from  $U' := U \setminus (U_0 \cup W)$ .



Observe that

$$|U'| \geq n - D - |U_0| - |W| \geq 0.5n.$$

But then, by the assertion of Lemma 5.3, there must be yet another copy of  $K_t$  in  $G_p$  between  $U_0$  and  $U'$ , contradicting the maximality of  $\mathcal{T}_0$ . Hence  $|\mathcal{T}_0| \geq 0.2n/(t-1)$ .

It remains to prove that every set  $U$  of  $0.11n/t$  vertices of  $G_p$  spans a clique  $K_t$ . But this is immediate since  $D < 0.01n/t$  and thus any two disjoint subsets of  $U$  of cardinality  $D$  and  $0.1n/t$  span a copy of  $K_t$ .  $\square$

Next we show that  $G_p$  is very likely to contain a fractional  $K_t$ -factor.

**Theorem 5.5.** *For any  $t \geq 3$  and  $\delta \in (0, 1/(2t-3))$  there exists  $n_0$  such that the following holds. Suppose  $G$  is an  $(n, d, \lambda)$ -graph with  $n \geq n_0$ ,  $d \leq n^{1-\delta}$  and  $\lambda \leq (1/(50t \cdot 4^{t-2}))d^{t-1}/n^{t-2}$ . Let  $p = d^{-\eta}$  for some  $\eta \in (0, \delta/(2t^2)]$ . Then  $G_p$  contains a fractional  $K_t$ -factor with probability at least  $1 - n \exp(-\sqrt{d}) - 2^{-n}$ .*

*Proof.* Let  $t$  and  $\delta$  as in the statement be fixed. Let  $n_0 = n_0(t, \delta)$  be given by Propositions 5.2 and 5.4 for the given parameters  $t$  and  $\delta$ . Then, for any fixed  $\eta \in (0, \delta/(2t^2)]$ , the conclusions in Propositions 5.2 and 5.4 hold with probability at least  $1 - n \exp(-\sqrt{d}) - 2^{-n}$ . Let  $D := p^{\binom{t}{2}}d/4$ . The conclusion in our theorem follows from Corollary 2.8 (with  $\alpha = 0$ ).  $\square$

We are now ready to prove Theorem 5.1.

**5.3. Proof of Theorem 5.1.** Let  $p = d^{-\eta}$  where  $\eta = \delta/(2t^2)$ . Let  $\ell := d^\eta = p^{-1}$ . By Fact 2.10, we have  $d \geq n^{1-1/(2t-3)}/2 \geq n^{1/2}$ . Thus  $\ell = d^\eta \geq n^{\delta/(4t^2)}$ . Consider the random variable  $I$  that takes values uniformly at random in  $[\ell] = \{1, \dots, \ell\}$ . For each  $e \in E(G)$ , let  $I_e \sim I$  be an independent copy of  $I$ . We randomly partition the edge set of  $G$  into spanning subgraphs  $G_1, \dots, G_\ell$  of  $G$ , where each  $e \in E(G)$  is put into  $G_{I_e}$ . Observe that each  $G_i$  is distributed as  $G_p$ . By Theorem 5.5, the probability that a given  $G_i$  should *not* contain a fractional  $K_t$ -factor is at most  $n \exp(-\sqrt{d}) + 2^{-n}$ . Since  $d \geq n^{1-1/(2t-3)}/2$ , it follows that, with probability  $1 - \ell(n \exp(-\sqrt{d}) + 2^{-n}) = 1 - o(1) > 0$ , there is a partition of  $G$  into deterministic edge-disjoint spanning subgraphs  $G_1, \dots, G_\ell$  such that each  $G_i$  ( $i \in [\ell]$ ) possesses a fractional  $K_t$ -factor  $f_i$ . Since  $G_1, \dots, G_\ell$  are edge-disjoint, condition (10) holds because for any  $T \in \mathcal{K}_t(G)$ , there is at most one  $i \in [\ell]$  such that  $f_i(T) > 0$ .  $\square$

## 6. PROOF OF THEOREM 1.3

Let  $t \geq 3$  be fixed and let  $G$  be an  $(n, d, \lambda)$ -graph as in the statement of Theorem 1.3. Let  $\delta = 4t^2/((4t^2 + 1)(2t - 3))$  and  $\beta = 1/((4t^2 + 1)(2t - 3))$ , so that  $n^\beta = n^{\delta/(4t^2)}$ . Let  $\ell = n^\beta$ . By Theorems 4.1 and 5.1 (depending on whether  $d \geq n^{1-1/(2t-3)+\beta}$  or  $d \leq n^{1-\delta}$ ), our graph  $G$  contains fractional  $K_t$ -factors  $f_1, \dots, f_\ell$  such that, for every edge  $uv \in E(G)$ , we have  $\sum_{i=1}^\ell \sum_{T \in \mathcal{K}_t(G): V(T) \supset \{u, v\}} f_i(T) \leq 1$ . Clearly,  $f(T) := \sum_{i=1}^\ell f_i(T) \leq 1$  for any  $T \in \mathcal{K}_t(G)$ . Let  $H := \mathcal{K}_t(G)$  be the  $t$ -uniform hypergraph with copies of  $K_t$  in  $G$  as hyperedges and  $V(H) = V(G)$ . Thus, for every  $v \in V = V(H)$  and  $i \in [\ell]$ , we have  $\sum_{T \in E(H): v \in T} f_i(T) = 1$  and thus  $\sum_{T \in E(H): v \in T} f(T) = \ell$ . Moreover, for any distinct vertices  $u$  and  $v$ , we have

$$\sum_{T \in E(H): \{u, v\} \subseteq T} f(T) = \sum_{i \in [\ell]} \sum_{T \in E(H): \{u, v\} \subseteq T} f_i(T) \leq 1.$$

Let  $H_f$  be the random spanning subhypergraph of  $H$  such that each edge  $T$  of  $H$  is included in  $H_f$  independently with probability  $f(T)$ .

Next we let  $k := 8\beta^{-1/2}$ . For  $v \in V$ , let  $X_v$  be the degree of  $v$  in  $H_f$ . We have  $\mathbb{E}X_v = \sum_{T \in E(H): v \in T} f(T) = \ell$ . Then Chernoff's inequality (Theorem 2.2), implies that

$$\mathbb{P}[|X_v - \ell| > (k/2)\sqrt{\ell \ln \ell}] \leq 2e^{-k^2(\ln \ell)/12} < 2e^{-5 \ln n} = 2n^{-5}.$$

On the other hand, for any distinct vertices  $u$  and  $v \in V$ , let  $Y_{uv}$  be the collective degree  $\deg_{H_f}(u, v)$  of  $u$  and  $v$  in  $H_f$ . Then  $\mathbb{E}Y_{uv} = \sum_{T \in E(H): \{u, v\} \subseteq T} f(T) \leq 1$  (in fact,  $Y_{uv} = 0$  if  $uv \notin E(G)$ ). By Chernoff's inequality again, we have  $\mathbb{P}[Y_{uv} \geq 1 + 3 \ln n] \leq e^{-3 \ln n} = n^{-3}$ . The union bound over all  $u \in V$  and all pairs  $\{u, v\} \in \binom{V}{2}$  yields that, with probability at least  $1 - 1/n$ , we have

$$\deg_{H_f}(u) = \ell \pm (k/2)\sqrt{\ell \ln \ell} \quad \text{and} \quad \deg_{H_f}(u, v) \leq 1 + 3 \ln n$$

for all  $u$  and  $v \in V$ .

It is easy to check now that the assumptions of Theorem 2.1 are satisfied with  $\delta' = 1/t$ ,  $\gamma = 0.9$ ,  $D = (1 + (k/2)\sqrt{(\ln \ell)/\ell})\ell \leq 2n^\beta$  and  $C = 1 + 3 \ln n$ . Thus an application of that theorem gives us a matching in  $H_f$  covering all but at most

$$O(n(C/D)^{(1-\delta')/(t-1)}) = O\left(n\left(\frac{1 + 3 \ln n}{2n^\beta}\right)^{1/t}\right) \leq n^{1-1/(8t^4)}$$

vertices. These edges correspond to cliques  $K_t$  in the original  $(n, d, \lambda)$ -graph  $G$  and thus they correspond to a collection of vertex-disjoint  $t$ -cliques covering all but at most  $n^{1-1/(8t^4)}$  vertices of the graph. This completes the proof of Theorem 1.3.  $\square$

## 7. CONCLUDING REMARKS

In this paper we have studied near-perfect  $K_t$ -factors in sparse  $(n, d, \lambda)$ -graphs. We have presented two different approaches for finding many ‘weight-disjoint’ fractional  $K_t$ -factors: one for ‘large’  $d$  (Theorem 4.1) and one for ‘small’  $d$  (Theorem 5.1).

We believe that the first approach is more powerful, since it can be extended to the whole range of  $d$  for all  $t \geq 4$  as follows. In the proof of Theorem 4.1 we used Proposition 2.11 to embed a copy of  $K_{t-1}$  in a set of  $\Omega(d)$  vertices, even after removing a small number of edges from each vertex, say  $o((d/n)^{t-2}d)$  many. We can allow to remove from each vertex even  $o(d^2/n)$  many edges if we just ask for the existence of a single copy of  $K_{t-1}$ , which would suffice for our first approach. To establish this, we could have used a Turán-type result in  $(n, d, \lambda)$ -graphs due to Sudakov, Szabó and Vu [22, Theorem 3.1]. However, a drawback is that the pseudorandomness condition in [22, Theorem 3.1] is not numerically explicit (it was stated as  $\lambda \ll d^{t-1}/n^{t-2}$ ). Thus we chose to present a self-contained and numerically explicit proof. Moreover, the first approach seems not to provide Theorem 1.3 for triangle-factors in the whole range, while triangle-factors can be considered as the most important testbed at the moment.

As for the second approach, we observe that it can be pushed to work in a range of  $d$  from  $\Omega(n^{1-1/(2t-3)})$  to  $n/\text{polylog } n$ , although near the upper bound we would only be able to assert that one can cover all but  $n/\text{polylog } n$  vertices with vertex-disjoint copies of  $K_t$ .

**Recent developments.** When finalizing this paper, we learned that very recently Nenadov [20] proved the existence of  $K_t$ -factors in  $(n, d, \lambda)$ -graphs with  $\lambda \leq \varepsilon d^{t-1}/(n^{t-2} \log n)$  for a suitably small  $\varepsilon = \varepsilon_t > 0$ , which is at most a  $\log n$  factor away from Conjecture 1.2. Independently, Morris

and the authors of this paper [12] proved a similar result with  $\lambda \leq \varepsilon_t d^t / n^{t-1}$ , which is a stronger requirement, except when  $d \geq c_t n / \log n$  for a suitable constant  $c_t > 0$ .

An  $n$ -vertex  $(\lambda, p)$ -bijumbled graph is a graph that satisfies inequality (1) of the expander mixing lemma (Theorem 1.1). This notion is slightly more general: any  $(n, d, \lambda)$ -graph is  $(\lambda, d/n)$ -bijumbled but in a bijumbled graph not every vertex has degree  $d$ . The result of [20] applies to  $(\varepsilon_t p^2 / (n \log n), p)$ -bijumbled graphs with minimum degree  $\Omega(pn)$ . The result of the present paper as well as the one in [12] can be easily shown to hold in appropriately bijumbled graphs as well.

## REFERENCES

- [1] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Blow-up lemmas for sparse graphs* (2016), available at [arXiv:1612.00622](https://arxiv.org/abs/1612.00622). ↑3
- [2] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Powers of Hamilton cycles in pseudorandom graphs*, *Combinatorica* **37** (2017), no. 4, 573–616. ↑2, 3
- [3] N. Alon and F. R. K. Chung, *Explicit construction of linear sized tolerant networks*, *Discrete Math.* **72** (1988), no. 1-3, 15–19. ↑1
- [4] N. Alon, *Explicit Ramsey graphs and orthonormal labelings*, *Electron. J. Combin.* **1** (1994), Research Paper 12, 8 pp (electronic). ↑1, 2
- [5] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov, *Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels*, *J. Combin. Theory Ser. A* **119** (2012), no. 6, 1200–1215. ↑9, 10
- [6] N. Alon and N. Kahale, *Approximating the independence number via the  $\vartheta$ -function*, *Mathematical Programming* **80** (1998), no. 3, 253–264. ↑2
- [7] N. Alon and J. H. Spencer, *The probabilistic method*, 4th edition, Hoboken, NJ: John Wiley & Sons, 2016. ↑1, 3
- [8] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics. 184. New York, NY: Springer, 1998. ↑3
- [9] D. Conlon, J. Fox, and Y. Zhao, *Extremal results in sparse pseudorandom graphs*, *Adv. Math.* **256** (2014), 206–290. ↑1
- [10] Z. Füredi, *Matchings and covers in hypergraphs*, *Graphs Combin.* **4** (1988), no. 2, 115–206. ↑10
- [11] J. Han, Y. Kohayakawa, and Y. Person, *Near-optimal clique-factors in sparse pseudorandom graphs*. Proceedings of Discrete Mathematics Days 2018, Sevilla, to appear. ↑2
- [12] J. Han, Y. Kohayakawa, P. Morris, and Y. Person, *Clique-factors in sparse pseudorandom graphs*, 2018. In preparation. ↑18
- [13] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience, New York, 2000. ↑3, 4
- [14] J. Komlós, G. N. Sárközy, and E. Szemerédi, *Blow-up lemma*, *Combinatorica* **17** (1997), no. 1, 109–123. ↑3
- [15] A. V. Kostochka and V. Rödl, *Partial Steiner systems and matchings in hypergraphs*, *Random Structures Algorithms* **13** (1998), no. 3-4, 335–347. ↑3, 10
- [16] M. Krivelevich, *Perfect fractional matchings in random hypergraphs*, *Random Structures & Algorithms* **9** (1996), no. 3, 317–334. ↑5
- [17] M. Krivelevich and B. Sudakov, *Pseudo-random graphs*, *More sets, graphs and numbers*, 2006, pp. 199–262. ↑1, 2, 9
- [18] M. Krivelevich, B. Sudakov, and T. Szabó, *Triangle factors in sparse pseudo-random graphs*, *Combinatorica* **24** (2004), no. 3, 403–426. ↑2, 6, 8, 9
- [19] J. Matoušek and B. Gärtner, *Understanding and using linear programming*, Berlin: Springer, 2007. ↑4
- [20] R. Nenadov, *Triangle-factors in pseudorandom graphs* (2018), available at [arXiv:1805.09710](https://arxiv.org/abs/1805.09710). ↑2, 17, 18
- [21] V. Rödl, *On a packing and covering problem*, *European J. Combin.* **6** (1985), no. 1, 69–78. ↑3
- [22] B. Sudakov, T. Szabó, and V. Vu, *A generalization of Turán’s theorem*, *Journal of Graph Theory* **49** (2005), no. 3, 187–195. ↑9, 17

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