A New Model for Evaluating Subjective Online Ratings with Uncertain Intervals

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Abstract

We formalize the information acquisition and choice structure of a decision maker (DM) when the main characteristics defining the alternatives are not directly observed but numerical evaluations of unknown quality are provided by external raters. The DM observes the overall numerical value assigned by the raters to an alternative and defines an uncertain interval within which the evaluation observed is contained. The width of the interval is determined by the subjective perception and evaluation differences existing between the DM and the raters transmitting the information. We analyze the incentives of the DM to improve upon an evaluation contained within an uncertain interval by retrieving further information from the raters of other alternatives. Different scenarios will be developed based on the ability of the DM to fully assimilate uncertainty and the introduction of heuristic approximations to account for the potential frictions arising from uncertainty. One of the main qualities of the current framework is its capacity to formalize interactions among alternatives determined by interval width differences across their characteristics, providing an analytical advantage over the operational complexity involved in the use of fuzzy and intuitionistic fuzzy sets. The same remark applies to the formalization of the interactions across attributes that must be considered when defining sequential decision processes or dynamical systems while dealing with multiple sources of uncertainty. Numerical simulations are provided to compare the different scenarios developed and describe the main consequences derived from ignoring the uncertainty inherent to the evaluations received. In particular, we illustrate the ranking consequences derived from increasing the spread of the evaluation uncertainty, an effect that can be easily combined with the risk attitude exhibited by DMs. The inclusion of both these features bridges the gap between economics, psychology and multiple criteria decision making, whose techniques do not generally account for these differences among DMs.

Keywords: uncertainty; online search; subjective perception; linguistic evaluations; heuristics.

1. Introduction

Consider a decision maker (DM) who must select an alternative when browsing online through a given search or recommender engine (Xiao and Benbasat, 2007; Hostler et al., 2011; Li et al., 2014). The DM must analyze the evaluations provided by other users regarding a set of characteristic categories defining the alternatives. Websites such as Amazon, Yelp, Reevoo, TripAdvisor and Trivago contain reviews of the different potential choice alternatives provided by different raters. The characteristics defining the alternatives are categorized, i.e. TripAdvisor offers a five-bullet rating schema for several categories, and an overall rating is generated for each alternative. Other websites, like Amazon, deliver a unique five-star-based rating where alternatives get an average general evaluation.

DMs have direct access to the numerical evaluations of other users as well as to more detailed linguistic descriptions of the alternatives. That is, the characteristics composing the alternatives are assigned both numerical and extended linguistic evaluations to define their rating summary, with the latter descriptions providing potentially superior evaluations (Tavana et al., 2015a). Opinion mining models have been developed to account for the subjectivity and imprecision inherent to linguistic evaluations, so as to categorize the expressions that consumers use to describe products (Huang et al., 2013). se

The DM aims at selecting the best alternative according to his preferences, which assigns a fundamental role in the selection process to the interpretation of the evaluations provided by the raters. The ratings provided are determined by the subjective perception of each alternative and its characteristics. It has been recognized that differences in the experience and learning capacity of the DMs condition the evaluations provided for the alternatives (Hertwig et al., 2004). In this regard, Tavana et al. (2015b) introduced a composite index that ranked the alternatives when the expected utility of the DM was determined by the potential perception differential arising between him and the raters.

In the current paper, a DM observes the overall numerical value assigned by the raters to an alternative, which may or may not coincide with the value that he would assign if he were to select it. As a result, the DM defines an uncertain interval within which the evaluation observed is contained. We formalize the incentives of the DM to improve upon an evaluation contained within an uncertain interval by retrieving further uncertain information from the raters of the same or other alternatives. In addition, we analyze the resulting effects on the information acquisition process and selection behavior of the DM. Different decision scenarios will be developed depending on the information assimilation capacity assumed on the DMs.

It will be assumed that the width of the uncertain interval subjectively defined by the DM is given exogenously. However, the spread of uncertainty could be directly related to the reputation of the raters, given its importance in online evaluation environments conditioned by the subjectivity and vagueness of the opinions provided (Fouss et al., 2010; Ashtiani and Azgomi, 2016), and the ability of the raters to report strategically (Peterson and Merino, 2003; Di Caprio and Santos Arteaga, 2011). In this regard, search and recommender engines generally include a classification of the raters, which allows the DM to discriminate among the reviewers and reports through a reliability index (Tavana et al., 2017).

All in all, the main aim of the formal environment introduced in the current paper is to define an evaluation framework that can serve as an alternative to fuzzy multiple criteria decision making (MCDM) models when DMs retrieve numerical ratings subject to multiple sources of uncertainty. In order to do so, the model incorporates the main formal requirements from economists and psychologists within a MCDM environment, namely:

- the beliefs of DMs regarding the quality of the evaluations retrieved from the raters;
- the interactions across alternatives whose characteristics are defined by uncertain evaluation intervals; and
- the risk attitude of DMs.

The paper proceeds as follows. Section 2 presents a literature review. Section 3 defines the basic assumptions on which the decision model is built. Section 4 describes the potential improvement scenarios considered, which are incorporated in the formal decision framework introduced in Section 5. Section 6 presents several numerical comparisons among the proposed improvement scenarios. These comparisons provide intuition required to analyze the real-life evaluation and ranking environments described in Section 7. Section 8 concludes and suggests future research directions. A complete formalization of the value functions determining the behavior of the DMs is provided in the Appendix.

2. Related literature

The emergence and rapid diffusion of online evaluation and recommendation engines has led to a recent and growing research field dealing with the increasing amount of potential choices available to DMs (Schwartz, 2004), the high degree of subjectivity inherent to the information available

(Zadeh, 1975), and the limited capacity of DMs to evaluate the information acquired and implement a utility maximizing choice strategy (Simon, 1997).

The necessity to develop suitable formal utility models that reflect real-life decision making problems, with vague preferences and imperfect information described through natural language, has been consistently emphasized by information scientists (Aliev et al., 2012b, 2013). Indeed, extensions of the standard expected utility models implemented in the economic and decision theoretical literature through fuzzy (Zadeh, 1975) and intuitionistic fuzzy sets (Atanasov, 1986; Szmidt, 2014) have already been defined (Aliev et al., 2012, 2013). Aiming at improving upon the economic decision-theoretical approach, Aliev et al. (2012a) defined a fuzzy-valued non-expected utility model designed to represent linguistic preference relations and imprecise beliefs.

Bounded rationality constitutes a second alternative to the expected utility normative approach (Simon, 1955). In this case, DMs define a heuristic rule that allows them to reach a given satisficing level from their choices (Bettman, 1979). Heuristic rules are generally determined through a subjective tradeoff defined between the accuracy of the decisions and the cognitive effort implemented (Payne et al., 1993; Gigerenzer and Selten, 2002). This branch of the literature has highlighted the limited information processing capacity of DMs, even when considering regular and not extremely complex decision settings (Payne et al., 1993; Samiee et al., 2005), with DMs focusing on particular attributes within small sets of alternatives.

Finally, cognitive science has recently taken the lead, given the acknowledgement of the importance that the subjective perception of the different alternatives and their characteristics has for the selection and decision processes of DMs (Bartels and Johnson, 2015). Cognitive science has overtaken the economic rational approach given the sophistication that it can provide when analyzing the cognitive abilities and limitations of the DMs (Chater, 2015). It has, for example, illustrated how the memory and attention processes affect the value of the alternatives perceived by the DM (Stewart et al., 2006), or that perception differs across genders, a fact documented when studying online reviews (Bae and Lee, 2011).

2.1. Fuzzy approaches developed within the systems literature

The fuzzy (and, in particular, the intuitionistic fuzzy) literature has consistently acknowledged the perception and evaluation differences existing across DMs. Indeed, when considering online evaluations or the receipt of reports from an unknown third person describing the characteristics of a given alternative, a DM is known to face three well-defined uncertainty levels. Firstly, DMs initiate search processes because they are uncertain about the distribution of the characteristics defining the different alternatives within their respective domains (Lanzetta 1963). Secondly, information senders deliver subjective evaluations of the characteristics based on their corresponding perception and reporting capabilities (Dimoka et al., 2012). In this case, the extent of these subjective inaccuracies must be determined by the DM. Thirdly, the perception and evaluations of the DM are also subjective and subject to the very same inaccuracies inherent to the evaluations of other agents. In other words, the evaluation of alternatives is a subjective process constrained by the uncertainty that follows from the limited cognitive capacities of the DMs (Lerner et al., 2015). As the psychology literature has illustrated, the resulting evaluations depend on personal factors such as experience (Kimmel, 2012), the beliefs of DMs (Fishbein, 1963) and personal emotions (Zeelenberg et al., 2008).

All in all, when evaluating an alternative, each one of its characteristics is subject to three different uncertainty levels, which arise from the distribution of its potential realizations, the perception and evaluation subjectivity of the information senders and those of the DMs. In this regard, when considering real-life applications of intuitionistic fuzzy sets, their comparability arises as a considerable drawback limiting their implementation across research fields. Despite their capacity to incorporate different uncertainty levels into the analysis, they fail to provide comparable evaluations (Ngan, 2016). This has been consistently emphasized in the systems literature, which is recently aiming at improving the applicability of these operators. For instance, Ngan (2017) emphasized the substantial efforts placed on the design of suitable fundamental operators and measures. Such efforts have led to a highly complex series of developments within the fuzzy set theoretical field, which constitutes a severe shortcoming when promoting their application to different research domains.

This is particularly the case when considering higher-order fuzzy sets, with the recent systems literature focusing on the design and implementation of computationally simple and efficient operators (Ngan, 2018). As a result, the design of similarity measures for intuitionistic fuzzy sets that can be easily implemented across different research fields prevails as one of the main concerns of the systems literature nowadays. Applications focus mainly on machine learning-based environments and encompass problems dealing with pattern recognition, classification and clustering (Milošević et al., 2017; Qian et al., 2019).

Particular emphasis has also been placed on multi-criteria and group decision-making frameworks. For instance, Pamučar et al. (2017) dealt with uncertainty in MCDM environments using interval rough numbers so as to focus on the interval knowledge inherent to the data provided by DMs. In a similar MCDM setting, Fan et al. (2019) applied intuitionistic fuzzy rough numbers to aggregate group information. Pamučar et al. (2018) formalized uncertainty through interval-valued fuzzy-rough numbers in order to eliminate the subjectivity that arises when defining the borders of fuzzy sets. Finally, Narayanamoorthy et al. (2019) applied interval-valued intuitionistic hesitant fuzzy entropy to determine the relative importance of the criteria within a VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) setting.

It should be emphasized that, given the operational complexities involved, these models do not formalize the interactions across attributes that must be considered when defining sequential decision processes or dynamical systems. This is particularly the case when incorporating multiple sources of uncertainty to the analysis, as we do in the current setting. Moreover, the strategic interactions taking place across different information sources are not accounted for when following the intuitionistic fuzzy approach currently being applied in the systems literature.

Finally, we would like to note that despite the substantial importance assigned to fuzzy multiple criteria decision scenarios, most of the evaluations provided in online environments by popular rating websites consist of crisp numerical values. In this regard, we retrieve a set of laptop (multiple criteria) evaluations from one of these rating websites, rank them by implementing the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) and illustrate the main effects from modifying the width of the uncertain evaluation intervals when considering the best potential use that may be given to the information acquisition process of DMs. We illustrate how different uncertainty spreads lead to ranking modifications that can be easily interpreted and analyzed.

3. Basic assumptions

The main intuition on which our methodological approach is built follows from the concept of information entropy and the imprecision inherent to the linguistic and numerical evaluations provided by raters. As illustrated through Section 2, subjective evaluations are inherently imprecise and do not necessarily coincide among DMs. The ability of a DM to acquire information so as to improve upon the evaluations received and select the best potential alternative should therefore depend on the width of the corresponding uncertain evaluation intervals. Combining this feature – together with the resulting interactions across characteristics and alternatives – and the risk attitude of DMs within a unified evaluation framework allows us to incorporate the main formal requirements from economists and psychologists.

In other words, besides its capacity to allow for uncertainty interactions across characteristics and alternatives, the current model can simultaneously account for the risk attitude of DMs as well as their beliefs regarding the quality of the evaluations retrieved from the different groups of reviewers rating the set of alternatives available. Note that if the standard fuzzy multiple criteria decision making models were to incorporate this latter feature, it would imply considering fuzzy numbers of variable widths – determined, for instance, by the number of reviews per alternative –, as well as the different interactions taking place across the resulting fuzzy numbers through characteristics and alternatives.

Let X_1 be the set of all possible values that can be taken by one of the main characteristics defining an alternative. In order to simplify the presentation, we identify a generic characteristic composing the different alternatives with the numerical value $x_1 \in X_1$. That is, we assume that alternatives are described by crisp numerical evaluations such as those provided in the standard search and recommender engines. Note that an evaluation does not necessarily describe a unique property of the characteristic being considered, but generally averages a series of them whose combination characterizes each alternative within X_1. We restrict our attention to the case where the set of potential evaluations is given by $X_1 = [x_1^m, x_1^M]$ with $x_1^m, x_1^M \in \mathbb{R}$, $x_1^m \neq x_1^M$. Henceforth, D will be used to denote a generic DM. Following the standard approach to choice under uncertainty (Mas-Collel et al., 1995), D is assumed to define

- a strictly increasing continuous utility function $u: X_1 \to \mathbb{R}$ to represent his preferences on X_1;
- a continuous probability density function $\mu: X_1 \to [0,1]$ to express his subjective beliefs regarding the potential values in X_1 that can be taken by a randomly selected alternative.

Without loss of generality, we assume that $Support(\mu) = X_1$, that is, the support of the probability function coincides with the set of potential characteristic values defining an alternative.

3.1. Uncertain evaluations

The current framework is based on the intuition describing the information content of an interval evaluation in terms of its entropy, with the uniform distribution displaying the lowest information content and highest entropy (Tavana et al., 2015). That is, each review is associated with a potential set of evaluations generated by the unknown quality of the information received. Given the uncertainty faced by D, such interval is associated with a uniform distribution, as is the case when the highest entropy is defined on an interval domain (Tavana et al., 2014).

We will assume that when the realization of the characteristic is given by x_1^r, D expects his evaluation to be contained within the interval $[x_1^r - \varepsilon, x_1^r + \varepsilon]$, for some $\varepsilon > 0$ defining the spread of uncertainty. In other words, the evaluation provided by the raters of an alternative and that of D do not necessarily coincide. D is aware of the subjectivity inherent to the evaluation received and defines an interval of potentially viable evaluations around it.

Ideally, D should account for the whole set of potential evaluations derived from the x_1^r realization as well as the probability density value associated to each one of them. As stated above, we will assume that the ε-based approximation is uniformly distributed over the interval of potential evaluations, maximizing both uncertainty and information entropy on the side of D

$$\vartheta(x_1 \mid \varepsilon) = \begin{cases} \dfrac{1}{2\varepsilon} & if \ x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon] \\ 0 & otherwise \end{cases} \tag{1}$$

Thus, when D is told that the characteristic of an alternative is given by x_1^r, he calculates the expected utility derived from this revealed characteristic following

$$\int_{x_1^r - \varepsilon}^{x_1^r + \varepsilon} \vartheta(x_1 \mid \varepsilon) u(x_1)\, dx_1 \tag{2}$$

where $\vartheta(x_1 \mid \varepsilon) = \dfrac{1}{2\varepsilon}$, though the distribution of the potential evaluations contained within the interval $[x_1^r - \varepsilon, x_1^r + \varepsilon]$ could be modified depending on whether D exhibits a certain degree of optimism or pessimism about the rater providing the evaluation.

It should be emphasized that the spread of the uncertain interval, i.e. the value assigned to ε, only has an effect on the value of the expected utility when D is not risk neutral, i.e. when the utility function is non-linear. Otherwise, the value of the expected utility would remain unchanged independently of the value assigned to ε. This will not be the case in the current setting, where potential improvements upon the realizations received will be determined by the spread of uncertainty.

In this regard, note that the ambiguity inherent to the evaluations received opens a second uncertain dimension on the side of D. The first dimension corresponds to the distribution of alternatives within the evaluation space X_1, while the second refers to the spread of the ambiguity inherent to the potential evaluations.

4. Potential improvement scenarios

Taking $x_o^r \in X_1$ as an initial realization, we focus on the subsets $I^+(x_o^r)$ composing X_1 and containing the values of all the new potential realizations $x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon]$ delivering a utility higher than $x_o \in [x_o^r - \varepsilon, x_o^r + \varepsilon]$. That is, we consider the values of any new realization x_1^r that delivers a utility higher than x_o^r within any of the potential evaluations contained in the domain of the respective uncertain intervals.

This scenario requires D to completely assimilate the uncertainty inherent to the realizations observed. Potential improvements must therefore be defined over the whole evaluation interval implied by the initial realization, $x_o \in [x_o^r - \varepsilon, x_o^r + \varepsilon]$, whose uncertainty as well as that inherent to any new realization, $x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon]$, must be taken into account by D

$$I^+(x_o^r) \overset{def}{=} \{x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon] \cap \mathrm{Support}(\mu) : u(x_1) > u(x_o),\ x_1^r \in X_1,\ x_o \in [x_o^r - \varepsilon, x_o^r + \varepsilon]\} \tag{3}$$

Note that, improvements become potentially plausible as soon as the intervals assigned to both observations overlap, i.e. as soon as $[x_1^r - \varepsilon, x_1^r + \varepsilon] \cap [x_o^r - \varepsilon, x_o^r + \varepsilon] \neq \varnothing$. The probability of improvement must however be adapted depending on the relative position of the x_o^r and x_1^r realizations.

In line with the bounded rationality literature (Simon, 1955, 1997), we will also assume that D may define a satisficing value that determines his search incentives and, therefore, his choices. That is, we will compare the above complete assimilation scenario with a heuristic mechanism defining improvements over the $[x_o^r - \varepsilon, x_o^r]$ evaluation interval of x_o^r while ignoring the uncertainty inherent to the realizations. Thus, D will aim at guaranteeing an ε-improvement over the observation x_o^r while disregarding the potential evaluations located within $[x_o^r - \varepsilon, x_o^r + \varepsilon]$

$$I^+(x_o^r) \overset{def}{=} \{x_1 \in X_1 \cap \mathrm{Support}(\mu) : u(x_1^r) > u(x_o^r + \varepsilon)\} \tag{4}$$

Finally, a limit case introduced for comparison purposes will consist of completely ignoring any uncertainty inherent to the realizations x_o^r and x_1^r. In this case, we eliminate the satisficing ε-based shift introduced in Equation (4), allowing D to improve upon x_o^r without imposing any evaluation interval requirement

$$I^{+}(x_o^r) \overset{def}{=} \{x_1 \in X_1 \cap \text{Support}(\mu) : u(x_1^r) > u(x_o^r)\} \tag{5}$$

Note that ignoring the inherent uncertainty, i.e. assuming that the perception and evaluations of the unknown raters equal those of D, can lead to suboptimal choices due precisely to the potential differences existing in the perception and evaluation of the alternatives.

5. Potential improvement sets: Complete uncertainty and heuristics

In the current section, we analyze the information acquisition and choice incentives of D under the three improvement scenarios defined in Equations (3)-(5). We start by describing the concept of value improvement, which is determined by the initial observation x_o^r and the potential realizations located within $I^{+}(x_o^r)$. The resulting value function $V : X_1 \times \mathbb{R}^{+} \to \mathbb{R}$ is defined as follows

$$V(x_o^r, \varepsilon) = \int_{I^{+}(x_o^r)} \mu(x_1^r) \left[\int_{x_1^r - \varepsilon}^{x_o^r + \varepsilon} \vartheta(x_1 \mid \varepsilon) \left(E[u(x_1)] \right) dx_1 + \int_{x_o^r + \varepsilon}^{x_1^r + \varepsilon} \vartheta(x_1 \mid \varepsilon) \left(u(x_1) \right) dx_1 \right] dx_1^r \tag{6}$$

The value of the function defined in Equation (6) is based on the potential improvements over the $[x_o^r - \varepsilon, x_o^r + \varepsilon]$ interval that can be obtained with the next observation, i.e. $x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon]$, $\forall x_1^r \in X_1$. In order to provide additional intuition on the working of this function for a given x_o^r value and a unique $x_1^r > x_o^r$, we have illustrated the overlapping of both evaluation intervals in Figure 1. The overlapping of both uncertain intervals is accounted for by the expectation operator defined in the first term within the brackets, i.e. $E[u(x_1)]$, while the remaining section of the domain defined by x_1^r constitutes a pure improvement over x_o^r. Note that the value function must be defined for each and every $x_o^r \in X_1$ and that for each one of these values a set of potential improvements must be defined via $I^{+}(x_o^r)$. We will develop the corresponding terms further through the next subsections.

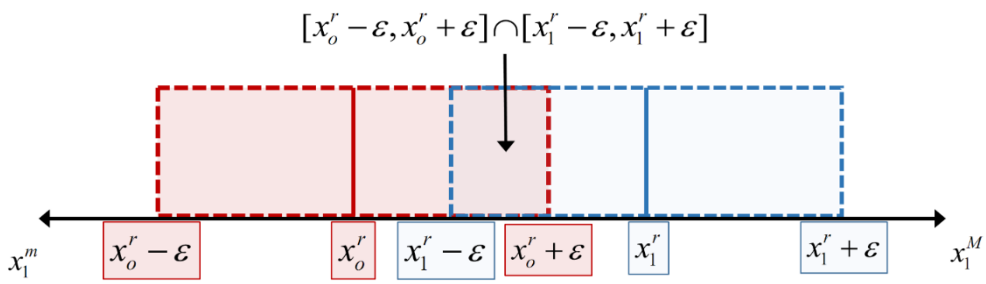

Figure 1. Generic potential improvements defined by any $x_1\in[x_1^r-\varepsilon, x_1^r+\varepsilon]$ over any initial $x_o\in[x_o^r-\varepsilon, x_o^r+\varepsilon]$

5.1. Lower and upper limits of the domain

Note that the domain of the potential evaluations defined by the uncertain quality of the x_1^r realizations will eventually exceed both the lower and upper limits of the domain of X_1 while interacting with the domain of potential evaluations defined by the initial observation x_o^r. As a result, several subcases must be explicitly accounted for when considering the lower and upper limits of the domain on which the characteristics observed are defined.

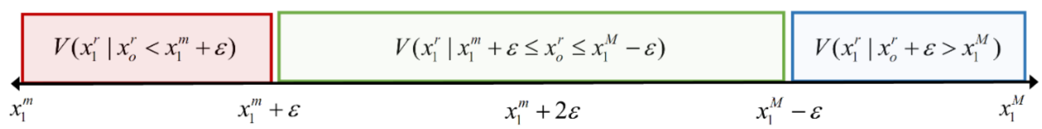

Figure 2. Different potential improvement areas composing the domain of X_1

Figure 2 has been introduced to provide additional intuition regarding the basic areas composing the domain of X_1 that should be analyzed when considering the initial realizations observed and their potential improvements. We will define seven different improvement sections through this section, while noting that further modifications can still be considered depending on the width of the uncertain domain defined by ε. We will illustrate this latter case through the numerical simulations provided in Section 6.

First, consider values of the initial realization located within the lower sections of the domain of X_1. Three particular subcases must be analyzed when describing the different $I^+(x_o^r)$ scenarios.

5.1.1. Lower interval section: $x_o^r < x_1^m+\varepsilon$

The value function defined for $x_o^r\in[x_1^m, x_1^m+\varepsilon]$, and such that $x_1^M > x_o^r+3\varepsilon$, is given by

$$
\begin{aligned}
&V(x_o^r \mid x_o^r < x_1^m + \varepsilon) = \\
&\int_{x_1^m}^{x_o^r} \mu(x_1^r)\left[\int_{x_1^m}^{x_1^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon)-x_1^m}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r + \\
&\int_{x_o^r}^{x_1^m+\varepsilon} \mu(x_1^r)\left[\int_{x_1^m}^{x_o^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon)-x_1^m}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon)-x_1^m}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_1^m+\varepsilon}^{x_o^r+2\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_o^r+2\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big[u(x_1)\big]dx_1\right]dx_1^r + \\
&\int_{x_1^M-\varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big[u(x_1)\big]dx_1\right]dx_1^r
\end{aligned}
\tag{7}
$$

Figure 3 has been introduced to provide additional intuition regarding the different spread of the domains – and the associated densities – when the initial observation and its potential improvements differ while exceeding the lower domain limit of X_1. Two distinct levels of uncertainty are accounted for by the value function:

- $\mu(x_1^r)$ considers the uncertainty inherent to the potential realizations of $x_1^r \in X_1$;
- $\vartheta(x_1 \mid \varepsilon)$ accounts for the ambiguity associated with the evaluations received; in this case, the uniform density adapts to the value of x_1^r and the domain limits of X_1.

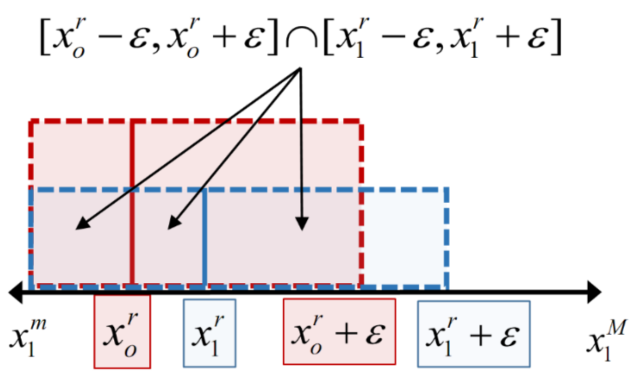

Figure 3. Potential improvements defined within the lower interval section

Since the value function has been designed to account for the potential improvements derived from observing x_1^r, it has been implicitly assumed that when the domains inherent to the x_o^r and x_1^r realizations overlap, D selects the newly observed alternative and computes the expected utility of having selected the more preferred alternative. The disutility derived from selecting the less preferred alternative has been defined as a constant, $c>0$, though it can be endogenized as a function of the distance between x_1^r and x_o^r, depending on their relative position. The intuition justifying the inclusion of this constant follows from the psychology literature. Psychologists have analyzed the satisfaction derived from the acquisition of a product and the subsequent increment or decrement experienced by the utility of D when his evaluation is compared to the expected value of the characteristics (Anderson and Sullivan, 1993). A main conclusion derived from this analysis is that whenever quality falls below the level expected by D, its negative impact on satisfaction is greater than the one obtained if expectations are exceeded.

We have introduced the weights $\frac{x_1^r}{x_o^r + x_1^r}$ and $\frac{x_o^r}{x_o^r + x_1^r}$ to define the expected payoff received from selecting the x_1^r alternative when the domains inherent to the realizations overlap. These weights assign a probability to x_1^r being the preferred alternative based on the relative position of the x_o^r and x_1^r realizations within X_1. Note, for example, that the first term on the right hand side of Equation (7) implies that when D selects x_1^r instead of x_o^r he improves upon x_o^r with a very low probability since $x_1^r < x_o^r$ within the interval being considered.

Finally, the value function computed by D within the heuristic scenario described in Equation (4) considers only full improvements via its corresponding $I^+(x_o^r)$ set, leading to

$$V(x_o^r \mid x_o^r < x_1^m + \varepsilon) = \int_{x_o^r+\varepsilon}^{x_1^M} \mu(x_1)u(x_1)dx_1 \tag{8}$$

A similar intuition applies to the heuristic setting within the remaining subsets of the domain of X_1. In order to simplify the presentation of the full assimilation and heuristic settings, the middle sections of the corresponding value functions for potential realizations of x_o^r contained within the $[x_1^m+\varepsilon, x_1^m+2\varepsilon]$ to $[x_1^M-2\varepsilon, x_1^M-\varepsilon]$ intervals is provided in the Appendix section.

5.1.2. Upper interval section: $x_o^r + \varepsilon > x_1^M$

The value function defined for $x_o^r \in [x_1^M - \varepsilon, x_1^M]$, and such that $x_1^m < x_o^r - 3\varepsilon$, is given by

$$
\begin{aligned}
&V(x_o^r \mid x_o^r+\varepsilon > x_1^M) = \\
&\int_{x_o^r-2\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_0^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\left(u(x_1)\right)+\frac{x_o^r}{x_o^r+x_1^r}\left(u(x_1)-c\right)\right]dx_1\right]dx_1^r + \\
&\int_{x_1^M-\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\left[\frac{x_1^r}{x_o^r+x_1^r}\left(u(x_1)\right)+\frac{x_o^r}{x_o^r+x_1^r}\left(u(x_1)-c\right)\right]dx_1\right]dx_1^r + \\
&\int_{x_o^r}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\left[\frac{x_1^r}{x_o^r+x_1^r}\left(u(x_1)\right)+\frac{x_o^r}{x_o^r+x_1^r}\left(u(x_1)-c\right)\right]dx_1\right]dx_1^r
\end{aligned} \tag{9}
$$

Similarly to the previous interval cases, the first two terms on the right hand side of Equation (9) describe the limited improvement capacity of the new observation when D selects x_1^r instead of x_o^r, a tendency reversed as x_1^r overtakes x_o^r through the third and final right hand side term. Note also how the uniform density $\vartheta(x_1 \mid \varepsilon)$ adapts to the value of x_1^r as the upper domain limit of X_1 is reached. As was the case with Figure 3, Figure 4 has been introduced to provide additional intuition regarding the different spread of the domains – and the associated densities – when the initial observation and its potential improvements differ while exceeding the upper domain limit of X_1.

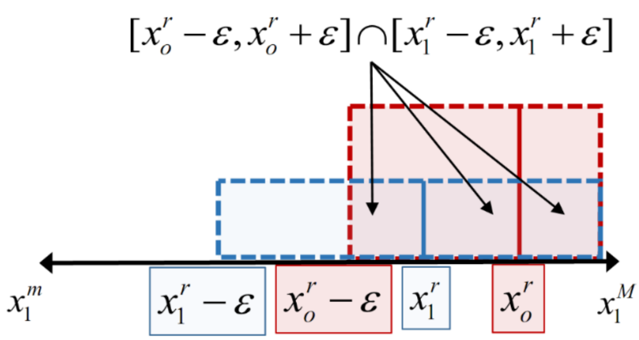

Figure 4. Potential improvements defined within the upper interval section

Given the fact that the value function computed by D within the heuristic scenario considers only full improvements via its corresponding $I^+(x_o^r)$ set, we have that

$V(x_o^r \mid x_o^r+\varepsilon > x_1^M) = \int_{x_o^r+\varepsilon}^{x_1^M} \mu(x_1)u(x_1)dx_1 = 0$, since $x_o^r+\varepsilon > x_1^M$.

To summarize, the information acquisition and evaluation processes of D are determined by the set of value functions defined above. Thus, after D observes x_o^r from an initial alternative, the decision of whether or not to consider other alternatives is based on the value taken by the function $V(\cdot)$ at x_o^r, which, at the same time, is determined by the width of the uncertain domain defined by ε. The evaluations obtained by D are illustrated numerically in the next section for a set of standard parameters defined in search and recommender engines and different values of ε.

6. Numerical evaluation

Consider a standard reference framework where D is endowed with a risk neutral utility function, $u(x_1) = x_1$, and uniform probabilities, $\mu(x_1) = \frac{1}{10}, \forall x_1 \in X_1$, are assumed on $X_1 = [0,10]$ to illustrate the uncertainty faced by D regarding the potential evaluations received on the set of alternatives. Consider also a disutility cost of $c = 1$ through all the simulations, an assumption that does not modify the qualitative results obtained.

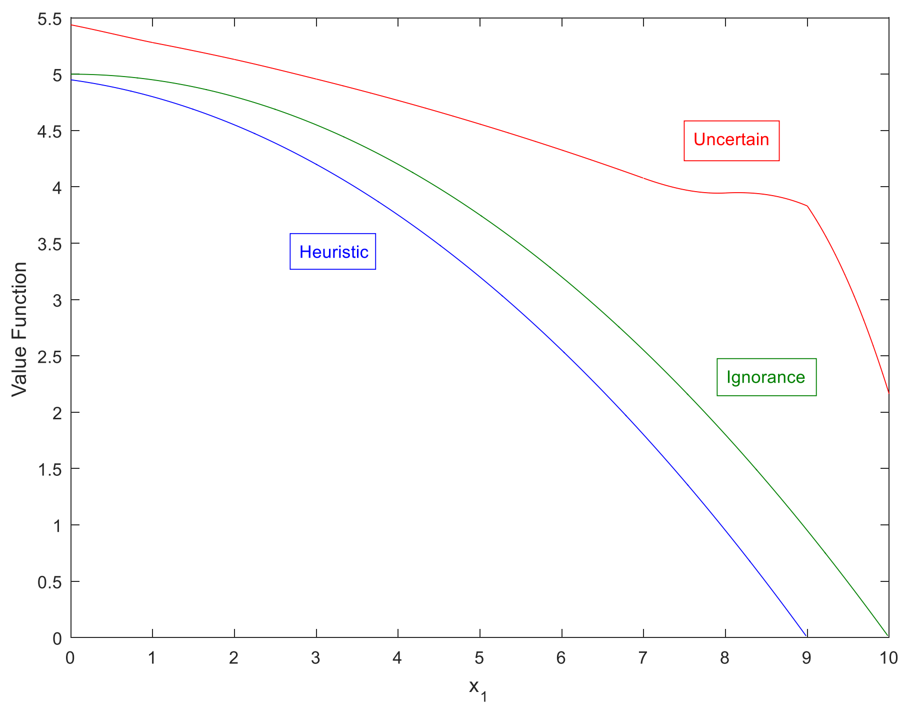

Figure 5. Risk neutral evaluation setting with $\varepsilon = 1$

The above reference framework is illustrated in Figure 5 for a spread of $\varepsilon = 1$. The horizontal axis represents the set of $x_1 \in X_1$ realizations that may be observed by D, while the subjective evaluations obtained from the value functions for the different improvement scenarios are defined on the vertical axis. The following simplifications have been introduced to eliminate the non-linearity of the value function and allow for its computation

- x_1^r has been removed from the $\vartheta(x_1 \mid \varepsilon)$ densities at the domain limits of X_1. As a result, $\frac{1}{(x_1^r+\varepsilon)-x_1^m}$ and $\frac{1}{x_1^M-(x_1^r-\varepsilon)}$ have been both converted in a function of ε, i.e. $\frac{1}{\varepsilon}$, with $\varepsilon > 0$.
- The denominator of the weights $\frac{x_1^r}{x_o^r+x_1^r}$ and $\frac{x_o^r}{x_o^r+x_1^r}$ has been modified from $x_o^r+x_1^r$ to 10, which corresponds to the width of the evaluation domain.

At the same time, Figure 6 illustrates the risk neutral reference framework with the spread of uncertainty increased to $\varepsilon = 2$, where a non-linear pattern more distinct that in the $\varepsilon = 1$ scenario can be observed. It should be highlighted that the $\varepsilon = 2$ scenario requires defining the following equation for the value function when considering the $x_o^r \in [4,6]$ interval

$$
\begin{aligned}
&V(x_o^r \mid x_1^m + 2\varepsilon \le x_o^r + \varepsilon \le x_1^M - 2\varepsilon) = \\
&\int_{x_o^r-2\varepsilon}^{x_1^m+\varepsilon} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon)-x_1^m}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r + \\
&\int_{x_1^m+\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r + \\
&\int_{x_o^r}^{x_o^r+\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_o^r+\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_1^M-\varepsilon}^{x_o^r+2\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_o^r+2\varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big[u(x_1)\big]dx_1\right]dx_1^r
\end{aligned}
\tag{10}
$$

Equation (10) combines the $x_1^m + 2\varepsilon \le x_o^r \le x_1^m + 3\varepsilon$ setting of Equation (A2) with the $x_1^M - 3\varepsilon \le x_o^r \le x_1^M - 2\varepsilon$ framework of Equation (A4). Note that both these equations are located in the Appendix section.

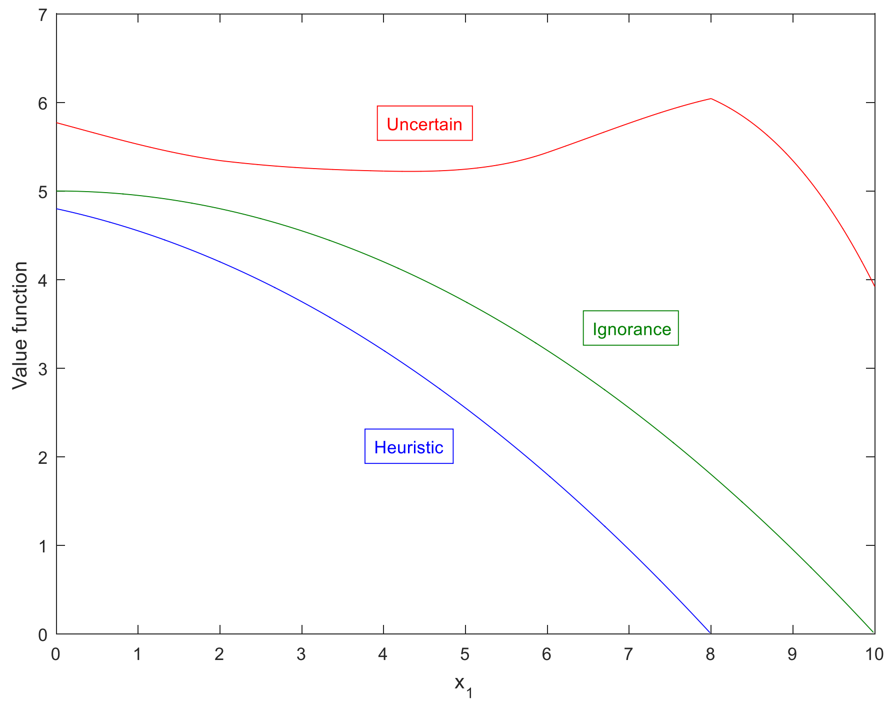

Figure 6. Risk neutral evaluation setting with $\varepsilon = 2$

Two main results follow from a direct comparison of Figures 5 and 6.

1. When considering the heuristic and ignorance settings that correspond to Equation (4) and Equation (5), respectively, the search incentives of D are considerably lower than in the uncertain scenario. That is, accounting for uncertainty increases the expected utility value derived from additional information, leading D to consider a wider set of potentially preferred alternatives. This effect is exacerbated if a relatively high search cost is imposed. As a result, regret is more prone to arise among those D who do not assimilate the uncertainty inherent to the evaluations.
2. Considerably different expected values are obtained in the $\varepsilon = 1$ and $\varepsilon = 2$ settings, which implies that given the same information retrieval and assimilation costs, the incentives to acquire information will differ substantially depending on the width of the uncertainty spread. In particular, as intuitively expected, a lower spread of uncertainty decreases the relative differences among the different improvement scenarios.

Figures 7 and 8 represent both uncertainty spread settings with a risk averse D defined by $u(x_1) = \sqrt{x_1}$. The same trends can be observed in the corresponding figures when compared to the linear utility case. Note, however, that if we were to consider information costs, the same value of the cost would lead to lower x_1 thresholds being reached in the risk averse case.

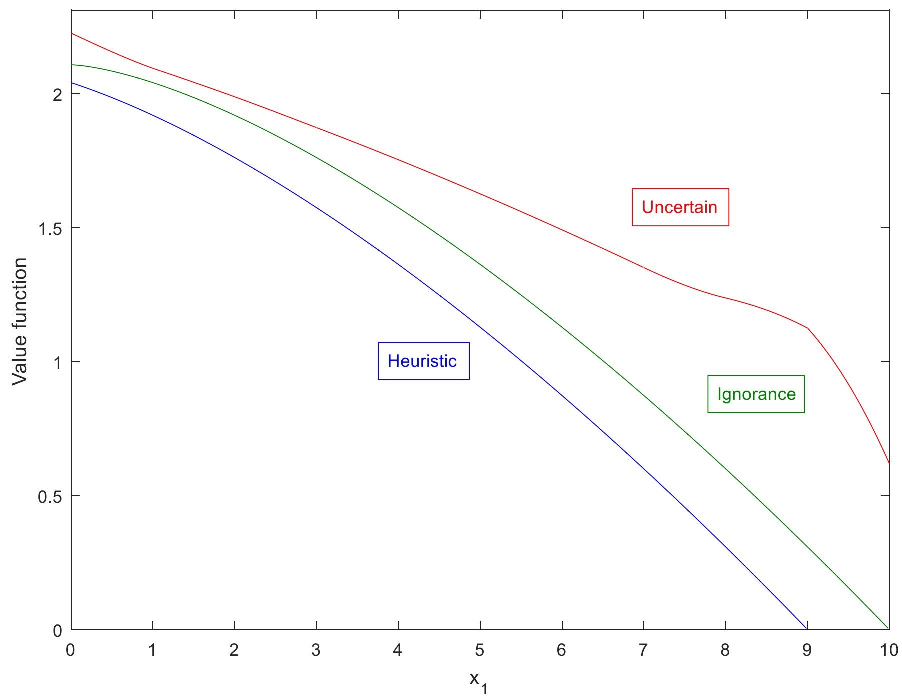

Figure 7. Risk averse evaluation setting with $\varepsilon = 1$

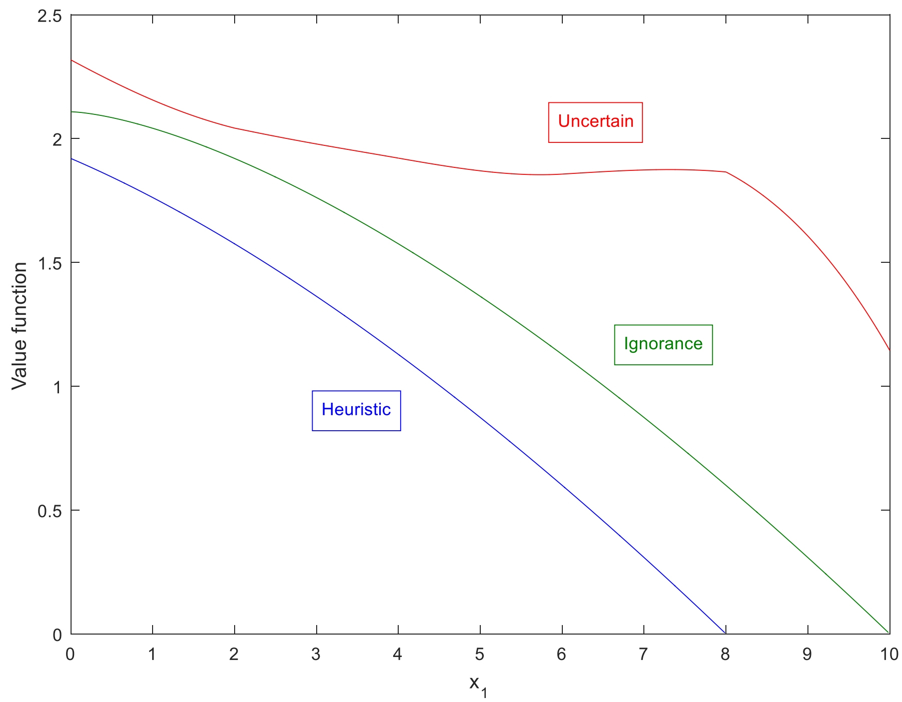

Figure 8. Risk averse evaluation setting with $\varepsilon = 2$

Thus, recommender engines can be used to increase both the incentives of D to consider additional alternatives as well as the resulting expected utility derived from an ampler set of potentially preferred alternatives, particularly when information retrieval and assimilation costs are imposed on D. Finally, we must highlight the non-linear shape taken by the value function as the uncertainty spread increases to $\varepsilon = 2$, leading to areas within the domain of X_1 where the search incentives of D are reversed.

We conclude this section by noting that a unique uncertain variable has been considered in the framework formalized, even though alternatives generally consist of several characteristics whose evaluation is subject to the same type of uncertainty. Thus, an immediate extension of the current model, which can be formalized due to its flexibility, would be to consider alternatives composed by several uncertain characteristics interacting within a sequential comparative framework. In this case, new sets of potential evaluations must be introduced in the analysis to account for the realizations of the additional characteristics defining the alternatives.

For instance, consider the sequential decision framework defined by Di Caprio et al. (2014) describing the incentives to either continue observing a given alternative or shift to a different alternative. Extending the analysis introduced through the current paper, the continuation function and the corresponding reference sets would be given by:

$$\begin{aligned} CT(x_0^r, y_1^r, \varepsilon) = & \int_{x_0^r-\varepsilon}^{x_0^r+\varepsilon} \vartheta(x_1 \mid \varepsilon)\left[\int_{P^+(x_0^r)} \eta(y_1^r)\left(\int_{y_1^r-\varepsilon}^{y_1^r+\varepsilon} \vartheta(y_1 \mid \varepsilon)\big(u(x_0)+v(y_1)\big)dy_1\right)dy_1^r\right]dx_1 + \\ & \int_{x_0^r-\varepsilon}^{x_0^r+\varepsilon} \vartheta(x_1 \mid \varepsilon)\left[\int_{P^-(x_0^r)} \eta(y_1^r)\left(\int_{y_1^r-\varepsilon}^{y_1^r+\varepsilon} \vartheta(y_1 \mid \varepsilon)\big(E_X+E_Y\big)dy_1\right)dy_1^r\right]dx_1 \end{aligned} \tag{11}$$

$$P^+(x_0^r) \overset{def}{=} \{y_1 \in [y_1^r-\varepsilon, y_1^r+\varepsilon] \cap \text{Support}(\eta) : v(y_1) > E_X + E_Y - u(x_0), y_1^r \in Y_1, x_0 \in [x_0^r-\varepsilon, x_0^r+\varepsilon]\} \tag{12}$$

$$P^-(x_0^r) \overset{def}{=} \{y_1 \in [y_1^r-\varepsilon, y_1^r+\varepsilon] \cap \text{Support}(\eta) : v(y_1) \le E_X + E_Y - u(x_0), y_1^r \in Y_1, x_0 \in [x_0^r-\varepsilon, x_0^r+\varepsilon]\} \tag{13}$$

where $Y_1 = [y_1^m, y_1^M]$ with $y_1^m, y_1^M \in \mathbb{R}$, $y_1^m \neq y_1^M$, denotes the set of potential evaluations for the second characteristic, $\eta : Y_1 \to [0,1]$ accounts for the subjective beliefs of D regarding the potential values in Y_1 that can be taken by the second characteristic of a randomly selected alternative, and $v : Y_1 \to \mathbb{R}$ is a strictly increasing continuous utility function representing the preferences of D on Y_1. The remaining notation follows intuitively from the one defined throughout the paper for X_1.

The starting function would be based on the potential improvements provided by the next X_1 evaluation relative to the $x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon]$ interval. The function and the corresponding reference sets would be defined as follows:

$$ST(x_0^r, \varepsilon) = \int_{Q^+(x_0^r)} \mu(x_1^r) \left[\int_{x_1^r - \varepsilon}^{x_1^r + \varepsilon} \vartheta(x_1 \mid \varepsilon) \left(E[u(x_1)] + E_Y \right) dx_1 \right] dx_1^r + \int_{Q^-(x_0^r)} \mu(x_1^r) \left[\int_{x_1^r - \varepsilon}^{x_1^r + \varepsilon} \vartheta(x_1 \mid \varepsilon) \left(\max\{u(x_0), E_X\} + E_Y \right) dx_1 \right] dx_1^r \tag{14}$$

$$Q^+(x_0^r) \overset{def}{=} \{x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon] \cap \text{Support}(\mu) : u(x_1) > \max\{u(x_0), E_x\}, x_1^r \in X_1, x_0 \in [x_0^r - \varepsilon, x_0^r + \varepsilon]\} \tag{15}$$

$$Q^-(x_0^r) \overset{def}{=} \{x_1 \in [x_1^r - \varepsilon, x_1^r + \varepsilon] \cap \text{Support}(\mu) : u(x_1) \le \max\{u(x_0), E_x\}, x_1^r \in X_1, x_0 \in [x_0^r - \varepsilon, x_0^r + \varepsilon]\} \tag{16}$$

Note that, in this case, we have to consider the different cut-off points existing between the domains on which the x_0 and x_1 variables are defined and the value of E_X. The above functions perform an analysis of the $X_1 \times Y_1$ space while comparing the potential observations and their combinations with the corresponding reference values. As such, the analysis must be adapted depending on the different areas of $X_1 \times Y_1$ being compared. Moreover, these equations allow for interactions among alternatives determined by domain and interval width differences across characteristics. The current framework would allow us to formalize and illustrate the results that can be obtained in these potential scenarios, which constitutes an advantage over the operational complexity involved in the use of fuzzy and intuitionistic fuzzy sets.

7. Empirical implementation

We illustrate the main empirical consequences derived from considering the perception differences existing between the raters and D by comparing the rankings obtained after implementing the TOPSIS technique to rank a list of laptops subjectively evaluated by different users. We start by providing a basic description of the standard TOPSIS framework below.

It should be emphasized that the uncertain evaluation framework introduced in the current paper can be applied to any MCDM or optimization technique making use of the subjective opinions of known or unknown raters. This is the case since the main modifications are performed on the numerical evaluations received from the raters, which generally constitute the opinions of experts in MCDM techniques such as, for example, VIKOR or PROMETHEE (Preference Ranking Organization METHod for Enrichment of Evaluations), and optimization models such as

DEA (Data Envelopment Analysis).

7.1. The TOPSIS ranking technique

The TOPSIS technique defines ideal positive and negative points and computes the relative distance of each alternative from both these values. After measuring the relative distances, a ranking is obtained to determine the importance of each of the alternatives available. We describe below the basic framework required to implement TOPSIS and rank the set of laptops under analysis.

Denote by $A_1, A_2, ..., A_m$ the m available alternatives and by $C_1, C_2, ..., C_n$ the corresponding set of criteria. The decision matrix defined in terms of x_{ij}, which denotes the performance of alternative A_i according to criterion C_j, is given by

$$\begin{array}{c|cccc} & C_1 & C_2 & \dots & C_n \\ \hline A_1 & x_{11} & x_{12} & \dots & x_{1n} \\ A_2 & x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_m & x_{m1} & x_{m2} & \dots & x_{mn} \\ \hline \end{array}$$

$$W = [w_1, w_2, ..., w_n]$$

where w_j is the (subjective) importance assigned by D to criterion j. After evaluating each alternative according to each criterion, which can be positive or negative, the decision process required to generate a ranking of the alternatives can be implemented.

Step 1. The first step consists of defining the normalized decision matrix where the different criteria are normalized so as to allow for direct comparisons among them. The scores achieved by each alternative are normalized as follows

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_i x_{ij}^2}}, \; i = 1, ..., m, \; j = 1, ..., n; \tag{17}$$

where x_{ij} denotes the performance of alternative A_i with respect to criterion C_j.

Step 2. In the next step, the weighted normalized decision matrix is calculated based on the subjective importance assigned by D to each criterion. Each column of the decision matrix is multiplied by its associated weight as follows

$$v_{ij} = w_j r_{ij}\,,\ i = 1,...,m\,,\ j = 1,...,n\,; \tag{18}$$

Step 3. In this step, the ideal positive and negative solutions are determined. The best, v_i^+, and the worst, v_i^-, values of the different criterion functions depend on whether the criterion being considered provides a benefit (a positive criterion) or a cost (a negative criterion) to D. The corresponding ideal values are given by

$$A^+ = \left(v_1^+,...,v_n^+\right) \tag{19}$$

where

$$v_i^+ = max_i \left\{(v_{ij}) \mid j \in \text{positive criterion}\right\} \tag{20}$$

$$v_i^+ = \min_i \left\{(v_{ij}) \mid j \in \text{negative criterion}\right\} \tag{21}$$

Similarly, the negative ideal value is given by

$$A^- = \left(v_1^-,...,v_n^-\right) \tag{22}$$

with

$$v_i^- = \min_i \left\{(v_{ij}) \mid j \in \text{positive criterion}\right\} \tag{23}$$

$$v_i^- = \max_i \left\{(v_{ij}) \mid j \in \text{negative criterion}\right\} \tag{24}$$

Step 4. We compute now the distance existing between every alternative and the corresponding ideal (and negative ideal) solutions

$$d_i^+ = \left[\sum_{j=1}^{n}\left(v_j^+ - v_{ij}\right)^2\right]^{1/2}, \quad i = 1,...,m \tag{25}$$

$$d_i^- = \left[\sum_{j=1}^{n}\left(v_j^- - v_{ij}\right)^2\right]^{1/2}, \quad i = 1,...,m \tag{26}$$

Step 5. After d_i^+ and d_i^- have been calculated for each alternative, the relative proximity to the ideal solution is used to determine the final ranking of the alternatives. The relative proximity of alternative A_i is defined as follows

$$R_i = \frac{d_i^-}{d_i^+ + d_i^-} \quad i = 1,...,m \tag{27}$$

Note that R_i represents the distance between alternative i and the negative ideal solution. That is, higher values of R_i correspond to alternatives located farther from the negative solution. The best- and worst-case scenarios are therefore defined by $R_i = 1$ and $R_i = 0$, respectively.

7.2. Applying the uncertain evaluation framework to subjective online ratings

In the current section, we illustrate empirically the main implications that follow from the application of the perception-based improvement framework to a real life scenario where a risk neutral D observes different ratings from a set of unknown previous users. The corresponding information has been retrieved from a website (https://www.reevoo.com) displaying the subjective evaluations provided by groups of unknown users about the different characteristics composing each alternative. The available ratings consist of numerical evaluations of the characteristics together with brief linguistic reports describing each alternative.

On the 14th of June 2019, we accessed the website, searched for laptops and selected the first 24 described within the first page of search outputs. We retrieved the summary information from these laptops, which allows us to perform a comprehensive comparison across the main six characteristics analyzed. The values of these characteristics for each laptop are described in Table 1. Note that these values are essentially a set of subjective numerical ratings averaged after being provided by other users who have previously purchased the laptop. Note also that these characteristics are provided as certain values assumed to follow from a series of objective predetermined criteria – though they are indeed subjective evaluations of the raters.

Table 1. Decision matrix built with the information retrieved from the rating website

Laptop	Battery	Design	Size	Perform	Money	Overall
1	8	8,1	8,7	9,1	8,8	8,7
2	8,5	9	9,2	8,8	8,3	9
3	8,2	9	8,7	8,8	8,5	9
4	2	4	4,5	4	4,5	4
5	8,3	8,5	8,7	9,1	8,8	8,8
6	8,8	9	9,3	8,9	9	8,8
7	7,6	7,8	7,9	7,4	7,5	7,5
8	8,1	8,8	8,8	7,9	8,9	8,4
9	7	9,8	9,8	10	9,6	9,8
10	7,2	7,7	8,3	7,5	8,4	7,8
11	8,6	8,8	8,9	8,4	8,9	8,6
12	7,9	8,2	8,4	8	8,4	8,2
13	7,8	8,5	9	8,7	8,8	8,8
14	8,1	8,3	8,8	8,2	8,2	8,2
15	8	10	10	7	10	10
16	8	8,1	8,6	7,1	7,5	6,9
17	8,7	8,8	9	8,7	8,5	8,8
18	8	9	8,5	8,6	8,5	8,5

19	7,6	8,7	8,8	8,1	8,2	8,1
20	8,4	9,3	9	8,6	8,5	9
21	7,3	9	8,5	9,2	8,3	8,8
22	8,5	8,4	8,9	8,7	8,6	8,7
23	7,6	8,7	8,5	8,9	8,4	8,7
24	7,1	8,4	8,8	9,2	8,9	8,7

The main characteristics (positive criteria) of the laptops evaluated by the reviewers are battery life, design, size and weight, performance, value for money and overall rating. We have assigned a weight of 0,15 to each characteristic except for the overall rating, which, giving its overview quality, has been assigned a weight of 0,25. As is generally the case with TOPSIS, the ratings obtained would differ if these values were modified by D. We are however interested on the effect that the different uncertain interval widths assigned to the evaluations of other users have on the rankings obtained by D.

Decision makers acknowledge the evaluation uncertainty they face and consider the effects from increments in its value. In this regard, seven different scenarios determined by the assimilation of this uncertainty by D when aiming at improving upon the evaluations received through additional information will be analyzed. The "no imp" scenario –where a ranking of the alternatives is computed based on the evaluations retrieved absent any improvement considerations – constitutes the main reference framework. Four different widths of the evaluation intervals – determined by different ε values –, a heuristic setting with $\varepsilon = 0,5$ and D ignoring the inherent uncertainty complete the set of scenarios analyzed. We should note that rankings of the heuristic case have been computed for each of the ε values considered in the evaluation interval scenarios. However, the results obtained are all quite similar and we have therefore only reported those corresponding to the $\varepsilon = 0,5$ case.

Table 2. Rankings obtained when accounting for different uncertainty evaluation scenarios

Laptop	no imp	ε =0,5	ε =1	ε =1,5	ε =2	h (ε =0,5)	ign
1	0,7960	0,3895	0,6403	0,8694	0,7907	0,2078	0,2958
2	0,8307	0,3616	0,6187	0,7857	0,7017	0,1517	0,2443
3	0,8232	0,3664	0,6348	0,8258	0,7377	0,1647	0,2577
4	0,0012	0,9994	0,9990	0,9271	0,6132	0,9997	0,9997
5	0,8165	0,3718	0,6366	0,8566	0,7668	0,1785	0,2714
6	0,8422	0,3501	0,6103	0,7729	0,6855	0,1317	0,2297
7	0,6307	0,5505	0,6717	0,9469	0,9594	0,4251	0,4945
8	0,7687	0,4098	0,6521	0,9051	0,8397	0,2317	0,3198
9	0,8910	0,1702	0,1820	0,2089	0,1928	0,0950	0,1062
10	0,6652	0,5102	0,6687	0,9475	0,9476	0,3687	0,4441
11	0,8031	0,3796	0,6465	0,8887	0,7992	0,1890	0,2827
12	0,7282	0,4532	0,6633	0,9728	0,9544	0,2948	0,3782
13	0,8055	0,3773	0,6428	0,8680	0,7777	0,1831	0,2756
14	0,7411	0,4402	0,6611	0,9473	0,9064	0,2756	0,3605

15	0,8332	0,2060	0,2311	0,2734	0,2792	0,1691	0,1874
16	0,6124	0,5583	0,6905	0,9196	0,8900	0,4355	0,5020
17	0,8169	0,3685	0,6408	0,8628	0,7661	0,1717	0,2665
18	0,7809	0,3995	0,6525	0,9133	0,8383	0,2216	0,3105
19	0,7304	0,4454	0,6594	0,9347	0,8882	0,2811	0,3645
20	0,8311	0,3595	0,6115	0,7789	0,6979	0,1486	0,2420
21	0,7903	0,3877	0,6310	0,8275	0,7516	0,2044	0,2884
22	0,8016	0,3822	0,6487	0,9076	0,8105	0,1962	0,2888
23	0,7843	0,3940	0,6503	0,9069	0,8188	0,2162	0,3041
24	0,7872	0,3875	0,6348	0,8368	0,7590	0,2031	0,2885

The rankings obtained after accounting for the different uncertainty evaluation scenarios are presented in Table 2. The main differences among the scenarios considered can be observed when comparing the different uncertain evaluation rankings with the "no imp" one. Intuitively, the ranking defined by TOPSIS – absent any potential improvement resulting from the acquisition of additional information – should oppose the one obtained when information can be acquired to improve upon the evaluations observed. In other words, it is harder to improve upon the highest ranked alternative than the lowest one, since the margin for improvement of the former is much narrower than that of the latter. In this regard, Figures 9 to 11 provide additional intuition by comparing the "no imp" ranking with selected pairs from the uncertain evaluation scenarios. These graphical comparisons complement the formal correlation analysis presented in Table 3.

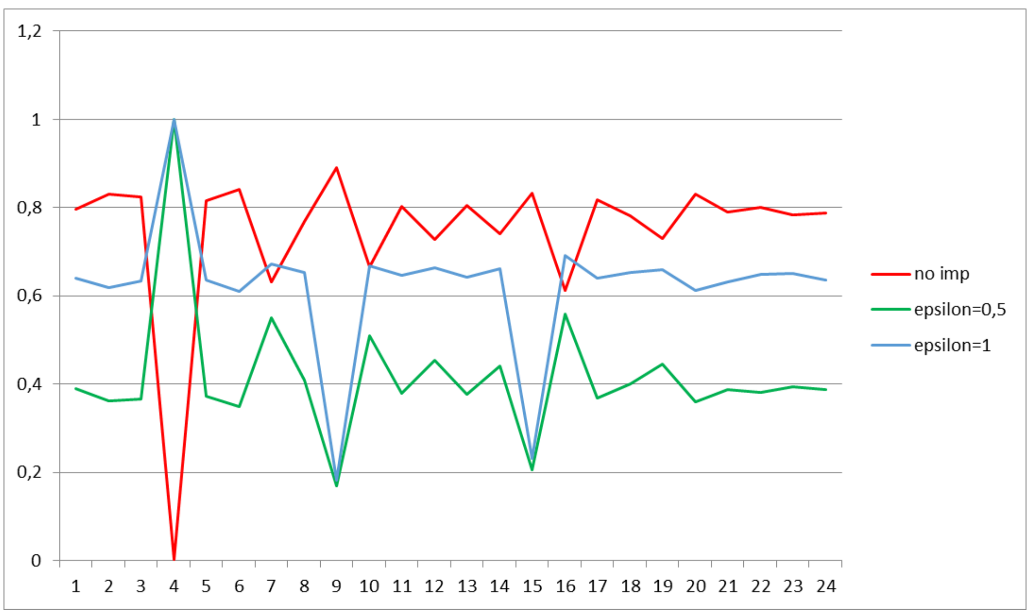

Figure 9. Rankings obtained in the "no imp", $\varepsilon = 0,5$ and $\varepsilon = 1$ evaluation scenarios

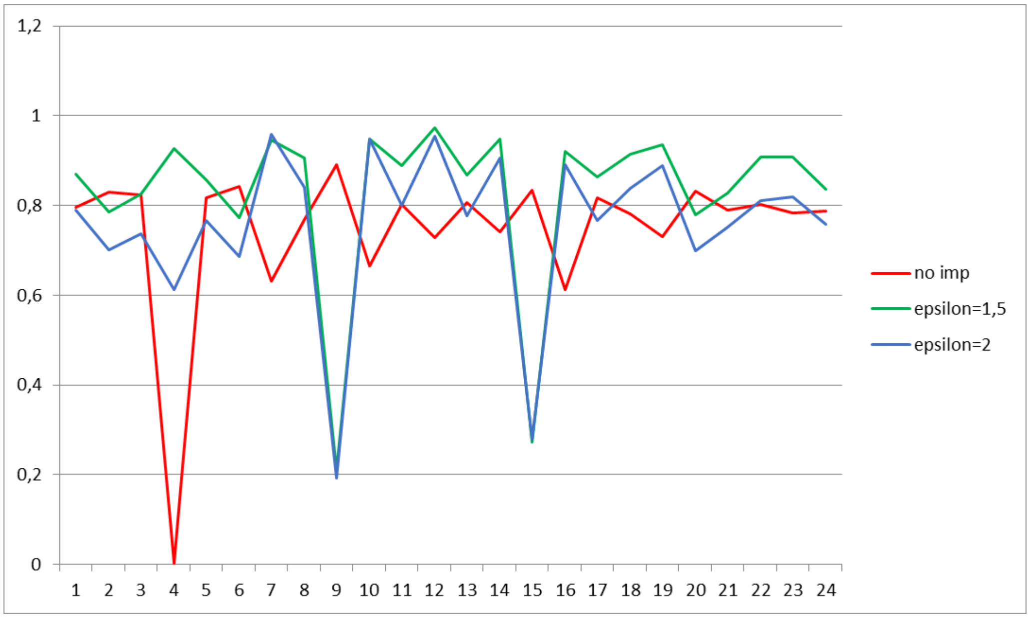

Figure 10. Rankings obtained in the “no imp”, $\varepsilon = 1{,}5$ and $\varepsilon = 2$ evaluation scenarios

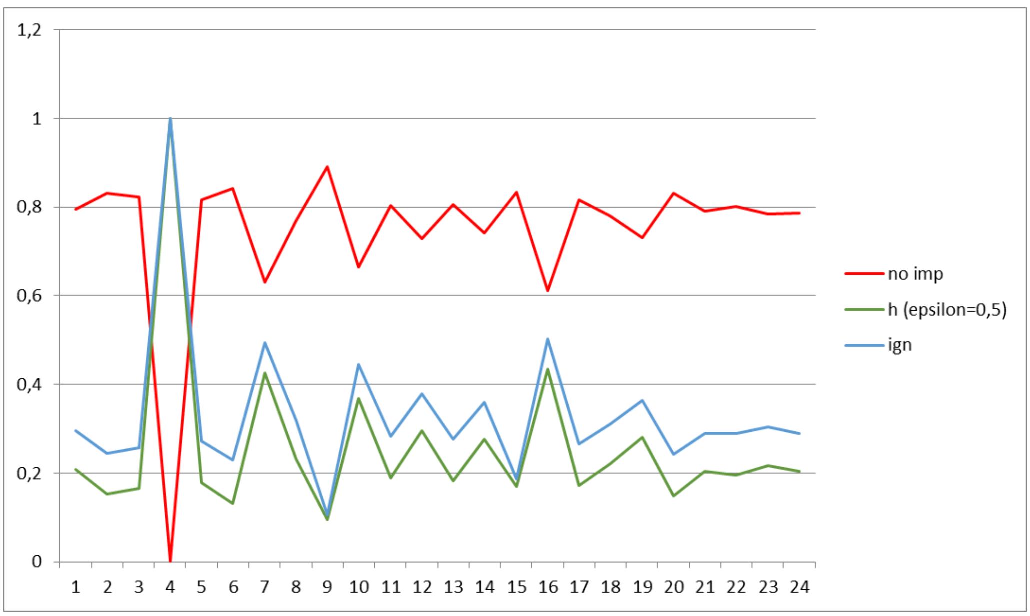

Figure 11. Rankings obtained in the “no imp”, $h(\varepsilon = 0{,}5)$ and ignoring uncertainty scenarios

We have run Spearman’s correlation test to identify similar trends in the rankings generated by the different scenarios analyzed. We focus particularly on the consequences from increasing the width of the uncertainty interval when compared to the heuristic simplification and ignoring uncertainty settings. As Table 3 illustrates, the correlation between the rankings obtained absent any improvement considerations and the different ε scenarios accounting for potential improvements upon the evaluations retrieved is highly negative. However, it decreases as the spread of the uncertainty interval increases. Thus, as illustrated in Figures 9 and 10, an increment in the spread of the uncertain evaluation intervals modifies the information acquisition behavior of D, who starts acquiring information on relatively better ranked alternatives. This is done to build on the potential improvements derived from the width of the corresponding uncertain intervals.

The rankings obtained when considering the heuristic and ignoring uncertainty settings, or relatively low spreads of the uncertain evaluation intervals, are considerably similar. These settings exhibit all highly positive correlations among themselves, which decrease for relatively high values of ε. In this regard, the $\varepsilon = 2$ scenario generates the most diverse ranking when compared to the other settings, with the spread of the uncertain interval leading to the selection of alternatives whose evaluations are located in the proximity of [7.5, 8]. Similarly, those evaluations located in the proximity of [8, 8.5] are prioritized in the $\varepsilon = 1{,}5$ scenario.

Note that the current numerical example illustrates the ranking effects derived from increasing the width of the interval defining the uncertainty faced by D regarding the quality of the information received. As such, all the alternatives have been evaluated in the exact same way, with the width assumed on each one of them being identical within the corresponding scenarios. However, this does not need to be the case and the width of the evaluation intervals could be assumed to be determined by the relative number of reviews available, with higher number of reviews leading to narrower intervals and lower uncertainty. In this regard, the number of reviews available per laptop ranges from 1 to 528, highlighting the differences in the width of the uncertainty intervals that could be generated. The resulting rankings would therefore be modified, with wider intervals favoring relatively more valued alternatives and narrower ones favoring less valued alternatives.

7.3. Potential extensions: Endogenizing the interval widths

An additional indicator of perception uncertainty could also be incorporated to the analysis when evaluating the different characteristics of the laptops, namely, the number of reviews available per laptop. In this regard, evaluation uncertainties could be assumed to decrease as larger numbers of independent reviews are provided. As a result, the relative number of reviews available per laptop could be used to determine the width of the evaluation interval as follows

$$\varepsilon = \gamma - \frac{\#reviews_i}{\max_i \left\{\#reviews_i\right\}} \tag{28}$$

with $i = 1,...,24$, and where γ is the reference value relative to which the interval width is defined. For instance, in the numerical example described in the previous section, γ could have been assigned a value of 1, 1,5 or 2, though the relative number of reviews could be weighted accordingly to allow for any reference value. Note that the width assigned to the evaluation interval would account for the number of reviews provided per laptop, $\#reviews_i$, relative to the largest number available among the laptops composing the sample. This operation must be performed for all the characteristics describing the laptops.

Finally, since a higher number of reviews can be assumed to smooth the uncertainty faced by D, the relative review value obtained must be subtracted from the main reference one that could be reached by a sample laptop. In this regard, the value $(\gamma - 1)$ could be interpreted as the basic uncertainty that D considers to exist even when large numbers of evaluations become available.

Table 3. Correlations among the rankings generated by the different evaluation scenarios

Evaluation Scenarios			no imp	e=0,5	e=1	e=1,5	e=2	h (e=0,5)	ign
Spearman's rho	no imp	Correlation Coefficient	1,000	-,996**	-,939**	-,904**	-,742**	-,991**	-,992**
		Sig. (2-tailed)	.	,000	,000	,000	,000	,000	,000
		N	24	24	24	24	24	24	24
	e=0,5	Correlation Coefficient	-,996**	1,000	,943**	,912**	,750**	,991**	,997**
		Sig. (2-tailed)	,000	.	,000	,000	,000	,000	,000
		N	24	24	24	24	24	24	24
	e=1	Correlation Coefficient	-,939**	,943**	1,000	,957**	,783**	,934**	,957**
		Sig. (2-tailed)	,000	,000	.	,000	,000	,000	,000
		N	24	24	24	24	24	24	24
	e=1,5	Correlation Coefficient	-,904**	,912**	,957**	1,000	,868**	,903**	,928**
		Sig. (2-tailed)	,000	,000	,000	.	,000	,000	,000
		N	24	24	24	24	24	24	24
	e=2	Correlation Coefficient	-,742**	,750**	,783**	,868**	1,000	,738**	,763**
		Sig. (2-tailed)	,000	,000	,000	,000	.	,000	,000
		N	24	24	24	24	24	24	24
	h (e=0,5)	Correlation Coefficient	-,991**	,991**	,934**	,903**	,738**	1,000	,988**
		Sig. (2-tailed)	,000	,000	,000	,000	,000	.	,000
		N	24	24	24	24	24	24	24
	ign	Correlation Coefficient	-,992**	,997**	,957**	,928**	,763**	,988**	1,000
		Sig. (2-tailed)	,000	,000	,000	,000	,000	,000	.
		N	24	24	24	24	24	24	24

**** Correlation is significant at the 0.01 level (2-tailed)**

8. Conclusion

The current paper has focused on the uncertainty inherent to the evaluations observed by D when retrieving and assimilating information online. We have studied the incentives of D to improve upon an evaluation contained within an uncertain interval using further uncertain information. Three different scenarios have been analyzed determined by the assumptions imposed on the information assimilation capacities of D. These scenarios have ranged from the full assimilation to the elimination of the uncertainty inherent to the perception and information transmission processes. The current model has illustrated how the widths of the subjective intervals and the attitudes towards risk of D determine his information retrieval incentives when trying to improve upon an alternative given any initial evaluation of its characteristics.

The analysis performed on this uncertain framework can be extended so as to consider the subjective degree of optimism or pessimism of D, or potential signals issued by the raters on which different levels of trust can be defined. The intuitionistic fuzzy environments implemented in the systems literature do not allow for credibility considerations and strategic reports, while the densities defined in our setting can be modified to reflect the subjective beliefs or degrees of trust inherent to D. The psychology literature has emphasized the importance that the trust in the provider of information has for D to either consider or disregard the advice received (Casaló and Guinalíu, 2011). Note that D lacks information about his own perception and reporting capacities and those of the raters, both of which add frictions to the initial search uncertainty regarding the characteristics being observed and evaluated.

When considering the applicability of the current perception-based setting to optimization environments, we can observe the differences existing relative to other uncertainty-based frameworks – such as the uncertain probabilistic and chance-constrained models – introduced in the systems literature (Liu, 2002, 2015). As emphasized through the paper, this remark is also applicable to the fuzzy and intuitionistic fuzzy developments introduced in the MCDM literature. In this regard, the current formal setting can be combined with MCDM techniques such as the analytical network process or DEMATEL to explicitly account for the effects of uncertainty when ranking different alternatives (Tseng, 2011). The online-based applications of such a formal environment are particularly relevant. For example, Yu et al. (2018) considered the problem of hotel selection from different websites as a MCDM model based on online linguistic reviews and designed a VIKOR-based framework to rank the potential alternatives. As already emphasized, the current setting can be easily adapted to complement and extend this type of analyses when evaluating the uncertain characteristic of the alternatives being considered.

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Appendix

We describe below the set of middle interval sections defining the function $V(x_o^r)$ both when D fully assimilates the uncertainty inherent to the perception and information transmission processes and when a heuristic assimilation mechanism is applied.

A.1. Middle interval section (I): $x_1^m + \varepsilon \le x_o^r \le x_1^m + 2\varepsilon$

The value function defined for $x_o^r \in [x_1^m + \varepsilon, x_1^m + 2\varepsilon]$, and such that $x_1^M > x_o^r + 3\varepsilon$, is given by

$$
\begin{aligned}
&V(x_o^r \mid x_1^m + \varepsilon \le x_o^r \le x_1^m + 2\varepsilon) = \\
&\int_{x_1^m}^{x_1^m+\varepsilon} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon)-x_1^m}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r + \\
&\int_{x_1^m+\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r + \\
&\int_{x_o^r}^{x_o^r+2\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon}\frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \\
&\int_{x_o^r+2\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big[u(x_1)\big]dx_1\right]dx_1^r + \\
&\int_{x_1^M-\varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big[u(x_1)\big]dx_1\right]dx_1^r
\end{aligned}
\tag{A1}
$$

Note that part of the $[x_1^r - \varepsilon, x_1^r + \varepsilon]$ interval inherent to the set of potential realizations of $x_1^r < x_1^m + \varepsilon$ may be defined below the lower limit of X_1, i.e. x_1^m, while providing potential improvements over x_o^r. Such a possibility is accounted for by the first right hand side expression of Equation (A1). The rest of the equation defines the progressive overtaking of x_o^r by x_1^r until the upper limit of X_1, i.e. x_1^M, is reached. The same type of intuition applies to the set of potential realizations defined by x_1^r and located below $x_1^m + \varepsilon$ when analyzing the next interval section of X_1.

Finally, as illustrated when describing the lower interval section, the value function defined by D within the heuristic scenario considers only full improvements via its $I^+(x_o^r)$ set. Thus, the same expression of the heuristic value function will prevail through the different middle interval sections with $x_o^r + \varepsilon \le x_1^M$, that is, $V(x_o^r \mid x_1^m + \varepsilon \le x_o^r \le x_1^M - \varepsilon) = \int_{x_o^r+\varepsilon}^{x_1^M} \mu(x_1) u(x_1) dx_1$.

A.2. Middle interval section (II): $x_1^m + 2\varepsilon \le x_o^r \le x_1^m + 3\varepsilon$

The value function defined for $x_o^r \in [x_1^m + 2\varepsilon, x_1^m + 3\varepsilon]$, and such that $x_1^M > x_o^r + 3\varepsilon$, is given by

$$
\begin{aligned}
& V(x_o^r \mid x_1^m + 2\varepsilon \le x_o^r \le x_1^m + 3\varepsilon) = \\
& \int_{x_o^r - 2\varepsilon}^{x_1^m+\varepsilon} \mu(x_1^r) \left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{(x_1^r+\varepsilon) - x_1^m} \left[\frac{x_1^r}{x_o^r + x_1^r} \big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r} \big(u(x_1) - c\big) \right] dx_1 \right] dx_1^r + \\
& \int_{x_1^m+\varepsilon}^{x_o^r} \mu(x_1^r) \left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon} \left[\frac{x_1^r}{x_o^r + x_1^r} \big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r} \big(u(x_1) - c\big) \right] dx_1 \right] dx_1^r + \\
& \int_{x_o^r}^{x_o^r+2\varepsilon} \mu(x_1^r) \left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon} \left[\frac{x_1^r}{x_o^r + x_1^r} \big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r} \big(u(x_1) - c\big) \right] dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon} \big(u(x_1)\big) dx_1 \right] dx_1^r + \\
& \int_{x_o^r+2\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r) \left[\int_{x_1^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon} \big[u(x_1) \big] dx_1 \right] dx_1^r + \\
& \int_{x_1^M-\varepsilon}^{x_1^M} \mu(x_1^r) \left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M - (x_1^r - \varepsilon)} \big[u(x_1) \big] dx_1 \right] dx_1^r
\end{aligned}
\tag{A2}
$$

The following middle interval value functions complement the previous ones through the domain of X_1 until the upper interval defined for $x_o^r \in [x_1^M - \varepsilon, x_1^M]$, and such that $x_1^m < x_o^r - 3\varepsilon$, is reached.

A.3. Middle interval section (III): $x_1^m + 3\varepsilon \le x_o^r \le x_1^M - 3\varepsilon$

The value function defined for $x_o^r \in [x_1^m + 3\varepsilon, x_1^M - 3\varepsilon]$ is given by

$$V(x_o^r \mid x_1^m + 3\varepsilon \le x_o^r \le x_1^M - 3\varepsilon) =$$

$$\int_{x_o^r-2\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r +$$

$$\int_{x_o^r}^{x_o^r+2\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon}\frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \tag{A3}$$

$$\int_{x_o^r+2\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\big[u(x_1)\big]dx_1\right]dx_1^r +$$

$$\int_{x_1^M-\varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big[u(x_1)\big]dx_1\right]dx_1^r$$

A.4. Middle interval section (IV): $x_1^M - 3\varepsilon \le x_o^r \le x_1^M - 2\varepsilon$

The value function defined for $x_o^r \in [x_1^M - 3\varepsilon, x_1^M - 2\varepsilon]$, and such that $x_1^m < x_o^r - 3\varepsilon$, is given by

$$V(x_o^r \mid x_1^M - 3\varepsilon \le x_o^r \le x_1^M - 2\varepsilon) =$$

$$\int_{x_o^r-2\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r-\varepsilon}^{x_1^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1\right]dx_1^r +$$

$$\int_{x_o^r}^{x_o^r+\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon}\frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r +$$

$$\int_{x_o^r+\varepsilon}^{x_1^M-\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^r+\varepsilon}\frac{1}{2\varepsilon}\big(u(x_1)\big)dx_1\right]dx_1^r + \tag{A4}$$

$$\int_{x_1^M-\varepsilon}^{x_o^r+2\varepsilon} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_o^r+\varepsilon} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\left[\frac{x_1^r}{x_o^r+x_1^r}\big(u(x_1)\big)+\frac{x_o^r}{x_o^r+x_1^r}\big(u(x_1)-c\big)\right]dx_1 + \int_{x_o^r+\varepsilon}^{x_1^M}\frac{1}{x_1^M-(x_1^r-\varepsilon)}\big(u(x_1)\big)dx_1\right]dx_1^r +$$

$$\int_{x_o^r+2\varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r-\varepsilon}^{x_1^M} \frac{1}{x_1^M-(x_1^r-\varepsilon)}\big[u(x_1)\big]dx_1\right]dx_1^r$$

A.5. Middle interval section (V): $x_1^M - \varepsilon \le x_o^r + \varepsilon \le x_1^M$

The value function defined for $x_o^r \in [x_1^M - 2\varepsilon, x_1^M - \varepsilon]$, and such that $x_1^m < x_o^r - 3\varepsilon$, is given by

$$
\begin{aligned}
&V(x_o^r \mid x_1^M - \varepsilon \le x_o^r + \varepsilon \le x_1^M) = \\
&\int_{x_o^r - 2\varepsilon}^{x_o^r} \mu(x_1^r)\left[\int_{x_o^r - \varepsilon}^{x_1^r + \varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r + x_1^r}\big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r}\big(u(x_1) - c\big)\right] dx_1\right] dx_1^r + \\
&\int_{x_o^r}^{x_1^M - \varepsilon} \mu(x_1^r)\left[\int_{x_1^r - \varepsilon}^{x_o^r + \varepsilon} \frac{1}{2\varepsilon}\left[\frac{x_1^r}{x_o^r + x_1^r}\big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r}\big(u(x_1) - c\big)\right] dx_1 + \int_{x_o^r + \varepsilon}^{x_1^r + \varepsilon} \frac{1}{2\varepsilon}\big(u(x_1)\big) dx_1\right] dx_1^r + \\
&\int_{x_1^M - \varepsilon}^{x_1^M} \mu(x_1^r)\left[\int_{x_1^r - \varepsilon}^{x_o^r + \varepsilon} \frac{1}{x_1^M - (x_1^r - \varepsilon)}\left[\frac{x_1^r}{x_o^r + x_1^r}\big(u(x_1)\big) + \frac{x_o^r}{x_o^r + x_1^r}\big(u(x_1) - c\big)\right] dx_1 + \int_{x_o^r + \varepsilon}^{x_1^M} \frac{1}{x_1^M - (x_1^r - \varepsilon)}\big(u(x_1)\big) dx_1\right] dx_1^r
\end{aligned}
\tag{A5}
$$