On nested code pairs from the Hermitian curve

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Abstract

Nested code pairs play a crucial role in the construction of ramp secret sharing schemes [15] and in the CSS construction of quantum codes [14]. The important parameters are (1) the codimension, (2) the relative minimum distance of the codes, and (3) the relative minimum distance of the dual set of codes. Given values for two of them, one aims at finding a set of nested codes having parameters with these values and with the remaining parameter being as large as possible. In this work we study nested codes from the Hermitian curve. For not too small codimension, we present improved constructions and provide closed formula estimates on their performance. For small codimension we show how to choose pairs of one-point algebraic geometric codes in such a way that one of the relative minimum distances is larger than the corresponding non-relative minimum distance.

Keywords: Algebraic geometric code, Asymmetric quantum code, Hermitian curve, ramp secret sharing, relative minimum distance

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1 Introduction

In this paper we study improved constructions of nested code pairs from the Hermitian curve. Here the phrase 'improved construction' refers to optimizing those parameters important for the corresponding linear ramp secret sharing schemes as well as stabilizer asymmetric quantum codes. Our work is a natural continuation of [7], where improved constructions of nested code pairs were defined from Cartesian product point sets. The analysis in the present paper includes a closed formula estimate on the dimension of order bound improved Hermitian codes, which is of interest in its own right, i.e. also without the above mentioned applications.

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A linear ramp secret sharing scheme is a cryptographic method to encode a secret message in \mathbb{F}_q^ℓ into n shares from \mathbb{F}_q . These shares are then distributed among a group of n parties and only specified subgroups are able to reconstruct the secret. A secret sharing scheme is characterized by its privacy number t and its reconstruction number r. The first is defined as the largest number such that no subgroup of this size can obtain any information on the secret. The second is defined to be the smallest number such that any subgroup of this size can reconstruct the entire secret. A linear ramp secret sharing scheme can be understood as the following coset construction. Consider linear codes $C_2 \subset C_1 \subseteq \mathbb{F}_q^n$. Let $\{\vec{b}_1,\ldots,\vec{b}_{k_2}\}$ be a basis for C_2 and extend it to a basis $\{\vec{b}_1,\ldots,\vec{b}_{k_2},\vec{b}_{k_2+1},\ldots,\vec{b}_{k_2+\ell}\}$ for C_1 . Here, of course, ℓ is the codimension of C_1 and C_2 . Choose elements a_1,\ldots,a_{k_2} uniformly and independent at random and encode the secret $\vec{s}=(s_1,\ldots,s_\ell)$ as the codeword $\vec{c}=(c_1,\ldots,c_n)=a_1\vec{b}_1+\cdots+a_{k_2}\vec{b}_{k_2}+s_1\vec{b}_{k_2+1}+\cdots+s_\ell\vec{b}_{k_2+\ell}$. Then use c_1,\ldots,c_n as the shares. The crucial parameters for the construction are the codimension of the pair of nested codes and their relative minimum distances $d(C_1,C_2)$ and $d(C_2^\perp,C_1^\perp)$. Recall that these are defined as

$$d(C_1, C_2) = \min\{w_H(\vec{c}) \mid \vec{c} \in C_1 \backslash C_2\}$$

and similar for the dual codes. The following well-known theorem (see for instance [15]) shows how to determine the privacy number and the reconstruction number.

Theorem 1. Given \mathbb{F}_q -linear codes $C_2 \subset C_1$ of length n and codimension ℓ , the corresponding ramp secret sharing scheme encodes secrets $\vec{s} \in \mathbb{F}_q^{\ell}$ into a set of n shares from \mathbb{F}_q . The privacy number equals $t = d(C_2^{\perp}, C_1^{\perp}) - 1$, and the reconstruction number is $r = n - d(C_1, C_2) + 1$.

A linear q-ary asymmetric quantum error-correcting code is a q^k dimensional subspace of the Hilbert space \mathbb{C}^{q^n} where error bases are defined by unitary operators Z and X, the first representing phase-shift errors, and the second representing bit-flip errors [2, 21, 14]. In [13] it was identified that in some realistic models phase-shift errors occur more frequently than bit-flip errors, and the asymmetric codes were therefore introduced [13, 19, 5, 16, 25] to balance the error correcting ability accordingly. For such codes we write the set of parameters as $[[n,k,d_z/d_x]]_q$ where d_z is the minimum distance related to phase-shift errors and d_x is the minimum distance related to bit-flip errors. The CSS construction transforms a pair of nested classical linear codes $C_2 \subset C_1 \subseteq \mathbb{F}_q^n$ into an asymmetric quantum code. From [19] we have

Theorem 2. Consider linear codes $C_2 \subset C_1 \subseteq \mathbb{F}_q^n$. Then the corresponding asymmetric quantum code defined using the CSS construction has parameters

$$[[n, \ell = \dim C_1 - \dim C_2, d_z/d_x]]_q$$

where $d_z = d(C_1, C_2)$ and $d_x = d(C_2^{\perp}, C_1^{\perp})$.

Quantum codes with $d(C_1, C_2) > d(C_1)$ or $d(C_2^{\perp}, C_1^{\perp}) > d(C_2^{\perp})$ are called impure, and they are desirable due to the fact that one can take advantage of this property in connection with the error-correction. More precisely, one can tolerate $\lfloor (d(C_1, C_2) - 1)/2 \rfloor$ phase-shift errors and $\lfloor (d(C_2^{\perp}, C_1^{\perp}) - 1)/2 \rfloor$ bit-flip errors, respectively, but in the decoding algorithms it is only necessary to correct up to $\lfloor (d(C_1) - 1)/2 \rfloor$ and $\lfloor (d(C_2^{\perp}) - 1)/2 \rfloor$ errors, respectively. Despite this observation, only few impure codes have been presented in the literature.

With the above two applications in mind, the challenge is to find nested codes $C_2 \subset C_1$ such that two of the parameters ℓ , $d(C_1, C_2)$, $d(C_2^{\perp}, C_1^{\perp})$ attain given prescribed values, and the remaining parameter is as large as possible. In this paper we analyse two good constructions from the Hermitian function field. In the first we consider code pairs such that C_1 is an order bound improved primary code [1, 10] and such that C_2 is the dual of an order bound improved dual code [12]. Considering in this case the minimum distances rather than the relative distances is no restriction due to the optimized choice of codes – the minimum distances $d(C_1)$ and $d(C_2^{\perp})$ being so good that there is essentially no room for $d(C_1, C_2) > d(C_1)$ or $d(C_2^{\perp}, C_1^{\perp}) > d(C_2^{\perp})$ to hold. For this construction to work, the codimension cannot be very small. For small codimension when $d(C_1)$ and $d(C_2^{\perp})$ are far from each other we then show how to choose ordinary one-point algebraic geometric codes such that one of the relative distances becomes larger than the corresponding ordinary minimum distance. In particular, this construction leads to impure asymmetric quantum codes.

The paper is organized as follows. In Section 2 we collect material from the literature on how to determine parameters of primary and dual codes coming from the Hermitian curve, and we introduce the order bound improved codes¹. In Section 3 we establish closed formula lower bounds on the dimension of order bound improved Hermitian codes of any designed minimum distance. We then continue in Section 4 by determining the pairs $(\delta_1, \delta_2) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ for which the order bound improved

¹This section also contains a collective treatment of the order bounds for general primary as well as dual (improved) one-point algebraic geometric codes which may not be easy to find in the literature.

primary code C_1 of designed distance δ_1 contains C_2 , the dual of an order bound improved dual code of designed distance δ_2 . This and the information from Section 3 is then translated into information on improved nested code pairs of not too small codimension in Section 5. Next, in Section 6 we determine parameters of nested one-point algebraic geometric code pairs of small codimension for which one of the relative distances is larger than the non-relative. Finally, in Section 7 samples of the given constructions are compared with known asymmetric quantum codes, with existence bounds on asymmetric quantum codes, and with non-existence bounds on linear ramp secret sharing schemes. Section 8 is the conclusion.

2 Hermitian codes and their parameters

Given an algebraic function field over a finite field, let $= P_1, \ldots, P_n, Q$ be rational places. By H(Q) we denote the Weierstrass semigroup of Q, and we write

$$H^*(Q) = \{ \lambda \in H(Q) \mid C_{\mathcal{L}}(D, \lambda Q) \neq C_{\mathcal{L}}(D, (\lambda - 1)Q) \}$$

where $D = P_1 + \cdots + P_n$. Recall that the dual code of $C_{\mathcal{L}}(D, \lambda Q)$ is written $C_{\Omega}(D, \lambda Q)$. The order bound [12, 4] then tells us that if

$$\vec{c} \in C_{\Omega}(D, (\lambda - 1)Q) \setminus C_{\Omega}(D, \lambda Q)$$

(which can only happen if $\lambda \in H^*(Q)$), then the Hamming weight of \vec{c} satisfies

$$w_H(\vec{c}) \ge \mu(\lambda) \tag{1}$$

where

$$\mu(\lambda) = \#\{\eta \in H(Q) \mid \lambda - \eta \in H(Q)\}.$$

The similar bound for the primary case [8, 1, 10] tells us that if

$$\vec{c} \in C_{\mathcal{L}}(D, \lambda Q) \setminus C_{\mathcal{L}}(D, (\lambda - 1)Q),$$

then

$$w_H(\vec{c}) \ge \sigma(\lambda) \tag{2}$$

where

$$\sigma(\lambda) = \#\{\eta \in H^*(Q) \mid \eta - \lambda \in H(Q)\}.$$

Besides implying that

$$d(C_{\mathcal{L}}(D, \lambda Q)) \ge \min\{\sigma(\gamma) \mid 0 \le \gamma \le \lambda, \gamma \in H^*(Q)\}\$$

$$d(C_{\Omega}(D, \lambda Q)) \ge \min\{\mu(\gamma) \mid \lambda < \gamma, \gamma \in H^*(Q)\},$$
 (3)

which are both as strong as the Goppa bound, it tells us that for $\epsilon, \lambda \in H^*(Q)$ with $\epsilon < \lambda$ it holds that

$$d(C_{\mathcal{L}}(D, \lambda Q), C_{\mathcal{L}}(D, \epsilon Q)) \ge \min\{\sigma(\gamma) \mid \epsilon < \gamma \le \lambda, \gamma \in H^*(Q)\}, \tag{4}$$

and similarly

$$d(C_{\Omega}(D, \epsilon Q), C_{\Omega}(D, \lambda Q)) \ge \min\{\mu(\gamma) \mid \epsilon < \gamma \le \lambda, \gamma \in H^*(Q)\}.$$
 (5)

Furthermore, for $i \in H^*(Q)$ let $f_i \in \mathcal{L}(iQ) \setminus \mathcal{L}((i-1)Q)$. Then we obtain the improved primary code

$$\widetilde{E}(\delta) = \operatorname{Span}\{(f_i(P_1), \dots, f_i(P_n)) \mid \sigma(i) \geq \delta\},\$$

which clearly has minimum distance at least δ and highest possible dimension for a primary code with that designed distance. Similarly, the improved dual code

$$\widetilde{C}(\delta) = \left(\operatorname{Span} \left\{ \left(f_i(P_1), \dots, f_i(P_n) \right) \mid \mu(i) < \delta \right\} \right)^{\perp}$$

has minimum distance at least δ and again the highest possible dimension for a dual code with that designed distance.

Turning to the Hermitian curve $x^{q+1} - y^q - y$ over \mathbb{F}_{q^2} where q is a prime power, it is well-known that the corresponding function field has exactly $q^3 + 1$ rational places P_1, \ldots, P_{q^3}, Q . Choosing $n = q^3$ one obtains $H(Q) = \langle q, q+1 \rangle$ and

$$H^*(Q) = \{ iq + j(q+1) \mid 0 \le i \le q^2 - 1, 0 \le j \le q - 1 \}.$$
 (6)

In [23] it was shown that

$$C_{\mathcal{L}}(D,\lambda Q) = C_{\Omega}(D,(q^3 + q^2 - q - 2 - \lambda)Q)$$
(7)

for any $\lambda \in H^*(Q)$, and the minimum distance was established for dimensions up to a certain value. The minimum distance for the remaining dimensions was then settled in [26]. In the present paper we shall need improved code constructions, and we will in some cases also be occupied with the relative distances rather than minimum distances. To this end we recall material from [8] on the functions μ and σ – stated there in the more general case of norm-trace curves, but adapted here to the Hermitian case.

Proposition 3. Consider the Hermitian curve. For $iq + j(q + 1) \in H^*(Q)$ we have

$$\sigma(iq+j(q+1)) = \begin{cases} q^3 - iq - j(q+1) & \text{if } 0 \le i < q^2 - q \\ (q^2 - i)(q - j) & \text{if } q^2 - q \le i \le q^2 - 1, \end{cases}$$
(8)

and $\mu((q^2-1-i)q+(q-1-j)(q+1))=\sigma(iq+j(q+1))$. For each $\lambda \in H^*(Q)$ there exists a word $\vec{c} \in (C_{\mathcal{L}}(D,\lambda Q)\backslash C_{\mathcal{L}}(D,(\lambda-1)Q))\cap \widetilde{E}(\sigma(\lambda))$ having Hamming weight equal to $\sigma(\lambda)$.

Proof. Given a numerical semigroup Λ with finitely many gaps and an element $\lambda \in \Lambda$, we know from [12, Lem. 5.15] that $\#(\Lambda \setminus (\lambda + \Lambda)) = \lambda$. As $\#H^*(Q) = q^3$ we therefore obtain $\sigma(iq + j(q+1)) \geq q^3 - (iq + j(q+1))$. On the other hand, it is clear that $\sigma(iq + j(q+1)) \geq (q^2 - i)(q - j)$ by the definition of σ . Taking the maximum between these two expressions, we obtain the right hand side of (8). That these estimates on σ are the true values and that the last part of the proposition holds true both follow as a consequence of [8, Lem. 4]. The details of applying [8, Lem. 4] are left for the reader. Finally, the relation between μ and σ is a consequence of $H^*(Q)$ being a box in the parameters i and j, see (6).

In Appendix A we list a series of lemmas which all follow as corollaries to Proposition 3 and which will be needed in Sections 3 and 4.

Throughout the rest of the paper we restrict to considering codes derived from the Hermitian curve, and we always assume the length to be $n=q^3$. From Proposition 3 we see that the bound (4) on the relative distance of $C_{\mathcal{L}}(D, \epsilon Q) \subset C_{\mathcal{L}}(D, \lambda Q)$ is sharp. A similar remark then holds for the bound (5) on the dual codes due to (7). Finally, we observe from [8, Sec. 4] that

$$\widetilde{E}(\delta) = \widetilde{C}(\delta) \tag{9}$$

holds. Proposition 3 therefore not only gives us the true value of the minimum distance of the improved primary codes (without loss of generality we may assume $\delta = \sigma(\lambda)$ for some $\lambda \in H^*(Q)$), but also does it for the improved dual codes.

We conclude the section with some information on the cases where the improved primary codes coincide with one-point algebraic geometric codes.

Corollary 4. For
$$\delta > q^2 - q$$
 we have $\widetilde{E}(\delta) = C_{\mathcal{L}}(D, (q^3 - \delta)Q)$, but $C_{\mathcal{L}}(D, (q^3 - (q^2 - q))Q)$ is strictly contained in $\widetilde{E}(q^2 - q)$.

This corollary implies that the dimension of $\widetilde{E}(\delta)$ can be determined from the usual one-point Hermitian codes whenever $\delta > q^2 - q$. For later reference we state these dimensions in terms of δ .

Proposition 5. Denote by g = q(q-1)/2 the genus of Hermitian function field. If $q^2 - q < \delta < q^3 - 2g + 2$, then the dimension of $\widetilde{E}(\delta)$ is given by

$$q^{3} - g + 1 - \delta$$
. If $q^{3} - 2g + 2 \le \delta \le q^{3}$, we have

$$\dim \widetilde{E}(\delta) = \sum_{s=0}^{a+b} (s+1) - \max\{a, 0\}$$

where $q^3 - \delta = aq + b(q+1)$ for -q < a < q and $0 \le b < q$.

Proof. First, note that in both cases Corollary 4 implies the equality $\widetilde{E}(\delta) = C_{\mathcal{L}}(D, (q^3 - \delta)Q)$. For the first case recall from [24, Cor. II.2.3] that the code $C_{\mathcal{L}}(D, \lambda G)$ has dimension $\lambda + 1 - g$ whenever $2g - 2 < \lambda < n$. By the assumptions on δ , we have $q^3 - \delta > q^3 - (q^3 - 2g + 2) = 2g - 2$, meaning that $C_{\mathcal{L}}(D, (q^3 - \delta)Q)$ has dimension $q^3 - \delta + 1 - g$. By the observation in the beginning of the proof, the same holds true for $\widetilde{E}(\delta)$.

To prove the second case, observe that the dimension of $C_{\mathcal{L}}(d, (q^3 - \delta)Q)$ is exactly the number of elements λ in $H^*(Q)$ with $\lambda \leq q^3 - \delta$. By (6) such elements have the form $\lambda = iq + j(q+1)$, but equivalently we can write $\lambda = i'q + j$ where i' = i + j. By the division algorithm this representation is unique for $0 \leq j < q$. For i'q + j to satisfy the requirements of (6), we also require $i' - j = i \geq 0$. That is, $H^*(Q)$ contains the integers whose quotients modulo q are at least their remainders modulo q.

Writing $q^3 - \delta = (a+b)q + b$ with $0 \le b < q$ using the division algorithm, the number of elements in $H^*(Q)$ less than (a+b+1)q is given by $\sum_{s=0}^{a+b} (s+1)$. If $b \ge a+b$, which happens only if $a \le 0$, this number is also the number of elements with value at most $q^3 - \delta$. Otherwise, the count includes b - (a+b) = a elements of $H^*(Q)$ that are greater than $q^3 - \delta$. Hence, subtracting $\max\{a,0\}$ gives the desired count in both cases.

It remains to establish information on the dimension of $\widetilde{E}(\delta)$ for $\delta \leq q^2 - q$ since the improved codes differ from the usual Hermitian codes in this case. This subject is treated in the next section.

3 The dimension of improved codes

As explained in the previous section, the dimension of $\widetilde{E}(\delta)$ can be determined from well-known methods as long as $\delta > q^2 - q$. In this section we present closed formula lower bounds on the dimension in the remaining cases. We start with an important lemma.

Lemma 6. Let $\delta \leq q^2$. The number of integer points

$$(x,y) \in \{q^2 - q, \dots, q^2 - 1\} \times \{0, \dots, q - 1\}$$

with $(q^2 - x)(q - y) \ge \delta$ is at least

$$q^2 - \left| \delta + \delta \ln(q^2/\delta) \right|. \tag{10}$$

If $\delta < q$, then the number of integer points is at least

$$q^2 - \lfloor \delta + \delta \ln(\delta) \rfloor, \tag{11}$$

which is stronger than (10).

Proof. The number of integer points in the given Cartesian product is at least that of the volume of

$$\{(x,y) \in [q^2 - q, q^2] \times [0,q] \mid (q^2 - x)(q - y) \ge \delta\},\$$

which equals

$$\int_{q^{2}-q}^{q^{2}-\frac{\delta}{q}} \int_{0}^{q-\frac{\delta}{q^{2}-x}} dy \, dx$$

$$= q(q^{2} - \frac{\delta}{q} - q^{2} + q) + \int_{q^{2}-\frac{\delta}{q}}^{q^{2}-q} \frac{\delta}{q^{2}-x} \, dx$$

$$= q^{2} - \delta - \delta[\ln(z)]_{\frac{\delta}{q}}^{q} = q^{2} - \delta - \delta \ln(q^{2}/\delta),$$

where we used the substitution $z = q^2 - x$. Since the number of integer points is integral, we obtain the bound $\lceil q^2 - \delta - \delta \ln(q^2/\delta) \rceil$, which is the same as (10).

If $\delta < q$, then the number of integer points is at least the combined volumes of

$$\{(x,y) \in [q^2 - q, q^2] \times [0,q] \mid x \le q^2 - \delta \text{ or } y \le q - \delta\}$$

and

$$\{(x,y) \in [q^2 - \delta, q^2] \times [q - \delta, q] \mid (q^2 - x)(q - y) \ge \delta\}.$$

The first mentioned volume equals $q^2 - \delta^2$. The latter volume is

$$\int_{q^2-\delta}^{q^2-1} \int_{q-\delta}^{q-\frac{\delta}{q^2-x}} dy \, dx$$

$$= \int_{q^2-\delta}^{q^2-1} \left(\delta - \frac{\delta}{q^2-x}\right) dx$$

$$= \delta(\delta - 1) - \delta[\ln(z)]_1^{\delta} = \delta(\delta - 1) - \delta\ln(\delta).$$

Adding up the two volumes, we obtain (11) by applying the ceiling function as above.

The dimension of the improved codes of designed distance at most $q^2 - q$ is covered by the following two propositions. Recall from (9) that the equality $\widetilde{C}(\delta) = \widetilde{E}(\delta)$ holds for codes defined from the Hermitian function field. Hence, the stated formulas for primary codes also hold for the dual codes.

Proposition 7. Given $q < \delta \le q^2 - q$ write

$$q^{3} - \delta = q^{3} - q^{2} + aq + b(q+1)$$

where -q < a < q and $0 \le b < q$. If 0 < a, then

$$\dim(\widetilde{E}(\delta)) \ge q^3 - \delta - g + 1 - \sum_{s=0}^{a+b} (s+1) + a + q^2 - \lfloor \delta + \delta \ln(q^2/\delta) \rfloor.$$

If $a \leq 0$, then

$$\dim(\widetilde{E}(\delta)) \ge q^3 - \delta - g + 1 - \sum_{s=0}^{a+b} (s+1) + q^2 - \lfloor \delta + \delta \ln(q^2/\delta) \rfloor.$$

Proof. Let g = q(q-1)/2 be the number of gaps in H(Q), i.e. the genus of the function field. As is well-known, for $2g \le \lambda < q^3 - 1$ the number of $\epsilon \in H^*(Q)$ with $\epsilon \le \lambda$ equals $\lambda - g + 1$. Therefore, by choosing $\lambda = q^3 - \delta$ the restriction on δ as given in the proposition implies that there are exactly $q^3 - \delta - g + 1$ elements $\epsilon \in H^*(Q)$ with $\epsilon \le q^3 - \delta$. These elements then satisfy $\sigma(\epsilon) \ge \delta$ by Lemma 25. From (8) we see that the additional elements in $H^*(Q)$ with $\sigma(\epsilon) \ge \delta$ must belong to

$${iq + j(q+1) \mid q^2 - q \le i \le q^2 - 1, \ 0 \le j \le q - 1}.$$
 (12)

Lemma 6 gives an estimate on the total number of elements ϵ in (12) with $\sigma(\epsilon) \geq \delta$. Adding this number to $q^3 - \delta - g + 1$, we have counted the elements ϵ in (12) with $\epsilon \leq q^3 - \delta$ twice. By using similar arguments as in the proof of Proposition 5, the number of such elements equals $\sum_{s=0}^{a+b} (s+1) - a$ when $0 \leq a < q$, and it equals $\sum_{s=0}^{a+b} (s+1)$ when -q < a < 0. This proves the proposition.

Proposition 8. Given $1 \leq \delta \leq q$ the dimension of the code $\widetilde{E}(\delta)$ satisfies

$$\dim(\widetilde{E}(\delta)) \ge q^3 - \lfloor \delta + \delta \ln(\delta) \rfloor.$$

Proof. By Lemma 30 the elements $\lambda \in H^*(Q)$ which do not satisfy $\sigma(\lambda) \geq \delta$ must belong to $\{iq+j(q+1) \mid q^2-\delta \leq i \leq q^2, q-\delta \leq j < q\}$. The number of elements in this set having $\sigma(\lambda) < \delta$ is bounded above by $\lfloor \delta + \delta \ln(\delta) \rfloor$ by Lemma 6. Since the total number of monomials in $H^*(Q)$ is q^3 , the result follows.

4 Inclusion of improved codes

As already mentioned our first construction of improved nested code pairs consists of choosing $\widetilde{C}(\delta_2)$ and $\widetilde{E}(\delta_1)$ such that $\widetilde{C}(\delta_2)^{\perp} \subset \widetilde{E}(\delta_1)$. To treat this construction we therefore need a clear picture of the pairs (δ_1, δ_2) of minimum distances that imply this inclusion. We establish this in the present section. As it turns out, the formulas for σ and μ given in Proposition 3 mean that several cases must be considered, and each case is presented as a separate proposition.

In the following, quantifiers on λ, ε are considered on the domain $H^*(Q)$. Given $\delta_1 \in \sigma(H^*(Q))$, define δ_2^{\max} to be the maximal value of δ_2 such that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ holds. This inclusion is equivalent to

$$\forall \lambda \colon \left[(\sigma(\lambda) < \delta_1) \to (\mu(\lambda) \ge \delta_2) \right]. \tag{13}$$

We first observe that if we can find a $\lambda_1 \in H^*(Q)$ such that

$$\left[\forall \varepsilon > \lambda_1 \colon \ \mu(\varepsilon) \ge \mu(\lambda_1)\right] \land \left[\forall \varepsilon < \lambda_1 \colon \ \sigma(\varepsilon) \ge \delta_1\right] \tag{14}$$

is true, then (13) is also true whenever $\delta_2 \leq \mu(\lambda_1)$. In particular, we therefore have

$$\delta_2^{\max} \ge \mu(\lambda_1). \tag{15}$$

On the other hand, we immediately see from (13) that a $\lambda_2 \in H^*(Q)$ with

$$\sigma(\lambda_2) < \delta_1 \tag{16}$$

implies the bound

$$\delta_2^{\max} \le \mu(\lambda_2). \tag{17}$$

In the proofs of each of the following propositions, our strategy therefore is to determine λ_1 and λ_2 satisfying (14) and (16), respectively, while also satisfying $\mu(\lambda_1) = \mu(\lambda_2)$. From (15) and (17) it then follows that $\delta_2^{\max} = \mu(\lambda_1)$. Note, however, that λ_1 and λ_2 need not be distinct. If $\lambda_1 = \lambda_2$, we shall simply use λ .

With this strategy in mind, the following lemmas will prove very useful.

Lemma 9. Let $\lambda = iq + j(q+1) \in H^*(Q)$, meaning that $0 \le i < q^2$ and $0 \le j < q$. In addition, assume that $i \le q^2 - q$, $i = q^2 - 1$, or j = 0. If $\varepsilon \in H^*(Q)$ satisfies $\varepsilon < \lambda$, then $\sigma(\varepsilon) \ge \sigma(\lambda)$.

Proof. For both the cases $i < q^2 - q$ and j = 0, the claim follows by Lemma 26. If $i = q^2 - q$, Lemma 28 implies that $\sigma(\lambda) = \sigma((i+j)q)$, and any $\varepsilon \in H^*(Q)$ satisfying $\lambda > \varepsilon > (i+j)q$ has $\sigma(\varepsilon) \ge \sigma((i+j)q)$ by Lemma 29. This also holds true for $\varepsilon < (i+j)q$ by the first part of the proof.

Finally, for $i=q^2-1$ consider $\varepsilon=i'q+j'(q+1)\in H^*(Q)$ with $\varepsilon<\lambda$. If $q^2-q\leq i'\leq q^2-1$, the claim follows by Lemmas 27 and 29. Otherwise, ε is at most $(q^2-q-1)q+(q-1)(q+1)=q^3-q-1$, meaning that $\sigma(\varepsilon)\geq q+1$ by Lemma 26. The proof follows by noting that $\sigma(\lambda)\leq q$.

Lemma 10. Let $\lambda = iq + j(q+1) \in H^*(Q)$, meaning that $0 \le i < q^2$ and $0 \le j < q$. In addition, assume that $i \ge q-1$ or j=0. If $\varepsilon \in H^*(Q)$ satisfies $\varepsilon > \lambda$, then $\mu(\varepsilon) \ge \mu(\lambda)$.

Proof. This proof is similar to the one of Lemma 9. Defining the notation $\lambda' = (q^2 - 1 - i)q + (q - 1 - j)(q + 1)$, the translation of Lemma 25 into information on μ gives $\mu(\lambda) = n - \lambda'$ whenever

$$i > q - 1 \text{ or } j = q - 1.$$
 (18)

Additionally, if $\varepsilon \in H^*(Q)$ with $\varepsilon > \lambda$, then $\varepsilon' < \lambda'$ where the ε' is defined in the same way as λ' . This implies that $\mu(\lambda) < \mu(\varepsilon)$ when $\lambda = iq + j(q+1)$ satisfies (18). This immediately proves the claim for i > q - 1.

For i = q - 1 the μ -equivalent of Lemma 28 gives $\mu(\lambda) = \mu((i - (q - 1 - j)) + (q - 1)(q + 1))$, and any elements between have greater μ -value by the translation of Lemma 29. The remaining elements greater than λ are covered by the first part of the proof.

The last part of the proof is j = 0 and i < q - 1, which follows the same procedure as the last part of the proof of Lemma 9.

Proposition 11. Let $2 \leq \delta_1 \leq q$. Then $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if $\delta_2 \leq q^3 - (\delta_1 - 2)(q + 1)$.

Proof. Let $\lambda' = (q^2 - 1)q + (q - \delta_1)(q + 1)$. We have $\sigma(\lambda') = \delta_1$ by (8), and Lemma 9 implies that $\sigma(\varepsilon) \geq \delta_1$ for all $\varepsilon < \lambda'$. Additionally, Lemmas 27 and 29 yield that $\lambda = \lambda' + q + 1$ is the smallest element in $H^*(Q)$ with a strictly smaller σ -value. Combining this with Lemma 10 applied to λ reveals that λ satisfies (14). However, (16) is satisfied as well since $\sigma(\lambda) < \delta_1$. Thus, $\delta_2^{\max} = \mu(\lambda) = q^3 - (\delta_1 - 2)(q + 1)$.

Proposition 12. Let $q < \delta_1 \leq q^2 - q$. Then $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if

$$\delta_2 \le \begin{cases} q^3 - q^2 + q - \delta_1 + 2 & \text{if } 0 \le b \le a \\ q^3 - q^2 - a(q+1) & \text{if } b > a \end{cases}$$

where $\delta_1 - (q+1) = aq + b$ for $a \ge 0$ and $0 \le b < q$.

Proof. First, observe that $aq + b \le (q - 3)q + (q - 1)$, meaning that a is at most q - 3.

Assume that b = 0. Then $\lambda = (q^2 - 1 - a)q$ has $\sigma(\lambda) = \delta_1 - 1$, meaning that it satisfies (16). Note that $\mu(\lambda) = q^3 - q^2 - aq + 1$, which can be rewritten to obtain the claimed expression.

If $0 < b \le a$, we can use $\lambda = (q^2 - q - 1 - a + (b - 1))q + (q - 1 - (b - 1))(q + 1)$, which satisfies (16) since $\sigma(\lambda) = \delta_1 - 1$. Here, we see that

$$\mu(\lambda) = q^3 - q^2 - aq - b + 1 = q^3 - q^2 + q - \delta_1 + 2.$$

Finally, for b > a we can let $\lambda = (q^2 - q - 1)q + (q - 1 - a)(q + 1)$ with $\sigma(\lambda) = (a+1)q + a + 1 < \delta_1$. Again, λ satisfies (16). Calculating the value of μ gives $\mu(\lambda) = q^3 - q^2 - a(q+1)$.

In all three of the above situations, the element immediately preceding λ in $H^*(Q)$ is given by $\lambda' = \lambda - 1$, and the reader may verify that $\sigma(\lambda') \geq \delta_1$. In each case applying Lemma 9 to λ' implies that $\sigma(\varepsilon) \geq \delta_1$ for all $\varepsilon < \lambda$. Lemma 10 applied to λ then shows that λ satisfies (14) as well. In conclusion, the specified values of λ satisfy both (14) and (16), and computing each value of $\mu(\lambda)$ gives the expression in the proposition.

Proposition 13. Let $q^2 - q < \delta_1 \le q^3 - 2q^2 + 2q$. Then $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if $\delta_2 \le q^3 - q^2 + q + 2 - \delta_1$.

Proof. Set $\lambda' = n - \delta_1$ and observe that $\lambda' \geq 2q^2 - 2q = 4g$ where g is the genus of the Hermitian function field. Thus, λ' is a non-gap in $H^*(Q)$, and $\sigma(\lambda') = \delta_1$ by (8). Lemma 9 implies that any smaller element of $H^*(Q)$ has σ -value at least δ_1 . We see, however, that $\lambda = \lambda' + 1$ has $\sigma(\lambda) = \delta_1 - 1$, and it must be the smallest such value. At the same time it meets the requirements of Lemma 10, implying that (14) is fulfilled. As already noted λ satisfies (16) as well, meaning that $\delta_2^{\max} = \mu(\lambda) = q^3 - q^2 + q + 2 - \delta_1$. \square

Proposition 14. Let $q^3 - 2q^2 + 2q < \delta_1 \le q^3 - q^2$. Then $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if

$$\delta_2 \le \begin{cases} (a+1)q + b + 2 & \text{if } b < a \\ (a+2)q & \text{if } a \le b < q - 1 \\ (a+2)q + 1 & \text{if } b = q - 1 \end{cases}$$

where $q^3 - q^2 - \delta_1 = aq + b$ for $a \ge 0$ and $0 \le b < q$.

Proof. First note that $aq + b \le (q - 2)q$, implying that a is at most q - 2. Assume that b < a and let $\lambda = (q + a - (b + 1))q + (b + 1)(q + 1)$. This element

satisfies the requirements of Lemma 10, and $\lambda - 1$ satisfies the requirements of Lemma 9. This means that λ fulfils (14). Simultaneously, (16) is met since $\sigma(\lambda) = \delta_1 - 1$. Thus, $\delta_2^{\max} = \mu(\lambda) = (a+1)q + b + 2$.

Let a = b and $\lambda = (q - 1)q + (a + 1)(q + 1)$. Applying Lemmas 9 and 10 to $\lambda - 1$ and λ as above, we see that λ satisfies (14). It also meets (16) since $\sigma(\lambda) = \delta_1 - 1$. Subsequently, $\delta_2^{\text{max}} = \mu(\lambda) = (a + 2)q$.

Now, consider a < b < q-1 and let $\lambda_1 = (q-1)q + (a+1)(q+1)$ and $\lambda_2 = (a+1)q + (q-1)(q+1)$. We can apply both Lemmas 9 and 10 to λ_1 to obtain that it satisfies (14). On the other hand, $\sigma(\lambda_2) < \delta_1$ implies that (16) is fulfilled. In addition, $\mu(\lambda_1) = \mu(\lambda_2)$, which gives $\delta_2^{\max} = \mu(\lambda_1) = (a+2)q$.

The remaining part is b = q-1. If this happens, note that $\lambda = (q+a+1)q$ has $\sigma(\lambda) = \delta_1 - 1$, whereas $\lambda - 1 = (a+1)q + (q-1)(q+1)$ has $\sigma(\lambda - 1) = \delta_1$. By the same arguments as in the first part of the proof, we obtain that $\delta_2^{\max} = \mu(\lambda) = (a+2)q + 1$.

Proposition 15. Let $q^3 - q^2 \le \delta_1 \le q^3$. Then $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if

$$\delta_2 \le \begin{cases} a+1 & \text{if } b < a \\ a+2 & \text{if } b \ge a \end{cases}$$

where $q^3 - \delta_1 = aq + b$ for $a \ge 0$ and $0 \le b < q$.

Proof. Assume first that b < a, and set $\lambda_1 = aq$ and $\lambda_2 = a(q+1)$. The latter meets the assumptions of Lemmas 9 and 10, meaning that λ_2 satisfies (14). Observe that $\sigma(\lambda_1) < \delta_1$ and $\mu(\lambda_1) = \mu(\lambda_2)$. From this we see that $\delta_2^{\max} = \mu(\lambda_1) = a + 1$.

Otherwise, if $b \geq a$, let $\lambda = (a+1)q$, which satisfies (16) by the observation that $\sigma(\lambda) < \delta_1$. The element of $H^*(Q)$ immediately preceding λ is $\lambda' = a(q+1)$, which has $\sigma(\lambda') \geq \delta_1$. As in the previous proofs, applying Lemmas 9 and 10 to λ' and λ , respectively, shows that λ fulfils (14). Hence, $\delta_2^{\max} = \mu(\lambda) = a + 2$.

It is worth noting that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if every $\lambda \in H^*(Q)$ with $\sigma(\lambda) < \delta_1$ also satisfies $\mu(\lambda) \geq \delta_2$. By Proposition 3 this may be rewritten as $\mu(\lambda) < \delta_1$ implying $\sigma(\lambda) \geq \delta_2$. Hence, the inclusion of codes is symmetric in the sense that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ if and only if $\widetilde{C}(\delta_1)^{\perp} \subseteq \widetilde{E}(\delta_2)$.

One could expect that this symmetry would show up in Propositions 11–15 as well. However, this is not the case since the propositions describe the maximal value of δ_2 such that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ for a given value of δ_1 . Although this implies that $\widetilde{C}(\delta_1)^{\perp} \subseteq \widetilde{E}(\delta_2)$, there may be a $\delta' > \delta_1$ such that $\widetilde{C}(\delta')^{\perp} \subseteq \widetilde{E}(\delta_2)$ as shown in Example 1 below.

Example 1. Let q = 4 and set $\delta_1 = 6$. Then considering the values of σ and μ as given in Table 4 of the Appendix reveals that the greatest possible value of δ_2 such that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$ is given by $\delta_2 = 48$. By the observations above we know that this implies $\widetilde{C}(6)^{\perp} \subseteq \widetilde{E}(48)$ as well. However, inspecting the tables again will reveal that the $\widetilde{C}(8)^{\perp}$ is also a subset of $\widetilde{E}(48)$. Notice that both of these observations agree with the formulas in Propositions 12 and 14.

5 Improved nested codes of not too small codimension

Based on our findings in Sections 3 and 4, we are now able to describe the parameters of our first construction of nested code pairs, namely the one where the codimension is not too small. If $\delta_1, \delta_2 \in H^*(Q)$ satisfy the conditions in one of the Propositions 11–15, it follows that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$. By the bounds (4) and (5) and the observation following Proposition 3, the relative distance of this code pair is exactly $d(\widetilde{E}(\delta_1)) = \delta_1$, and the relative distance of its dual is $d(\widetilde{C}(\delta_2)^{\perp}) = \delta_2$.

For each possible pair of designed distances described in Propositions 11–15, we can combine the dimensions of the usual Hermitian codes with the dimension bounds of Propositions 7 and 8. This gives bounds on the codimension, ℓ , of $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$.

Proposition 16. Let $\delta_1, \delta_2 \in H^*(Q)$, and $\delta_1 \leq q$. Further, let δ_2 satisfy the conditions of Proposition 11, meaning that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$. Denote their codimension by ℓ .

If $\delta_2 \leq q$, then

$$\ell \ge q^3 - \lfloor \delta_1 + \delta_1 \ln(\delta_1) \rfloor - \lfloor \delta_2 + \delta_2 \ln(\delta_2) \rfloor.$$

If $q < \delta_2 \le q^2 - q$, then

$$\ell \ge q^3 + q^2 - g + 1 - \lfloor \delta_1 + \delta_1 \ln(\delta_1) \rfloor - \delta_2 - \sum_{s=0}^{a+b} (s+1) - \lfloor \delta_2 + \delta_2 \ln(q^2/\delta_2) \rfloor + \max\{a, 0\}$$

where a and b are as in Proposition 7 applied to δ_2 .

If
$$q^2 - q < \delta_2 < q^3 - 2g + 2$$
, then

$$\ell \ge q^3 - g + 1 - \lfloor \delta_1 + \delta_1 \ln(\delta_1) \rfloor - \delta_2.$$

Finally, if $q^3 - 2g + 2 \le \delta_2$, we have

$$\ell \ge \sum_{s=0}^{a+b} (s+1) - \lfloor \delta_1 + \delta_1 \ln(\delta_1) \rfloor - \max\{a, 0\}$$

where $q^3 - \delta_2 = aq + b(q+1)$ for -q < a < q and $0 \le b < q$.

Proof. By Proposition 8 we have $\dim \widetilde{E}(\delta_1) \geq q^3 - \lfloor \delta_1 + \delta_1 \ln(\delta_1) \rfloor$. In each case, we can obtain a bound on the codimension ℓ by subtracting an upper bound on $\widetilde{C}(\delta_2)^{\perp} = q^3 - \dim \widetilde{E}(\delta_2)$. In turn, such a bound can be obtained via a lower bound on $\dim \widetilde{E}(\delta_2)$.

In the case $\delta_2 \leq q$ the dimension of $\widetilde{E}(\delta_2)$ can again be bounded by Proposition 8. In the second case the bound on $\dim \widetilde{E}(\delta_2)$ follows by Proposition 7. Proposition 5 delivers the bounds in the two final cases.

Proposition 17. Let $\delta_1, \delta_2 \in H^*(Q)$ and $q < \delta_1 \leq q^2 - q$. Further, let δ_2 satisfy the conditions of Proposition 11, meaning that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$. Denote their codimension by ℓ and let a_1, b_1 be as in Proposition 7 applied to δ_1 .

If $\delta_2 \leq q$, then

$$\ell \ge q^3 + q^2 - g + 1 - \delta_1 - \sum_{s=0}^{a_1 + b_1} (s+1) + \max\{a_1, 0\} - |\delta_1 + \delta_1 \ln(q^2/\delta_1)| - |\delta_2 + \delta_2 \ln(\delta_2)|.$$

If $q < \delta_2 \le q^2 - q$, then

$$\ell \ge q^3 + 2q^2 - 2g + 2 - (\delta_1 + \delta_2) - \sum_{s=0}^{a_1 + b_1} (s+1) + \max\{a_1, 0\}$$
$$- \sum_{s=0}^{a_2 + b_2} (s+1) + \max\{a_2, 0\} - \lfloor \delta_1 + \delta_1 \ln(q^2/\delta_1) \rfloor - \lfloor \delta_2 + \delta_2 \ln(q^2/\delta_2) \rfloor$$

where a_2 and b_2 are as in Proposition 7 applied to δ_2 . Finally, if $q^2 - q < \delta_2$, then

$$\ell \ge q^3 + q^2 - 2g + 2 - (\delta_1 + \delta_2) - \sum_{s=0}^{a_1 + b_1} (s+1) - \lfloor \delta_1 + \delta_1 \ln(q^2/\delta_1) \rfloor + \max\{a_1, 0\}.$$

Proof. We use the same strategy as in the proof of Proposition 16. A bound for the dimension of $\widetilde{E}(\delta_1)$ can be found in Proposition 7. For

 $\delta_2 \leq q$ the bound on $\dim \widetilde{E}(\delta_2)$ comes from Proposition 8, and in the case $q < \delta_2 \leq q^2 - q$ it comes from Proposition 7. In the final case the bound follows from Proposition 5, where we note that $\delta_2 \leq q^3 - 2g + 2$ by Proposition 12 and the assumption on δ_1 .

Proposition 18. Let $\delta_1, \delta_2 \in H^*(Q)$ and $q^2 - q < \delta_1 < q^3 - 2g + 2$. Further, let δ_2 satisfy the conditions of Proposition 11, meaning that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$. Denote their codimension by ℓ .

If $\delta_2 \leq q$, we have

$$\ell \ge q^3 - g + 1 - \delta_1 - \lfloor \delta_2 + \delta_2 \ln(\delta_2) \rfloor.$$

If $q < \delta_2 \le q^2 - q$, then

$$\ell \ge q^3 + q^2 - 2g + 2 - (\delta_1 + \delta_2) - \sum_{s=0}^{a+b} (s+1) - \lfloor \delta_2 + \delta_2 \ln(q^2/\delta_2) \rfloor + \max\{a, 0\}$$

where a and b are as in Proposition 7 applied to δ_2 .

Finally, for $q^2 - q < \delta_2$ we have

$$\ell = q^3 - \delta_1 - \delta_2 - 2g + 2.$$

Proof. Again, the the strategy is the same as in the proof of Proposition 16. The exact dimension of $\widetilde{E}(\delta_1)$ is given by Proposition 5. For $\delta_2 \leq q$ the dimension of $\widetilde{E}(\delta_2)$ can be bounded by applying Proposition 8, and in the case $q < \delta_2 \leq q^2 - q$ the bound follows by Proposition 7. For the final case we note by Proposition 13 that $\delta_2 < q^3 - q^2 + q + 2 - (q^2 - q) = q^3 - 4g + 2$. Hence, the exact dimension of $\widetilde{E}(\delta_2)$ is given by the first part of Proposition 5 in this case.

Proposition 19. Let $\delta_1, \delta_2 \in H^*(Q)$ and $q^3 - 2g + 2 \leq \delta_1$. Further, let δ_2 satisfy the conditions of Proposition 11, meaning that $\widetilde{C}(\delta_2)^{\perp} \subseteq \widetilde{E}(\delta_1)$. Denote their codimension by ℓ . Then

$$\ell \ge \sum_{s=0}^{a+b} (s+1) - \max\{a,0\} - \lfloor \delta_2 + \delta_2 \ln(\delta_2) \rfloor$$

where $q^3 - \delta_1 = aq + b(q+1)$ for -q < a < q and $0 \le b < q$.

Proof. The dimension of $\widetilde{E}(\delta_1)$ is given by the last part of Proposition 5. To obtain a bound on the maximal value of δ_2 , note that the minimal value of $q^3 - \delta_1$ can be written as $q^2 - q - 2 = q(q - 2) + (q - 2)$. Proposition 15 now implies $\delta_2 \leq q$. Hence, dim $\widetilde{E}(\delta_2) \geq q^3 - \lfloor \delta_2 + \delta_2 \ln(\delta_2) \rfloor$ by Proposition 8. \square

The application of Theorem 1 or 2 translates Propositions 16–19 into information on improved linear ramp secret sharing schemes and improved asymmetric quantum codes, respectively. The details are left for the reader.

6 Improved information on nested codes of small codimension

We will now consider a second construction which in general gives nested code pairs of smaller codimension than the construction in Section 5. This construction bears some resemblance to the one given in [7, Sec. IV], but in the setting of Hermitian codes.

From the definition of the codes, $C_{\mathcal{L}}(D, \lambda_2 Q) \subsetneq C_{\mathcal{L}}(D, \lambda_1 Q)$ whenever $\lambda_2 < \lambda_1$ and both λ_1 and λ_2 belongs to $H^*(Q)$. Our second construction is captured by the following two propositions.

Proposition 20. Let $\lambda_1 = iq + j(q+1) \in H^*(Q)$ where $i \leq j < q$, and define $\lambda_2 = jq + i(q+1) - 1$. Then $C_{\mathcal{L}}(D, \lambda_2 Q) \subsetneq C_{\mathcal{L}}(D, \lambda_1 Q)$ have codimension $\ell = j - i + 1$, and their relative distances satisfy

$$d(C_{\mathcal{L}}(D,\lambda_1 Q), C_{\mathcal{L}}(D,\lambda_2 Q)) = q^3 - \lambda_1 = d(C_{\mathcal{L}}(D,\lambda_1 Q)), \tag{19}$$

and

$$d(C_{\mathcal{L}}(D,\lambda_2 Q)^{\perp}, C_{\mathcal{L}}(D,\lambda_1 Q)^{\perp}) = (i+1)(j+1) \ge d(C_{\mathcal{L}}(D,\lambda_2 Q)^{\perp}).$$
 (20)

The inequality in (20) is strict if and only if $i \neq 0$ and $j \neq q-1$.

Proof. The codimension of $C_{\mathcal{L}}(D, \lambda_1 Q)$ and $C_{\mathcal{L}}(D, \lambda_2 Q)$ is given by the number of elements ε in $H^*(Q)$ with $\lambda_2 < \varepsilon \le \lambda_1$. By (6) $H^*(Q)$ contains every integer between λ_2 and λ_1 , meaning that the codimension is exactly $\lambda_1 - \lambda_2 = j - i + 1$.

To prove the first equalities in (19) and (20), we use Proposition 3 to obtain $\sigma(\lambda_1) = q^3 - \lambda_1$ and $\mu(\lambda_2) = (i+1)(j+1)$. Applying (4) and (5) then implies that the relative distances are at least $q^3 - \lambda_1$ and (i+1)(j+1), respectively. That these are indeed equalities follows from the observations following Proposition 3.

In (19) the last equality follows from the last part of Proposition 3. For (20) the observations in [8, Rem. 4] imply that (3) is in fact an equality. Thus, the minimal distance of $d\left(C_{\mathcal{L}}(D, \lambda_2 Q)^{\perp}\right)$ is given by $\mu((i+j)(q+1)) = i+j+1$ if i+j < q and

$$\mu((i - (q - 1 - j))q + (q - 1)(q + 1)) = q(i + j - q + 2)$$

otherwise. In the first case equality with (i+1)(j+1) occurs if and only if i=0, and in the second if and only if j=q-1.

In the above construction we only consider values of i less than q. A similar technique can be used for $q^2 - q \le i < q^2$. We state the proposition, but omit the proof since it follows by similar arguments as above.

Proposition 21. Let $\lambda_1 = (q^2 - 1 - i)q + (q - 1 - j)(q + 1) \in H^*(Q)$ where $i \leq j < q$, and define $\lambda_2 = (q^2 - 1 - j)q + (q - 1 - i)(q + 1) - 1$. Then $C_{\mathcal{L}}(D, \lambda_2 Q) \subsetneq C_{\mathcal{L}}(D, \lambda_1 Q)$ have codimension $\ell = j - i + 1$, and their relative distances satisfy

$$d(C_{\mathcal{L}}(D,\lambda_1 Q), C_{\mathcal{L}}(D,\lambda_2 Q)) = (i+1)(j+1) \ge d(C_{\mathcal{L}}(D,\lambda_1 Q)), \quad (21)$$

and

$$d(C_{\mathcal{L}}(D, \lambda_2 Q)^{\perp}, C_{\mathcal{L}}(D, \lambda_1 Q)^{\perp}) = q^3 - iq - j(q+1) = d(C_{\mathcal{L}}(D, \lambda_2 Q)^{\perp}).$$

The inequality in (21) is strict if $i \neq 0$ and $j \neq q-1$.

By applying one of Theorems 1 and 2, we can transform Propositions 20 and 21 into information on improved linear ramp secret sharing schemes and improved asymmetric quantum codes, respectively. The details of this translation are left for the reader.

7 Comparison with bounds and existing constructions

Having presented two improved constructions of nested code pairs in Sections 5 and 6, this section is devoted to the comparison between the corresponding asymmetric quantum codes and codes that already exist in the literature. The codes are also compared with the Gilbert-Varshamov bound for asymmetric quantum codes. Moreover, we compare the corresponding secret sharing schemes with a recent lower bound on the threshold gap [3]. When presenting code parameters we give the actual codimension rather than using the estimates in Section 5 which rely on the bounds in Propositions 7 and 8.

Since the codes obtained in Sections 5 and 6 are relatively long compared to the field size, the literature does not contain many immediately comparable codes. Yet, one way to obtain such codes is by using Construction II of La Guardia [17, Thm. 7.1], which gives asymmetric quantum generalized Reed-Solomon codes. Adjusting the theorem to codes over \mathbb{F}_{q^2} gives the following result.

Theorem 22. Let q be a prime power. There exist asymmetric quantum generalized Reed-Solomon codes with parameters

$$[[m_1m_2, m_1(2k - m_2 + c), \ge d/ \ge d - c]]_{q^2}$$

where $1 < k < m_2 < 2k + c \le q^{2m_1}$ and $k = m_2 - d + 1$, and where $m_2, d > c + 1, c \ge 1$, and $m_1 \ge 1$ are integers.

Example 2. By using different values for the parameters in Theorem 22, we obtain asymmetric quantum codes of varying lengths. If the chosen parameters give a code of length less than q^3 , we can pad each codeword with zeroes in order to obtain the correct length. Note that this does not change the relative distance of the nested codes nor of their duals.

For q=3 Table 1 lists the best code parameters that can be obtained in this way together with the comparable codes from the constructions in Sections 5 and 6. In the third column, the parameter d_z is maximized under the condition that the dimension and the distance d_x are at least as high as in [17]. In the fourth, the dimension is maximized, keeping at least the same minimal distances. As is evident, the codes of the present paper perform very favourably.

We further note that all presented new codes exceed the Gilbert-Varshamov bound for asymmetric quantum codes [18, Thm. 4]. Additionally, we remark that nesting usual one-point Hermitian codes and using the Goppa bound does not provide asymmetric quantum codes as good as the ones in columns three and four.

The two constructions in Theorem 24 and Corollary 29 of [7] based on codes defined from Cartesian product point sets provide another way to obtain asymmetric quantum codes that can be compared to the ones in this paper. We summarize these constructions in the following two theorems.

Theorem 23. Consider integers $m \geq 2$ and $s \leq q$ where q is a prime power. Given $\delta_1 \in \{1, 2, ..., s^m\}$ define $v \in \{0, 1, ..., m-1\}$ such that $s^v \leq \delta \leq s^{v+1}$, and choose an integer $\delta_2 \leq \lfloor (s - \delta_1/s^v + 1)s^{m-v+1} \rfloor$. Then there exists an asymmetric quantum code with parameters

$$[[s^m,\ell,\delta_1/\delta_2]]_q$$

where

$$\ell \ge s^m - \sum_{t=1}^m \frac{1}{(t-1)!} \left(\delta_1 \left(\ln \left(\frac{s^m}{\delta_1} \right) \right)^{t-1} + \delta_2 \left(\ln \left(\frac{s^m}{\delta_2} \right) \right)^{t-1} \right).$$

Theorem 24. Consider integers $1 < s \le q$ where q is a prime power, and let $m \in \{0, 1, ..., s-1\}$. Then for any $\ell \le m+1$ such that ℓ is even if and only if m is odd, there exists an asymmetric quantum code with parameters

$$[[s^2, \ell, d_z/d_x]]_q$$

where the distances are $d_z = \frac{1}{4} (2s - (m - \ell + 1)) (2s - (m + \ell - 1))$ and $d_x = \frac{1}{4} (m - \ell + 3) (m + \ell + 1)$. The two distances may also be interchanged.

Construction of [17, Thm. 7.1]		This paper				
(m_1,m_2,k,c)	Code	d_z maximized	ℓ maximized			
(2, 13, 2, 10)	$[[27, 2, 12/2]]_9$	$[[27, 2, 23/2]]_9$	$[[27, 12, 12/2]]_9$			
(2, 13, 3, 8)	$[[27, 2, 11/3]]_9$	$[[27, 2, 18/4]]_9$	$[[27, 11, 11/3]]_9$			
(2, 13, 4, 6)	$[[27, 2, 10/4]]_9$	$[[27, 2, 18/4]]_9$	$[[27, 10, 10/4]]_9$			
(2, 13, 5, 4)	$[[27, 2, 9/5]]_9$	$[[27, 2, 16/6]]_9$	$[[27, 9, 9/6]]_9$			
(2, 13, 6, 2)	$[[27, 2, 8/6]]_9$	$[[27, 2, 16/6]]_9$	$[[27, 10, 8/6]]_9$			
(3, 9, 3, 4)	$[[27, 3, 7/3]]_9$	$[[27, 3, 19/3]]_9$	$[[27, 15, 7/3]]_9$			
(3, 9, 4, 2)	$[[27, 3, 6/4]]_9$	$[[27, 3, 17/4]]_9$	$[[27, 15, 6/4]]_9$			
(2, 13, 3, 9)	$[[27, 4, 11/2]]_9$	$[[27, 4, 20/2]]_9$	$[[27, 13, 11/2]]_9$			
(2, 13, 4, 7)	$[[27, 4, 10/3]]_9$	$[[27, 4, 18/3]]_9$	$[[27, 12, 10/3]]_9$			
(2, 13, 5, 5)	$[[27, 4, 9/4]]_9$	$[[27, 4, 16/4]]_9$	$[[27, 11, 9/4]]_9$			
(2, 13, 6, 3)	$[[27, 4, 8/5]]_9$	$[[27, 4, 14/6]]_9$	$[[27, 10, 8/6]]_9$			
(2, 13, 7, 1)	$[[27, 4, 7/6]]_9$	$[[27, 4, 14/6]]_9$	$[[27, 11, 7/6]]_9$			
(2, 13, 4, 8)	$[[27, 6, 10/2]]_9$	$[[27, 6, 18/2]]_9$	$[[27, 14, 10/2]]_9$			
(2, 13, 5, 6)	$[[27, 6, 9/3]]_9$	$[[27, 6, 16/3]]_9$	$[[27, 13, 9/3]]_9$			
(2, 13, 6, 4)	$[[27, 6, 8/4]]_9$	$[[27, 6, 14/4]]_9$	$[[27, 12, 8/4]]_9$			
(2, 13, 7, 2)	$[[27, 6, 7/5]]_9$	$[[27, 6, 12/6]]_9$	$[[27, 11, 7/6]]_9$			
(2, 13, 5, 7)	$[[27, 8, 9/2]]_9$	$[[27, 8, 16/2]]_9$	$[[27, 15, 9/2]]_9$			
(2, 13, 6, 5)	$[[27, 8, 8/3]]_9$	$[[27, 8, 14/3]]_9$	$[[27, 14, 8/3]]_9$			
(2, 13, 7, 3)	$[[27, 8, 7/4]]_9$	$[[27, 8, 12/4]]_9$	$[[27, 13, 7/4]]_9$			
(2, 13, 8, 1)	$[[27, 8, 6/5]]_9$	$[[27, 8, 10/6]]_9$	$[[27, 13, 6/6]]_9$			
(2, 13, 6, 6)	$[[27, 10, 8/2]]_9$	$[[27, 10, 14/2]]_9$	$[[27, 16, 8/2]]_9$			
(2, 13, 7, 4)	$[[27, 10, 7/3]]_9$	$[[27, 10, 12/3]]_9$	$[[27, 15, 7/3]]_9$			
(2, 13, 8, 2)	$[[27, 10, 6/4]]_9$	$[[27, 10, 10/4]]_9$	$[[27, 15, 6/4]]_9$			
(2, 13, 7, 5)	$[[27, 12, 7/2]]_9$	$[[27, 12, 12/2]]_9$	$[[27, 17, 7/2]]_9$			
(2, 13, 8, 3)	$[[27, 12, 6/3]]_9$	$[[27, 12, 10/3]]_9$	$[[27, 17, 6/3]]_9$			
(2, 13, 9, 1)	$[[27, 12, 5/4]]_9$	$[[27, 12, 8/4]]_9$	$[[27, 15, 6/4]]_9$			
(2, 13, 8, 4)	$[[27, 14, 6/2]]_9$	$[[27, 14, 10/2]]_9$	$[[27, 19, 6/2]]_9$			
(2, 13, 9, 2)	$[[27, 14, 5/3]]_9$	$[[27, 14, 8/3]]_9$	$[[27, 14, 8/3]]_9$			
(2,13,9,3)	$[[27, 16, 5/2]]_9$	$[[27, 16, 8/2]]_9$	$[[27, 19, 6/2]]_9$			
(2, 13, 10, 1)	$[[27, 16, 4/3]]_9$	$[[27, 17, 6/3]]_9$	$[[27, 19, 4/3]]_9$			
(2, 13, 10, 2)	$[[27, 18, 4/2]]_9$	$[[27, 19, 6/2]]_9$	$[[27, 21, 4/2]]_9$			
(2, 13, 11, 1)	$[[27, 20, 3/2]]_9$	$[[27, 21, 4/2]]_9$	$[[27, 23, 3/2]]_9$			

Table 1. Asymmetric quantum codes of length 27 over \mathbb{F}_9 . The first column states the parameters used in Theorem 22 to obtain the codes in the second. If necessary, these have been padded with zeroes to obtain length 27. The codes in the third and fourth columns are based on the construction in Sections 5 and 6.

Example 3. By using different parameters in Theorems 23 and 24 and padding with zeroes if necessary, we obtain the asymmetric quantum codes

presented in Table 2. This table also shows comparable codes from the constructions in Sections 5 and 6 which have either d_z or ℓ maximized as in Example 2. From the table it is evident that the codes of the present paper perform very favourably.

Again, we further note that all presented new codes exceed the Gilbert-Varshamov bound [18, Thm. 4], and that these codes cannot be constructed using information from the Goppa bound applied to nested one-point Hermitian codes.

Example 4. A few codes of length 8 over \mathbb{F}_4 are given in [6]. The construction in Section 5 can match – but not improve on – the codes $[[8,1,4/3]]_4$, [[8,2,5/2]], $[[8,2,3/3]]_4$, $[[8,3,4/2]]_4$, and $[[8,4,3/2]]_4$. Additionally, [6] presents a code with parameters $[[8,1,6/2]]_4$, where we can construct an $[[8,1,4/3]]_4$ -code instead. All of these codes exceed the quantum Gilbert-Varshamov bound [18, Thm. 4], and the Goppa bound applied to nested one-point Hermitian codes cannot provide such parameters.

Some codes over \mathbb{F}_9 are presented as well. These codes do, however, have a length that is at least 36.

When presenting constructions of codes, it is customary to compare it to tables of 'best known' linear codes such as [20, 11]. Unfortunately, similar tables do not exist for asymmetric quantum codes. As we shall recall in a moment, however, one can still measure asymmetric quantum codes against the usual bounds on linear codes.

Before doing so, we observe that the tables in [11] only contains alphabets up to \mathbb{F}_9 , whereas [20] has \mathbb{F}_{256} as its largest alphabet. As indicated by the following example, however, the latter tables are generally not as optimized as the ones by [11].

Example 5. In some cases the improved codes $\widetilde{E}(\delta)$ exceed the codes given by [20]. For instance, when considering codes over \mathbb{F}_{16} , the codes $\widetilde{E}(12)$, $\widetilde{E}(9)$, and $\widetilde{E}(8)$ with parameters $[64, 48, 12]_{16}$, $[64, 51, 9]_{16}$, and $[64, 53, 8]_{16}$, respectively, all provide a minimal distance that is one higher than the corresponding code in the table.

Over \mathbb{F}_{25} the same is true for the codes $\widetilde{E}(20)$, $\widetilde{E}(16)$, and $\widetilde{E}(12)$, which have parameters $[125, 97, 20]_{25}$, $[125, 101, 16]_{25}$, and $[125, 106, 12]_{25}$. Additionally, $\widetilde{E}(15)$ has parameters $[125, 103, 15]_{25}$, exceeding the table distance by 2.

Recall from Section 6 that our second construction of nested code pairs (which are code pairs of small codimension) gives impure asymmetric quantum codes. This is already an advantage as the error-correcting algorithms can

take advantage of the impurity. Another advantage of considering relative distances rather than only minimal distances emerges when analysing the error-correcting ability of asymmetric quantum codes. In order to illustrate this advantage, we can compare nested codes from the construction in Section 6 with pairs of best known linear codes from the tables in [11, 20]. Note that the pairs of best known linear codes from such tables generally do not result in nested code pairs; that is, they are not guaranteed to satisfy the requirement that the dual of one code is contained in the other. The comparison with tables of best known linear codes is done in the following example. Whenever the tables of [20] are considered, we will use the minimum distance of an improved algebraic geometric Goppa code from the Hermitian curve if this exceeds the table value as in Example 5.

Example 6. Having fixed a code pair $C_2 \subset C_1 \subseteq \mathbb{F}_q^n$ of codimension ℓ and $d(C_1) = \delta_1$, we consider the greatest value $g(\ell, \delta)$ such that the tables of best known linear codes ensure the existence of $C, C' \subseteq \mathbb{F}_q^n$ with $\dim C - \dim C' = \ell$, $d(C) = \delta_1$, and $d(C'^{\perp}) \geq g(\ell, \delta_1)$. This is the same method as used in [7], and bears resemblance to the idea in [6, Thm. 2]. Using this procedure it is in no way guaranteed that $C' \subset C$. However, as shown in Table 3 the construction in Section 6 is in many cases on par with the best known codes, while simultaneously guaranteeing the inclusion $C_2 \subset C_1$. In some cases the use of relative distances will even exceed the values obtained from the best known codes. As in the previous examples, the codes in Table 3 all exceed the Gilbert-Varshamov bound for asymmetric quantum codes [18, Thm. 4].

Turning to secret sharing schemes, [3] presents a lower bound on the threshold gap r-t. That is, the authors bound the smallest possible difference between the reconstruction number r and the privacy number t for q-ary linear ramp secret sharing schemes with n shares and secrets from from \mathbb{F}_q^{ℓ} . For linear ramp secret sharing schemes over \mathbb{F}_{q^2} , they show that

$$r \ge t + \frac{(q^{2m} - 1)(n+2) + (q^{2m+2} - q^{2m})(\ell - 2m)}{q^{2m+2} - 1}$$
 (22)

for every $m \in \{0, 1, \dots, \ell - 1\}$. Of course, one should choose the m that gives the best bound. Comparing the secret sharing schemes obtained in this paper with the bound (22) helps to quantify how optimal the construction is. This is done in the following example, which also illustrates the advantage of using the improved codes from Section 5 and the improved information from Section 6 rather than relying solely on the Goppa bound applied to nested one-point Hermitian codes.

Constructions of [7]			This paper		
Type	(s,m)	Code	d_z maximized	ℓ maximized	
Thm. 24 Thm. 23 Thm. 23 Thm. 23 Thm. 23 Thm. 23 Thm. 23	(5, 2) (5, 4) (5, 1) (5, 3) (5, 2) (5, 4) (5, 3) (5, 2) (5, 4) (5, 2) (5, 2) (5, 2) (5, 2) (5, 2)		$ \begin{bmatrix} [27, 1, 20/4]]_9 \\ [27, 1, 13/9]]_9 \\ [27, 2, 23/2]]_9 \\ [27, 2, 16/6]]_9 \\ [27, 3, 19/3]]_9 \\ [27, 3, 12/8]]_9 \\ [27, 4, 16/4]]_9 \\ [27, 5, 19/2]]_9 \\ [27, 5, 13/6]]_9 \\ [27, 7, 17/2]]_9 \\ [27, 7, 17/2]]_9 \\ [27, 7, 15/3]]_9 \\ [27, 8, 16/2]]_9 $	$ \begin{array}{c} [[27,\ 4,16/4]]_9 \\ [27,\ 5,\ 9/9]]_9 \\ [27,\ 4,20/2]]_9 \\ [27,\ 6,12/6]]_9 \\ [27,\ 7,15/3]]_9 \\ [27,\ 7,\ 8/8]]_9 \\ [27,\ 10,10/4]]_9 \\ [27,\ 17,\ 7/2]]_9 \\ [27,\ 13,\ 6/6]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 21,\ 4/2]]_9 \\ [27,\ 21,\ 3/3]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 6/2]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10]_9 \\ [27,\ 19,\ 10$	
Thm. 23 Thm. 23 Thm. 23 Thm. 23 Thm. 23 Thm. 23	$\begin{array}{c} (5,2) \\ (5,2) \\ (5,2) \\ (5,2) \\ (5,2) \\ (5,2) \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{c} [[27, \ 9, 15/2]]_9 \\ [[27, 10, 14/2]]_9 \\ [[27, 11, 13/2]]_9 \\ [[27, 12, 12/2]]_9 \\ [[27, 14, 10/2]]_9 \\ [[27, 17, \ 7/2]]_9 \end{array} $	$ \begin{array}{c} [27,23,\ 3/2]_9 \\ [27,21,\ 4/2]_9 \\ [27,25,\ 2/2]_9 \\ [27,23,\ 3/2]_9 \\ [27,25,\ 2/2]_9 \\ [27,25,\ 2/2]_9 \end{array} $	

Table 2. Asymmetric quantum codes of length 27 over \mathbb{F}_9 . The first column indicates whether Theorem 23 or 24 was used in the codes given in the third column. The second states the parameters used, except for those that can be read off directly from the code. The codes in the fourth and fifth columns are based on the construction in Sections 5 and 6.

(i,j)	Parameters	$g(\ell,\delta_1)$	(i,j)	Parameters	$g(\ell,\delta_1)$
(2, 2)	$[[27, 1, 13/9]]_9$	9	(4, 4)	$[[125, 1, 81/25]]_{25}$	25
(1, 1)	$[[27, 1, 20/4]]_9$	4	(3, 3)	$[[125, 1, 92/16]]_{25}$	14
(1, 2)	$[[27, 2, 16/6]]_9$	6	(2, 2)	$[[125, 1, 103/9]]_{25}$	_
(0, 1)	$[[27, 2, 23/2]]_9$	2	(1, 1)	$[[125, 1, 114/4]]_{25}$	_
(0, 2)	$[[27, 3, 19/3]]_9$	3	(3, 4)	$[[125, 2, 86/20]]_{25}$	19
(2.2)	[[64 1 27/16]]	16	(2,3)	$[[125, 2, 97/12]]_{25}$	10
(3,3)	$[[64, 1, 37/16]]_{16}$	8	(1, 2)	$[[125, 2, 108/6]]_{25}$	_
(2,2)	$[[64, 1, 46/9]]_{16}$	8 4	(0,1)	$[[125, 2, 119/2]]_{25}$	_
(1,1)	$[[64, 1, 55/4]]_{16}$	4 12*	(2,4)	$[[125, 3, 91/15]]_{25}$	13
(2,3)	$[[64, 2, 41/12]]_{16}$		(1,3)	$[[125, 3, 102/8]]_{25}$	_
(1,2)	$[[64, 2, 50/6]]_{16}$	6	(0, 2)	$[[125, 3, 113/3]]_{25}$	_
(0,1)	$[[64, 2, 59/2]]_{16}$	2	(1,4)	$[[125, 4, 96/10]]_{25}$	10
(1,3)	$[[64, 3, 45/8]]_{16}$	8*	(0,3)	$[[125, 4, 107/4]]_{25}$	_
(0,2)	$[[64, 3, 54/3]]_{16}$	3	(0, 4)	$[[125, 5, 101/5]]_{25}$	6
(0,3)	$[[64, 4, 49/4]]_{16}$	5		11 / / 1120	

Table 3. Comparing the asymmetric quantum codes from Section 6 with the best known codes. For q=3 [11] is used, and for the remaining values of q, [20] is used. The codes marked in bold exceed $g(\ell,\delta_1)$, and the values of $g(\ell,\delta_1)$ marked with an asterisk stem from the improvements in Example 5. A dash indicates that the tables do not contain enough information to determine $g(\ell,\delta_1)$.

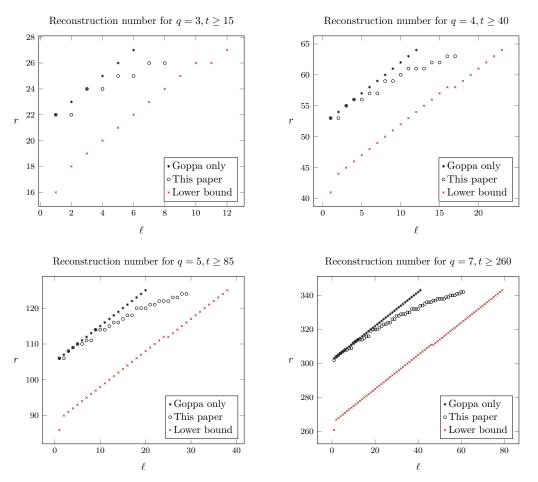


Figure 1. The minimal achievable reconstruction number r given a desired privacy number t. The plots show the constructions from Sections 5 and 6, a construction using the Goppa bound only, and the lower bound from (22).

Example 7. In each of the plots in Figure 1, a desired privacy number t has been fixed. For each codimension the plots then show the minimal reconstruction number r achievable with the constructions from Sections 5 and 6 when privacy number at least t is required. The plots also show the lower bound in (22).

Recall that the codes under consideration have length q^3 , meaning that the four corresponding secret sharing schemes support 27, 64, 125, and 343 participants, respectively. As the plots demonstrate, the constructions of this paper provide secret sharing schemes with parameters that could not be obtained by using nested Hermitian one-point codes and the Goppa bound alone.

We remark that the four given examples use a relatively large privacy

parameter t. Yet, it is also possible to obtain improved reconstruction numbers for small values of t.

8 Concluding remarks

In this paper we presented two improved constructions of nested code pairs from the Hermitian curve, and gave a detailed analysis of their performance when applied to the concepts of secret sharing and asymmetric quantum codes. Regarding information leakage in secret sharing we studied the reconstruction number r and the privacy number t, which give information on full recovery and full privacy, respectively. We note that it is possible to obtain information about partial information leakage by studying relative generalized Hamming weights rather than just relative minimum distances [15, 9]. For asymmetric quantum codes we applied the CSS construction. Applying the method of Steane's enlargements [22] is a future research agenda.

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Appendix A Additional results on σ and μ

In this section we state a number of lemmas that are needed in Sections 3 and 4. The lemmas all follow as corollaries to Proposition 3. To aid the reader in understanding them more easily, we first give an example for reference.

Example 8. In Table 4 we list $H^*(Q)$, $\sigma(H^*(Q))$, and $\mu(H^*(Q))$ for the Hermitian function field over \mathbb{F}_{16} , i.e. for q=4. Entries are ordered according to (i,j) where $\lambda=iq+j(q+1)$.

The first lemma explains when (2) equals the Goppa bound and when it is sharper.

Lemma 25. For all $\lambda = iq + j(q+1) \in H^*(Q)$ it holds that $\sigma(\lambda) \ge n - \lambda$ where $n = q^3$. The inequality is strict if and only if $q^2 - q \le i < q^2$ and $1 \le j < q$.

The next five lemmas give information on the relation between the values $\sigma(iq + j(q + 1))$ and $\sigma(i'q + j'(q + 1))$ for different constellations of i, j, i', j'. Using the translation from σ to μ as given in Proposition 3, this simultaneously implies relations on μ .

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Table 4. Upper table: $H^*(Q)$. Middle table: $\sigma(H^*(Q))$. Lower table: $\mu(H^*(Q))$

Lemma 26. For $0 < i \le q^2 - q - 1$ and $0 \le j < q - 1$ it holds that $\sigma(iq + j(q + 1)) = \sigma((i - 1)q + (j + 1)(q + 1)) + 1$. Furthermore, for $0 \le i \le q^2 - q - 1$ it holds that $\sigma(iq + (q - 1)(q + 1)) = \sigma((i + q)q) + 1$.

Lemma 27. The sequence

$$(\sigma(0\cdot q), \sigma(q), \dots, \sigma((q^2-1)q), \sigma((q^2-1)q+(q+1)), \\ \sigma((q^2-1)q+2(q+1)), \dots, \sigma((q^2-1)q+(q-1)(q+1)))$$

is strictly decreasing.

Lemma 28. We have $\sigma((q^2 - q + s)q + t(q + 1)) = \sigma((q^2 - q + t)q + s(q + 1))$ for $0 \le s, t \le q - 1$.

Lemma 29. Given $q^2 - q \le i \le q^2 - 1$ then for non-negative s such that $q^2 - q \le i - s$ we have $\sigma((i - s)q + s(q + 1)) \ge \sigma(iq)$. Similarly, given $0 \le j \le q^2 - 1$ then for non-negative s such that $j + s \le q - 1$ we have $\sigma((q^2 - 1 - s)q + (j + s)(q + 1)) \ge \sigma((q^2 - 1)q + j(q + 1))$.

Lemma 30. If $\sigma(iq + j(q + 1)) \le q$ then $q^2 - q \le i < q^2$.

Finally, we present a lemma on the relation between $\sigma(\lambda)$ and $\mu(\lambda)$ for λ belonging to a certain window.

Lemma 31. For $\lambda = iq + j(q+1) \in H^*(Q)$ with $q \leq i < q^2 - q$ and j arbitrary; or $q^2 - q < i \leq q^2 - 1$ and j = 0; or $0 \leq i < q$ and j = q - 1, we have $\mu(\lambda) + \sigma(\lambda) = q^3 - (q^2 - q - 1)$.

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