ON THE LAST FALL DEGREE OF WEIL DESCENT POLYNOMIAL SYSTEMS

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ABSTRACT. Given a polynomial system \mathcal{F} over a finite field k which is not necessarily of dimension zero, we consider the Weil descent \mathcal{F}' of \mathcal{F} over a subfield k'. We prove a theorem which relates the last fall degrees of \mathcal{F}_1 and \mathcal{F}'_1 , where the zero set of \mathcal{F}_1 corresponds bijectively to the set of k-rational points of \mathcal{F} , and the zero set of \mathcal{F}'_1 is the set of k'-rational points of the Weil descent \mathcal{F}' . As an application we derive upper bounds on the last fall degree of \mathcal{F}'_1 in the case where \mathcal{F} is a set of linearized polynomials.

1. INTRODUCTION

Let k be a field and let $\mathcal{F} \subset R = k[X_0, \dots, X_{m-1}]$ be a finite subset which generates an ideal. Let $R_{\leq i}$ be the set of polynomials in R of degree at most i.

For $i \in \mathbb{Z}_{\geq 0}$, we let $V_{\mathcal{F},i}$ be the smallest k-vector space of $R_{\leq i}$ such that

- (1) $\{f \in \mathcal{F} : \deg(f) \leq i\} \subseteq V_{\mathcal{F},i};$
- (2) if $g \in V_{\mathcal{F},i}$ and if $h \in R$ with $\deg(hg) \leq i$, then $hg \in V_{\mathcal{F},i}$.

We write $f \equiv_i g \pmod{\mathcal{F}}$, for $f, g \in R$, if $f - g \in V_{\mathcal{F},i}$.

The last fall degree as defined in [6] (see also [7]) is the largest d such that $V_{\mathcal{F},d} \cap R_{\leq d-1} \neq V_{\mathcal{F},d-1}$. We denote the last fall degree of \mathcal{F} by $d_{\mathcal{F}}$.

As shown in [6, 7] the last fall degree is intrinsic to a polynomial system, independent of the choice of a monomial order, always bounded by the degree of regularity, and invariant under linear change of variables and linear change of equations. In [6, 7] complexity bounds on solving zero dimensional polynomial systems were proven based on the last fall degree. It was shown in [6] that the polynomial systems arising from the Hidden Field Equations (HFE) public key crypto-system [1, 2] have bounded last fall degree if the degree of the defining polynomial and the cardinality of the base field are fixed (the bound was improved in [4]), and it follows that the HFE polynomials systems can be solved unconditionally in polynomial time.

For $\mathcal{F} \subset R = k[X_0, \ldots, X_{m-1}]$, let $Z_k(\mathcal{F})$ denote the set of solutions of \mathcal{F} over k; let $Z(\mathcal{F})$ denote the set of solutions of \mathcal{F} over \overline{k} , where \overline{k} is an algebraic closure of k. If \mathcal{F} is zero-dimensional then determining $Z(\mathcal{F})$ reduces to computing $V_{\mathcal{F},\max(d_{\mathcal{F}},e)}$ where e is the cardinality of $Z(\mathcal{F})$ [7].

Suppose k is a finite field of cardinality q^n with subfield k' of cardinality q. The Weil descent system of \mathcal{F} to k' is a polynomial system obtained when one expresses all equation with the help of a basis of k'/k. Let $\alpha_0, \ldots, \alpha_{n-1}$ be a basis of k/k'. For $f \in \mathcal{F}$ and $j = 0, \ldots, n-1$,

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we define $f_j \in k'[X_{ij}, i = 0, ..., m - 1, j = 0, ..., n - 1]$ by

$$f\left(\sum_{j=0}^{n-1} \alpha_j X_{0j}, \dots, \sum_{j=0}^{n-1} \alpha_j X_{m-1 \ j}\right) = \sum_{j=0}^{n-1} f_j \alpha_j.$$

We note that $\deg f_j \leq \deg f$. The system

$$\mathcal{F}' = \{f_j: f \in \mathcal{F}, j = 0, \dots, n-1\}$$

is called the *Weil descent system* of \mathcal{F} with respect to $\alpha_0, \ldots, \alpha_{n-1}$.

There is a bijection between $Z_k(\mathcal{F})$ and $Z_{k'}(\mathcal{F}') = Z(\mathcal{F}'_1)$, where \mathcal{F}'_1 is \mathcal{F}' together with the field equations of k', that is,

$$\mathcal{F}'_1 = \mathcal{F}' \cup \{X^q_{ij} - X_{ij}, i = 0, \dots, m - 1, j = 0, \dots, n - 1\}.$$

The HFE polynomial system is constructed by forming the Weil descent of some \mathcal{F} consisting of a single univariate polynomial, followed by linear change of variables and linear change of equations [6]. Multivariate-HFE systems can be constructed similarly except \mathcal{F} is replaced by a finite set of multivariate polynomials of dimension zero. In [7] upper bounds on the last fall degree degree of \mathcal{F}'_1 were proven in terms of q, m, the last fall degree of \mathcal{F} , the degree of \mathcal{F} and the number of solutions of \mathcal{F} , but not on n. The result implies that multi-HFE cryptosystems giving rise to multi-HFE polynomial systems as described above are vulnerable to attack as well.

In this paper we consider the situation where \mathcal{F} is not necessarily zero-dimensional. Let

$$\mathcal{F}_{1} = \mathcal{F} \cup \{X_{i}^{q} - Y_{i1}, \dots, Y_{i n-2}^{q} - Y_{i n-1}, Y_{i n-1}^{q} - X_{i} : i = 0, \dots, m-1\}$$

$$\subset k[X_{i}, Y_{ij} : i = 0, \dots, m-1; j = 1, \dots, n-1].$$

We observe that $Z_k(\mathcal{F})$ can easily be identified with $Z(\mathcal{F}_1)$. So there is a bijection between $Z(\mathcal{F}_1)$ and $Z(\mathcal{F}'_1)$. Note also that the ideals generated by \mathcal{F}_1 and \mathcal{F}'_1 are radical ideals. The following theorem relates the last fall degrees of \mathcal{F}_1 and \mathcal{F}'_1 .

Theorem 1.1. $\max(d_{\mathcal{F}_1}, q \deg \mathcal{F}) = \max(d_{\mathcal{F}'_1}, q \deg \mathcal{F}).$

Theorem 1.1 is closely related to Proposition 2 of [6] and Proposition 4.1 of [7]. In comparison, the bound established in Theorem 1.1 is a bit weaker. However the set \mathcal{F}_1 stated in the theorem directly contains \mathcal{F} as a subset. This makes it easier to apply the theorem both conceptually and technically. When \mathcal{F} consists of a univariate polynomial or more generally when $Z(\mathcal{F})$ is finite, it is not hard to bound $d_{\mathcal{F}_1}$. From this an easier and more conceptual proof of the theorems in [6, 7] can be constructed based on Theorem 1.1. However in this paper we will focus on applying the theorem to the situation where \mathcal{F} is not zero-dimensional, especially when \mathcal{F} consists of linearized polynomials.

1.1. **Proof of Theorem 1.1.** For non-negative integers i, let σ_i denote the automorphism of \overline{k} over k' such that $\sigma_i(x) = x^{q^i}$ for $x \in \overline{k}$. For every multivariate polynomial h with coefficients from \overline{k} , let h^{σ_i} denote the polynomial obtained from h by acting on each coefficient of h by σ_i .

Let Γ be the *n* by *n* matrix with rows and columns indexed by $0, \ldots, n-1$, so that $\alpha_j^{\sigma_i}$ is the (i, j)-th entry of Γ for $i, j = 0, \ldots, n-1$.

Let

$$g_f = f\left(\sum_{j=0}^{n-1} \alpha_j X_{0j}, \dots, \sum_{j=0}^{n-1} \alpha_j X_{m-1 \ j}\right) = \sum_{j=0}^{n-1} f_j \alpha_j$$

where $f_j \in k'[X_{ij}, i = 0, \dots, m-1, j = 0, \dots, n-1].$
Let $\hat{f} = \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_{n-1} \end{pmatrix}$ be the column vector with f_i as the *i*-th entry for $i = 0, \dots, n-1$.
$$\begin{pmatrix} g_f^{\sigma_0} \\ \vdots \end{pmatrix}$$

Then
$$g_f^{\sigma_i} = \sum_{j=0}^{n-1} f_j \alpha_j^{\sigma_i}$$
, and $\Gamma \hat{f} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ g_f^{\sigma_{n-1}} \end{pmatrix}$. Let $\mathcal{G} = \{g_f^{\sigma_0}, \dots, g_f^{\sigma_{n-1}} : f \in \mathcal{F}\}$ and

 $\mathcal{G}_1 = \mathcal{G} \cup \{X_{ij}^q - X_{ij}, i = 0, \dots, m-1, j = 0, \dots, n-1\}$. Since Γ is invertible, it follows from Proposition 2.6 (part v) of [7] that $d_{\mathcal{G}} = d_{\mathcal{F}'}$, and $d_{\mathcal{G}_1} = d_{\mathcal{F}'_1}$.

Let Z_{ij} , i = 0, ..., m - 1 and j = 0, ..., n - 1, be defined by the following change of coordinates:

$$\begin{pmatrix} X_i \\ Y_{i1} \\ \vdots \\ Y_{i n-1} \end{pmatrix} = \Gamma \begin{pmatrix} Z_{i0} \\ Z_{i1} \\ \vdots \\ Z_{i n-1} \end{pmatrix}.$$

Under the change of coordinates, \mathcal{F}_1 becomes \mathcal{G}_2 where

$$\mathcal{G}_{2} = \{ f(\sum_{j=0}^{n-1} \alpha_{j} Z_{0j}, \dots, \sum_{j=0}^{n-1} \alpha_{j} Z_{m-1 \ j}) : f \in \mathcal{F} \} \cup \{ Z_{ij}^{q} - Z_{ij} : i, j = 0, \dots, n-1 \}$$

= $\{ g_{f}(Z_{01}, \dots, Z_{m,n-1}) : f \in \mathcal{F} \} \cup \{ Z_{ij}^{q} - Z_{ij} : i, j = 0, \dots, n-1 \},$

which we identify as a subset of \mathcal{G}_1 . Since $g_f^q \equiv g_f^\sigma \mod I$ where I is the deal generated by $X_{ij}^q - X_{ij}, i = 0, \ldots, m-1, j = 0, \ldots, n-1$, we see that $g_f^\sigma \in V_{\mathcal{G}_2,qd}$ where $d = \deg \mathcal{F} \ge \deg g_f$. It follows inductively that $g_f^{\sigma_i} \in V_{\mathcal{G}_2,qd}$ for all i, hence $\mathcal{G}_1 \subset V_{\mathcal{G}_2,qd}$. Hence $V_{\mathcal{G}_1,i} = V_{\mathcal{G}_2,i}$ for $i \ge qd$. Therefore $\max(d_{\mathcal{G}_1}, qd) = \max(d_{\mathcal{G}_2}, qd)$. Since $d_{\mathcal{G}_1} = d_{\mathcal{F}_1'}$ and $d_{\mathcal{F}_1} = d_{\mathcal{G}_2}$, we conclude that $\max(d_{\mathcal{F}_1}, qd) = \max(d_{\mathcal{F}_1'}, qd)$. Theorem 1.1 follows.

2. Systems of linearized polynomials

A k'-linearized polynomial in $R = k[x_0 \dots x_{m-1}]$ is an element of the k-submodule of R generated by $x_i^{q^j}$ where $q = |k'|, i = 0 \dots m-1$ and $j \ge 0$. As before let n = [k : k']. Let $Q = \{x_i^{q^n} - x_i : i = 0 \dots m-1\}.$

For $g = \sum_{i=0}^{d} a_i x^i \in k[x]$, let $L(g) = \sum_{i=0}^{d} a_i x^{q^i}$. More generally we consider the k-linear map from $\bigoplus_{i=0}^{m-1} k[x_i]$ onto the k-module of k'-linearized polynomials such that $L(x_i^j) = x_i^{q^j}$ for $i = 0, \ldots, m-1$ and $j \ge 0$.

Let $S = k[x_{ij} : i = 0 \dots m - 1, j = 0, \dots, n - 1]$. We also write $S = k[\hat{x}_i : i = 0, \dots, m - 1]$. where $\hat{x}_i = x_{i0}, \dots, x_{i n-1}$. Let $S_1 \subset S$ be the k-module of linear forms over x_{ij} , $i = 0 \dots m - 1, j = 0, \dots, n - 1$.

For $g \in k[x]$ and $h \in k[x_i]$, let $g \circ h \in k[x_i]$ be defined as $(g \circ h)(x_i) = g(h(x_i))$. For $f = \sum_{i=0}^{m-1} f_i \in \bigoplus_{i=0}^{m-1} k[x_i] \text{ with } f_i \in k[x_i], \text{ let } g \circ f \in \bigoplus_{i=0}^{m-1} k[x_i] \text{ be defined as } g \circ f = \sum_{i=0}^{m-1} g \circ f_i. \text{ Hence } (g \circ f)(x_0, \dots, x_{m-1}) = \sum_{i=0}^{m-1} g(f_i(x_i)). \text{ Note also that } L(g) \circ L(f) \text{ is a}$ k'-linearized polynomial in $k[x_0, \ldots, x_{m-1}]$.

Similarly for $g \in k[x]$ and $f = \sum_{i,j} f_{ij} \in \bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k[x_{ij}]$ with $f_{ij} \in k[x_{ij}]$, let $g \circ f \in$ $\bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k[x_{ij}]$ be defined as $g \circ f = \sum_{i,j} g \circ f_{ij}$.

Consider the map ℓ from $\bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k[x_i^j]$ to S_1 such that $\ell(x_i^j) = x_{ij}$. Let $\bar{Q} = \{x_{ij}^q - x_{ij+1} : i = 0 \dots m-1, j = 0 \dots n-1\}$ where j+1 is taken mod n. Consider the k-algebra isomorphism from $R/\langle Q \rangle$ to $S/\langle \bar{Q} \rangle$ sending $x_i^{q^j}$ to x_{ij} for $i = 0, \ldots, m-1$, $j = 0, \ldots, n-1$. (Note that $x_i^{q^{j+1}} = (x_i^{q^j})^q$ maps to x_{ij}^q and $x_{ij}^q \equiv x_{ij+1} \mod \overline{Q}$.)

For $f \in R$ where the degree of f in x_i is less than q^n for all i, let $\overline{f} \in S$ denote the image of f in $S/\langle Q \rangle$ under the isomorphism. We note that elements of S_1 are all distinct mod Q. Let $f = \sum_{i=0}^{n-1} f_i(x_i)$ with deg $f_i < n$ for all *i*. Let $f_i = \sum_{j=0}^{n-1} a_{ij} x_i^j$. Then $L(f) \in R$ corresponds to $\ell(f)$. If we identify with $x_i \in R$ with $x_{i0} \in S$ for $i = 0, \ldots, m-1$. Then $L(f) \equiv_d \ell(f)$ (mod Q) where $d = \deg L(f)$.

Suppose \mathcal{F} is a finite set of k'-linearized polynomials of maximum degree $d = q^c$ for some c > 0. We may identify $x_i \in R$ with $x_{i0} \in S$ and consider $\mathcal{F} \subset S$. Let \mathcal{F}' be the Weil descent system of (F) with respect to a k/k' basis. We are interested in the last fall degree of $\mathcal{F}'_{1} = \mathcal{F}' \cup \{x^{q}_{ij} - x_{ij} : i = 0, \dots, m-1, j = 0, \dots, n-1\}$. Let $\mathcal{G} = \mathcal{F} \cup \bar{Q} \subset S$. By Theorem1.1, $\max(d_{\mathcal{G}}, q \deg \mathcal{F}) = \max(d_{\mathcal{F}'_1}, q \deg \mathcal{F}).$

Recall that $Z(\mathcal{F}'_1) = Z_{k'}(\mathcal{F}')$, which corresponds to $Z_k(\mathcal{F}) = Z(\mathcal{F} \cup Q)$, the set of k-rational points of $Z(\mathcal{F})$. In what follows we consider a more general situation where instead of $Z_k(\mathcal{F})$ we are interested in $Z_W(\mathcal{F}) = Z(\mathcal{F}) \cap W^m$ where W is a τ -invariant subspace of k and τ is the Frobenius map over $k': x \to x^q$ for all $x \in k$. Note that every τ -invariant subspace W of k is of the form $Z(L(\mathfrak{f}_W))$ where \mathfrak{f}_W divides $x^n - 1$. In fact W is the kernel of $\mathfrak{f}_W(\tau)$, and f_W is the characteristic polynomial of τ as a linear map on W. In particular $f_W = x^n - 1$ corresponds to W = k and $\mathfrak{f}_W = x - 1$ corresponds to W = k'. Suppose $d_W = \deg \mathfrak{f}_W$.

In this more general situation we let $S = k[x_{ij} : i = 0 \dots m - 1, j = 0, \dots, d_W - 1]$. We also write $S = k[\hat{x}_i : i = 0, \dots, m - 1]$. where $\hat{x}_i = x_{i0}, \dots, x_{i \ d_W - 1}$. Let $f = \sum_{i=0}^{m-1} f_i(x_i)$ with deg $f_i < d_W$ for all *i*. Suppose $f_i = \sum_{j=0}^{d_W-1} a_{ij} x_i^j$. Then $L(f) = \sum_{i=1}^{m-1} L(f_i)$ where $L(f_i) = \sum_{j=0}^{d_W-1} a_{ij} x_i^{q^j} \text{ and } \ell(f) = \sum_{i=0}^{m-1} \ell(f_i) \text{ where } \ell(f_i) = \sum_{j=0}^{d_W-1} a_{ij} x_{ij}.$

Below we fix W and let $n' = d_W$. Write $\mathfrak{f}_W(x) = x^{n'} - \mathfrak{g}_W(x)$ with $\deg \mathfrak{g}_W < n'$. Let $Q = \{x_i^{q^{n'}} - L(\mathfrak{g}_W(x_i)) : i = 0, \dots, m-1\}, \text{ and correspondingly we let } \bar{Q} = \{x_{i n'-1}^q - \ell(\mathfrak{g}_W(x_i)), x_{ij}^q - x_{i j+1} : i = 0, \dots, m-1, j = 0, \dots, n'-2\}.$ Then we have an isomorphism from $R/\langle Q \rangle$ to $S/\langle \bar{Q} \rangle$ sending $x_i^{q^j}$ to x_{ij} for $i = 0, \ldots, m-1, j = 0, \ldots, n'-1$. Let $S_1 \subset S$ be the k-module of linear forms over $x_{ij}, i = 0, \ldots, m-1, j = 0, \ldots, n'-1$. We note that elements of S_1 are all distinct mod Q.

For $f \in R$, we have $f \equiv_d f_1 \pmod{Q}$ where $d = \deg f$ and the degree of f_1 in x_i is less than q^{d_W} for all *i*. Let $\bar{f} \in S$ denote the image of f_1 in $S/\langle \bar{Q} \rangle$ under the isomorphism. Let $f = \sum_{i=0}^{m-1} f_i(x_i)$ with deg $f_i < n'$ for all *i*. Then $\overline{L(f)} = \ell(f)$. If we identify with $x_i \in R$ with $x_{i0} \in S$ for i = 0, ..., m - 1. Then $L(f) \equiv_d \ell(f) \pmod{\bar{Q}}$ where $d = \deg L(f)$.

Lemma 2.1. Suppose $f = \sum_{i,j} a_{ij} x_{ij} \in S_1$ with $a_{ij} \in k$. Then with respect to \bar{Q} , $f_i^q \equiv_q f_{i+1}$ (mod Q), for $i \ge 0$ where $f = f_0$ and $f_i \in S_1$ for $i \ge 0$.

Proof For $r \ge 0$, we have inductively $f_r = \sum b_{ij}x_{ij} \in S_1$. Now $f_r^q = \sum_{ij} b_{ij}^q x_{ij}^q$, and since for all $i, x_{ij}^q \equiv_q x_{ij+1} \pmod{\bar{Q}}$ for $j = 0, \ldots, n'-2$, and $x_{in'-1}^q \equiv_q \ell(\mathfrak{g}(x_i))$, it follows that $f_r^q \equiv_q f_{r+1} \pmod{\bar{Q}}$ with $f_{r+1} \in S_1$. \Box

Lemma 2.2. Let \mathcal{H} be a finite set of S and suppose $\bar{Q} \subset \mathcal{H}$. Suppose $f \in S_1$ and $f \equiv_i 0$ (mod \mathcal{H}) for some i > 0. Let $g \in k[x]$. Then $L(g) \circ f - f' \in \langle \bar{Q} \rangle$ for some $f' \in S_1$, where $\langle \bar{Q} \rangle$ denotes the ideal generated by \bar{Q} , and $f' \equiv_r 0 \pmod{\mathcal{H}}$ where $r = \max(i, q)$.

Proof The lemma follows by applying Lemma 2.1 inductively. More specifically assume inductively $L(x^i) \circ f \equiv_q f'_i \pmod{\bar{Q}}$ with $f'_i \in S_1$, then $L(x^{i+1}) \circ f \equiv_q f'_i \equiv_q f'_{i+1} \pmod{\bar{Q}}$ for some $f'_{i+1} \in S_1$. From this the lemma easily follows. \Box

For $f \in \mathcal{F}$, $f \equiv_d \bar{f} \pmod{\bar{Q}}$ with $\bar{f} \in S_1$. Let \bar{F} consist of all such $\bar{f} \in S_1$ with $f \in \mathcal{F}$. Let $\bar{\mathcal{G}} = \bar{\mathcal{F}} \cup \bar{Q}$. Then $\bar{\mathcal{G}} \subset V_{\mathcal{G},d}$ and $\bar{\mathcal{G}} \subset V_{\bar{\mathcal{G}},q}$.

Let $S_{1i} = S_1 \cap k[\hat{x}_j : j = i, \dots, m-1]$, that is, the submodule containing all k-linear forms in $x_{ij}, i = i, \dots, m-1, j = 0, \dots, n'-1$. In particular $S_1 = S_{10}$. Let $\bar{Q}_r = \bar{Q} \cap k[\hat{x}_i : i = r, \dots, m-1]$ for $r = 1, \dots, m-1$.

Lemma 2.3. Consider a k'-linearized polynomial of the form L(f) with $f = \sum_{i=0}^{m-1} f_i$ and $f_i \in k[x_i]$ of degree less than n', for i = 0, ..., m-1. Suppose $\ell(f) \in V_{\bar{g},q}$ and the GCD of f_0 and f_W is 1. Then $x_{00} - \ell_0 \in V_{\bar{g},q}$ for some linear form $\ell_0 \in S_{11}$. Moreover for i = 1, ..., n'-1, $x_{0i} - \ell_i \in V_{\bar{d},q}$ for some linear form $\ell_i \in S_{11}$, and $\ell_{i-1}^q \equiv \ell_i \pmod{\bar{Q}_1}$.

Proof Since the GCD of f_0 and \mathfrak{f}_W is 1, $A(x)f_0(x) + B(x)\mathfrak{f}_W(x) = 1$ for some $A(x), B(x) \in k[x]$ where deg A < n and deg $B < \deg f_0$. Now

$$L(A(x)) \circ L(f_0(x_0)) + L(B(x)) \circ L(\mathfrak{f}_W(x_0)) = x_0$$
$$L(A(x)) \circ L(\sum_{i=1}^{m-1} f_i(x_i)) = L(g)$$

for some $g = \sum_{i=1}^{m-1} g_i$ where $g_i \in k[x_i]$. So

$$L(A(x)) \circ L(f) + L(B(x)) \circ L(\mathfrak{f}_W) = x_0 + L(g).$$

We have

$$L(A(x)) \circ \ell(f) \equiv x_{00} + \ell(g) \pmod{\langle Q \rangle}$$

Note that $\ell(g) \in S_{11}$. By Lemma 2.2 there is some $f' \in S_1$ such that $L(A) \circ \ell(f) - f' \in \langle \bar{Q} \rangle$ and $f' \equiv_q 0 \pmod{\mathcal{G}}$. So put $\ell_0 = -\ell(g)$. Then $f' \equiv x_{00} - \ell_0 \pmod{\langle \mathcal{G} \rangle}$, and since f' and $x_{00} - \ell_0$ are both in S_1 , we have $f' = x_{00} - \ell_0$. Let $\ell_1 \in S_{11}$ such that $\ell_0^q \equiv_q \ell_1 \pmod{\bar{Q}_1}$. Then $x_{01} \equiv_q x_{00}^q \equiv_q \ell_0^q \equiv_q \ell_1 \pmod{\bar{\mathcal{G}}}$, and inductively we have $x_{0i} \equiv_q \ell_i$ for some linear form $\ell_i \in S_{11}$, with $\ell_{i-1}^q \equiv_q \ell_i \pmod{\bar{Q}_1}$. \Box

When the condition in Lemma 2.3 is satisfied, $x_{0i} \equiv_q \ell_i$ for some $\ell_i \in S_{11}$. Substituting he variable x_{0j} by ℓ_j , for $j = 0, \ldots, n' - 1$, gives reduction from $S_1 \cap V_{\bar{\mathcal{G}},q}$ to $S_{11} \cap V_{\bar{\mathcal{G}},q}$. More explicitly, for $g \in S_1$, write $g = g_0 + g_1$ where g_0 is a linear form in $x_{00}, \ldots, x_{0n'-1}$, and $g_1 \in S_{11}$. Then $g \equiv_1 g'$ where $g' = g_0(\ell_0, \ldots, \ell_{n'-1}) + g_1 \in S_{11}$. Therefore for all $g \in S_1 \cap V_{\bar{\mathcal{G}},q}$, there is some $g' \in S_{11}$ such that $0 \equiv_q g \equiv_1 g' \pmod{\bar{\mathcal{G}}}$. A similar condition will give reduction from $S_{11} \cap V_{\bar{\mathcal{G}},q}$ to $S_{12} \cap V_{\bar{\mathcal{G}},q}$, and so on. This leads to the following definition.

We say that \mathcal{F} is *reducible* for W if for $i = 0, \ldots, m-2$, either $V_{\bar{\mathcal{G}},q} \cap S_{1i} = V_{\bar{\mathcal{G}},q} \cap S_{1i+1}$, or else there is a k'-linearized polynomial of the form $L(f_i)$ with $f_i = \sum_{j=i}^{m-1} g_{ij}, g_{ij} \in k[x_j]$ of degree less than n', for $j = i, \ldots, m-1$, and $\ell(f_i) \in V_{\bar{\mathcal{G}},q} \cap S_{1i}$ and the GCD of g_{ii} and $\mathfrak{f}(x_i)$ is 1.

In particular if \mathfrak{f}_W is irreducible over k' then the GCD of every nonzero polynomial of degree less than $n' = \deg \mathfrak{f}_W$ is relatively prime to \mathfrak{f}_W . Therefore we have the following:

Lemma 2.4. If \mathfrak{f}_W is irreducible over k' then \mathcal{F} is reducible for W.

Theorem 2.5. Suppose \mathcal{F} is a finite set of k'-linearized polynomials, and W is a τ -invariant subspace of k where τ is the Frobenius map over k'. Let $\overline{\mathcal{G}} = \overline{\mathcal{F}} \cup \overline{Q}$. If \mathcal{F} is reducible for W, then $d_{\overline{\mathcal{G}}} \leq (q-1)m+1$. Moreover a basis of $Z_W(\mathcal{F})$ can be constructed in time $(n'm)^{O(q)}$ where $n' = \deg f_W$.

Theorem 2.6. Suppose \mathcal{F} is a finite set of k'-linearized polynomials of maximum degree $d = q^c$ for some c > 0. Let \mathcal{F}' be the Weil descent system of \mathcal{F} with respect to a k/k' basis, and $\mathcal{F}'_1 = \mathcal{F}' \cup \{x^q_{ij} - x_{ij} : i = 0, \dots, m-1, j = 0, \dots, n-1\}$. If \mathcal{F} is reducible for k, then $d_{\mathcal{F}'_1} \leq \max((q-1)m+1, qd)$.

Example Consider the case where \mathcal{F} consists of a bivariate linearized polynomial

$$F(x,y) = ax^{q^2} + bx^q + cx + uy^{q^2} + vy^q + wy = L(ax^2 + bx + c) + L(uy^2 + vy + w),$$

with $a, b, c, u, v, w \in k = \mathbb{F}_{q^n}$. By Lemma 2.3 (with $f = ax^2 + bx + c + uy^2 + vy + w$), if either $GCD(ax^2 + bx + c, x^n - 1) = 1$ or $GCD(uy^2 + vy + w, y^n - 1) = 1$, then \mathcal{F} is reducible for k. By Theorem 2.6, $d_{\mathcal{F}'_1} \leq 2q$. \Box

Since $\overline{\mathcal{G}} \subset V_{\mathcal{G},d}$, Theorem 2.6 follows from Theorem 1.1 and Theorem 2.5. The rest of this section is devoted to the proof of Theorem 2.5.

2.1. Proof of Theorem 2.5.

Lemma 2.7. Suppose \mathcal{F} is reducible for W. For $i = 0, \ldots, m-2$, if $V_{\bar{\mathcal{G}},q} \cap S_{1i} \neq V_{\bar{\mathcal{G}},q} \cap S_{1i+1}$, then $x_{ij} \equiv_q \gamma_{ij} \pmod{\bar{\mathcal{G}}}$ for some linear form $\gamma_{ij} \in S_{1\ m-1}$, for $j = 0, \ldots, n'-1$; moreover $\gamma_{ij}^q \equiv_q \gamma_{i\ j+1} \pmod{\bar{\mathcal{Q}}_{m-1}}$ for $j = 0, \ldots, n'-2$.

Proof For i = 0, ..., m-2, if $V_{\bar{\mathcal{G}},q} \cap S_{1i} \neq V_{\bar{\mathcal{G}},q} \cap S_{1i+1}$, then there is a k'-linearized polynomial of the form $L(f_i)$ with $f_i = \sum_{j=i}^{m-1} g_{ij}$, where $g_{ij} \in k[x_j]$ of degree less than n', for $j = i, ..., m-1, \ell(f_i) \in V_{\bar{\mathcal{G}},q} \cap S_{1i}$ and the GCD of g_{ii} and $\mathfrak{f}(x_i)$ is 1.

By Lemma 2.3 we have the following: for $j = 0, \ldots, n' - 1$, $x_{ij} \equiv_q \ell_{ij} \pmod{\bar{\mathcal{G}}}$ for some linear form $\ell_{ij} \in S_1_{i+1}$, moreover $\ell_{ij}^q \equiv_q \ell_{ij+1} \pmod{\bar{Q}_{i+1}}$. From this it is easy to see by induction (proceeding from i = m - 2 to i = 0) that $x_{ij} \equiv_q \gamma_{ij}$ for some linear form $\gamma_{ij} \in S_1_{m-1}$, moreover $\gamma_{ij}^q \equiv_q \gamma_{ij+1} \pmod{\bar{Q}_{m-1}}$ for $i = 0, \ldots, m-1, j = 0, \ldots, n'-2$. \Box

Lemma 2.8. Let $\mathcal{N} = \{i \in \{0, \ldots, m-2\} : V_{\bar{\mathcal{G}},q} \cap S_{1i} \neq V_{\bar{\mathcal{G}},q} \cap S_{1i+1}\}$. Let $\Gamma = \{x_{ij} - \gamma_{ij} : \gamma_{ij} \in S_{1\ m-1}, x_{ij} \equiv_q \gamma_{ij} \pmod{\bar{\mathcal{G}}}, i \in \mathcal{N}, j = 0, \ldots, n'-1\}$. Let $H_{\bar{\mathcal{F}}} = \{\ell(h_f) : f \in \bar{\mathcal{F}}\}$. Then there exist $H_1 = \{\ell(h_{ij}) : i = 0, \ldots, m-2, j = 0, \ldots, n'-1\}$ where $h_{ij} \in k[x_{m-1}]$ with $\deg h_{ij} < n'$ such that letting $H = H_{\bar{\mathcal{F}}} \cup H_1$, then $\Gamma \cup H \subset V_{\bar{\mathcal{G}},q}, \ \bar{\mathcal{G}} = \bar{\mathcal{F}} \cup \bar{\mathcal{Q}} \subset V_{H \cup \bar{\mathcal{Q}}_{m-1} \cup \Gamma,q}, \langle \bar{\mathcal{G}} \rangle = \langle H \cup \bar{\mathcal{Q}}_{m-1} \cup \Gamma \rangle$. **Proof** By Lemma 2.7, $\Gamma \subset V_{\bar{\mathcal{G}},q}$. For $f \in \mathcal{F} \subset S_1$, let $f' \in S_1_{m-1}$ be obtained from f by substituting x_{ij} with γ_{ij} for $i = 0, \ldots, m-2, j = 0, \ldots, n'-1$. Then $f' \equiv_1 f \pmod{\Gamma}$, and $f' = \ell(h_f)$ for some $h_f \in k[x_{m-1}]$. We have $\ell(h_f) \in V_{\Gamma,1} \subset V_{\bar{\mathcal{G}},q}$, and $f \in V_{H_{\bar{\mathcal{F}}} \cup \Gamma,1}$ where $H_{\bar{\mathcal{F}}} = \{\ell(h_f) : f \in \bar{\mathcal{F}}\}.$

For i = 0, ..., m-2 and $j = 0, ..., n'-2, x_{ij}^q - x_{ij+1} \in \bar{Q}$. Lemma 2.7 implies that $x_{ij}^q - x_{ij+1} \equiv_q \gamma_{ij}^q - \gamma_{ij+1} \equiv_q \ell(h_{ij}) \pmod{\Gamma \cup \bar{Q}_{m-1}}$ for some $h_{ij} \in k[x_{m-1}]$.

For $i = 0, \ldots, m-2, x_{i n'-1}^q - \ell(\mathfrak{g}_W(x_i)) \in \overline{Q}$. Let $\ell(\mathfrak{g}_W(x_i)) = \sum_{j=0}^{n'-1} a_{ij} x_{ij}$ with $a_{ij} \in k$ Lemma 2.7 implies that

$$x_{i\ n'-1}^{q} - \ell(\mathfrak{g}_{W}(x_{i})) \equiv_{q} \gamma_{i\ n'-2}^{q} - \sum_{j=0}^{n'-1} a_{ij}\gamma_{ij} \pmod{\Gamma}.$$

Since $\gamma_{i n'-2}^q \equiv_q \gamma_{i n'-1} \pmod{\bar{Q}_{m-1}}$, we have

$$\gamma_{i \ n'-2}^{q} - \sum_{j=0}^{n'-1} a_{ij} \gamma_{ij} \equiv_{q} \ell(h_{i \ n'-1}) \pmod{\bar{Q}_{m-1}}$$

with $h_{i n'-1} \in k[x_{m-1}]$ of degree less than n'.

To summarize, we have

$$x_{ij}^q - x_{i \ j+1} \equiv_q \ell(h_{ij}) \pmod{\Gamma \cup \bar{Q}_{m-1}}$$

for i = 0, ..., m - 2 and j = 0, ..., n' - 2, and

 $x_{i\ n'-1}^q - \ell(\mathfrak{g}_W(x_i)) \equiv_q \ell(h_{i\ n'-1}) \pmod{\Gamma \cup \bar{Q}_{m-1}}$

for $i = 0, \ldots, m-2$. Let $H_1 = \{\ell(h_{ij}) : i = 0, \ldots, m-2, j = 0, \ldots, n'-1\}$. It follows that $\bar{Q} \subset V_{H_1 \cup \Gamma \cup \bar{Q}_{m-1}, q}$ and on the other hand $H_1 \subset V_{\bar{Q} \cup \Gamma, q}$, and since $\Gamma \subset V_{\bar{G}, q}$, we have $H_1 \subset V_{\bar{G}, q}$.

Let $H = H_{\bar{\mathcal{F}}} \cup H_1$. Then we conclude that $\Gamma \cup H \subset V_{\bar{\mathcal{G}},q}$, and on the other hand $\bar{\mathcal{G}} = \bar{\mathcal{F}} \cup \bar{Q} \subset V_{H \cup \bar{Q}_{m-1} \cup \Gamma,q}$. In particular, we have $\langle \bar{\mathcal{G}} \rangle = \langle H \cup \bar{Q}_{m-1} \cup \Gamma \rangle$. \Box

Note that $H \cup Q_{m-1} \subset k[\hat{x}_{m-1}]$ where $\hat{x}_{m-1} = x_{m-1}, \dots, x_{m-1}, x_{m-1}$.

Lemma 2.9. Let H be as in Lemma 2.8. Suppose $H = \{\ell(h_i) : i = 1, ..., s\}$ and let $h_0 = \mathfrak{f}_W(x_{m-1})$. Let g be the GCD of h_i , i = 0, ..., s. Then $\langle \overline{\mathcal{G}} \rangle = \langle \Gamma \cup \{\ell(g)\} \cup \overline{Q}_{m-1} \rangle$, moreover $\Gamma \cup \{\ell(g)\} \cup \overline{Q}_{m-1} \subset V_{\overline{\mathcal{G}},q}$.

Proof We have $g = \sum_{i=0}^{s} a_i h_i$ with $a_i \in k[x_{m-1}]$, so

$$L(g) = \sum_{i} L(a_i) \circ L(h_i).$$

So

$$\ell(g) \equiv \sum_{i} L(a_i) \circ \ell(h_i) \mod \bar{Q}_{m-1}.$$

Apply Lemma 2.2 to $H \cup \bar{Q}_{m-1} \subset k[\hat{x}_{m-1}]$ it follows that there is $h'_i \in S_{1,m-1}$ such that $h'_i \equiv_q 0 \pmod{H \cup \bar{Q}_{m-1}}$ and $L(a_i) \circ \ell(h_i) \equiv h'_i \pmod{\bar{Q}_{m-1}}$. So

$$\ell(g) \equiv \sum_{i} h'_i \equiv_q 0 \pmod{H \cup \bar{Q}_{m-1}}.$$

Since $\ell(g)$ and h'_i are all in $S_{1,m-1}$, we have

$$\ell(g) = \sum_{i} h'_{i} \equiv_{q} 0 \pmod{H \cup \bar{Q}_{m-1}},$$

in particular, $\ell(g) \equiv_q 0 \pmod{\overline{\mathcal{G}}}$. It follows that

$$L(\mathfrak{f}_W) = L(h_0) \in \langle \{L(h_i) : i = 0, \dots, s\} \rangle = \langle L(g) \rangle.$$

Under the isomorphism from $k[\hat{x}_{m-1}]/\langle \bar{Q}_{m-1} \rangle \rightarrow k[x_{m-1}]/\langle L(\mathfrak{f}_W(x_{m-1})) \rangle$, $\ell(h_i)$ corresponds to $L(h_i)$, hence the ideal generated by $H \cup \bar{Q}_{m-1}$ corresponds to the ideal generated by L(g). Since by Lemma 2.8, $\langle \bar{\mathcal{G}} \rangle = \langle \Gamma \cup H \cup \bar{Q}_{m-1} \rangle$, it follows that $\langle \bar{\mathcal{G}} \rangle = \langle \Gamma \cup \{\ell(g)\} \cup \bar{Q}_{m-1} \rangle$.

Moreover from the discussion above we have $\Gamma \cup \{\ell(g)\} \cup \overline{Q}_{m-1} \subset V_{\overline{\mathcal{G}},q}$. \Box

Under the isomorphism from $k[\hat{x}_0, \ldots, \hat{x}_{m-1}]/\langle Q \rangle \to k[x_0, \ldots, x_{m-1}]/\langle Q \rangle$, $x_{i0} - \gamma_{i0}$ corresponds to $x_i - L(g_i)$ where $\ell(g_i) = \gamma_{i0}$ for $i \in \mathcal{N}$. Under the isomorphism the ideal determined by $\overline{\mathcal{G}}$ corresponds to the ideal determined by $\mathcal{F} \cup Q$. Since, by Lemma 2.9, $\langle \overline{\mathcal{G}} \rangle = \langle \Gamma \cup \{\ell(g)\} \cup \overline{Q}_{m-1} \rangle$ and $g|f_W$, it follows that $\langle \mathcal{F} \cup Q \rangle$ is generated by L(g) and $x_i - L(g_i)$ where $i \in \mathcal{N}$. By Lemma 2.9 $\Gamma \cup \{\ell(g)\} \cup \overline{Q}_{m-1} \subset V_{\overline{\mathcal{G}},q}$, it follows from Proposition 2.3 of [7] that $\ell(g)$ and γ_{i0} , hence L(g) and $x_i - L(g_i)$ can be constructed in time $(mn')^{O(q)}$ time. From this a basis of $Z_W(\mathcal{F})$ over k' can be easily written down.

It is easy to see that if $f \in k[\hat{x}_{m-1}]$ and $f \in \langle \{\ell(g)\} \cup \bar{Q}_{m-1} \rangle$ then $f \equiv_{\deg f+1} \ell(g)f_1 \pmod{\bar{Q}_{m-1}}$ for some $f_1 \in k[\hat{x}_{m-1}]$. Suppose $f \in \langle \bar{\mathcal{G}} \rangle$. Then $f \equiv_{\deg f} f_1 \pmod{\bar{Q}}$ where the degree of x_{ij} in f_1 is less than q for all i, j. Using $x_{ij} \equiv \gamma_{ij} \pmod{\Gamma}$, we have $f_1 \equiv_{\deg f_1} h \pmod{\Gamma \cup \bar{Q}_{m-1}}$ where $h \in k[\hat{x}_{m-1}]$. It follows that $h \in \langle \{\ell(g)\} \cup \bar{Q}_{m-1} \rangle$, hence $h \equiv_{\deg h+1} \ell(g)h_1 \pmod{\bar{Q}_{m-1}}$, so $h \equiv_{\deg h+1} 0 \pmod{\{\ell(g)\} \cup \bar{Q}_{m-1}}$. If $\deg f > (q-1)m$, then $\deg f > \deg f_1$, and since $\Gamma \cup \{\ell(g)\} \cup \bar{Q}_{m-1} \subset V_{\bar{\mathcal{G}},q}$, we conclude that $f \in V_{\bar{\mathcal{G}},\deg f}$. Therefore $d_{\bar{\mathcal{G}}} \leq (q-1)m + 1$. Theorem 2.5 follows.

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