# ON THE LAST FALL DEGREE OF WEIL DESCENT POLYNOMIAL SYSTEMS 

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#### Abstract

Given a polynomial system $\mathcal{F}$ over a finite field $k$ which is not necessarily of dimension zero, we consider the Weil descent $\mathcal{F}^{\prime}$ of $\mathcal{F}$ over a subfield $k^{\prime}$. We prove a theorem which relates the last fall degrees of $\mathcal{F}_{1}$ and $\mathcal{F}_{1}^{\prime}$, where the zero set of $\mathcal{F}_{1}$ corresponds bijectively to the set of $k$-rational points of $\mathcal{F}$, and the zero set of $\mathcal{F}_{1}^{\prime}$ is the set of $k^{\prime}$-rational points of the Weil descent $\mathcal{F}^{\prime}$. As an application we derive upper bounds on the last fall degree of $\mathcal{F}_{1}^{\prime}$ in the case where $\mathcal{F}$ is a set of linearized polynomials.


## 1. Introduction

Let $k$ be a field and let $\mathcal{F} \subset R=k\left[X_{0}, \ldots, X_{m-1}\right]$ be a finite subset which generates an ideal. Let $R_{\leq i}$ be the set of polynomials in $R$ of degree at most $i$.

For $i \in \mathbb{Z}_{\geq 0}$, we let $V_{\mathcal{F}, i}$ be the smallest $k$-vector space of $R_{\leq i}$ such that
(1) $\{f \in \mathcal{F}: \operatorname{deg}(f) \leq i\} \subseteq V_{\mathcal{F}, i}$;
(2) if $g \in V_{\mathcal{F}, i}$ and if $h \in R$ with $\operatorname{deg}(h g) \leq i$, then $h g \in V_{\mathcal{F}, i}$.

We write $f \equiv_{i} g(\bmod \mathcal{F})$, for $f, g \in R$, if $f-g \in V_{\mathcal{F}, i}$.
The last fall degree as defined in [6] (see also [7]) is the largest $d$ such that $V_{\mathcal{F}, d} \cap R_{\leq d-1} \neq$ $V_{\mathcal{F}, d-1}$. We denote the last fall degree of $\mathcal{F}$ by $d_{\mathcal{F}}$.

As shown in [6, 7] the last fall degree is intrinsic to a polynomial system, independent of the choice of a monomial order, always bounded by the degree of regularity, and invariant under linear change of variables and linear change of equations. In [6, 7 , complexity bounds on solving zero dimensional polynomial systems were proven based on the last fall degree. It was shown in [6] that the polynomial systems arising from the Hidden Field Equations (HFE) public key crypto-system [1, 2] have bounded last fall degree if the degree of the defining polynomial and the cardinality of the base field are fixed (the bound was improved in [4]), and it follows that the HFE polynomials systems can be solved unconditionally in polynomial time.

For $\mathcal{F} \subset R=k\left[X_{0}, \ldots, X_{m-1}\right]$, let $Z_{k}(\mathcal{F})$ denote the set of solutions of $\mathcal{F}$ over $k$; let $Z(\mathcal{F})$ denote the set of solutions of $\mathcal{F}$ over $\bar{k}$, where $\bar{k}$ is an algebraic closure of $k$. If $\mathcal{F}$ is zero-dimensional then determining $Z(\mathcal{F})$ reduces to computing $V_{\mathcal{F}, \max \left(d_{\mathcal{F}}, e\right)}$ where $e$ is the cardinality of $Z(\mathcal{F})$ [7].

Suppose $k$ is a finite field of cardinality $q^{n}$ with subfield $k^{\prime}$ of cardinality $q$. The Weil descent system of $\mathcal{F}$ to $k^{\prime}$ is a polynomial system obtained when one expresses all equation with the help of a basis of $k^{\prime} / k$. Let $\alpha_{0}, \ldots, \alpha_{n-1}$ be a basis of $k / k^{\prime}$. For $f \in \mathcal{F}$ and $j=0, \ldots, n-1$,

[^0]we define $f_{j} \in k^{\prime}\left[X_{i j}, i=0, \ldots, m-1, j=0, \ldots, n-1\right]$ by
$$
f\left(\sum_{j=0}^{n-1} \alpha_{j} X_{0 j}, \ldots, \sum_{j=0}^{n-1} \alpha_{j} X_{m-1} j\right)=\sum_{j=0}^{n-1} f_{j} \alpha_{j} .
$$

We note that $\operatorname{deg} f_{j} \leq \operatorname{deg} f$. The system

$$
\mathcal{F}^{\prime}=\left\{f_{j}: f \in \mathcal{F}, j=0, \ldots, n-1\right\}
$$

is called the Weil descent system of $\mathcal{F}$ with respect to $\alpha_{0}, \ldots, \alpha_{n-1}$.
There is a bijection between $Z_{k}(\mathcal{F})$ and $Z_{k^{\prime}}\left(\mathcal{F}^{\prime}\right)=Z\left(\mathcal{F}_{1}^{\prime}\right)$, where $\mathcal{F}_{1}^{\prime}$ is $\mathcal{F}^{\prime}$ together with the field equations of $k^{\prime}$, that is,

$$
\mathcal{F}_{1}^{\prime}=\mathcal{F}^{\prime} \cup\left\{X_{i j}^{q}-X_{i j}, i=0, \ldots, m-1, j=0, \ldots, n-1\right\} .
$$

The HFE polynomial system is constructed by forming the Weil descent of some $\mathcal{F}$ consisting of a single univariate polynomial, followed by linear change of variables and linear change of equations [6]. Multivariate-HFE systems can be constructed similarly except $\mathcal{F}$ is replaced by a finite set of multivariate polynomials of dimension zero. In 7] upper bounds on the last fall degree degree of $\mathcal{F}_{1}^{\prime}$ were proven in terms of $q, m$, the last fall degree of $\mathcal{F}$, the degree of $\mathcal{F}$ and the number of solutions of $\mathcal{F}$, but not on $n$. The result implies that multi-HFE cryptosystems giving rise to multi-HFE polynomial systems as described above are vulnerable to attack as well.

In this paper we consider the situation where $\mathcal{F}$ is not necessarily zero-dimensional.
Let

$$
\begin{array}{r}
\mathcal{F}_{1}=\mathcal{F} \cup\left\{X_{i}^{q}-Y_{i 1}, \ldots,\right. \\
\left.Y_{i}^{q}{ }_{n-2}-Y_{i}{ }_{n-1}, Y_{i}^{q}{ }_{n-1}-X_{i}: i=0, \ldots, m-1\right\} \\
\end{array} \subset k\left[X_{i}, Y_{i j}: i=0, \ldots, m-1 ; j=1, \ldots, n-1\right] .
$$

We observe that $Z_{k}(\mathcal{F})$ can easily be identified with $Z\left(\mathcal{F}_{1}\right)$. So there is a bijection between $Z\left(\mathcal{F}_{1}\right)$ and $Z\left(\mathcal{F}_{1}^{\prime}\right)$. Note also that the ideals generated by $\mathcal{F}_{1}$ and $\mathcal{F}_{1}^{\prime}$ are radical ideals.

The following theorem relates the last fall degrees of $\mathcal{F}_{1}$ and $\mathcal{F}_{1}^{\prime}$.
Theorem 1.1. $\max \left(d_{\mathcal{F}_{1}}, q \operatorname{deg} \mathcal{F}\right)=\max \left(d_{\mathcal{F}_{1}^{\prime}}, q \operatorname{deg} \mathcal{F}\right)$.
Theorem 1.1 is closely related to Proposition 2 of [6] and Proposition 4.1 of 7]. In comparison, the bound established in Theorem 1.1 is a bit weaker. However the set $\mathcal{F}_{1}$ stated in the theorem directly contains $\mathcal{F}$ as a subset. This makes it easier to apply the theorem both conceptually and technically. When $\mathcal{F}$ consists of a univariate polynomial or more generally when $Z(\mathcal{F})$ is finite, it is not hard to bound $d_{\mathcal{F}_{1}}$. From this an easier and more conceptual proof of the theorems in [6, 7] can be constructed based on Theorem 1.1] However in this paper we will focus on applying the theorem to the situation where $\mathcal{F}$ is not zero-dimensional, especially when $\mathcal{F}$ consists of linearized polynomials.
1.1. Proof of Theorem 1.1. For non-negative integers $i$, let $\sigma_{i}$ denote the automorphism of $\bar{k}$ over $k^{\prime}$ such that $\sigma_{i}(x)=x^{q^{i}}$ for $x \in \bar{k}$. For every multivariate polynomial $h$ with coefficients from $\bar{k}$, let $h^{\sigma_{i}}$ denote the polynomial obtained from $h$ by acting on each coefficient of $h$ by $\sigma_{i}$.

Let $\Gamma$ be the $n$ by $n$ matrix with rows and columns indexed by $0, \ldots, n-1$, so that $\alpha_{j}^{\sigma_{i}}$ is the $(i, j)$-th entry of $\Gamma$ for $i, j=0, \ldots, n-1$.

Let

$$
g_{f}=f\left(\sum_{j=0}^{n-1} \alpha_{j} X_{0 j}, \ldots, \sum_{j=0}^{n-1} \alpha_{j} X_{m-1} j\right)=\sum_{j=0}^{n-1} f_{j} \alpha_{j}
$$

where $f_{j} \in k^{\prime}\left[X_{i j}, i=0, \ldots, m-1, j=0, \ldots, n-1\right]$.
Let $\hat{f}=\left(\begin{array}{c}f_{0} \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1}\end{array}\right)$ be the column vector with $f_{i}$ as the $i$-th entry for $i=0, \ldots, n-1$.
Then $g_{f}^{\sigma_{i}}=\sum_{j=0}^{n-1} f_{j} \alpha_{j}^{\sigma_{i}}$, and $\Gamma \hat{f}=\left(\begin{array}{c}g_{f}^{\sigma_{0}} \\ \cdot \\ \cdot \\ g_{f}^{\sigma_{n-1}}\end{array}\right)$. Let $\mathcal{G}=\left\{g_{f}^{\sigma_{0}}, \ldots, g_{f}^{\sigma_{n-1}}: f \in \mathcal{F}\right\}$ and $\mathcal{G}_{1}=\mathcal{G} \cup\left\{X_{i j}^{q}-X_{i j}, i=0, \ldots, m-1, j=0, \ldots, n-1\right\}$. Since $\Gamma$ is invertible, it follows from Proposition 2.6 (part v) of $\left[7\right.$ that $d_{\mathcal{G}}=d_{\mathcal{F}^{\prime}}$, and $d_{\mathcal{G}_{1}}=d_{\mathcal{F}^{\prime} 1}$.

Let $Z_{i j}, i=0, \ldots, m-1$ and $j=0, \ldots, n-1$, be defined by the following change of coordinates:

$$
\left(\begin{array}{c}
X_{i} \\
Y_{i 1} \\
\cdot \\
\cdot \\
Y_{i n-1}
\end{array}\right)=\Gamma\left(\begin{array}{c}
Z_{i 0} \\
Z_{i 1} \\
\cdot \\
\cdot \\
Z_{i n-1}
\end{array}\right)
$$

Under the change of coordinates, $\mathcal{F}_{1}$ becomes $\mathcal{G}_{2}$ where

$$
\begin{aligned}
\mathcal{G}_{2} & =\left\{f\left(\sum_{j=0}^{n-1} \alpha_{j} Z_{0 j}, \ldots, \sum_{j=0}^{n-1} \alpha_{j} Z_{m-1}\right): f \in \mathcal{F}\right\} \cup\left\{Z_{i j}^{q}-Z_{i j}: i, j=0, \ldots, n-1\right\} \\
& =\left\{g_{f}\left(Z_{01}, \ldots, Z_{m, n-1}\right): f \in \mathcal{F}\right\} \cup\left\{Z_{i j}^{q}-Z_{i j}: i, j=0, \ldots, n-1\right\},
\end{aligned}
$$

which we identify as a subset of $\mathcal{G}_{1}$. Since $g_{f}^{q} \equiv g_{f}^{\sigma} \bmod I$ where $I$ is the deal generated by $X_{i j}^{q}-X_{i j}, i=0, \ldots, m-1, j=0, \ldots, n-1$, we see that $g_{f}^{\sigma} \in V_{\mathcal{G}_{2}, q d}$ where $d=\operatorname{deg} \mathcal{F} \geq \operatorname{deg} g_{f}$. It follows inductively that $g_{f}^{\sigma_{i}} \in V_{\mathcal{G}_{2}, q d}$ for all $i$, hence $\mathcal{G}_{1} \subset V_{\mathcal{G}_{2}, q d}$. Hence $V_{\mathcal{G}_{1}, i}=V_{\mathcal{G}_{2}, i}$ for $i \geq q d$. Therefore $\max \left(d_{\mathcal{G}_{1}}, q d\right)=\max \left(d_{\mathcal{G}_{2}}, q d\right)$. Since $d_{\mathcal{G}_{1}}=d_{\mathcal{F}_{1}^{\prime}}$ and $d_{\mathcal{F}_{1}}=d_{\mathcal{G}_{2}}$, we conclude that $\max \left(d_{\mathcal{F}_{1}}, q d\right)=\max \left(d_{\mathcal{F}^{\prime} 1}, q d\right)$. Theorem 1.1 follows.

## 2. Systems of linearized polynomials

A $k^{\prime}$-linearized polynomial in $R=k\left[x_{0} \ldots x_{m-1}\right]$ is an element of the $k$-submodule of $R$ generated by $x_{i}^{q^{j}}$ where $q=\left|k^{\prime}\right|, i=0 \ldots m-1$ and $j \geq 0$. As before let $n=\left[k: k^{\prime}\right]$. Let $Q=\left\{x_{i}^{q^{n}}-x_{i}: i=0 \ldots m-1\right\}$.

For $g=\sum_{i=0}^{d} a_{i} x^{i} \in k[x]$, let $L(g)=\sum_{i=0}^{d} a_{i} x^{q^{i}}$. More generally we consider the $k$-linear map from $\oplus_{i=0}^{m-1} k\left[x_{i}\right]$ onto the $k$-module of $k^{\prime}$-linearized polynomials such that $L\left(x_{i}^{j}\right)=x_{i}^{q^{j}}$ for $i=0, \ldots, m-1$ and $j \geq 0$.

Let $S=k\left[x_{i j}: i=0 \ldots m-1, j=0, \ldots, n-1\right]$. We also write $S=k\left[\hat{x}_{i}: i=0, \ldots, m-1\right]$. where $\hat{x}_{i}=x_{i 0}, \ldots, x_{i n-1}$. Let $S_{1} \subset S$ be the $k$-module of linear forms over $x_{i j}, i=$ $0 \ldots m-1, j=0, \ldots, n-1$.

For $g \in k[x]$ and $h \in k\left[x_{i}\right]$, let $g \circ h \in k\left[x_{i}\right]$ be defined as $(g \circ h)\left(x_{i}\right)=g\left(h\left(x_{i}\right)\right)$. For $f=\sum_{i=0}^{m-1} f_{i} \in \bigoplus_{i=0}^{m-1} k\left[x_{i}\right]$ with $f_{i} \in k\left[x_{i}\right]$, let $g \circ f \in \bigoplus_{i=0}^{m-1} k\left[x_{i}\right]$ be defined as $g \circ f=$ $\sum_{i=0}^{m-1} g \circ f_{i}$. Hence $(g \circ f)\left(x_{0}, \ldots, x_{m-1}\right)=\sum_{i=0}^{m-1} g\left(f_{i}\left(x_{i}\right)\right)$. Note also that $L(g) \circ L(f)$ is a $k^{\prime}$-linearized polynomial in $k\left[x_{0}, \ldots, x_{m-1}\right]$.

Similarly for $g \in k[x]$ and $f=\sum_{i, j} f_{i j} \in \bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k\left[x_{i j}\right]$ with $f_{i j} \in k\left[x_{i j}\right]$, let $g \circ f \in$ $\bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k\left[x_{i j}\right]$ be defined as $g \circ f=\sum_{i, j} g \circ f_{i j}$.

Consider the map $\ell$ from $\bigoplus_{i=0}^{m-1} \bigoplus_{j=0}^{n-1} k\left[x_{i}^{j}\right]$ to $S_{1}$ such that $\ell\left(x_{i}^{j}\right)=x_{i j}$.
Let $\bar{Q}=\left\{x_{i j}^{q}-x_{i j+1}: i=0 \ldots m-1, j=0 \ldots n-1\right\}$ where $j+1$ is taken $\bmod n$. Consider the $k$-algebra isomorphism from $R /\langle Q\rangle$ to $S /\langle\bar{Q}\rangle$ sending $x_{i}^{q^{j}}$ to $x_{i j}$ for $i=0, \ldots, m-1$, $j=0, \ldots, n-1$. (Note that $x_{i}^{q^{j+1}}=\left(x_{i}^{q^{j}}\right)^{q}$ maps to $x_{i j}^{q}$ and $x_{i j}^{q} \equiv x_{i j+1} \bmod \bar{Q}$.)

For $f \in R$ where the degree of $f$ in $x_{i}$ is less than $q^{n}$ for all $i$, let $\bar{f} \in S$ denote the image of $f$ in $S /\langle\bar{Q}\rangle$ under the isomorphism. We note that elements of $S_{1}$ are all distinct $\bmod \bar{Q}$. Let $f=\sum_{i=0}^{m-1} f_{i}\left(x_{i}\right)$ with $\operatorname{deg} f_{i}<n$ for all $i$. Let $f_{i}=\sum_{j=0}^{n-1} a_{i j} x_{i}^{j}$. Then $L(f) \in R$ corresponds to $\ell(f)$. If we identify with $x_{i} \in R$ with $x_{i 0} \in S$ for $i=0, \ldots, m-1$. Then $L(f) \equiv_{d} \ell(f)$ $(\bmod \bar{Q})$ where $d=\operatorname{deg} L(f)$.

Suppose $\mathcal{F}$ is a finite set of $k^{\prime}$-linearized polynomials of maximum degree $d=q^{c}$ for some $c>0$. We may identify $x_{i} \in R$ with $x_{i 0} \in S$ and consider $\mathcal{F} \subset S$. Let $\mathcal{F}^{\prime}$ be the Weil descent system of $(F)$ with respect to a $k / k^{\prime}$ basis. We are interested in the last fall degree of $\mathcal{F}_{1}^{\prime}=\mathcal{F}^{\prime} \cup\left\{x_{i j}^{q}-x_{i j}: i=0, \ldots, m-1, j=0, \ldots, n-1\right\}$. Let $\mathcal{G}=\mathcal{F} \cup \bar{Q} \subset S$. By Theorem 1.1, $\max \left(d_{\mathcal{G}}, q \operatorname{deg} \mathcal{F}\right)=\max \left(d_{\mathcal{F}_{1}^{\prime}}, q \operatorname{deg} \mathcal{F}\right)$.

Recall that $Z\left(\mathcal{F}_{1}^{\prime}\right)=Z_{k^{\prime}}\left(\mathcal{F}^{\prime}\right)$, which corresponds to $Z_{k}(\mathcal{F})=Z(\mathcal{F} \cup Q)$, the set of $k$-rational points of $Z(\mathcal{F})$. In what follows we consider a more general situation where instead of $Z_{k}(\mathcal{F})$ we are interested in $Z_{W}(\mathcal{F})=Z(\mathcal{F}) \cap W^{m}$ where $W$ is a $\tau$-invariant subspace of $k$ and $\tau$ is the Frobenius map over $k^{\prime}: x \rightarrow x^{q}$ for all $x \in k$. Note that every $\tau$-invariant subspace $W$ of $k$ is of of the form $Z\left(L\left(\mathfrak{f}_{W}\right)\right)$ where $\mathfrak{f}_{W}$ divides $x^{n}-1$. In fact $W$ is the kernel of $\mathfrak{f}_{W}(\tau)$, and $f_{W}$ is the characteristic polynomial of $\tau$ as a linear map on $W$. In particular $\mathfrak{f}_{W}=x^{n}-1$ corresponds to $W=k$ and $\mathfrak{f}_{W}=x-1$ corresponds to $W=k^{\prime}$. Suppose $d_{W}=\operatorname{deg} \mathfrak{f}_{W}$.

In this more general situation we let $S=k\left[x_{i j}: i=0 \ldots m-1, j=0, \ldots, d_{W}-1\right]$. We also write $S=k\left[\hat{x}_{i}: i=0, \ldots, m-1\right]$. where $\hat{x}_{i}=x_{i 0}, \ldots, x_{i} d_{W-1}$. Let $f=\sum_{i=0}^{m-1} f_{i}\left(x_{i}\right)$ with $\operatorname{deg} f_{i}<d_{W}$ for all $i$. Suppose $f_{i}=\sum_{j=0}^{d_{W}-1} a_{i j} x_{i}^{j}$. Then $L(f)=\sum_{i=1}^{m-1} L\left(f_{i}\right)$ where $L\left(f_{i}\right)=\sum_{j=0}^{d_{W}-1} a_{i j} x_{i}^{q^{j}}$ and $\ell(f)=\sum_{i=0}^{m-1} \ell\left(f_{i}\right)$ where $\ell\left(f_{i}\right)=\sum_{j=0}^{d_{W}-1} a_{i j} x_{i j}$.

Below we fix $W$ and let $n^{\prime}=d_{W}$. Write $\mathfrak{f}_{W}(x)=x^{n^{\prime}}-\mathfrak{g}_{W}(x)$ with $\operatorname{deg} \mathfrak{g}_{W}<n^{\prime}$. Let $Q=\left\{x_{i}^{q^{n^{\prime}}}-L\left(\mathfrak{g}_{W}\left(x_{i}\right)\right): i=0, \ldots, m-1\right\}$, and correspondingly we let $\bar{Q}=\left\{x_{i n^{\prime}-1}^{q}-\right.$ $\left.\ell\left(\mathfrak{g}_{W}\left(x_{i}\right)\right), x_{i j}^{q}-x_{i j+1}: i=0, \ldots, m-1, j=0, \ldots, n^{\prime}-2\right\}$. Then we have an isomorphism from $R /\langle Q\rangle$ to $S /\langle\bar{Q}\rangle$ sending $x_{i}^{q^{j}}$ to $x_{i j}$ for $i=0, \ldots, m-1, j=0, \ldots, n^{\prime}-1$. Let $S_{1} \subset S$ be the $k$-module of linear forms over $x_{i j}, i=0 \ldots m-1, j=0, \ldots, n^{\prime}-1$. We note that elements of $S_{1}$ are all distinct $\bmod \bar{Q}$.

For $f \in R$, we have $f \equiv_{d} f_{1}(\bmod Q)$ where $d=\operatorname{deg} f$ and the degree of $f_{1}$ in $x_{i}$ is less than $q^{d_{W}}$ for all $i$. Let $\bar{f} \in S$ denote the image of $f_{1}$ in $S /\langle\bar{Q}\rangle$ under the isomorphism. Let $f=\sum_{i=0}^{m-1} f_{i}\left(x_{i}\right)$ with $\operatorname{deg} f_{i}<n^{\prime}$ for all $i$. Then $\overline{L(f)}=\ell(f)$. If we identify with $x_{i} \in R$ with $x_{i 0} \in S$ for $i=0, \ldots, m-1$. Then $L(f) \equiv_{d} \ell(f)(\bmod \bar{Q})$ where $d=\operatorname{deg} L(f)$.

Lemma 2.1. Suppose $f=\sum_{i, j} a_{i j} x_{i j} \in S_{1}$ with $a_{i j} \in k$. Then with respect to $\bar{Q}, f_{i}^{q} \equiv_{q} f_{i+1}$ $(\bmod \bar{Q})$, for $i \geq 0$ where $f=f_{0}$ and $f_{i} \in S_{1}$ for $i \geq 0$.

Proof For $r \geq 0$, we have inductively $f_{r}=\sum b_{i j} x_{i j} \in S_{1}$. Now $f_{r}^{q}=\sum_{i j} b_{i j}^{q} x_{i j}^{q}$, and since for all $i, x_{i j}^{q} \equiv_{q} x_{i} \underline{j+1}(\bmod \bar{Q})$ for $j=0, \ldots, n^{\prime}-2$, and $x_{i n^{\prime}-1}^{q} \equiv_{q} \ell\left(\mathfrak{g}\left(x_{i}\right)\right)$, it follows that $f_{r}^{q} \equiv_{q} f_{r+1}(\bmod \bar{Q})$ with $f_{r+1} \in S_{1}$.

Lemma 2.2. Let $\mathcal{H}$ be a finite set of $S$ and suppose $\bar{Q} \subset \mathcal{H}$. Suppose $f \in S_{1}$ and $f \equiv_{i} 0$ $(\bmod \mathcal{H})$ for some $i>0$. Let $g \in k[x]$. Then $L(g) \circ f-f^{\prime} \in\langle\bar{Q}\rangle$ for some $f^{\prime} \in S_{1}$, where $\langle\bar{Q}\rangle$ denotes the ideal generated by $\bar{Q}$, and $f^{\prime} \equiv_{r} 0(\bmod \mathcal{H})$ where $r=\max (i, q)$.

Proof The lemma follows by applying Lemma 2.1 inductively. More specifically assume inductively $L\left(x^{i}\right) \circ f \equiv_{q} f_{i}^{\prime}(\bmod \bar{Q})$ with $f_{i}^{\prime} \in S_{1}$, then $L\left(x^{i+1}\right) \circ f \equiv_{q} f_{i}^{\prime q} \equiv_{q} f_{i+1}^{\prime}(\bmod \bar{Q})$ for some $f_{i+1}^{\prime} \in S_{1}$. From this the lemma easily follows.

For $f \in \mathcal{F}, f \equiv_{d} \bar{f}(\bmod \bar{Q})$ with $\bar{f} \in S_{1}$. Let $\bar{F}$ consist of all such $\bar{f} \in S_{1}$ with $f \in \mathcal{F}$. Let $\overline{\mathcal{G}}=\overline{\mathcal{F}} \cup \bar{Q}$. Then $\overline{\mathcal{G}} \subset V_{\mathcal{G}, d}$ and $\overline{\mathcal{G}} \subset V_{\overline{\mathcal{G}}, q}$.

Let $S_{1 i}=S_{1} \cap k\left[\hat{x}_{j}: j=i, \ldots, m-1\right]$, that is , the submodule containing all $k$-linear forms in $x_{i j}, i=i, \ldots, m-1, j=0, \ldots, n^{\prime}-1$. In particular $S_{1}=S_{10}$. Let $\bar{Q}_{r}=\bar{Q} \cap k\left[\hat{x}_{i}: i=\right.$ $r, \ldots, m-1]$ for $r=1, \ldots, m-1$.
Lemma 2.3. Consider a $k^{\prime}$-linearized polynomial of the form $L(f)$ with $f=\sum_{i=0}^{m-1} f_{i}$ and $f_{i} \in k\left[x_{i}\right]$ of degree less than $n^{\prime}$, for $i=0, \ldots, m-1$. Suppose $\ell(f) \in V_{\overline{\mathcal{G}}, q}$ and the $G C D$ of $f_{0}$ and $\mathfrak{f}_{W}$ is 1. Then $x_{00}-\ell_{0} \in V_{\overline{\mathcal{G}}, q}$ for some linear form $\ell_{0} \in S_{11}$. Moreover for $i=1, \ldots, n^{\prime}-1, x_{0 i}-\ell_{i} \in V_{\overline{\mathcal{G}}, q}$ for some linear form $\ell_{i} \in S_{11}$, and $\ell_{i-1}^{q} \equiv \ell_{i}\left(\bmod \bar{Q}_{1}\right)$.

Proof Since the GCD of $f_{0}$ and $\mathfrak{f}_{W}$ is $1, A(x) f_{0}(x)+B(x) \mathfrak{f}_{W}(x)=1$ for some $A(x), B(x) \in$ $k[x]$ where $\operatorname{deg} A<n$ and $\operatorname{deg} B<\operatorname{deg} f_{0}$. Now

$$
\begin{gathered}
L(A(x)) \circ L\left(f_{0}\left(x_{0}\right)\right)+L(B(x)) \circ L\left(\mathfrak{f}_{W}\left(x_{0}\right)\right)=x_{0} \\
L(A(x)) \circ L\left(\sum_{i=1}^{m-1} f_{i}\left(x_{i}\right)\right)=L(g)
\end{gathered}
$$

for some $g=\sum_{i=1}^{m-1} g_{i}$ where $g_{i} \in k\left[x_{i}\right]$. So

$$
L(A(x)) \circ L(f)+L(B(x)) \circ L\left(\mathfrak{f}_{W}\right)=x_{0}+L(g)
$$

We have

$$
L(A(x)) \circ \ell(f) \equiv x_{00}+\ell(g) \quad(\bmod \langle\bar{Q}\rangle)
$$

Note that $\ell(g) \in S_{11}$. By Lemma 2.2 there is some $f^{\prime} \in S_{1}$ such that $L(A) \circ \ell(f)-f^{\prime} \in\langle\bar{Q}\rangle$ and $f^{\prime} \equiv{ }_{q} 0(\bmod \mathcal{G})$. So put $\ell_{0}=-\ell(g)$. Then $f^{\prime} \equiv x_{00}-\ell_{0}(\bmod \langle\mathcal{G}\rangle)$, and since $f^{\prime}$ and $x_{00}-\ell_{0}$ are both in $S_{1}$, we have $f^{\prime}=x_{00}-\ell_{0}$. Let $\ell_{1} \in S_{11}$ such that $\ell_{0}^{q} \equiv_{q} \ell_{1}\left(\bmod \bar{Q}_{1}\right)$. Then $x_{01} \equiv_{q} x_{00}^{q} \equiv_{q} \ell_{0}^{q} \equiv_{q} \ell_{1}(\bmod \overline{\mathcal{G}})$, and inductively we have $x_{0 i} \equiv_{q} \ell_{i}$ for some linear form $\ell_{i} \in S_{11}$, with $\ell_{i-1}^{q} \equiv_{q} \ell_{i}\left(\bmod \bar{Q}_{1}\right)$.

When the condition in Lemma 2.3 is satisfied, $x_{0 i} \equiv_{q} \ell_{i}$ for some $\ell_{i} \in S_{11}$. Substituting he variable $x_{0 j}$ by $\ell_{j}$, for $j=0, \ldots, n^{\prime}-1$, gives reduction from $S_{1} \cap V_{\overline{\mathcal{G}}, q}$ to $S_{11} \cap V_{\overline{\mathcal{G}}, q}$. More explicitly, for $g \in S_{1}$, write $g=g_{0}+g_{1}$ where $g_{0}$ is a linear form in $x_{00}, \ldots, x_{0} n_{n^{\prime}-1}$, and $g_{1} \in S_{11}$. Then $g \equiv_{1} g^{\prime}$ where $g^{\prime}=g_{0}\left(\ell_{0}, \ldots, \ell_{n^{\prime}-1}\right)+g_{1} \in S_{11}$. Therefore for all $g \in S_{1} \cap V_{\overline{\mathcal{G}}, q}$, there is some $g^{\prime} \in S_{11}$ such that $0 \equiv_{q} g \equiv_{1} g^{\prime}(\bmod \overline{\mathcal{G}})$. A similar condition will give reduction from $S_{11} \cap V_{\overline{\mathcal{G}}, q}$ to $S_{12} \cap V_{\overline{\mathcal{G}}, q}$, and so on. This leads to the following definition.

We say that $\mathcal{F}$ is reducible for $W$ if for $i=0, \ldots, m-2$, either $V_{\overline{\mathcal{G}}, q} \cap S_{1 i}=V_{\overline{\mathcal{G}}, q} \cap S_{1 i+1}$, or else there is a $k^{\prime}$-linearized polynomial of the form $L\left(f_{i}\right)$ with $f_{i}=\sum_{j=i}^{m-1} g_{i j}, g_{i j} \in k\left[x_{j}\right]$ of degree less than $n^{\prime}$, for $j=i, \ldots, m-1$, and $\ell\left(f_{i}\right) \in V_{\overline{\mathcal{G}}, q} \cap S_{1 i}$ and the GCD of $g_{i i}$ and $\mathfrak{f}\left(x_{i}\right)$ is 1 .

In particular if $\mathfrak{f}_{W}$ is irreducible over $k^{\prime}$ then the GCD of every nonzero polynomial of degree less than $n^{\prime}=\operatorname{deg} \mathfrak{f}_{W}$ is relatively prime to $\mathfrak{f}_{W}$. Therefore we have the following:
Lemma 2.4. If $\mathfrak{f}_{W}$ is irreducible over $k^{\prime}$ then $\mathcal{F}$ is reducible for $W$.
Theorem 2.5. Suppose $\mathcal{F}$ is a finite set of $k^{\prime}$-linearized polynomials, and $W$ is a $\tau$-invariant subspace of $k$ where $\tau$ is the Frobenius map over $k^{\prime}$. Let $\overline{\mathcal{G}}=\overline{\mathcal{F}} \cup \bar{Q}$. If $\mathcal{F}$ is reducible for $W$, then $d_{\overline{\mathcal{G}}} \leq(q-1) m+1$. Moreover a basis of $Z_{W}(\mathcal{F})$ can be constructed in time $\left(n^{\prime} m\right)^{O(q)}$ where $n^{\prime}=\operatorname{deg} \mathfrak{f}_{W}$.

Theorem 2.6. Suppose $\mathcal{F}$ is a finite set of $k^{\prime}$-linearized polynomials of maximum degree $d=q^{c}$ for some $c>0$. Let $\mathcal{F}^{\prime}$ be the Weil descent system of $\mathcal{F}$ with respect to a $k / k^{\prime}$ basis, and $\mathcal{F}_{1}^{\prime}=\mathcal{F}^{\prime} \cup\left\{x_{i j}^{q}-x_{i j}: i=0, \ldots, m-1, j=0, \ldots, n-1\right\}$. If $\mathcal{F}$ is reducible for $k$, then $d_{\mathcal{F}_{1}^{\prime}} \leq \max ((q-1) m+1, q d)$.

Example Consider the case where $\mathcal{F}$ consists of a bivariate linearized polynomial

$$
\begin{aligned}
F(x, y) & =a x^{q^{2}}+b x^{q}+c x+u y^{q^{2}}+v y^{q}+w y \\
& =L\left(a x^{2}+b x+c\right)+L\left(u y^{2}+v y+w\right),
\end{aligned}
$$

with $a, b, c, u, v, w \in k=\mathbb{F}_{q^{n}}$. By Lemma 2.3 (with $f=a x^{2}+b x+c+u y^{2}+v y+w$ ), if either $G C D\left(a x^{2}+b x+c, x^{n}-1\right)=1$ or $G C D\left(u y^{2}+v y+w, y^{n}-1\right)=1$, then $\mathcal{F}$ is reducible for $k$. By Theorem [2.6, $d_{\mathcal{F}_{1}^{\prime}} \leq 2 q$.

Since $\overline{\mathcal{G}} \subset V_{\mathcal{G}, d}$, Theorem 2.6 follows from Theorem 1.1 and Theorem [2.5. The rest of this section is devoted to the proof of Theorem 2.5,

### 2.1. Proof of Theorem 2.5.

Lemma 2.7. Suppose $\mathcal{F}$ is reducible for $W$. For $i=0, \ldots, m-2$, if $V_{\overline{\mathcal{G}}, q} \cap S_{1 i} \neq V_{\overline{\mathcal{G}}, q} \cap S_{1 i+1}$, then $x_{i j} \equiv_{q} \gamma_{i j}(\bmod \overline{\mathcal{G}})$ for some linear form $\gamma_{i j} \in S_{1 m-1}$, for $j=0, \ldots, n^{\prime}-1$; moreover $\gamma_{i j}^{q} \equiv_{q} \gamma_{i j+1}\left(\bmod \bar{Q}_{m-1}\right)$ for $j=0, \ldots, n^{\prime}-2$.

Proof For $i=0, \ldots, m-2$, if $V_{\overline{\mathcal{G}}, q} \cap S_{1 i} \neq V_{\overline{\mathcal{G}}, q} \cap S_{1 i+1}$, then there is a $k^{\prime}$-linearized polynomial of the form $L\left(f_{i}\right)$ with $f_{i}=\sum_{j=i}^{m-1} g_{i j}$, where $g_{i j} \in k\left[x_{j}\right]$ of degree less than $n^{\prime}$, for $j=$ $i, \ldots, m-1, \ell\left(f_{i}\right) \in V_{\overline{\mathcal{G}}, q} \cap S_{1 i}$ and the GCD of $g_{i i}$ and $\mathfrak{f}\left(x_{i}\right)$ is 1 .

By Lemma 2.3 we have the following: for $j=0, \ldots, n^{\prime}-1, x_{i j} \equiv_{q} \ell_{i j}(\bmod \overline{\mathcal{G}})$ for some linear form $\ell_{i j} \in S_{1+1}$, moreover $\ell_{i j}^{q} \equiv_{q} \ell_{i j+1}\left(\bmod \bar{Q}_{i+1}\right)$. From this it is easy to see by induction (proceeding from $i=m-2$ to $i=0$ ) that $x_{i j} \equiv_{q} \gamma_{i j}$ for some linear form $\gamma_{i j} \in S_{1 m-1}$, moreover $\gamma_{i j}^{q} \equiv_{q} \gamma_{i j+1}\left(\bmod \bar{Q}_{m-1}\right)$ for $i=0, \ldots, m-1, j=0, \ldots, n^{\prime}-2$.
Lemma 2.8. Let $\mathcal{N}=\left\{i \in\{0, \ldots, m-2\}: V_{\overline{\mathcal{G}}, q} \cap S_{1 i} \neq V_{\overline{\mathcal{G}}, q} \cap S_{1 i+1}\right\}$. Let $\Gamma=\left\{x_{i j}-\gamma_{i j}\right.$ : $\left.\gamma_{i j} \in S_{1 m-1}, x_{i j} \equiv_{q} \gamma_{i j}(\bmod \overline{\mathcal{G}}), i \in \mathcal{N}, j=0, \ldots, n^{\prime}-1\right\}$. Let $H_{\overline{\mathcal{F}}}=\left\{\ell\left(h_{f}\right): f \in \overline{\mathcal{F}}\right\}$. Then there exist $H_{1}=\left\{\ell\left(h_{i j}\right): i=0, \ldots, m-2, j=0, \ldots, n^{\prime}-1\right\}$ where $h_{i j} \in k\left[x_{m-1}\right]$ with $\operatorname{deg} h_{i j}<n^{\prime}$ such that letting $H=H_{\overline{\mathcal{F}}} \cup H_{1}$, then $\Gamma \cup H \subset V_{\overline{\mathcal{G}}, q}, \overline{\mathcal{G}}=\overline{\mathcal{F}} \cup \bar{Q} \subset V_{H \cup \bar{Q}_{m-1} \cup \Gamma, q}$, $\langle\overline{\mathcal{G}}\rangle=\left\langle H \cup \bar{Q}_{m-1} \cup \Gamma\right\rangle$.

Proof By Lemma 2.7. $\Gamma \subset V_{\overline{\mathcal{G}}, q}$. For $f \in \overline{\mathcal{F}} \subset S_{1}$, let $f^{\prime} \in S_{1 m-1}$ be obtained from $f$ by substituting $x_{i j}$ with $\gamma_{i j}$ for $i=0, \ldots, m-2, j=0, \ldots, n^{\prime}-1$. Then $f^{\prime} \equiv_{1} f(\bmod \Gamma)$, and $f^{\prime}=\ell\left(h_{f}\right)$ for some $h_{f} \in k\left[x_{m-1}\right]$. We have $\ell\left(h_{f}\right) \in V_{\Gamma, 1} \subset V_{\overline{\mathcal{G}}, q}$, and $f \in V_{H_{\overline{\mathcal{F}}} \cup \Gamma, 1}$ where $H_{\overline{\mathcal{F}}}=\left\{\ell\left(h_{f}\right): f \in \overline{\mathcal{F}}\right\}$.

For $i=0, \ldots, m-2$ and $j=0, \ldots, n^{\prime}-2, x_{i j}^{q}-x_{i j+1} \in \bar{Q}$. Lemma 2.7 implies that $x_{i j}^{q}-x_{i j+1} \equiv_{q} \gamma_{i j}^{q}-\gamma_{i j+1} \equiv_{q} \ell\left(h_{i j}\right)\left(\bmod \Gamma \cup \bar{Q}_{m-1}\right)$ for some $h_{i j} \in k\left[x_{m-1}\right]$.

For $i=0, \ldots, m-2, x_{i}^{q}{ }_{n^{\prime}-1}-\ell\left(\mathfrak{g}_{W}\left(x_{i}\right)\right) \in \bar{Q}$. Let $\ell\left(\mathfrak{g}_{W}\left(x_{i}\right)\right)=\sum_{j=0}^{n^{\prime}-1} a_{i j} x_{i j}$ with $a_{i j} \in k$ Lemma 2.7 implies that

$$
x_{i{ }^{\prime}-1}^{q}-\ell\left(\mathfrak{g}_{W}\left(x_{i}\right)\right) \equiv_{q} \gamma_{i n^{\prime}-2}^{q}-\sum_{j=0}^{n^{\prime}-1} a_{i j} \gamma_{i j} \quad(\bmod \Gamma) .
$$

Since $\gamma_{i n^{\prime}-2}^{q} \equiv_{q} \gamma_{i n^{\prime}-1}\left(\bmod \bar{Q}_{m-1}\right)$, we have

$$
\gamma_{i n^{\prime}-2}^{q}-\sum_{j=0}^{n^{\prime}-1} a_{i j} \gamma_{i j} \equiv_{q} \ell\left(h_{i n^{\prime}-1}\right) \quad\left(\bmod \bar{Q}_{m-1}\right)
$$

with $h_{n^{\prime}-1} \in k\left[x_{m-1}\right]$ of degree less than $n^{\prime}$.
To summarize, we have

$$
x_{i j}^{q}-x_{i j+1} \equiv_{q} \ell\left(h_{i j}\right) \quad\left(\bmod \Gamma \cup \bar{Q}_{m-1}\right)
$$

for $i=0, \ldots, m-2$ and $j=0, \ldots, n^{\prime}-2$, and

$$
x_{i n^{\prime}-1}^{q}-\ell\left(\mathfrak{g}_{W}\left(x_{i}\right)\right) \equiv_{q} \ell\left(h_{i n^{\prime}-1}\right) \quad\left(\bmod \Gamma \cup \bar{Q}_{m-1}\right)
$$

for $i=0, \ldots, m-2$. Let $H_{1}=\left\{\ell\left(h_{i j}\right): i=0, \ldots, m-2, j=0, \ldots, n^{\prime}-1\right\}$. It follows that $\bar{Q} \subset V_{H_{1} \cup \Gamma \cup \bar{Q}_{m-1}, q}$ and on the other hand $H_{1} \subset V_{\bar{Q} \cup \Gamma, q}$, and since $\Gamma \subset V_{\overline{\mathcal{G}}, q}$, we have $H_{1} \subset V_{\overline{\mathcal{G}}, q}$.

Let $H=H_{\overline{\mathcal{F}}} \cup H_{1}$. Then we conclude that $\Gamma \cup H \subset V_{\overline{\mathcal{G}}, q}$, and on the other hand $\overline{\mathcal{G}}=$ $\overline{\mathcal{F}} \cup \bar{Q} \subset V_{H \cup \bar{Q}_{m-1} \cup \Gamma, q}$. In particular, we have $\langle\overline{\mathcal{G}}\rangle=\left\langle H \cup \bar{Q}_{m-1} \cup \Gamma\right\rangle$.

Note that $H \cup \bar{Q}_{m-1} \subset k\left[\hat{x}_{m-1}\right]$ where $\hat{x}_{m-1}=x_{m-1} 0, \ldots, x_{m-1 n^{\prime}-1}$.
Lemma 2.9. Let $H$ be as in Lemma 2.8. Suppose $H=\left\{\ell\left(h_{i}\right): i=1, \ldots, s\right\}$ and let $h_{0}=\mathfrak{f}_{W}\left(x_{m-1}\right)$. Let $g$ be the $G C D$ of $h_{i}, i=0, \ldots, s$. Then $\langle\overline{\mathcal{G}}\rangle=\left\langle\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1}\right\rangle$, moreover $\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1} \subset V_{\overline{\mathcal{G}}, q}$.

Proof We have $g=\sum_{i=0}^{s} a_{i} h_{i}$ with $a_{i} \in k\left[x_{m-1}\right]$, so

$$
L(g)=\sum_{i} L\left(a_{i}\right) \circ L\left(h_{i}\right) .
$$

So

$$
\ell(g) \equiv \sum_{i} L\left(a_{i}\right) \circ \ell\left(h_{i}\right) \quad \bmod \bar{Q}_{m-1} .
$$

Apply Lemma 2.2 to $H \cup \bar{Q}_{m-1} \subset k\left[\hat{x}_{m-1}\right]$ it follows that there is $h_{i}^{\prime} \in S_{1 m-1}$ such that $h_{i}^{\prime} \equiv_{q} 0\left(\bmod H \cup \bar{Q}_{m-1}\right)$ and $L\left(a_{i}\right) \circ \ell\left(h_{i}\right) \equiv h_{i}^{\prime}\left(\bmod \left\langle\bar{Q}_{m-1}\right\rangle\right)$. So

$$
\ell(g) \equiv \sum_{i} h_{i}^{\prime} \equiv_{q} 0 \quad\left(\bmod H \cup \bar{Q}_{m-1}\right)
$$

Since $\ell(g)$ and $h_{i}^{\prime}$ are all in $S_{1 m-1}$, we have

$$
\ell(g)=\sum_{i} h_{i}^{\prime} \equiv_{q} 0 \quad\left(\bmod H \cup \bar{Q}_{m-1}\right)
$$

in particular, $\ell(g) \equiv_{q} 0(\bmod \overline{\mathcal{G}})$. It follows that

$$
L\left(\mathfrak{f}_{W}\right)=L\left(h_{0}\right) \in\left\langle\left\{L\left(h_{i}\right): i=0, \ldots, s\right\}\right\rangle=\langle L(g)\rangle
$$

Under the isomorphism from $k\left[\hat{x}_{m-1}\right] /\left\langle\bar{Q}_{m-1}\right\rangle \rightarrow k\left[x_{m-1}\right] /\left\langle L\left(\mathfrak{f}_{W}\left(x_{m-1}\right)\right)\right\rangle, \ell\left(h_{i}\right)$ corresponds to $L\left(h_{i}\right)$, hence the ideal generated by $H \cup \bar{Q}_{m-1}$ corresponds to the ideal generated by $L(g)$.

Since by Lemma 2.8, $\langle\overline{\mathcal{G}}\rangle=\left\langle\Gamma \cup H \cup \bar{Q}_{m-1}\right\rangle$, it follows that $\langle\overline{\mathcal{G}}\rangle=\left\langle\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1}\right\rangle$. Moreover from the discussion above we have $\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1} \subset V_{\overline{\mathcal{G}}, q}$.

Under the isomorphism from $k\left[\hat{x}_{0}, \ldots, \hat{x}_{m-1}\right] /\langle\bar{Q}\rangle \rightarrow k\left[x_{0}, \ldots, x_{m-1}\right] /\langle Q\rangle, x_{i 0}-\gamma_{i 0}$ corresponds to $x_{i}-L\left(g_{i}\right)$ where $\ell\left(g_{i}\right)=\gamma_{i 0}$ for $i \in \mathcal{N}$. Under the isomorphism the ideal determined by $\overline{\mathcal{G}}$ corresponds to the ideal determined by $\mathcal{F} \cup Q$. Since, by Lemma 2.9, $\langle\overline{\mathcal{G}}\rangle=\left\langle\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1}\right\rangle$ and $g \mid \mathfrak{f}_{W}$, it follows that $\langle\mathcal{F} \cup Q\rangle$ is generated by $L(g)$ and $x_{i}-L\left(g_{i}\right)$ where $i \in \mathcal{N}$. By Lemma $2.9 \Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1} \subset V_{\overline{\mathcal{G}}, q}$, it follows from Proposition 2.3 of [7] that $\ell(g)$ and $\gamma_{i 0}$, hence $L(g)$ and $x_{i}-L\left(g_{i}\right)$ can be constructed in time $\left(m n^{\prime}\right)^{O(q)}$ time. From this a basis of $Z_{W}(\mathcal{F})$ over $k^{\prime}$ can be easily written down.

It is easy to see that if $f \in k\left[\hat{x}_{m-1}\right]$ and $f \in\left\langle\{\ell(g)\} \cup \bar{Q}_{m-1}\right\rangle$ then $f \equiv_{\operatorname{deg} f+1} \ell(g) f_{1}$ $\left(\bmod \bar{Q}_{m-1}\right)$ for some $f_{1} \in k\left[\hat{x}_{m-1}\right]$. Suppose $f \in\langle\overline{\mathcal{G}}\rangle$. Then $f \equiv \operatorname{deg} f f_{1}(\bmod \bar{Q})$ where the degree of $x_{i j}$ in $f_{1}$ is less than $q$ for all $i, j$. Using $x_{i j} \equiv \gamma_{i j}(\bmod \Gamma)$, we have $f_{1} \equiv{ }_{\operatorname{deg} f_{1}} h$ $\left(\bmod \Gamma \cup \bar{Q}_{m-1}\right)$ where $h \in k\left[\hat{x}_{m-1}\right]$. It follows that $h \in\left\langle\{\ell(g)\} \cup \bar{Q}_{m-1}\right\rangle$, hence $h \equiv_{\operatorname{deg} h+1}$ $\ell(g) h_{1}\left(\bmod \bar{Q}_{m-1}\right)$, so $h \equiv_{\operatorname{deg} h+1} 0\left(\bmod \{\ell(g)\} \cup \bar{Q}_{m-1}\right)$. If $\operatorname{deg} f>(q-1) m$, then $\operatorname{deg} f>$ $\operatorname{deg} f_{1}$, and since $\Gamma \cup\{\ell(g)\} \cup \bar{Q}_{m-1} \subset V_{\overline{\mathcal{G}}, q}$, we conclude that $f \in V_{\overline{\mathcal{G}}, \operatorname{deg} f}$. Therefore $d_{\overline{\mathcal{G}}} \leq$ $(q-1) m+1$. Theorem 2.5 follows.

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