

NEWTON POLYGONS OF L-FUNCTIONS ASSOCIATED TO DELIGNE POLYNOMIALS

JIIYOU LI

ABSTRACT. A conjecture of Le says that the Deligne polytope Δ_d is generically ordinary if $p \equiv 1 \pmod{D(\Delta_d)}$, where $D(\Delta_d)$ is a combinatorial constant determined by Δ_d . In this paper a counterexample is given to show that the conjecture is not true in general.

1. INTRODUCTION

Let \mathbb{F}_q be the finite field of q elements of characteristic p . Let ψ be a fixed nontrivial additive character on \mathbb{F}_q . For a Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, define its associated L -function by

$$L^*(f, T) = \exp \left(\sum_{k=1}^{\infty} \sum_{x \in (\mathbb{F}_{q^k})^*} \psi(f(x)) \frac{T^k}{k} \right).$$

A celebrated theorem of Dwork-Bombieri-Grothendieck implies that $L^*(f, T)$ is a rational function in $\mathbb{Q}[\zeta_p](x)$ [5], [9]. Furthermore, Deligne's theorem on the Riemann hypothesis [4] gives very nice descriptions of the Archimedean absolute values of its zeroes and poles.

Adolphson and Sperber [2] show that if the Laurent polynomial f satisfies a non-degenerate geometric condition, then $L^*(f, T)^{(-1)^{n-1}}$ is indeed a polynomial in $\mathbb{Z}[\zeta_p][x]$ of degree $n!V(\Delta(f))$. Here $\Delta(f)$ is the *Newton polytope* (see [8] for details) of f , defined to be the convex closure in \mathbb{R}^n generated by the origin and all exponents of the nonzero monomials (viewed as lattice points in \mathbb{R}^n), and $V(\Delta(f))$ is the volume.

Since the Archimedean absolute values of the zeroes of $L^*(f, T)^{(-1)^{n-1}}$ are determined explicitly, it remains to study the most intriguing problem of determining the non-Archimedean absolute values. Equivalently, it suffices to study the slopes of the Newton polygon of $L^*(f, T)^{(-1)^{n-1}}$, which is defined as follows.

Definition 1.1. Let $g(x) = \sum_{i=0}^d c_i x^i \in 1 + x\mathbb{Z}[\zeta_p]$. The (q -adic) Newton polygon of $g(x)$, denoted by $NP(f)$, is defined to be the lower

convex hull of the points $(i, \text{ord}_q c_i)$ in the plane \mathbb{R}^2 . An example is given in Figure 1.

Unfortunately, the computation of the Newton polygon of a given L-function is very hard in general. However, as shown by the Dwork theory, in many cases the combinatorial geometrical properties of the Newton polytope may give very nice algebraic properties of f . For instance, when f is non-degenerate, Adolphson and Sperber [2] show that the Newton polygon of $L^*(f, T)^{(-1)^{n-1}}$ lies above a topological lower bound called Hodge polygon, which will be briefly introduced as follows.

Let Δ be an n -dimensional convex integral polytope in \mathbb{R}^n . Let $C(\Delta)$ be the cone generated by Δ and the origin. For a vector $u \in \mathbb{R}^n$, $w(u)$ is the smallest nonnegative real number c satisfying $u \in c\Delta$. If such c does not exist, then we define $w(u) = \infty$.

For a co-dimension 1 face δ not containing 0 in Δ , let $\sum_{i=1}^n e_i X_i = 1$ be the equation of the hyperplane containipng δ in \mathbb{R}^n . Since the coordinates of vertices spanning Δ are integers, the coefficients e_i are uniquely determined rational numbers. Let $D(\delta)$ be the least common denominator of $\{e_i, 1 \leq i \leq n\}$ and $D = D(\Delta)$ be the least common denominator of all $D(\delta)$, where δ runs over all the co-dimension 1 faces of Δ which do not contain the origin.

For an integer k , let $W_\Delta(k) = \#\{u \in \mathbb{Z}^n \mid w(u) = \frac{k}{D}\}$. Let

$$H_\Delta(k) = \sum_{i=0}^n (-1)^i \binom{n}{i} W_\Delta(k - iD).$$

Definition 1.2. The Hodge polygon $HP(\Delta)$ of Δ is defined to be the lower convex polygon in \mathbb{R}^2 with vertices

$$\left(\sum_{k=0}^m H_\Delta(k), \frac{1}{D} \sum_{k=0}^m k H_\Delta(k) \right), 0 \leq m \leq nD.$$

Equivalently, $HP(\Delta)$ is the polygon starting from the origin with a side of slope k/D with horizontal length $H_\Delta(k)$ for each integer $0 \leq k \leq nD$.

For a finite integral polytope $\Delta \subset \mathbb{R}^n$, let $\mathcal{M}_p(\Delta)$ be the set of non-degenerate f over $\overline{\mathbb{F}_p}$ with $\Delta(f) = \Delta$. To study how $NP(f)$ varies for $f \in \mathcal{M}_p(\Delta)$, by the Grothendieck specialization theorem [16], the generic Newton polygon is given by

$$GNP(\Delta, p) := \inf_{f \in \mathcal{M}_p(\Delta)} NP(f).$$

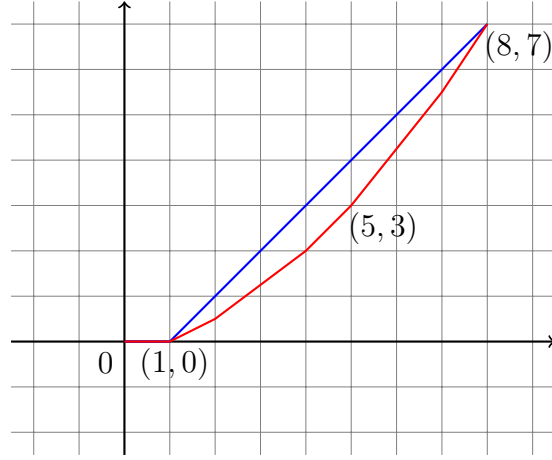


FIGURE 1. The Newton polygon (blue) and the Hodge polygon (red) of $f(x_1, x_2) = x_1x_2^3 + x_1^3x_2 + x_1x_2$.

Some work of computing $GNP(\Delta, p)$ in one-dimensional cases has been done [18] and in certain two variable cases in [13].

In [2] Adolphson and Sperber proved that for $f \in \mathcal{M}_p(\Delta)$, the graph of $NP(f)$ lies above the graph of $HP(\Delta)$ with the same endpoints, which can be stated as the following inequality $NP(f) \geq HP(\Delta)$. See Figure 1 for an example.

Combining these inequalities we have

Proposition 1.3. *For every prime p and every $f \in \mathcal{M}_p(\Delta)$,*

$$NP(f) \geq GNP(\Delta, p) \geq HP(\Delta).$$

A natural question is then to decide when the second inequality holds. For this purpose, we have the following definitions.

Definition 1.4. *A Laurent polynomial f is called ordinary if $NP(f) = HP(\Delta)$.*

Definition 1.5. *The family $\mathcal{M}_p(\Delta)$ is called generically ordinary if $GNP(\Delta, p) = HP(\Delta)$.*

It is known that a necessary condition for $GNP(\Delta, p) = HP(\Delta)$ is $p \equiv 1 \pmod{D(\Delta)}$. Adolphson and Sperber [1] conjectured that the converse is also true.

Conjecture 1.6 (Adolphson-Sperber). *If $p \equiv 1 \pmod{D(\Delta)}$, then $\mathcal{M}_p(\Delta)$ is generically ordinary.*

This is a generalization of a conjecture of Dwork [6] and Mazur [12]. Wan showed in [15] that Conjecture 1.6 is true for $n \leq 3$. Furthermore, he proved the following theorem.

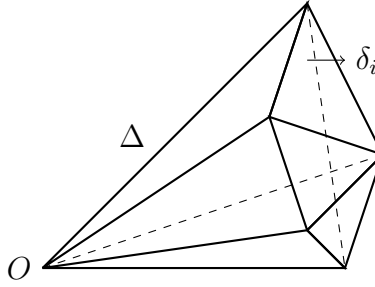


FIGURE 2. A facial decomposition for a 3 dimensional polytope.

Theorem 1.7 (Wan). *For any $n \geq 4$, there are explicitly constructed counterexamples for the A-S conjecture. However, a weaker version holds: there is an effectively computable integer $D^*(\Delta)$ such that if $p \equiv 1 \pmod{D^*(\Delta)}$, then $GNP(\Delta, p) = HP(\Delta)$.*

Roughly speaking, one of the most important techniques of Wan's proof is a decomposition theorem which decomposes Δ into small nice pieces and reduce the problem to small problems which are easier to be checked (see Figure 2 for an illustration). Then Wan developed several very useful decomposition theorems including the star decomposition, the boundary decomposition [15] and the collapsing decomposition [17].

Suppose $\Delta = \Delta(f)$ and thus we can write f as

$$f = \sum_{I \in \Delta \cap Z^n} a_I x^I.$$

Similarly, for a face $\delta \subseteq \Delta$, define

$$f^\delta = \sum_{I \in \delta \cap Z^n} a_I x^I.$$

Theorem 1.8 (Wan's facial decomposition theorem). *Suppose f is non-degenerate (See [2] for the definition). Let $\{\delta_i, 1 \leq i \leq s\}$ be the set all co-dimension 1 faces of $\Delta(f)$. Then f is ordinary if and only if each f^{δ_i} is ordinary.*

In particular, it turns out that if all f^{δ_i} 's are diagonal, then we have effective methods to determine if they are ordinary. Since the cases for diagonal Laurent polynomials are very easy to deal with, this decomposition theorem will immediately give good nontrivial applications.

Conjecture 1.6 is true in many important cases as showed by Wan [15] and more interesting examples are yet to be classified. Motivated by this, Le [11] studied an important class of Laurant polynomials, in

which f is given by

$$(1.1) \quad f(x_0, x_1, \dots, x_n) = x_0 h(x_1, x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) + 1/x_0,$$

where $\deg h = d$, $\deg g < d/2$, and h is an n -variable Deligne polynomial, which will be introduced in Section 2. We are interested in the A-S conjecture for this family of polynomials.

Conjecture 1.9. *Let f be a generic polynomial given in 1.1 and let $\Delta = \Delta(f)$. Then $GNP(\Delta, p) = HP(\Delta)$ if and only if $p \equiv 1 \pmod{D(\Delta)}$.*

Recall $\mathcal{M}_p(\Delta)$ is the set of non-degenerate f over $\overline{\mathbb{F}_p}$ with $\Delta(f) = \Delta$. Let $\widetilde{\mathcal{M}}_p(\Delta) \subseteq \mathcal{M}_p(\Delta)$ be the set of non-degenerate f of shape 1.1 with $\Delta(f) = \Delta$, and let

$$\widetilde{GNP}(\Delta, p) := \inf_{f \in \widetilde{\mathcal{M}}_p(\Delta)} NP(f).$$

Since clearly $\widetilde{GNP}(\Delta, p) \geq GNP(\Delta, p) \geq HP(\Delta)$, a slightly stronger conjecture arises.

Conjecture 1.10 (Le, 2013). *$\widetilde{GNP}(\Delta, p) = HP(\Delta)$ if and only if $p \equiv 1 \pmod{D(\Delta)}$.*

Le proved this conjecture for odd d and this implies that the A-S conjecture for this family holds for odd d . He then conjectured that it is also true for even d .

In this paper we give a counterexample to this conjecture.

Theorem 1.11. *Le's conjecture is false in general.*

Note that the reason that Le's conjecture fails is because when d is even, the family polynomials $\widetilde{\mathcal{M}}_p(\Delta)$ studied in this paper and the family of polynomials $\mathcal{M}_p(\Delta)$ are not identical. In other words, the family Le considered is slightly smaller and thus is not a universal family. A natural question is to determine whether Le's conjecture is true when the family of f defined by 1.1 becomes larger, i.e., the condition " $\deg g < d/2$ " is replaced by " $\deg g \leq d/2$ ".

2. A COUNTEREXAMPLE TO LE'S CONJECTURE

A polynomial $f(x) = f(x_1, x_2, \dots, x_n)$ in $\mathbb{F}_q[x_1, x_2, \dots, x_n]$ is called a *Deligne polynomial* if its leading homogeneous form f_d defines a smooth projective hypersurface in the projective space \mathbb{P}^{n-1} , where d is the degree of f . Deligne polynomials were extensively studied by Deligne [4], Katz [10], Browning and Heath-Brown [3], Le [11], Fu and Wan [7].

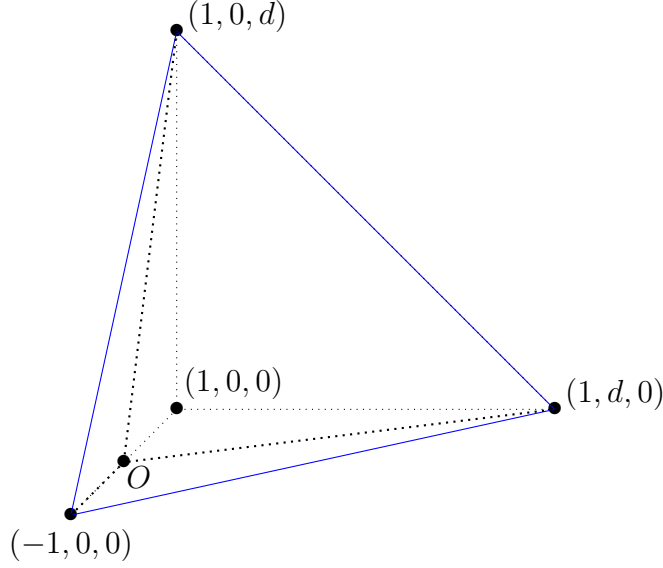


FIGURE 3. $\Delta \supseteq \Delta_d \supseteq \delta_d$ (δ_d is the blue triangle)

Deligne shows that if f is a Deligne polynomial, then $L^*(f, T)^{(-1)^{n-1}}$ is polynomial of degree $(d-1)^n$ and this fact immediately gives a fundamental estimate

$$\left| \sum_{x \in \mathbb{F}_{q^k}} \psi(f(x)) \right| \leq (d-1)^n q^{\frac{k}{2}}.$$

Motivated by the study of exponential sums, it is natural to consider a general class of Laurent polynomials of the following form

$$f(x_0, x_1, \dots, x_n) = x_0 h(x_1, x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) + 1/x_0,$$

where $\deg h = d$, $\deg g < d/2$, and h is a Deligne polynomial in n variables.

From now on, Δ is always used to denote the Newton Polytope of f and e_0, e_1, \dots, e_n be the standard basis for \mathbb{R}^{n+1} . Let δ_d be the polytope spanned by $-e_0, e_0 + de_1, \dots, e_0 + de_n$, and δ'_d spanned by $e_0, e_0 + de_1, \dots, e_0 + de_n$. They are the only two co-dimension 1 faces of Δ that do not contain the origin. Let $\Delta_d(\Delta'_d, \text{ respectively})$ be the convex hull of $\delta_d(\delta'_d, \text{ respectively})$ and the origin. Please see Figure 3 for an example.

Since the defining equation of δ_d is

$$X_1 - \frac{2}{d}X_2 - \frac{2}{d}X_3 - \dots - \frac{2}{d}X_{n+1} = 1,$$

one computes that

$$D(\Delta_d) = \begin{cases} d & d \text{ is odd;} \\ \frac{d}{2} & d \text{ is even.} \end{cases}$$

For simplicity we denote $D(\Delta_d)$ by D . The Hodge polygon $HP(\Delta_d)$ can be determined explicitly. We would like to know if Δ_d satisfies Conjecture 1.6.

Let $\widetilde{\mathcal{M}}_p(\Delta_d) \subseteq \mathcal{M}_p(\Delta_d)$ be the set of non-degenerate f of shape 1.1 with $\Delta(f) = \Delta_d$. Let

$$\widetilde{GNP}(\Delta_d, p) := \inf_{f \in \widetilde{\mathcal{M}}_p(\Delta_d)} NP(f).$$

Note that $\widetilde{GNP}(\Delta_d, p) \geq GNP(\Delta_d, p) \geq HP(\Delta_d)$.

By Wan's facial decomposition theorem, Le's conjecture is then equivalent to

$$\widetilde{GNP}(\Delta_d, p) = HP(\Delta_d) \iff p \equiv 1 \pmod{D}.$$

Applying the regular decomposition theorem, Le [11] proved that

Theorem 2.1. *If $p \not\equiv 1 \pmod{D}$ then $GNP(\Delta_d, p)$ lies strictly above $HP(\Delta_d)$. In particular, for any odd d , $p \equiv 1 \pmod{D}$ if and only if $\widetilde{GNP}(\Delta_d, p) = HP(\Delta_d)$.*

This implies that A-S conjecture for the family of polynomials (1.1) holds for odd d . Le then conjectured that it also holds for even d . We now give a counterexample to this conjecture.

Theorem 2.2. *Le's conjecture is false in general.*

Proof. Recall $\widetilde{\mathcal{M}}_p(\Delta_d) \subseteq \mathcal{M}_p(\Delta_d)$ be the set of non-degenerate f of shape 1.1 with $\Delta(f) = \Delta_d$. It suffices to prove

$$\widetilde{GNP}(\Delta_d, p) \neq HP(\Delta_d).$$

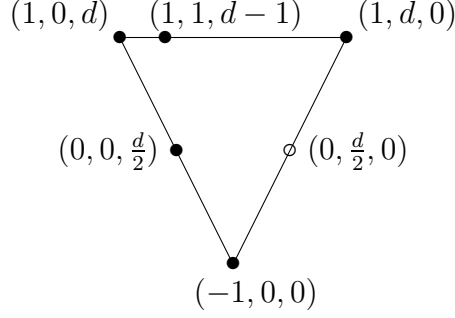
Let f be a polynomial in $\widetilde{\mathcal{M}}_p(\Delta_d)$. If we define

$$f^{\delta_d} = x_0 h(x_1, x_2, \dots, x_n) + 1/x_0,$$

and

$$f^{\delta'_d} = x_0 h(x_1, x_2, \dots, x_n),$$

then equivalently the facial decomposition theorem implies that f is generically ordinary if and only if both f^{δ_d} and $f^{\delta'_d}$ are generically ordinary. Since Le has already showed that $f^{\delta'_d}$ is generically ordinary, his conjecture is then equivalent to the statement that the polynomial f^{δ_d} is also generically ordinary.

FIGURE 4. A face $\tau \subseteq \delta_d$ for $n = 2$

Suppose now $n = 2$, $4 \mid d$ and $p \equiv \frac{d}{2} + 1 \pmod{d}$. Let τ be a 1-dimensional face spanned by $(1, d, 0)$ and $(-1, 0, 0)$ on the two dimensional face δ_d (see Figure 4 for the example). Its matrix M is given by

$$M = \begin{bmatrix} -1 & 1 \\ 0 & d \end{bmatrix}.$$

Clearly $\det M = -d$.

Let $r = (r_1, r_2)$. Solving the system of linear equations

$$M \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \equiv 0 \pmod{1}, r_i \in \mathbb{Q} \cap [0, 1),$$

we have

$$r_1 = r_2 = \frac{k}{d} (0 \leq k \leq d-1).$$

This gives $w(r) = \frac{2k}{d}$ and hence r is stable for even k .

However, since $p \equiv \frac{d}{2} + 1 \pmod{d}$, $\{pr_1\} = \{pr_2\} = \{\frac{(1+d/2)k}{d}\}$, for odd k and $k < \frac{d}{2}$ we have $\{pr_1\} = \{pr_2\} = \frac{k}{d} + \frac{1}{2}$. This shows that $w(r)$ is not stable for odd k and $k < \frac{d}{2}$ and hence δ is not generically ordinary.

Let Π_1 be the sub polytope spanned by $(0, 0, 0)$, $(-1, 0, 0)$, $(1, d, 0)$ and $(1, d-1, 1)$ (please see the figure below for an illustration). Since its face τ is not generically ordinary, Π_1 is also not generically ordinary. By the boundary decomposition theorem ([15], Theorem 5.1, see also Figure 5 for the graph) Δ_d is not generically ordinary. The proof is complete.

□

Acknowledgement. The author wishes to thank Professor Daqing Wan for his instructive suggestions.

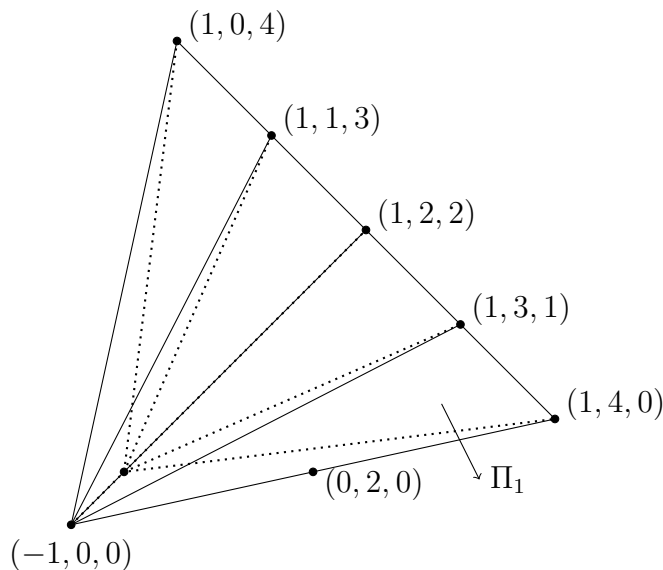


FIGURE 5. A boundary decomposition for the polytope Δ_d , $d = 4$, $n = 2$

REFERENCES

- [1] A. Adolphson and S. Sperber, *Newton Polyhedra and the degree of the L-function associated to an exponential sum*. Invent. Math., 555–569, 1987.
- [2] A. Adolphson and S. Sperber, *Exponential sums and Newton polyhedra: Cohomology and estimates*. Ann. Math., 367–406, 1989.
- [3] T. Browning and R. Heath-Brown, *Integral points on cubic hypersurfaces*, Analytic number theory, Cambridge University Press, 75–90, 2009.
- [4] P. Deligne, *La conjecture de Weil. II. (French) [[Weil’s conjecture. II]]*, Inst. Hautes Études Sci. Publ. Math. No. 52 (1980), 137–252.
- [5] B. Dwork, *On the zeta function of a hypersurface*, Publ. Math. IHES, 12 (1962), 5–68.
- [6] B. Dwork, *Normalized period matrices*, Ann. of Math., 98 (1973), 1–57.
- [7] L. Fu and D. Wan, *On Katz’s (A,B)-exponential sums*, arXiv:2003.08796.
- [8] I. Gelfand, M. Kapranov, and A. Zelevinsky, *Discriminants, Resultants, And Multidimensional Determinants*. Birkhauser Verlag Ag (Switzerland), 1994.
- [9] A. Grothendieck, *Formule de Lefschetz et rationalité des fonctions L*, Séminaire Bourbaki, Vol. 9, Exp. No. 279, 41–55, Soc. Math. France, Paris, 1995.
- [10] N. Katz, *On a question of Browning and Heath-Brown*, Analytic number theory, 267–288, Cambridge Univ. Press, Cambridge, 2009.
- [11] P. Le, *Regular decomposition of ordinarity in generic exponential sums*, J. Number Theory 133 (2013), 2648–2683.
- [12] B. Mazur, *Frobenius and the Hodge Filtration*, Bull. A.M.S., 78 (1972), 653–667.

- [13] C. Niu, *Generic Exponential Sums Associated with Polynomials of Degree 3 in Two Variables*, Acta Mathematica Sinica (English Series), 2301–2312, 2012.
- [14] SageMath, *the Sage Mathematics Software System (Version 8.1)*, The Sage Developers, 2018, <https://www.sagemath.org>.
- [15] D. Wan, *Newton polygons of zeta functions and L-functions*, Ann. of Math, 2 (1993) 249–293.
- [16] D. Wan, *Higher rank case of Dwork’s conjecture*. J. Amer. Math. Soc., 13 (2000) 807–852.
- [17] D. Wan, *Variation of p -adic Newton polygons for L-functions of exponential sums*, Asian Journal of Mathematics, 8 (2004), 427–472.
- [18] H. Zhu, *p -adic variation of L-functions of one variable exponential sums, I*, Amer. J. Math, 125 (2003), 669–690.

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI, P.R. CHINA

Email address: lijyou@sjtu.edu.cn