

Complete weight enumerators for several classes of two-weight and three-weight linear codes

Canze Zhu and Qunying Liao ^{*}

College of Mathematical Science, Sichuan Normal University, Chengdu Sichuan, 610066

Abstract. In this paper, for an odd prime p , by extending Li et al.’s construction [17], several classes of two-weight and three-weight linear codes over the finite field \mathbb{F}_p are constructed from a defining set, and then their complete weight enumerators are determined by using Weil sums. Furthermore, we show that some examples of these codes are optimal or almost optimal with respect to the Griesmer bound. Our results generalize the corresponding results in [15, 17].

Keywords. Linear codes; Complete weight enumerators; Character sums; Weil sums

1 Introduction

Let \mathbb{F}_{p^m} be the finite field with p^m elements and $\mathbb{F}_{p^m}^* = \mathbb{F}_{p^m} \setminus \{0\}$, where p is an odd prime and m is a positive integer. An $[n, k, d]$ linear code over \mathbb{F}_p is a k -dimensional subspace of \mathbb{F}_p^n with minimum distance d . In addition, the weight enumerator and complete weight enumerator are the important parameters for a linear code [21], especially, few-weight linear codes have better applications [1–3, 6, 22]. Motivated by Ding et al.’s work [8], a number of two-weight or three-weight linear codes have been constructed from defining sets [7, 11–17, 19, 20, 23, 24].

In 2015, Ding et al. gave a method to construct a class of two-weight or three-weight linear codes via the trace function from defining sets [8]. Let $D = \{d_1, d_2, \dots, d_n\} \subseteq \mathbb{F}_{p^m}^*$ and Tr_m denote the trace function from \mathbb{F}_{p^m} onto \mathbb{F}_p , a p -ary linear code is defined by

$$\mathcal{C}_D = \{\mathbf{c}(x) = (\text{Tr}_m(xd_1), \text{Tr}_m(xd_2), \dots, \text{Tr}_m(xd_n)) \mid x \in \mathbb{F}_p^m\}.$$

Motivated by the above construction, Li et al. defined a linear code

$$\mathcal{C}_{\bar{D}} = \{(a, b) = (\text{Tr}_m(ax + by))_{(x,y) \in D} \mid (a, b) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}\} \quad (1.1)$$

with $\bar{D} \subseteq \mathbb{F}_{p^m}^2$ [17]. Later, Jian et al. obtained several classes of two-weight and three-weight linear codes $\mathcal{C}_{\bar{D}}$ from (1.1) by choosing the defining set

$$\bar{D} = \{(x, y) \in \mathbb{F}_{p^m}^2 \mid \text{Tr}_m(x^2 + y^{p^u+1}) = 0\}, \quad (1.2)$$

^{*}Corresponding author.

E-mail. qunyingliao@sicnu.edu.cn (Q. Liao), canzezhu@163.com (C. Zhu).

Supported by National Natural Science Foundation of China (Grant No. 12071321).

where u is a positive integer [15].

In this paper, we define a linear code

$$\mathcal{C}_{D_\lambda} = \left\{ (\text{Tr}_{m_1}(ax) + \text{Tr}_{m_2}(by))_{(x,y) \in D_\lambda} \mid (a, b) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} \right\} \quad (1.3)$$

with

$$D_\lambda = \left\{ (x, y) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} \setminus \{(0, 0)\} \mid \text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^u+1}) = \lambda \right\}, \quad (1.4)$$

where $\lambda \in \mathbb{F}_p$, m_1 and m_2 are positive integers. We determine the parameters and the complete weight enumerators of \mathcal{C}_{D_λ} basing on Weil sums. In addition, for some examples, \mathcal{C}_{D_λ} is optimal or almost optimal with respect to the Griesmer bound [9]. Obviously, if $m_1 = m_2$ and $\lambda = 0$, then $D_\lambda = \bar{D}$ and $\mathcal{C}_{D_\lambda} = \mathcal{C}_{\bar{D}}$. Thus, we extend Li et al.'s construction [17], and generalize the corresponding results in [15, 17].

This paper is organized as follows. In section 2, some related basic notations and results of Weil sums are given. In section 3, the complete weight enumerators of several classes of two-weight and three-weight linear codes are presented. In section 4, the proofs of the main results are given. In section 5, some examples are obtained by using Magma, which are accordant with the main results. In section 6, we conclude the whole paper.

2 Preliminaries

Throughout the paper, we denote some notations as follows.

- $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is a primitive p -th root of the unity.
- u , m_1 and m_2 are positive integers, and $s = \frac{m_2}{2}$, $v = \gcd(m_2, u)$.
- $K = m_1 + m_2$.
- $L = (-1)^{\frac{(p-1)^2}{8}}$.
- For each $b \in \mathbb{F}_{p^m}$, $\chi_b(x) = \zeta_p^{\text{Tr}_m(bx)}$ ($x \in \mathbb{F}_{p^m}$) is the additive characters.
- η_m is the quadratic characters of \mathbb{F}_{p^m} , and it is extended by letting $\eta_m(0) = 0$.
- G_m is the quadratic Gauss sums over \mathbb{F}_{p^m} , i.e., $G_m = \sum_{c \in \mathbb{F}_{p^m}} \eta_m(c) \chi_1(c)$.

2.1 Group characters and Gauss sums

In this subsection, some properties for the additive characters, quadratic characters and quadratic Gauss sums are given.

Lemma 2.1 ([8, 18]) $G_m = (-1)^{m-1} L^m p^{\frac{m}{2}}$, and for $b \in \mathbb{F}_{p^m}$,

$$\sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_m(bx)} = \begin{cases} p^m, & b = 0; \\ 0, & \text{otherwise}; \end{cases}$$

for $x \in \mathbb{F}_p^*$,

$$\eta_m(x) = \begin{cases} 1, & 2 \mid m; \\ \eta_1(x), & \text{otherwise.} \end{cases}$$

2.2 Weil sums

It is well known that Weil sums are defined by $\sum_{x \in \mathbb{F}_{p^{m_2}}} \chi(f(x))$ with $f(x) \in \mathbb{F}_{p^{m_2}}[x]$, and there are many results for the Weil sum [4, 5]

$$S_{m_2,u}(a, b) = \sum_{x \in \mathbb{F}_{p^{m_2}}} \chi(ax^{p^u+1} + bx) \quad (a \in \mathbb{F}_{p^{m_2}}^*, b \in \mathbb{F}_{p^{m_2}}).$$

Lemma 2.2 If $\frac{m_2}{v}$ is odd, then

$$S_{m_2,u}(a, 0) = G_{m_2} \eta_{m_2}(a).$$

If $\frac{m_2}{v}$ is even, then

$$S_{m_2,u}(a, 0) = \begin{cases} p^s, & \text{if } \frac{s}{v} \text{ is even and } a^{\frac{p^{m_2}-1}{p^v+1}} \neq (-1)^{\frac{s}{v}}; \\ -p^{s+v}, & \text{if } \frac{s}{v} \text{ is even and } a^{\frac{p^{m_2}-1}{p^v+1}} = (-1)^{\frac{s}{v}}; \\ -p^s, & \text{if } \frac{s}{v} \text{ is odd and } a^{\frac{p^{m_2}-1}{p^v+1}} \neq (-1)^{\frac{s}{v}}; \\ p^{s+v}, & \text{if } \frac{s}{v} \text{ is odd and } a^{\frac{p^{m_2}-1}{p^v+1}} = (-1)^{\frac{s}{v}}. \end{cases}$$

Lemma 2.3 Fixed $a \in \mathbb{F}_{p^{m_2}}$, then the equation

$$a^{p^u} X^{p^{2u}} + aX = 0$$

is solvable in $\mathbb{F}_{p^{m_2}}^*$ if and only if both $\frac{m_2}{v}$ is even and $a^{\frac{p^{m_2}-1}{p^v+1}} = (-1)^{\frac{s}{v}}$. Furthermore, there are exactly $p^{2v} - 1$ non-zero solutions in this case.

Remark 2.1 By Lemma 2.3, it is easy to see that $f(X) = a^{p^u} X^{p^{2u}} + aX$ is a permutation polynomial over $\mathbb{F}_{p^{m_2}}^*$ if and only if $\frac{m_2}{v}$ is odd, or both $\frac{m_2}{v}$ is even and $a^{\frac{p^{m_2}-1}{p^v+1}} \neq (-1)^{\frac{s}{v}}$.

Lemma 2.4 Suppose that $f(X) = a^{p^u} X^{p^{2u}} + aX$ is a permutation polynomial over $\mathbb{F}_{p^{m_2}}$, then, $f(X) = -b^{p^u}$ has an unique solution in $\mathbb{F}_{p^{m_2}}$. Furthermore,

$$S_{m_2,u}(a, b) = \begin{cases} G_{m_2} \eta_{m_2}(a) \zeta_p^{\text{Tr}_{m_2}(-ax_0^{p^u+1})}, & \text{if } \frac{m_2}{v} \text{ is odd;} \\ (-1)^{\frac{s}{v}} p^s \zeta_p^{\text{Tr}_{m_2}(-ax_0^{p^u+1})}, & \text{if } \frac{m_2}{v} \text{ is even.} \end{cases}$$

Lemma 2.5 For the non-permutation polynomial $f(X) = a^{p^u}X^{p^{2u}} + aX$ over $\mathbb{F}_{p^{m_2}}$, suppose that the equation $f(X) = -b^{p^u}$ has a solution x_0 in $\mathbb{F}_{p^{m_2}}$, then,

$$S_{m_2,u}(a, b) = -(-1)^{\frac{s}{v}} p^{s+v} \zeta_p^{\text{Tr}_{m_2}(-ax_0^{p^u+1})},$$

otherwise,

$$S_{m_2,u}(a, b) = 0.$$

Taking $u = 0$ in Lemmas 2.2 and 2.4, we can get

Lemma 2.6 For $a \in \mathbb{F}_{p^{m_1}}^*$ and $b \in \mathbb{F}_{p^{m_1}}$,

$$Q_{m_1}(a, b) = \sum_{x \in \mathbb{F}_p^{m_1}} \zeta_p^{\text{Tr}_{m_1}(ax^2+bx)} = G_{m_1} \eta_{m_1}(a) \zeta_p^{\text{Tr}_{m_1}(-\frac{b^2}{4a})}.$$

In order to prove our main results, we need the evaluation of $S_{m_2,u}(z_1, z_2b)$, where $z_1, z_2 \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_{p^{m_2}}$. The following lemma is necessary.

Lemma 2.7 ([15], Lemma 10) If $z \in \mathbb{F}_p^*$ and $\frac{m_2}{v}$ is even, then

$$z^{\frac{p^{m_2}-1}{p^v+1}} = 1.$$

By Lemmas 2.3 and 2.7, the equation

$$X^{p^{2u}} + X = -b^{p^u} \tag{2.1}$$

is not always solvable in $\mathbb{F}_{p^{m_2}}$ when $\frac{m_2}{v} \equiv 0 \pmod{4}$ and has a unique solution otherwise. Now, suppose that (2.1) has a solution $\gamma_b \in \mathbb{F}_{p^{m_2}}$, then, $\frac{z_2}{z_1}\gamma_b$ is a solution of the equation

$$z_1^{p^u} X^{p^{2u}} + z_1 X = -(z_2 b)^{p^u}. \tag{2.2}$$

Thus, by Lemmas 2.3-2.6, the evaluation of $S_{m_2,u}(z_1, z_2b)$ is given in the following

Lemma 2.8 For $z_1, z_2 \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_{p^{m_2}}$,

$$S_{m_2,u}(z_1, z_2b) = \begin{cases} G_{m_2} \eta_{m_2}(z_1) \zeta_p^{-\frac{z_2^2}{z_1} \text{Tr}_{m_2}(\gamma_b^{p^u+1})}, & \text{if } \frac{m_2}{v} \text{ is odd;} \\ -p^s \zeta_p^{-\frac{z_2^2}{z_1} \text{Tr}_{m_2}(\gamma_b^{p^u+1})}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4}; \\ -p^{s+v} \zeta_p^{-\frac{z_2^2}{z_1} \text{Tr}_{m_2}(\gamma_b^{p^u+1})}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and (2.1) is solvable;} \\ 0, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and (2.1) is not solvable.} \end{cases}$$

Especially, for $b = 0$, $\gamma_0 = 0$ is a solution of (2.2). Thus, we have the following

Lemma 2.9 For $z_1 \in \mathbb{F}_p^*$,

$$S_{m_2,u}(z_1, 0) = \begin{cases} G_{m_2} \eta_{m_2}(z_1), & \text{if } \frac{m_2}{v} \text{ is odd;} \\ -p^s, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4}; \\ -p^{s+v}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4}. \end{cases}$$

2.3 The Pless power moments

The following lemma is necessary to calculate the weight enumerator of \mathcal{C}_{D_λ} .

Lemma 2.10 ([10], p.259, The Pless power moments) *For an $[n, k, d]$ code \mathcal{C} over \mathbb{F}_p with weight distribution $(1, A_1, \dots, A_n)$, suppose that the weight distribution of its dual code is $(1, A_1^\perp, \dots, A_n^\perp)$, then the first two Pless power moments are*

$$\sum_{j=0}^n A_j = p^k$$

and

$$\sum_{j=0}^n jA_j = p^{k-1}(pn - n - A_1^\perp).$$

For C_{D_λ} defined by (1.3), if $(0, 0) \notin D_\lambda$, by the nondegenerate property of the trace function, one has $A_1^\perp = 0$.

3 Main results

In this subsection, for D_λ and \mathcal{C}_{D_λ} given by (1.4) and (1.3), respectively, we present the complete weight enumerators of \mathcal{C}_{D_λ} by classifying $\lambda = 0$ or not, m_1 is odd or even, and $\frac{m_2}{v} \pmod{4}$. Furthermore, for any given $c \in \mathbb{F}_p^*$,

$$\text{Tr}_{m_1}((cx)^2) + \text{Tr}_{m_2}((cy)^{p^u+1}) = c^2(\text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^u+1})),$$

hence, D_0 can be expressed as

$$D_0 = \cup_{c \in \mathbb{F}_p^*} \tilde{D}_0 \quad (3.1)$$

with $\tilde{D}_0 \subsetneq D_0$. Thus, $\mathcal{C}_{\tilde{D}_0}$ defined by (1.3) is just the punctured version of \mathcal{C}_{D_0} .

For $\lambda \in \mathbb{F}_p^*$, since

$$\text{Tr}_{m_1}((-x)^2) + \text{Tr}_{m_2}((-y)^{p^u+1}) = \text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^u+1}),$$

and then,

$$D_\lambda = \cup_{c \in \mathbb{F}_p^*} \tilde{D}_\lambda \quad (3.2)$$

with $\tilde{D}_\lambda \subsetneq D_\lambda$. Thus, $\mathcal{C}_{\tilde{D}_\lambda}$ defined by (1.3) is just the punctured version of \mathcal{C}_{D_λ} .

The parameters of \mathcal{C}_{D_λ} and $\mathcal{C}_{\tilde{D}_\lambda}$ are given in the following theorems.

Theorem 3.1 If $\frac{m_2}{v}$ and K are both odd, or $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is odd, then \mathcal{C}_{D_0} is a $[p^{K-1}-1, K]$ code with weight enumerator in Table 1, and the complete weight enumerator is

$$\begin{aligned} W(\mathcal{C}_{D_0}) = & w_0^{p^{K-1}-1} + (p^{K-1}-1)w_0^{p^{K-2}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\ & + \frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + 1) w_0^{p^{K-2} + (p-1)p^{\frac{K-3}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2} - p^{\frac{K-3}{2}}} \\ & + \frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - 1) w_0^{p^{K-2} - (p-1)p^{\frac{K-3}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2} + p^{\frac{K-3}{2}}}. \end{aligned} \quad (3.3)$$

Table 1 The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$(p^{K-1}-1)$
$(p-1)(p^{K-2} - p^{\frac{K-3}{2}})$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + 1)$
$(p-1)(p^{K-2} + p^{\frac{K-3}{2}})$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - 1)$

Furthermore, $\mathcal{C}_{\tilde{D}_0}$ is a $[\frac{p^{K-1}-1}{p-1}, K]$ code with weight enumerator in Table 1°.

Table 1° The weight enumerator of $\mathcal{C}_{\tilde{D}_0}$

weight w	frequency A_w
0	1
p^{K-2}	$(p^{K-1}-1)$
$p^{K-2} - p^{\frac{K-3}{2}}$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + 1)$
$p^{K-2} + p^{\frac{K-3}{2}}$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - 1)$

Theorem 3.2 If $\frac{m_2}{v}$ is odd and K is even, then \mathcal{C}_{D_0} is a $[p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1, K]$ code with weight enumerator in Table 2, and the complete weight enumerator is

$$\begin{aligned} W(\mathcal{C}_{D_0}) = & w_0^{p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1} + (p-1)p^{\frac{K-2}{2}} (p^{\frac{K}{2}} - L^K) w_0^{p^{K-2}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{\frac{K-2}{2}} (p^{\frac{K-2}{2}} + L^K)} \\ & + (p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1) w_0^{p^{K-2} + L^K(p-1)p^{\frac{K-2}{2}} - 1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}}. \end{aligned} \quad (3.4)$$

Table 2 The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$(p-1)p^{\frac{K-2}{2}} (p^{\frac{K-2}{2}} + L^K)$	$(p-1)p^{\frac{K-2}{2}} (p^{\frac{K}{2}} - L^K)$
$(p-1)p^{K-2}$	$p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1$

Furthermore, $\mathcal{C}_{\tilde{D}_0}$ is a $[\frac{p^{K-1}-1}{p-1} + L^K p^{\frac{K-2}{2}}, K]$ code with weight enumerator in Table 2°.

Table 2° The weight enumerator of $\mathcal{C}_{\tilde{D}_0}$

weight w	frequency A_w
0	1
$p^{\frac{K-2}{2}}(p^{\frac{K-2}{2}} + L^K)$	$(p-1)p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^K)$
p^{K-2}	$p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1$

Theorem 3.3 If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is even, then \mathcal{C}_{D_0} is a $[p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1, K]$ code with weight enumerator in Table 3, and the complete weight enumerator is

$$\begin{aligned} & W(\mathcal{C}_{D_0}) \\ &= w_0^{p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1} + (p-1)p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^{m_1})w_0^{p^{K-2}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{\frac{K-2}{2}}(p^{\frac{K-2}{2}} + L^{m_1})} \\ &+ (p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1)w_0^{p^{K-2} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}}. \end{aligned} \quad (3.5)$$

Table 3 The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$(p-1)p^{\frac{K-2}{2}}(p^{\frac{K-2}{2}} + L^{m_1})$	$(p-1)p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^{m_1})$
$(p-1)p^{K-2}$	$p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1$

Furthermore, $\mathcal{C}_{\tilde{D}_0}$ is a $[p^{\frac{K-1}{2}-1} + L^{m_1}p^{\frac{K-2}{2}}, K]$ code with weight enumerator in Table 3°.

Table 3° The weight enumerator of $\mathcal{C}_{\tilde{D}_0}$

weight w	frequency A_w
0	1
$p^{\frac{K-2}{2}}(p^{\frac{K-2}{2}} + L^{m_1})$	$(p-1)p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^{m_1})$
p^{K-2}	$p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1$

Theorem 3.4 If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, then \mathcal{C}_{D_0} is a $[p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}+v} - 1, K]$ code with weight enumerator in Table 4, and the complete weight enumerator is

$$\begin{aligned} & W(\mathcal{C}_{D_0}) \\ &= w_0^{p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}+v}-1} + (p-1)p^{\frac{K-2}{2}-v}(p^{\frac{K}{2}-v} - L^{m_1})w_0^{p^{K-2}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2} + L^{m_1}p^{\frac{K-2}{2}+v}} \\ &+ (p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v} - 1)w_0^{p^{K-2} + L^{m_1}(p-1)p^{\frac{K}{2}+v}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\ &+ p^K(1 - p^{-2v})w_0^{p^{K-2} + L^{m_1}(p-1)p^{\frac{K-4}{2}+v}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2} + L^{m_1}(p-1)p^{\frac{K-4}{2}+v}}. \end{aligned} \quad (3.6)$$

Table 4 The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$(p-1)(p^{K-2} + L^{m_1} p^{\frac{K-2}{2}+v})$	$(p-1)p^{\frac{K-2}{2}-v}(p^{\frac{K}{2}-v} - L^{m_1})$
$(p-1)p^{K-2}$	$p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v} - 1$
$(p-1)(p^{K-2} + L^{m_1}(p-1)p^{\frac{K-4}{2}+v})$	$p^K(1 - p^{-2v})$

Furthermore, $\mathcal{C}_{\tilde{D}_0}$ is a $[\frac{p^{K-1}-1}{p-1} + L^{m_1} p^{\frac{K-2}{2}+v}, K]$ code with weight enumerator in Table 4°.

Table 4° The weight enumerator of $\mathcal{C}_{\tilde{D}_0}$

weight w	frequency A_w
0	1
$p^{K-2} + L^{m_1} p^{\frac{K-2}{2}+v}$	$(p-1)p^{\frac{K-2}{2}-v}(p^{\frac{K}{2}-v} - L^{m_1})$
p^{K-2}	$p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v} - 1$
$p^{K-2} + L^{m_1}(p-1)p^{\frac{K-4}{2}+v}$	$p^K(1 - p^{-2v})$

Theorem 3.5 If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is odd, then \mathcal{C}_{D_0} is a $[p^{K-1} - 1, K]$ code with weight enumerator in Table 5, and the complete weight enumerator is

$$\begin{aligned}
 & W(\mathcal{C}_{D_0}) \\
 &= w_0^{p^{K-1}-1} + (p^K - (p-1)p^{K-2v-1} - 1)w_0^{p^{K-2}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\
 &+ \frac{p-1}{2} (p^{K-2v-1} - L^{m_1+1} p^{\frac{K-1}{2}-v}) w_0^{p^{K-2}-(p-1)p^{\frac{K-3}{2}+v}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}+p^{\frac{K-3}{2}+v}} \quad (3.7) \\
 &+ \frac{p-1}{2} (p^{K-2v-1} + L^{m_1+1} p^{\frac{K-1}{2}-v}) w_0^{p^{K-2}+(p-1)p^{\frac{K-3}{2}+v}-1} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}-p^{\frac{K-3}{2}+v}}.
 \end{aligned}$$

Table 5 The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$p^K - (p-1)p^{K-2v-1} - 1$
$(p-1)(p^{K-2} + p^{\frac{K-3}{2}+v})$	$\frac{p-1}{2}(p^{K-2v-1} - \eta_1(-1)L^{m_1+1} p^{\frac{K-1}{2}-v})$
$(p-1)(p^{K-2} - p^{\frac{K-3}{2}+v})$	$\frac{p-1}{2}(p^{K-2v-1} + \eta_1(-1)L^{m_1+1} p^{\frac{K-1}{2}-v})$

Furthermore, $\mathcal{C}_{\tilde{D}_0}$ is a $[\frac{p^{K-1}-1}{p-1}, K]$ code with weight enumerator in Table 5°.

Table 5° The weight enumerator of $\mathcal{C}_{\tilde{D}_0}$

weight w	frequency A_w
0	1
p^{K-2}	$p^K - (p-1)p^{K-2v-1} - 1$
$p^{K-2} + p^{\frac{K-3}{2}+v}$	$\frac{p-1}{2}(p^{K-2v-1} - \eta_1(-1)L^{m_1+1} p^{\frac{K-1}{2}-v})$
$p^{K-2} - p^{\frac{K-3}{2}+v}$	$\frac{p-1}{2}(p^{K-2v-1} + \eta_1(-1)L^{m_1+1} p^{\frac{K-1}{2}-v})$

Theorem 3.6 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v}$ and K are both odd, then \mathcal{C}_{D_λ} is a $[p^{K-1} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-1}{2}}, K]$ code with weight enumerator in Table 6, and the complete weight enumerator is

$$\begin{aligned} W(\mathcal{C}_{D_\lambda}) &= w_0^{p^{K-1} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-1}{2}}} + (p^{K-1} - 1)w_0^{p^{K-2} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-1}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\ &\quad + \frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + \eta_1(-\lambda)L^{K+1}) \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-3}{2}}} \\ &\quad + p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - \eta_1(-\lambda)L^{K+1}) \sum_{\substack{j \in \mathbb{F}_p^* \\ \eta_1(j) = \eta_1(\lambda)}} w_{i=\pm\sqrt{4\lambda j}}^{p^{K-2} - \eta_1(-\lambda)L^{K+1}(p-1)p^{\frac{K-3}{2}}} \prod_{\substack{i \in \mathbb{F}_p \\ i \neq \pm\sqrt{4\lambda j}}} w_i^{p^{K-2} + \eta_1(-\lambda)L^{K+1}p^{\frac{K-3}{2}}}. \end{aligned} \quad (3.8)$$

Table 6 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$(p^{K-1}-1)$
$(p-1)(p^{K-2} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-3}{2}})$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + \eta_1(-\lambda)L^{K+1})$
$(p-1)p^{K-2} - \eta_1(-\lambda)(p+1)L^{K+1}p^{\frac{K-3}{2}}$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - \eta_1(-\lambda)L^{K+1})$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-1}{2}}), K]$ code with weight enumerator in Table 6°.

Table 6° The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$\frac{p-1}{2} p^{K-2}$	$(p^{K-1}-1)$
$\frac{p-1}{2} (p^{K-2} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-3}{2}})$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + \eta_1(-\lambda)L^{K+1})$
$\frac{p-1}{2} p^{K-2} - \frac{p+1}{2} \eta_1(-\lambda)L^{K+1}p^{\frac{K-3}{2}}$	$\frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - \eta_1(-\lambda)L^{K+1})$

Theorem 3.7 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v}$ is odd and K is even, then \mathcal{C}_{D_λ} is a $[p^{K-1} - L^K p^{\frac{K-2}{2}}, K]$ code with weight enumerator in Table 7, and the complete weight enumerator is

$$\begin{aligned} W(\mathcal{C}_{D_\lambda}) &= w_0^{p^{K-1} - L^K p^{\frac{K-2}{2}}} + (p^{K-1} + (p-1)L^K p^{\frac{K-2}{2}} - 1)w_0^{p^{K-2} - L^K p^{\frac{K-2}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{\frac{K-2}{2}}} \\ &\quad + (p^{K-1} - L^K p^{\frac{K-2}{2}}) \sum_{j \in \mathbb{F}_p^*} \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - \eta_1(i^2 - 4\lambda j)L^K p^{\frac{K-2}{2}}}. \end{aligned} \quad (3.9)$$

Table 7 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$\frac{p+1}{2} p^{K-1} + \frac{p-1}{2} L^K p^{\frac{K-2}{2}} - 1$
$(p-1)p^{K-2} - 2L^K p^{\frac{K-2}{2}}$	$\frac{p-1}{2} p^{\frac{K-2}{2}} (p^{\frac{K}{2}} - L^K)$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - L^K p^{\frac{K-2}{2}}), K]$ code with weight enumerator in Table 7°.

Table 7° The weight enumerator of $\mathcal{C}_{\tilde{D}_\lambda}$

weight w	frequency A_w
0	1
$\frac{p-1}{2}p^{K-2}$	$\frac{p+1}{2}p^{K-1} + \frac{p-1}{2}L^K p^{\frac{K-2}{2}} - 1$
$\frac{p-1}{2}p^{K-2} - L^K$	$\frac{p-1}{2}p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^K)$

Theorem 3.8 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v} \equiv 2 \pmod{4}$ and K is odd, then \mathcal{C}_{D_λ} is a $[p^{K-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}, K]$ code with weight enumerator in Table 8, and the complete weight enumerator is

$$\begin{aligned}
& W(\mathcal{C}_{D_\lambda}) \\
&= w_0^{p^{K-1}-\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}} + (p^{K-1} - 1)w_0^{p^{K-2}-\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\
&\quad + \frac{p-1}{2}p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} - L^{m_1+1}(p-1)) \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2}-\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}} \\
&\quad + p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} + L^{m_1+1}(p-1)) \sum_{\substack{j \in \mathbb{F}_p^* \\ \eta_1(j) = \eta_1(\lambda)}} w_{i=\pm\sqrt{4\lambda j}}^{p^{K-2}-\eta_1(-\lambda)L^{m_1+1}(p-1)p^{\frac{K-3}{2}}} \prod_{\substack{i \in \mathbb{F}_p^* \\ i \neq \pm\sqrt{4\lambda j}}} w_i^{p^{K-2}+\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}}}. \tag{3.10}
\end{aligned}$$

Table 8 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$(p^{K-1} - 1)$
$(p-1)(p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}})$	$\frac{p-1}{2}p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} - L^{m_1+1}(p-1))$
$(p-1)p^{K-2} - \eta_1(-\lambda)(p+1)L^{m_1+1}p^{\frac{K-3}{2}}$	$\frac{p-1}{2}p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} + L^{m_1+1}(p-1))$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}), K]$ code with weight enumerator in Table 8°.

Table 8° The weight enumerator of $\mathcal{C}_{\tilde{D}_\lambda}$

weight w	frequency A_w
0	1
$\frac{p-1}{2}p^{K-2}$	$(p^{K-1} - 1)$
$\frac{p-1}{2}(p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}})$	$\frac{p-1}{2}p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} - L^{m_1+1}(p-1))$
$\frac{p-1}{2}p^{K-2} + \eta_1(-\lambda)\frac{p+1}{2}L^{m_1+1}p^{\frac{K-3}{2}}$	$\frac{p-1}{2}p^{\frac{K-1}{2}}(p^{\frac{K-1}{2}} + L^{m_1+1}(p-1))$

Theorem 3.9 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v} \equiv 2 \pmod{4}$ and K is even, then \mathcal{C}_{D_λ} is a $[p^{K-1} - L^{m_1}p^{\frac{K-2}{2}}, K]$ code with weight enumerator in Table 9, and the complete weight enumerator

is

$$\begin{aligned}
W(\mathcal{C}_{D_\lambda}) = & w_0^{p^{K-1} - L^{m_1} p^{\frac{K-2}{2}}} + (p^{K-1} + (p-1)L^{m_1} p^{\frac{K-2}{2}} - 1)w_0^{p^{K-2} - L^{m_1} p^{\frac{K-2}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{\frac{K-2}{2}}} \\
& + (p^{K-1} - L^{m_1} p^{\frac{K-2}{2}}) \sum_{j \in \mathbb{F}_p^*} \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - \eta_1(i^2 - 4\lambda j)L^{m_1} p^{\frac{K-2}{2}}}.
\end{aligned} \tag{3.11}$$

Table 9 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$\frac{p+1}{2}p^{K-1} + \frac{p-1}{2}L^{m_1} p^{\frac{K-2}{2}} - 1$
$(p-1)p^{K-2} - 2L^{m_1} p^{\frac{K-2}{2}}$	$\frac{p-1}{2}p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^{m_1})$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - L^{m_1} p^{\frac{K-2}{2}}), K]$ code with weight enumerator in Table 9°.

Table 9° The weight enumerator of $\mathcal{C}_{\tilde{D}_\lambda}$

weight w	frequency A_w
0	1
$\frac{p-1}{2}p^{K-2}$	$\frac{p+1}{2}p^{K-1} + \frac{p-1}{2}L^{m_1} p^{\frac{K-2}{2}} - 1$
$\frac{p-1}{2}p^{K-2} - L^{m_1} p^{\frac{K-2}{2}}$	$\frac{p-1}{2}p^{\frac{K-2}{2}}(p^{\frac{K}{2}} - L^{m_1})$

Theorem 3.10 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, then \mathcal{C}_{D_λ} is a $[p^{K-1} - L^{m_1} p^{\frac{K-2}{2}+v}, K]$ code with weight enumerator in Table 10, and the complete weight enumerator is

$$\begin{aligned}
W(\mathcal{C}_{D_\lambda}) = & w_0^{p^{K-1} - L^{m_1} p^{\frac{K-2}{2}+v}} + (p^K - p^{K-2v}) \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - L^{m_1} p^{\frac{K-4}{2}+v}} \\
& + (p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v} - 1)w_0^{p^{K-2} - L^{m_1} p^{\frac{K-2}{2}+v}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\
& (p^{K-2v-1} - L^{m_1} p^{\frac{K-2}{2}-v}) \sum_{j \in \mathbb{F}_p^*} \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - \eta_1(i^2 - 4\lambda j)L^K p^{\frac{K-2}{2}}}.
\end{aligned} \tag{3.12}$$

Table 10 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$\frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}L^{m_1} p^{\frac{K-2}{2}-v} - 1$
$(p-1)(p^{K-2} - L^{m_1} p^{\frac{K-4}{2}+v})$	$p^K - p^{K-2v}$
$(p-1)p^{K-2} - 2L^{m_1} p^{\frac{K-2}{2}+v}$	$\frac{p-1}{2}(p^{K-2v-1} - L^{m_1} p^{\frac{K-2}{2}-v})$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - L^{m_1} p^{\frac{K-2}{2}+v}), K]$ code with weight enumerator in Table 10°.

Table 10° The weight enumerator of $\mathcal{C}_{\tilde{D}_\lambda}$.

weight w	frequency A_w
0	1
$\frac{p-1}{2}p^{K-2}$	$\frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}L^{m_1}p^{\frac{K-2}{2}-v} - 1$
$\frac{p-1}{2}(p^{K-2} - L^{m_1}p^{\frac{K-4}{2}+v})$	$p^K - p^{K-2v}$
$\frac{p-1}{2}p^{K-2} - L^{m_1}p^{\frac{K-2}{2}+v}$	$\frac{p-1}{2}(p^{K-2v-1} - L^{m_1}p^{\frac{K-2}{2}-v})$

Theorem 3.11 For $\lambda \in \mathbb{F}_p^*$, if $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is odd, then \mathcal{C}_{D_λ} is a $[p^{K-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}+v}, K]$ code with weight enumerator in Table 11, and the complete weight enumerator is

$$\begin{aligned}
& W(\mathcal{C}_{D_\lambda}) \\
&= w_0^{p^{K-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}+v}} + (p^{K-2v-1} - 1)w_0^{p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{K-2}} \\
&+ (p^K - \frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}-v}) \prod_{i \in \mathbb{F}_p} w_i^{p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}}} \\
&+ (p^{K-2v-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}-v}) \sum_{\substack{j \in \mathbb{F}_p^* \\ \eta_1(j) = \eta_1(\lambda)}} w_{i=\pm\sqrt{4\lambda j}}^{p^{K-2} - \eta_1(-\lambda)L^{m_1+1}(p-1)p^{\frac{K-1}{2}+v}} \prod_{\substack{i \in \mathbb{F}_p \\ i \neq \pm\sqrt{4\lambda j}}} w_i^{p^{K-2} + \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v}}.
\end{aligned} \tag{3.13}$$

Table 11 The weight enumerator of \mathcal{C}_{D_λ}

weight w	frequency A_w
0	1
$(p-1)p^{K-2}$	$p^{K-2v-1} - 1$
$(p-1)(p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v})$	$p^K - \frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}-v}$
$(p-1)p^{K-2} - (p+1)\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v}$	$\frac{p-1}{2}(p^{K-2v-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}-v})$

Furthermore, $\mathcal{C}_{\tilde{D}_\lambda}$ is a $[\frac{1}{2}(p^{K-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}+v}), K]$ code with weight enumerator in Table 11°.

Table 11° The weight enumerator of \mathcal{C}_{D_0}

weight w	frequency A_w
0	1
$\frac{p-1}{2}p^{K-2}$	$p^{K-2v-1} - 1$
$\frac{p-1}{2}(p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v})$	$p^K - \frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}-v}$
$\frac{1}{2}((p-1)p^{K-2} - (p+1)\eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v})$	$\frac{p-1}{2}(p^{K-2v-1} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}-v})$

4 Proofs of main results

4.1 Some auxiliary lemmas

Lemmas 4.1-4.2 are useful for calculating the length and the weights for \mathcal{C}_{D_λ} .

Lemma 4.1 For $\lambda \in \mathbb{F}_p$ and

$$N_\lambda = \#\{(x, y) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} \setminus \{(0, 0)\} \mid \text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^{u+1}}) = \lambda\},$$

the following assertions hold.

(1) For $\lambda = 0$,

$$N_\lambda = \begin{cases} p^{K-1} - 1, & \text{if } K \text{ is odd;} \\ p^{K-1} + (p-1)L^K p^{\frac{K-2}{2}} - 1, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} + (p-1)L^{m_1} p^{\frac{K-2}{2}} - 1, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even;} \\ p^{K-1} + (p-1)L^{m_1} p^{\frac{K-2}{2}+v} - 1, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \quad (4.1)$$

(2) For $\lambda \neq 0$,

$$N_\lambda = \begin{cases} p^{K-1} - \eta_1(-\lambda)L^{K+1} p^{\frac{K-1}{2}}, & \text{if } \frac{m_2}{v} \text{ and } K \text{ are both odd;} \\ p^{K-1} - L^K p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} - \eta_1(-\lambda)L^{m_1+1} p^{\frac{K-1}{2}}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} - L^{m_1} p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even;} \\ p^{K-1} - L^{m_1} p^{\frac{K-2}{2}+v}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} - \eta_1(-\lambda)L^{m_1+1} p^{\frac{K-1}{2}}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \quad (4.2)$$

Proof. It follows from Lemma 2.9 that

$$\begin{aligned} N &= \sum_{x \in \mathbb{F}_{p^{m_1}}} \sum_{y \in \mathbb{F}_{p^{m_2}}} \left(p^{-1} \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1(\text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^{u+1}}) - \lambda)} \right) \\ &= p^{K-1} - 1 + p^{-1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \sum_{x \in \mathbb{F}_{p^{m_1}}} \zeta_p^{\text{Tr}_{m_1}(z_1 x^2)} \sum_{y \in \mathbb{F}_{p^{m_2}}} \zeta_p^{\text{Tr}_{m_2}(z_1 y^{p^{u+1}})} \\ &= p^{K-1} + p^{-1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} Q_{m_1}(z_1, 0) S_{m_2, u}(z_1, 0) \\ &= p^{K-1} + p^{-1} G_{m_1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) S_{m_2, u}(z_1, 0) \\ &= \begin{cases} p^{K-1} + p^{-1} G_{m_1} G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) \eta_{m_2}(z_1), & \text{if } \frac{m_2}{v} \text{ is odd;} \\ p^{K-1} - p^{-1} G_{m_1} p^s \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4}; \\ p^{K-1} - p^{-1} G_{m_1} p^{s+v} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

If $\lambda = 0$, then

$$\begin{aligned}
N_\lambda &= N - 1 \\
&= \begin{cases} p^{K-1} - 1 + p^{-1}G_{m_1}G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \eta_{m_1}(z_1)\eta_{m_2}(z_1), & \text{if } \frac{m_2}{v} \text{ is odd;} \\ p^{K-1} - 1 - p^{-1}G_{m_1}p^s \sum_{z_1 \in \mathbb{F}_p^*} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4}; \\ p^{K-1} - 1 - p^{-1}G_{m_1}p^{s+v} \sum_{z_1 \in \mathbb{F}_p^*} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4}; \end{cases} \\
&= \begin{cases} p^{K-1} - 1, & \text{if } K \text{ is odd;} \\ p^{K-1} + (p-1)p^{-1}G_{m_1}G_{m_2} - 1, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} - (p-1)p^{s-1}G_{m_1} - 1, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even;} \\ p^{K-1} - (p-1)p^{s+v-1}G_{m_1} - 1, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \tag{4.3}
\end{aligned}$$

If $\lambda \neq 0$, then

$$\begin{aligned}
N_\lambda &= \begin{cases} p^{K-1} + p^{-1}G_{m_1}G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1)\eta_1(z_1), & \text{if } m_2 \text{ is odd;} \\ p^{K-1} + p^{-1}G_{m_1}G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if } m_2 \text{ is even and } \frac{m_2}{v} \text{ is odd;} \\ p^{K-1} - p^{-1}G_{m_1}p^s \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4}; \\ p^{K-1} - p^{-1}G_{m_1}p^{s+v} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4}; \end{cases} \\
&= \begin{cases} p^{K-1} + p^{-1}G_{m_1}G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1), & \text{if } \frac{m_2}{v} \text{ and } K \text{ are both odd;} \\ p^{K-1} + p^{-1}G_{m_1}G_{m_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1}, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} - p^{-1}G_{m_1}p^s \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1), & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} - p^{-1}G_{m_1}p^s \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even;} \\ p^{K-1} - p^{-1}G_{m_1}p^{s+v} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1), & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} - p^{-1}G_{m_1}p^{s+v} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \tag{4.4} \\
&= \begin{cases} p^{K-1} + \eta_1(-\lambda)p^{-1}G_1G_{m_1}G_{m_2}, & \text{if } \frac{m_2}{v} \text{ and } K \text{ are both odd;} \\ p^{K-1} - p^{-1}G_{m_1}G_{m_2}, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} - \eta_1(-\lambda)G_1G_{m_1}p^{s-1}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} + G_{m_1}p^{s-1}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even;} \\ p^{K-1} - \eta_1(-\lambda)G_1G_{m_1}p^{s+v-1}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} + G_{m_1}p^{s+v-1}, & \text{if } \frac{m_2}{v} \equiv 0 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases}
\end{aligned}$$

By (4.3)-(4.4) and Lemma 2.1, we complete the proof of Lemma 4.1. \square

Lemma 4.2 For $\lambda, \rho \in \mathbb{F}_p$, $(a, b) \in \mathbb{F}_{q^{m_1}} \times \mathbb{F}_{q^{m_2}} / \{(0, 0)\}$ and

$$\begin{aligned}
&N_{\lambda, \rho}(a, b) \\
&= \#\{(x, y) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} \mid \text{Tr}_{m_1}(x^2) + \text{Tr}_{m_2}(y^{p^{u+1}}) = \lambda \text{ and } \text{Tr}_{m_1}(ax) + \text{Tr}_{m_2}(by) = \rho\},
\end{aligned}$$

denote $T(a, b) = \text{Tr}_{m_1}\left(\frac{a^2}{4}\right) + \text{Tr}_{m_2}\left(\gamma_b^{p^{u+1}}\right)$, then the following assertions hold.

(I) For $\lambda = 0$,

(1) if $\frac{m_2}{v}$ and K are both odd, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2}, & \text{if } T(a, b) = 0; \\ p^{K-2} + \eta_1(-T(a, b))(p-1)L^{K+1}p^{\frac{K-3}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho = 0; \\ p^{K-2} - \eta_1(-T(a, b))L^{K+1}p^{\frac{K-3}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.5)$$

(2) If $\frac{m_2}{v}$ is odd and K is even, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2} + (p-1)L^Kp^{\frac{K-2}{2}}, & \text{if } T(a, b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if } T(a, b) = 0 \text{ and } \rho \neq 0, \\ & \text{or } T(a, b) \neq 0 \text{ and } \rho = 0; \\ p^{K-2} + L^Kp^{\frac{K-2}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.6)$$

(3) If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is odd, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2}, & \text{if } T(a, b) = 0; \\ p^{K-2} - \eta_1(-T(a, b))(p-1)L^{m_1+1}p^{\frac{K-3}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho = 0; \\ p^{K-2} + \eta_1(-T(a, b))L^{m_1+1}p^{\frac{K-3}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.7)$$

(4) If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is even, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2} + (p-1)L^{m_1}p^{\frac{K-2}{2}}, & \text{if } T(a, b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if } T(a, b) \neq 0 \text{ and } \rho = 0, \\ & \text{or } T(a, b) = 0 \text{ and } \rho \neq 0; \\ p^{K-2} + L^{m_1}p^{\frac{K-2}{2}}, & \text{if } T(a, b) \neq 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.8)$$

(5) If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is odd, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2}, & \text{if (2.1) is not solvable,} \\ & \text{or (2.1) is solvable and } T(a, b) = 0; \\ p^{K-2} - \eta_1(-T(a, b))(p-1)L^{m_1+1}p^{\frac{K-3}{2}+v}, & \text{if (2.1) is solvable, } T(a, b) \neq 0 \text{ and } \rho = 0; \\ p^{K-2} + \eta_1(-T(a, b))L^{m_1+1}p^{\frac{K-3}{2}+v}, & \text{if (2.1) is solvable, } T(a, b) \neq 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.9)$$

(6) If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, then

$$N_{\lambda,\rho}(a, b) = \begin{cases} p^{K-2} + (p-1)L^{m_1}p^{\frac{K-4}{2}+v}, & \text{if (2.1) is not solvable;} \\ p^{K-2} + (p-1)L^{m_1}p^{\frac{K-2}{2}+v}, & \text{if (2.1) is solvable, } T(a, b) = 0 \text{ and } \rho = 0; \\ p^{K-2} + L^{m_1}p^{\frac{K-2}{2}+v}, & \text{if (2.1) is solvable, } T(a, b) \neq 0 \text{ and } \rho \neq 0; \\ p^{K-2}, & \text{if (2.1) is solvable, } T(a, b) \neq 0 \text{ and } \rho = 0, \\ & \text{or (2.1) is solvable, } T(a, b) = 0 \text{ and } \rho \neq 0. \end{cases} \quad (4.10)$$

(II) For $\lambda \neq 0$,

(1) if $\frac{m_2}{v}$ and K are odd, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} - \eta_1(-\lambda)L^{K+1}p^{\frac{K-1}{2}}, & \text{if } \rho = 0 \text{ and } T(a,b) = 0; \\ p^{K-2}, & \text{if } \rho \neq 0 \text{ and } T(a,b) = 0; \\ p^{K-2} - \eta_1(-T(a,b))(p-1)L^{K+1}p^{\frac{K-3}{2}}, & \text{if } \lambda \neq 0, T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(-T(a,b))L^{K+1}p^{\frac{K-3}{2}}, & \text{if } \lambda \neq 0, T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.11)$$

(2) If $\frac{m_2}{v}$ is odd and K is even, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} - L^K p^{\frac{K-2}{2}}, & \text{if } T(a,b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if } T(a,b) = 0 \text{ and } \rho \neq 0, \\ & \text{or } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(\rho^2 - 4\lambda T(a,b))L^K p^{\frac{K-2}{2}}, & \text{if } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.12)$$

(3) If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is odd, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}}, & \text{if } \rho = 0 \text{ and } T(a,b) = 0; \\ p^{K-2}, & \text{if } \rho \neq 0 \text{ and } T(a,b) = 0; \\ p^{K-2} - \eta_1(-T(a,b))(p-1)L^{m_1+1}p^{\frac{K-3}{2}}, & \text{if } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(-T(a,b))L^{m_1+1}p^{\frac{K-3}{2}}, & \text{if } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.13)$$

(4) If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is even, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} + L^{m_1}p^{\frac{K-2}{2}}, & \text{if } T(a,b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if } T(a,b) = 0 \text{ and } \rho \neq 0, \\ & \text{or } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(\rho^2 - 4\lambda T(a,b))L^{m_1}p^{\frac{K-2}{2}}, & \text{if } T(a,b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.14)$$

(5) If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is odd, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-3}{2}+v}, & \text{if (2.1) is not solvable,} \\ p^{K-2} - \eta_1(-\lambda)L^{m_1+1}p^{\frac{K-1}{2}+v}, & \text{if (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho \neq 0; \\ p^{K-2} - \eta_1(-T(a,b))(p-1)L^{m_1+1}p^{\frac{K-3}{2}+v}, & \text{if (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(-T(a,b))L^{m_1+1}p^{\frac{K-3}{2}+v}, & \text{if (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.15)$$

(6) If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, then

$$N_{\lambda,\rho}(a,b) = \begin{cases} p^{K-2} - L^{m_1}p^{\frac{K-4}{2}+v}, & \text{if (2.1) is not solvable;} \\ p^{K-2} - L^{m_1}p^{\frac{K-2}{2}+v}, & \text{if (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho = 0; \\ p^{K-2}, & \text{if (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho \neq 0; \\ & \quad \text{or (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) = 0; \\ p^{K-2} + \eta_1(\rho^2 - 4\lambda T(a,b))L^{m_1}p^{\frac{K-2}{2}+v}, & \text{if (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases} \quad (4.16)$$

Proof. By calculating directly, we have

$$\begin{aligned} N_{\lambda,\rho}(a,b) &= \sum_{x \in \mathbb{F}_p^{m_1}} \sum_{y \in \mathbb{F}_p^{m_2}} \left(p^{-1} \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1(\mathrm{Tr}_{m_1}(x^2) + \mathrm{Tr}_{m_2}(y^{p^u+1}) - \lambda)} \right) \left(p^{-1} \sum_{z_2 \in \mathbb{F}_p} \zeta_p^{z_2(\mathrm{Tr}_{m_1}(ax) + \mathrm{Tr}_{m_2}(by) - \rho)} \right) \\ &= p^{K-2} + p^{-2} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{-\rho z_2} \sum_{x \in \mathbb{F}_p^{m_1}} \zeta_p^{\mathrm{Tr}_{m_1}(z_2 ax)} \sum_{y \in \mathbb{F}_p^{m_2}} \zeta_p^{\mathrm{Tr}_{m_2}(z_2 by)} \\ &\quad + p^{-2} \sum_{z_2 \in \mathbb{F}_p} \zeta_p^{-\rho z_2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \sum_{x \in \mathbb{F}_p^{m_1}} \zeta_p^{\mathrm{Tr}_{m_1}(z_1 x^2 + z_2 ax)} \sum_{y \in \mathbb{F}_p^{m_2}} \zeta_p^{\mathrm{Tr}_{m_2}(z_1 y^{p^u+1} + z_2 by)} \\ &= p^{K-2} + p^{-2} \Omega, \end{aligned}$$

where

$$\Omega = \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in \mathbb{F}_p} \zeta_p^{-\rho z_2} Q_{m_1}(z_1, z_2 a) S_{m_2, u}(z_1, z_2 b).$$

Now by Lemmas 2.1 and 2.8-2.9, we calculate Ω as follows.

Case 1. For odd $\frac{m_2}{v}$,

$$\begin{aligned}
\Omega &= \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} \left(G_{m_1} \zeta_p^{\text{Tr}_{m_1}(-\frac{z_2^2 a^2}{4z_1})} \eta_{m_1}(z_1) G_{m_2} \zeta_p^{\text{Tr}_{m_2}(-z_1(\frac{z_2}{z_1}\gamma_b)^{p^u+1})} \eta_{m_2}(z_1) \right) \\
&= G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} \eta_{m_1}(z_1) \eta_{m_2}(z_1) \zeta_p^{-z_1 \left(\text{Tr}_{m_1}((\frac{z_2}{z_1})^2 \frac{a^2}{4}) + \text{Tr}_{m_2}((\frac{z_2}{z_1})^2 \gamma_b^{p^u+1}) \right)} \right) \\
&= G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) \eta_{m_2}(z_1) \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3} \right).
\end{aligned}$$

If K is odd, then

$$\begin{aligned}
\Omega &= G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3} \right) \\
&= \begin{cases} G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if } T(a, b) = 0; \\ \eta_1(-T(a, b)) G_1 G_{m_1} G_{m_2} \sum_{z_1 \in F_p^*} \zeta_p^{\frac{\rho^2 - 4T(a,b)\lambda}{4T(a,b)} z_1}, & \text{if } T(a, b) \neq 0; \end{cases} \\
&= \begin{cases} 0, & \text{if } \lambda = 0 \text{ and } T(a, b) = 0; \\ \eta_1(-T(a, b))(p-1) G_1 G_{m_1} G_{m_2}, & \text{if } \lambda = 0, \rho = 0 \text{ and } T(a, b) \neq 0; \\ -\eta_1(-T(a, b)) G_1 G_{m_1} G_{m_2}, & \text{if } \lambda = 0, \rho \neq 0 \text{ and } T(a, b) \neq 0; \\ \eta_1(-\lambda) p G_1 G_{m_1} G_{m_2}, & \text{if } \lambda \neq 0, \rho = 0 \text{ and } T(a, b) = 0; \\ 0, & \text{if } \lambda \neq 0, \rho \neq 0 \text{ and } T(a, b) = 0; \\ \eta_1(-T(a, b))(p-1) G_1 G_{m_1} G_{m_2}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) = 0; \\ -\eta_1(-T(a, b)) G_1 G_{m_1} G_{m_2}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) \neq 0. \end{cases}
\end{aligned}$$

If K is even, then

$$\begin{aligned}
\Omega &= G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3} \right) \\
&= \begin{cases} G_{m_1} G_{m_2} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in F_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if } T(a, b) = 0; \\ \eta_1(-1) G_1 G_{m_1} G_{m_2} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1 \left(\frac{z_1}{4T(a,b)} \right) \zeta_p^{\frac{\rho^2}{4T(a,b)} z_1}, & \text{if } T(a, b) \neq 0; \end{cases} \\
&= \begin{cases} p(p-1) G_{m_1} G_{m_2}, & \text{if } \lambda = 0, T(a, b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda = 0, T(a, b) = 0 \text{ and } \rho \neq 0, \\ & \quad \text{or } \lambda = 0, T(a, b) \neq 0 \text{ and } \rho = 0; \\ \eta_1(-1) G_1^2 G_{m_1} G_{m_2}, & \text{if } \lambda = 0, T(a, b) \neq 0 \text{ and } \rho \neq 0; \\ -p G_{m_1} G_{m_2}, & \text{if } \lambda \neq 0, T(a, b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda \neq 0, T(a, b) = 0 \text{ and } \rho \neq 0, \\ & \quad \text{or } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) = 0; \\ \eta_1(4\lambda T(a, b) - \rho^2) G_1^2 G_{m_1} G_{m_2}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) = 0. \end{cases}
\end{aligned}$$

Case 2. For $\frac{m_2}{v} \equiv 2 \pmod{4}$,

$$\begin{aligned}
\Omega &= \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} Q_{m_1}(z_1, z_2 a) S_{m_2, u}(z_1, z_2 b) \\
&= -p^s \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} \left(G_{m_1} \zeta_p^{\text{Tr}_{m_1}(-\frac{z_2 a^2}{z_1})} \eta_{m_1}(z_1) \zeta_p^{\text{Tr}_{m_2}(-z_1(\frac{z_2}{z_1}\gamma_b)^{p^u+1})} \right) \\
&= -p^s G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} \zeta_p^{-z_1(\text{Tr}_{m_1}((\frac{z_2}{z_1})^2 a^2) + \text{Tr}_{m_2}((\frac{z_2}{z_1}\gamma_b)^{p^u+1}))} \\
&= -p^s G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3}.
\end{aligned}$$

If m_1 is odd, then

$$\begin{aligned}
\Omega &= -p^s G_{m_1} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3} \right) \\
&= \begin{cases} -p^s G_{m_1} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if } T(a, b) = 0; \\ -\eta_1(-T(a, b)) p^s G_1 G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \zeta_p^{\frac{\rho^2}{4T(a,b)} z_1}, & \text{if } T(a, b) \neq 0; \end{cases} \\
&= \begin{cases} 0, & \text{if } \lambda = 0 \text{ and } T(a, b) = 0; \\ -\eta_1(-T(a, b))(p-1)p^s G_1 G_{m_1}, & \text{if } \lambda = 0, \rho = 0 \text{ and } T(a, b) \neq 0; \\ \eta_1(-T(a, b))p^s G_1 G_{m_1}, & \text{if } \lambda = 0, \rho \neq 0 \text{ and } T(a, b) \neq 0; \\ -\eta_1(-\lambda)p^{s+1}G_1 G_{m_1}, & \text{if } \lambda \neq 0, \rho = 0 \text{ and } T(a, b) = 0; \\ 0, & \text{if } \lambda \neq 0, \rho \neq 0 \text{ and } T(a, b) = 0; \\ -\eta_1(-T(a, b))(p-1)p^s G_1 G_{m_1}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) = 0; \\ \eta_1(-T(a, b))p^s G_1 G_{m_1}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) \neq 0. \end{cases}
\end{aligned}$$

If m_1 is even, then

$$\begin{aligned}
\Omega &= -p^s G_{m_1} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in F_p} \zeta_p^{-T(a,b)z_1 z_3^2 - \rho z_1 z_3} \right) \\
&= \begin{cases} -p^s G_{m_1} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in F_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if } T(a, b) = 0; \\ -\eta_1(-1)p^s G_1 G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1\left(\frac{z_1}{4T(a,b)}\right) \zeta_p^{\frac{\rho^2}{4T(a,b)} z_1}, & \text{if } T(a, b) \neq 0; \end{cases} \\
&= \begin{cases} -(p-1)p^{s+1}G_{m_1}, & \text{if } \lambda = 0, T(a, b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda = 0, T(a, b) = 0 \text{ and } \rho \neq 0, \\ & \quad \text{or } \lambda = 0, T(a, b) \neq 0 \text{ and } \rho = 0; \\ -\eta_1(-1)p^s G_1^2 G_{m_1}, & \text{if } \lambda = 0, T(a, b) \neq 0 \text{ and } \rho \neq 0; \\ -p^{s+1}G_{m_1}, & \text{if } \lambda \neq 0, T(a, b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda \neq 0, T(a, b) = 0 \text{ and } \rho \neq 0, \\ & \quad \text{or } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) = 0; \\ -\eta_1(4\lambda T(a, b) - \rho^2)p^s G_1^2 G_{m_1}, & \text{if } \lambda \neq 0, T(a, b) \neq 0 \text{ and } \rho^2 - 4\lambda T(a, b) \neq 0. \end{cases}
\end{aligned}$$

Case 3. For $\frac{m_2}{v} \equiv 0 \pmod{4}$,

$$\begin{aligned} & \Omega \\ = & \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \sum_{z_2 \in F_p} \zeta_p^{-\rho z_2} \zeta_p^{\text{Tr}_{m_1}(-\frac{z_2^2 a^2}{z_1})} \eta_{m_1}(z_1) \zeta_p^{\text{Tr}_{m_2}(-z_1(\frac{z_2}{z_1} \gamma_b)^{p^u+1})}, & \text{if (2.1) is solvable;} \end{cases} \\ = & \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1), & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_{m_1}(z_1) \sum_{z_3 \in F_p} \zeta_p^{-z_1 z_3^2 T(a,b) - \rho z_1 z_3}, & \text{if (2.1) is solvable.} \end{cases} \end{aligned}$$

If m_1 is odd, then

$$\begin{aligned} & \Omega \\ = & \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1), & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-z_1 T(a,b) z_3^2 - \rho z_1 z_3}, & \text{if (2.1) is solvable;} \end{cases} \\ = & \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1), & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \left(\sum_{z_1 \in F_p^*} \zeta_p^{-\lambda z_1} \eta_1(z_1) \sum_{z_3 \in F_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if (2.1) is solvable and } T(a,b) = 0; \\ -p^{s+v} \eta_1(-T(a,b)) G_1 G_{m_1} \sum_{z_1 \in F_p^*} \zeta_p^{\frac{\rho^2 - 4\lambda T(a,b)}{4T(a,b)} z_1}, & \text{if (2.1) is solvable and } T(a,b) \neq 0; \\ 0, & \begin{array}{l} \text{if } \lambda = 0, \text{ (2.1) is not solvable,} \\ \text{or } \lambda = 0, \text{ (2.1) is solvable and } T(a,b) = 0; \end{array} \\ -\eta_1(-T(a,b))(p-1)p^{s+v} G_1 G_{m_1}, & \text{if } \lambda = 0, \text{ (2.1) is solvable, } T(a,b) \neq 0 \text{ and } \rho = 0; \\ \eta_1(-T(a,b))p^{s+v} G_1 G_{m_1}, & \text{if } \lambda = 0, \text{ (2.1) is solvable, } T(a,b) \neq 0 \text{ and } \rho \neq 0; \\ -\eta_1(-\lambda)p^{s+v} G_1 G_{m_1}, & \text{if } \lambda \neq 0, \text{ (2.1) is not solvable;} \\ -\eta_1(-\lambda)p^{s+v+1} G_1 G_{m_1}, & \text{if } \lambda \neq 0, \text{ (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda \neq 0, \text{ (2.1) is solvable, } T(a,b) = 0 \text{ and } \rho \neq 0; \\ -\eta_1(-T(a,b))(p-1)p^{s+v} G_1 G_{m_1}, & \text{if } \lambda \neq 0, \text{ (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) = 0; \\ \eta_1(-T(a,b))p^{s+v} G_1 G_{m_1}, & \text{if } \lambda \neq 0, \text{ (2.1) is solvable, } T(a,b) \neq 0 \\ & \quad \text{and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{array} \end{cases} \end{aligned}$$

If m_1 is even, then

$$\Omega$$

$$\begin{aligned}
&= \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1}, & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in \mathbb{F}_p} \zeta_p^{-z_1 z_3^2 T(a,b) - \rho z_1 z_3}, & \text{if (2.1) is solvable;} \end{cases} \\
&= \begin{cases} -p^{s+v} G_{m_1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1}, & \text{if (2.1) is not solvable;} \\ -p^{s+v} G_{m_1} \left(\sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-\lambda z_1} \sum_{z_3 \in \mathbb{F}_p} \zeta_p^{-\rho z_1 z_3} \right), & \text{if (2.1) is solvable and } T(a,b) = 0; \\ -\eta_1 (-T(a,b)) p^{s+v} G_1 G_{m_1} \sum_{z_1 \in \mathbb{F}_p^*} \eta_1(z_1) \zeta_p^{\frac{\rho^2 - 4\lambda T(a,b)}{4T(a,b)} z_1}, & \text{if (2.1) is solvable and } T(a,b) \neq 0; \end{cases} \\
&= \begin{cases} -(p-1)p^{s+v} G_{m_1}, & \text{if } \lambda = 0, (2.1) \text{ is not solvable;} \\ -(p-1)p^{s+v+1} G_{m_1}, & \text{if } \lambda = 0, (2.1) \text{ is solvable, } T(a,b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda = 0, (2.1) \text{ is solvable, } T(a,b) \neq 0 \text{ and } \rho = 0, \\ & \text{or } \lambda = 0, (2.1) \text{ is solvable, } T(a,b) = 0 \text{ and } \rho \neq 0; \\ -\eta_1(-1)p^{s+v} G_1^2 G_{m_1}, & \text{if } \lambda = 0, (2.1) \text{ is solvable, } T(a,b) \neq 0 \text{ and } \rho \neq 0. \\ p^{s+v} G_{m_1}, & \text{if } \lambda \neq 0, (2.1) \text{ is not solvable;} \\ p^{s+v+1} G_{m_1}, & \text{if } \lambda \neq 0, (2.1) \text{ is solvable, } T(a,b) = 0 \text{ and } \rho = 0; \\ 0, & \text{if } \lambda \neq 0, (2.1) \text{ is solvable, } T(a,b) = 0 \text{ and } \rho \neq 0, \\ & \text{or } \lambda \neq 0, (2.1) \text{ is solvable, } T(a,b) \neq 0 \\ & \text{and } \rho^2 - 4\lambda T(a,b) = 0; \\ -\eta_1(4\lambda T(a,b) - \rho^2)p^{s+v} G_1^2 G_{m_1}, & \text{if } \lambda \neq 0, (2.1) \text{ is solvable, } T(a,b) \neq 0 \\ & \text{and } \rho^2 - 4\lambda T(a,b) \neq 0. \end{cases}
\end{aligned}$$

So far, by the cases 1-3 and Lemma 2.1, we complete the proof of Lemma 4.2. \square

Lemmas 4.3-4.5 are important to calculate the weight enumerators for \mathcal{C}_{D_λ} .

Lemma 4.3 For $t \in \mathbb{F}_p$ and

$$\tilde{A}_t = \#\{(a,b) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} | T(a,b) = t\},$$

if $\frac{m_2}{v}$ is odd or $\frac{m_2}{v} \equiv 2 \pmod{4}$, then the following assertions hold.

(1) For $t = 0$,

$$\tilde{A}_t = \begin{cases} p^{K-1}, & \text{if } K \text{ is odd;} \\ p^{K-1} + (p-1)L^K p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} + (p-1)L^{m_1} p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \quad (4.17)$$

(2) For $t \neq 0$,

$$\tilde{A}_t = \begin{cases} p^{K-1} - \eta_1(-t)L^{K+1}p^{\frac{K-1}{2}}, & \text{if } \frac{m_2}{v} \text{ and } K \text{ are odd;} \\ p^{K-1} - L^K p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \text{ is odd and } K \text{ is even;} \\ p^{K-1} - \eta_1(-t)L^{m_1+1}p^{\frac{K-1}{2}}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is odd;} \\ p^{K-1} - L^K p^{\frac{K-2}{2}}, & \text{if } \frac{m_2}{v} \equiv 2 \pmod{4} \text{ and } m_1 \text{ is even.} \end{cases} \quad (4.18)$$

Proof. For odd $\frac{m_2}{v}$, or $\frac{m_2}{v} \equiv 2 \pmod{4}$, it follows from Lemma 2.3 that $X^{p^{2u}} + X$ is a permutation polynomial over $\mathbb{F}_{p^{m_2}}[x]$ and then (2.1) has an unique solution in $\mathbb{F}_{p^{m_2}}$, thus,

$$\tilde{A}_t = \#\{(a, b) \in \mathbb{F}_{p^{m_1}} \times \mathbb{F}_{p^{m_2}} \mid \text{Tr}_{m_1}(a^2) + \text{Tr}_{m_2}(b^{p^u+1}) = t\}.$$

Now by Lemma 4.2, we can obtain (4.17)-(4.18). \square

Lemma 4.4 ([15], Lemma 13) For $\frac{m_2}{v} \equiv 0 \pmod{4}$, and

$$B = \{c \in \mathbb{F}_{p^m} \mid X^{p^{2u}} + X = c^{p^u} \text{ is solvable in } \mathbb{F}_{p^m}\},$$

one has

$$\#B = p^{m-2v}.$$

Lemma 4.5 For $\frac{m_2}{v} \equiv 0 \pmod{4}$, $t \in \mathbb{F}_p$, and

$$\bar{A}_t = \{(a, b) \in \mathbb{F}_{p^{m_1}} \times B \mid \text{Tr}_{m_1}\left(\frac{a^2}{4}\right) + \text{Tr}_{m_2}(\gamma_b^{p^u+1}) = t\},$$

the following two assertions hold.

(1) If m_1 is even, then

$$\#\bar{A}_t = \begin{cases} p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v}, & t = 0; \\ p^{K-2v-1} - L^{m_1}p^{\frac{K-2}{2}-v}, & \text{otherwise.} \end{cases} \quad (4.19)$$

(2) If m_1 is odd, then

$$\#\bar{A}_t = \begin{cases} p^{K-2v-1}, & t = 0; \\ p^{K-2v-1} - \eta_1(-t)L^{m_1+1}p^{\frac{K-1}{2}-v}, & \text{otherwise.} \end{cases} \quad (4.20)$$

To prove Lemma 4.5, we need Tables 4 and 10, which are given in subsection 4.4.

4.2 The proofs for Theorems 3.1-3.3 and 3.6-3.9

The proofs for Theorems 3.1-3.3.

By Lemmas 4.1-4.3, we have the following three cases.

Case 1. If $\frac{m_2}{v}$ and K are both odd, or $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is odd, then the length of \mathcal{C}_{D_0} is $N_0 = p^{K-1} - 1$. It follows from (4.5) and (4.7) that the nonzero weights of \mathcal{C}_{D_0} are

$$w_1 = (p-1)(p^{K-2} - p^{\frac{K-3}{2}}), \quad w_2 = (p-1)p^{K-2}, \quad w_3 = (p-1)(p^{K-2} + p^{\frac{K-3}{2}}).$$

By Lemma 4.3, we know that $A_{w_2} = \tilde{A}_0 - 1 = p^{K-1} - 1$, which combines first two Pless power moments leads to

$$A_{w_1} = \frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} + 1), \quad A_{w_3} = \frac{p-1}{2} p^{\frac{K-1}{2}} (p^{\frac{K-1}{2}} - 1).$$

Case 2. If $\frac{m_2}{v}$ is odd and K is even, the length of \mathcal{C}_{D_0} is $N_0 = p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1$. It follows from (4.6) that the nonzero weights of \mathcal{C}_{D_0} are

$$w_1 = (p-1)p^{\frac{K-2}{2}} (p^{\frac{K-2}{2}} + L^K), \quad w_2 = (p-1)p^{K-2}.$$

Then, by the first two Pless power moments, one has

$$A_{w_1} = (p-1)p^{\frac{K-2}{2}} (p^{\frac{K}{2}} - L^K), \quad A_{w_2} = p^{K-1} + L^K(p-1)p^{\frac{K-2}{2}} - 1.$$

Case 3. If $\frac{m_2}{v} \equiv 2 \pmod{4}$ and m_1 is even, then the length of \mathcal{C}_{D_0} is $N_0 = p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1$, the nonzero weights of \mathcal{C}_{D_0} are

$$w_1 = (p-1)p^{\frac{K-2}{2}} (p^{\frac{K-2}{2}} + L^{m_1}), \quad w_2 = (p-1)p^{K-2},$$

and by the first two Pless power moments, one has

$$A_{w_1} = (p-1)p^{\frac{K-2}{2}} (p^{\frac{K}{2}} - L^{m_1}), \quad A_{w_2} = p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}} - 1.$$

By the cases 1-3, we can obtain Tables 1-3, correspondingly, which combines Lemma 4.2 leads to the complete weight enumerator of \mathcal{C}_{D_0} directly.

So far, we complete the proofs for Theorems 3.1-3.3. \square

The proofs for Theorems 3.6-3.9.

Using Lemmas 4.1-4.3, in the similar proofs as those of Theorems 3.1-3.3, correspondingly, one can obtain Theorems 3.6-3.9 immediately. \square

4.3 The proofs for Theorems 3.4-3.5 and 3.10-3.11

In this subsection, Theorems 3.4-3.5 and 3.10-3.11 are obtained from Lemmas 2.10, 4.1-4.2 and 4.4-4.5. To this aim, we firstly prove Lemma 4.5.

Proof of Lemma 4.5.

If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, by Lemmas 4.1-4.2 and 4.4, the length of \mathcal{C}_{D_0} is $N_0 - 1 = p^{K-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}+v} - 1$, the nonzero weights of \mathcal{C}_{D_0} are

$$\begin{aligned} w_1 &= (p-1)(p^{K-2} + L^{m_1}p^{\frac{K-2}{2}+v}), \\ w_2 &= (p-1)p^{K-2}, \\ w_3 &= (p-1)(p^{K-2} + L^{m_1}(p-1)p^{\frac{K-4}{2}+v}), \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} A_{w_1} &= (p-1)p^{\frac{K-2}{2}-v}(p^{\frac{K}{2}-v} - L^{m_1}), \\ A_{w_2} &= p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v} - 1, \\ A_{w_3} &= p^K(1 - p^{-2v}). \end{aligned} \quad (4.22)$$

Then, by Lemma 4.2, one has

$$\#\bar{A}_0 = A_{w_2} + 1 = p^{K-2v-1} + L^{m_1}(p-1)p^{\frac{K-2}{2}-v}. \quad (4.23)$$

Similarly, for $\lambda \in \mathbb{F}_p^*$, the length of \mathcal{C}_{D_λ} is $N_0 - 1 = p^{K-1} - L^{m_1}p^{\frac{K-2}{2}+v}$, the nonzero weights of \mathcal{C}_{D_λ} are

$$\begin{aligned} w_1 &= (p-1)p^{K-2}, \\ w_2 &= (p-1)(p^{K-2} - L^{m_1}p^{\frac{K-4}{2}+v}), \\ w_3 &= (p-1)p^{K-2} - 2L^{m_1}p^{\frac{K-2}{2}+v}, \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} A_{w_1} &= \frac{p+1}{2}p^{K-2v-1} + \frac{p-1}{2}L^{m_1}p^{\frac{K-2}{2}-v} - 1, \\ A_{w_2} &= p^K - p^{K-2v}, \\ A_{w_3} &= \frac{p-1}{2}(p^{K-2v-1} - L^{m_1}p^{\frac{K-2}{2}-v}). \end{aligned} \quad (4.25)$$

Then, by Lemma 4.2, one has

$$\sum_{\substack{t \in \mathbb{F}_p^* \\ \eta_1(t) = \eta_1(-\lambda)}} \#\bar{A}_t = A_{w_3} = \frac{p-1}{2}(p^{K-2v-1} - L^{m_1}p^{\frac{K-2}{2}-v}), \quad (4.26)$$

which leads to

$$\sum_{\substack{t \in \mathbb{F}_p^* \\ \eta_1(t) = 1}} \#\bar{A}_t = \sum_{\substack{t \in \mathbb{F}_p^* \\ \eta_1(t) = -1}} \#\bar{A}_t = \frac{p-1}{2}(p^{K-2v-1} - L^{m_1}p^{\frac{K-2}{2}-v}). \quad (4.27)$$

For any given $\alpha \in \mathbb{F}_p$ with $\eta_1(\alpha) = -1$, we have

$$\begin{aligned}
& \#\bar{A}_t \\
&= \frac{1}{p} \sum_{a \in \mathbb{F}_p^{m_1}} \sum_{b \in B} \sum_{z \in \mathbb{F}_p} \zeta_p^{z \left(\text{Tr}_{m_1}\left(\frac{a^2}{4}\right) + \text{Tr}_{m_2}(\gamma_b^{p^u+1}) - t \right)} \\
&= p^{K-2v-1} + \frac{1}{p} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \sum_{a \in \mathbb{F}_p^{m_1}} \zeta_p^{\text{Tr}_{m_1}(za^2)} \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \left(\sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=1}} \zeta_p^{-tz} \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} + \sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=-1}} \zeta_p^{-tz} \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} \right) \quad (4.28) \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \left(\sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=1}} \zeta_p^{-tz} \sum_{b \in B} \zeta_p^{\text{Tr}_{m_2}(\gamma_b^{p^u+1})} + \sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=-1}} \zeta_p^{-tz} \sum_{b \in B} \zeta_p^{\alpha \text{Tr}_{m_2}(\gamma_b^{p^u+1})} \right) \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \left(\sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \frac{(\eta_1(z)+1)}{2} A_+ + \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \frac{(-\eta_1(z)+1)}{2} A_- \right) \\
&= p^{K-2v-1} + \frac{1}{2p} G_{m_1} \left((A_+ - A_-) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \eta_1(z) + (A_+ + A_-) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \right),
\end{aligned}$$

where

$$A_+ = \sum_{b \in B} \zeta_p^{\text{Tr}_{m_2}(\gamma_b^{p^u+1})}, \quad A_- = \sum_{b \in B} \zeta_p^{\alpha \text{Tr}_{m_2}(\gamma_b^{p^u+1})}.$$

Then, by (4.28) and Lemma 2.1, one has

$$\#\bar{A}_0 = p^{K-2v-1} - \frac{p-1}{2p} L^{m_1} p^{\frac{m_1}{2}} (A_+ + A_-), \quad (4.29)$$

and for $t \in \mathbb{F}_p^*$,

$$\#\bar{A}_t = p^{K-2v-1} + \frac{1}{2p} L^{m_1} p^{\frac{m_1}{2}} (\eta_1(-t) G_1 (A_+ - A_-) - (A_+ + A_-)). \quad (4.30)$$

It follows from (4.23), (4.27) and (4.29)-(4.30) that

$$A_+ = A_- = -p^{\frac{m_2}{2}-v}. \quad (4.31)$$

Now by (4.29)-(4.31), we can obtain (4.19).

Similarly, if m_1 is odd, by (4.31), we have

$$\begin{aligned}
& \#\bar{A}_t \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \left(\sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=1}} \zeta_p^{-tz} \eta_1(z) \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} + \sum_{\substack{z \in \mathbb{F}_p^* \\ \eta_1(z)=-1}} \zeta_p^{-tz} \eta_1(z) \sum_{b \in B} \zeta_p^{z \text{Tr}_{m_2}(\gamma_b^{p^u+1})} \right) \\
&= p^{K-2v-1} + \frac{1}{p} G_{m_1} \left(\sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \frac{(\eta_1(z)+1)}{2} A_+ - \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \frac{(-\eta_1(z)+1)}{2} A_- \right) \\
&= p^{K-2v-1} + \frac{1}{2p} G_{m_1} \left((A_+ + A_-) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \eta_1(z) + (A_+ - A_-) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-tz} \right) \\
&= \begin{cases} p^{K-2v-1}, & t = 0; \\ p^{K-2v-1} - \eta_1(-t) L^{m_1+1} p^{\frac{K-1}{2}-v}, & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.32}$$

So far, we complete the proof of Lemma 4.5. \square

The proofs for Theorems 3.4-3.5 and 3.10-3.11.

If $\frac{m_2}{v} \equiv 0 \pmod{4}$ and m_1 is even, Tables 4 and 10 are given from (4.21)-(4.22) and (4.24)-(4.25), respectively, and then the complete weight enumerators of \mathcal{C}_{D_0} and \mathcal{C}_{D_λ} are obtained from lemmas 4.2 and 4.5 directly. So far, we complete the proofs for Theorems 3.4 and 3.10.

By Lemmas 4.1-4.2 and 4.4-4.5, in the similar proofs as those of Theorems 3.4 and 3.10, we can obtain Theorems 3.5 and 3.11 immediately. \square

5 Examples

In this section, we give some examples for the main results.

Example 5.1 For $p = 3$, by using Magma, we obtain \mathcal{C}_{D_0} for some special cases in Table 12, which are accordant with Theorems 3.1-3.5.

Table 12 Some \mathcal{C}_{D_0} for special cases

m_1	m_2	u	$\frac{m_2}{v}$	m_1+m_2	parameter	weight enumerator
3	2	2	1	5	[80, 5, 48]	$1 + 90z^{48} + 80z^{54} + 72z^{60}$
2	2	2	1	4	[32, 4, 18]	$1 + 32z^{18} + 48z^{24}$
2	2	1	2	4	[20, 4, 12]	$1 + 60z^{12} + 20z^{18}$
2	4	2	2	4	[224, 6, 144]	$1 + 504z^{144} + 224z^{162}$
2	4	3	4	6	[188, 6, 108]	$1 + 60z^{108} + 648z^{126} + 20z^{162}$
3	4	1	4	7	[728, 7, 432]	$1 + 90z^{432} + 2024z^{486} + 72z^{540}$

According to the Griesmer bound [9], the code [20, 4, 12] is optimal.

Example 5.2 For $p = 3$, by Table 12 and Theorems 3.1-3.5, we obtain Furthermore, $\mathcal{C}_{\tilde{D}_0}$ for some special cases in Table 12°.

Table 12° Some Furthermore, $\mathcal{C}_{\tilde{D}_0}$ for special cases

m_1	m_2	u	$\frac{m_2}{v}$	m_1+m_2	parameter	weight enumerator
3	2	2	1	5	[40, 5, 24]	$1 + 90z^{24} + 80z^{27} + 72z^{30}$
2	2	2	1	4	[16, 4, 9]	$1 + 32z^9 + 48z^{12}$
2	2	1	2	4	[10, 4, 6]	$1 + 60z^6 + 20z^9$
2	4	2	6	4	[112, 6, 72]	$1 + 504z^{72} + 224z^{81}$
2	4	3	4	6	[94, 6, 54]	$1 + 60z^{54} + 648z^{63} + 20z^{81}$
3	4	1	4	7	[364, 7, 216]	$1 + 90z^{216} + 2024z^{243} + 72z^{270}$

According to the Griesmer bound [9], the codes [16, 4, 9], [10, 4, 6] and [112, 6, 72] are all optimal.

Example 5.3 For $p = 3$, by using Magma, we obtain \mathcal{C}_{D_λ} for some special cases in Table 13, which are accordant with Theorems 3.6-3.11.

Table 13 Some \mathcal{C}_{D_λ} for special cases

λ	m_1	m_2	u	$\frac{m_2}{v}$	m_1+m_2	parameter	weight enumerator
-1	3	2	2	1	5	[90, 5, 54]	$1 + 80z^{54} + 72z^{60} + 90z^{66}$
1	2	2	2	1	4	[24, 4, 12]	$1 + 24z^{12} + 56z^{18}$
-1	2	2	1	2	4	[30, 4, 18]	$1 + 50z^{18} + 30z^{24}$
1	2	4	2	2	4	[252, 6, 162]	$1 + 476z^{162} + 252z^{180}$
-1	2	4	1	4	6	[270, 6, 162]	$1 + 50z^{162} + 648z^{180} + 30z^{216}$
-1	3	4	1	4	7	[648, 7, 378]	$1 + 72z^{378} + 2034z^{432} + 80z^{486}$

According to the Griesmer bound [9], the code [30, 4, 18] is almost optimal.

Example 5.4 For $p = 3$, by Table 13 and Theorems 3.6-3.11, we obtain $\mathcal{C}_{\tilde{D}_\lambda}$ for some special cases in Table 13°.

Table 13° Some $\mathcal{C}_{\tilde{D}_\lambda}$ for special cases

λ	m_1	m_2	u	$\frac{m_2}{v}$	m_1+m_2	parameter	weight enumerator
-1	3	2	2	1	5	[45, 5, 27]	$1 + 80z^{27} + 72z^{30} + 90z^{33}$
1	2	2	2	1	4	[12, 4, 6]	$1 + 24z^6 + 56z^9$
-1	2	2	1	2	4	[15, 4, 9]	$1 + 50z^9 + 30z^{12}$
1	2	4	2	2	4	[126, 6, 81]	$1 + 476z^{81} + 252z^{90}$
-1	2	4	1	4	6	[135, 6, 81]	$1 + 50z^{81} + 648z^{90} + 30z^{108}$
-1	3	4	1	4	7	[324, 7, 189]	$1 + 72z^{189} + 2034z^{216} + 80z^{243}$

According to the Griesmer bound [9], the codes [45, 5, 27] and [12, 4, 6] are both almost optimal, [15, 4, 9] and [126, 6, 81] are both optimal.

6 Conclusion

Note that codes in [15, 17] are always even dimension. In this paper, we construct several classes of two-weight and three-weight linear codes with any dimension over the finite field \mathbb{F}_p (p is an odd prime) by extending the construction in [15, 17], and we determine their complete weight enumerators by using Weil sums. Furthermore, according to the Griesmer bound, some examples of these codes are optimal or almost optimal, respectively.

References

- [1] A. Calderbank, J. Goethals, Three-weight codes and association schemes, Philips J. Res. 39(4-5) (1984) 143-152.
- [2] R. Calderbank, W. Kantor, The geometry of two-weight codes, Bull. Lond. Math. Soc. 18(2) (1986) 97-122.
- [3] C. Carlet, C. Ding, J. Yuan, Linear codes from perfect nonlinear mappings and their secret sharing schemes, IEEE Trans. Inf. Theory. 51(6) (2005) 2089-2102.
- [4] R. Coulter, Explicit evaluations of some Weil sums, Acta Arith. 83 (1998) 241-251.
- [5] R. Coulter, Further evaluations of Weil sums, Acta Arith. 86 (1998) 217-226.
- [6] C. Ding, X. Wang, A coding theory construction of new systematic authentication codes, Theor. Comput. Sci. 330(1) (2005) 81-99.
- [7] K. Ding, C. Ding, Binary linear codes with three weights, IEEE Commun. Lett. 18(11) (2014) 1879-1882.
- [8] K. Ding, C. Ding, A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Trans. Inf. Theory. 61(11) (2015) 5835-5842.
- [9] J. Griesmer, A bound for error-correcting codes, IBM J. Res. Dev. 4(5) (1960) 532-542.
- [10] W. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press. 2010.
- [11] Z. Heng, Q. Yue, A class of binary linear codes with at most three weights, IEEE Commun. Lett. 19(9) (2015) 1488-1491.
- [12] Z. Heng, Q. Yue, Two classes of two-weight linear codes, Finite Fields Appl. 38 (2016) 72-92.
- [13] Z. Heng, Q. Yue, A construction of q -ary linear codes with two weights, Finite Fields Appl. 48 (2017) 20-42.
- [14] Z. Heng, Q. Yue, C. Li, Three classes of linear codes with two or three weights, Discrete Math. 339(11) (2016) 2832-2847.
- [15] G. Jian, Z. Lin, R. Feng, Two-weight and three-weight linear codes based on Weil sums, Finite Fields Appl. 57 (2019) 92-107.
- [16] C. Li, S. Bae, S. Yang, Some two-weight and three-weight linear codes, Advances in Mathematics of Communications. 13(1) (2019) 195-211.
- [17] C. Li, Q. Yue, F. Fu, A construction of several classes of two-weight and three-weight linear codes, Appl. Algebra Eng. Commun. Comput. (2016) 1-20.
- [18] R. Lidl, H. Niederreiter, Cohn F.M.. Finite Fields. Cambridge University Press, Cambridge. (1997)

- [19] G. Luo, X. Cao, S. Xu, J. Mi, Binary linear codes with two or three weights from niho exponents, *Cryptogr. Commun.* 10(2) (2018) 301-318 .
- [20] C. Tang, N. Li, Y. Qi, Z. Zhou, T. Helleseth, Linear codes with two or three weights from weakly regular bent functions, *IEEE Trans. Inf. Theory.* 62(3) (2016) 1166-1176.
- [21] K. Torleiv, *Codes for Error Detection*, vol. 2, World Scientific. (2007).
- [22] J. Yuan, C. Ding, Secret sharing schemes from three classes of linear codes, *IEEE Trans. Inf. Theory.* 52(1) (2006) 206-212.
- [23] S. Yang, Z. Yao, Complete weight enumerators of a family of three-weight linear codes, *Des. Codes Cryptogr.* (2017) 663-674.
- [24] Z. Zhou, N. Li, C. Fan, T. Helleseth, Linear codes with two or three weights from quadratic bent functions, *Des. Codes Cryptogr.* (2015) 1-13.