ON THE CONSTRUCTIONS OF MDS SELF-DUAL CODES VIA CYCLOTOMY

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ABSTRACT. MDS self-dual codes over finite fields have attracted a lot of attention in recent years by their theoretical interests in coding theory and applications in cryptography and combinatorics. In this paper we present a series of MDS self-dual codes with new length by using generalized Reed-Solomon codes and extended generalized Reed-Solomon codes as the candidates of MDS codes and taking their evaluation sets as an union of cyclotomic classes. The conditions on such MDS codes being self-dual are expressed in terms of cyclotomic numbers.

1. Introduction

Let \mathbb{F}_q be the finite field with q elements. An $[n,k,d]_q$ linear code \mathcal{C} is a k-dimensional subspace of \mathbb{F}_q^n with minimum (Hamming) distance d. It is well known that n,k,d need satisfy the Singleton bound $d \leq n-k+1$. If the equality is attained then the code is called MDS code. The dual code \mathcal{C}^{\perp} of \mathcal{C} is defined by

$$\mathcal{C}^{\perp} = \{ v \in \mathbb{F}_q^n : (v, c) = 0 \text{ for all } c \in \mathcal{C} \}$$

where for $v = (v_1, v_2, \dots, v_n)$ and $c = (c_1, c_2, \dots, c_n)$, $(v, c) = \sum_{i=1}^n v_i c_i \in \mathbb{F}_q$ is the Euclidean inner product in \mathbb{F}_q^n . The code \mathcal{C} is called self-dual if $\mathcal{C} = \mathcal{C}^{\perp}$. If \mathcal{C} is both MDS and self-dual, \mathcal{C} is called MDS self-dual code.

MDS codes and self-dual codes are important families of classical codes in coding theory. Therefore, it is of interests to investigate MDS self-dual codes. Since the dimension and distance are determined by the length of an MDS self-dual code, thus we usually focus on the length and the field size of MDS self-dual codes. The problem is completely solved by Grassl and Gulliver [4] when q is even, but not for odd q.

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One of the basic problems of this topic is to determine the existence of MDS self-dual codes for length n and a fixed finite field \mathbb{F}_q .

In the literature, there are many known constructions of MDS self-dual codes. One of the method to construct MDS self-dual codes is based on the generalized Reed-Solomon (GRS for short) codes or extended generalized Reed-Solomon (EGRS for short) codes [2], [3], [6], [10], [11], since they are MDS codes. The codewords of GRS codes and EGRS codes are made by evaluation of polynomials in $\mathbb{F}_q[x]$ at \mathcal{S} for a certain subset \mathcal{S} of the projective line $\mathbb{F}_q \bigcup \{\infty\}$. The conditions on \mathcal{S} have been provided in order that such MDS codes constructed with \mathcal{S} are self-dual by Jin and Xing [6] for GRS case ($\mathcal{S} \subseteq \mathbb{F}_q$) and Yan [10] for EGRS case ($\infty \in \mathcal{S}$) respectively. In most of previous works, the set \mathcal{S} is chosen as a union of cosets of a subgroup of \mathbb{F}_q^* or a subspace of \mathbb{F}_q .

In this paper, we consider the construction of MDS self-dual codes over \mathbb{F}_q by using the first approach. Namely, we take \mathcal{S} as a union of cosets of a subgroup of \mathbb{F}_q^* . Let $\mathbb{F}_q^* = \langle \theta \rangle$ where θ is a primitive element of \mathbb{F}_q^* . Any subgroup of \mathbb{F}_q^* is a cyclic group $\mathcal{D} = \langle \theta^e \rangle$ where $q-1=ef, |\mathcal{D}|=f$ and all cosets of \mathcal{D} in \mathbb{F}_q^* are the e-th cyclotomic classes

$$\mathcal{D}_i^{(e)} = \theta^i \mathcal{D} = \{ \theta^{i+e\lambda} : 0 \le \lambda \le f - 1 \} \quad (0 \le i \le e - 1), \quad \mathcal{D} = \mathcal{D}_0^{(e)}.$$

In the previous works [2], [3], [7], [11], q is a square, $q=r^2, r=p^m$ $(p\geq 3)$ and $\mathcal S$ is a union of $\mathcal D_i^{(e)}$ with several particular i satisfying $r-1\mid i$. In this paper, we consider the case q, which is any prime power and we take $\mathcal S$ being a union of cosets in more flexible way, so that we get many new series of MDS self-dual codes with length n. For doing this we use the properties and computations on cyclotomic numbers.

This paper is organized as follows. In Section 2, we introduce the basic results given in [6] and [10] on criteria of MDS self-dual codes constructed by GRS and EGRS codes being self-dual. We also introduce the basic properties of cyclotomic numbers in Section 2 which are main machinary of this paper. In Section 3, we present our general results on constructing MDS self-dual codes over \mathbb{F}_q by using cyclotomic classes of \mathbb{F}_q^* . Then we show several particular cases as applications of our general results in Section 4.

2. Preliminaries

2.1. MDS Self-dual codes Constructed by GRS Codes and EGRS codes. In this subsection, we briefly review some basic results on GRS codes and EGRS codes. For the details, the reader may refer to [5] and [8].

Definition 2.1. Let $q = p^m$, $S = \{a_1, a_2, \dots, a_n\}$ be a subset of \mathbb{F}_q with n distinct elements, v_1, v_2, \dots, v_n be nonzero elements in \mathbb{F}_q (not necessarily distinct), $v = (v_1, v_2, \dots, v_n)$. For $1 \le k \le n-1$, the GRS code is defined by

$$C_{grs}(\mathcal{S}, v, q) = \{c_f = (v_1 f(a_1), v_2 f(a_2), \dots, v_n f(a_n)) \in \mathbb{F}_q^n : f(x) \in \mathbb{F}_q[x], \deg f \le k - 1\}.$$

This is an MDS (linear) code over \mathbb{F}_q with parameters $[n, k, d]_q, d = n - k + 1$. The extended GRS code is defined by

$$C_{egrs}(S, v, q) = \{c_f = (v_1 f(a_1), v_2 f(a_2), \dots, v_n f(a_n), f_{k-1}) \in \mathbb{F}_q^{n+1} : f(x) \in \mathbb{F}_q[x], \deg f \le k-1\}$$

where f_{k-1} is the coefficient of x^{k-1} in f(x). This is also an MDS (linear) code over \mathbb{F}_q with parameters $[n+1,k,d]_q, d=n-k+2$.

For $S = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{F}_q$, we denote

$$\Delta_{\mathcal{S}}(a_i) = \prod_{\substack{1 \le j \le n \\ i \ne i}} (a_i - a_j) \in \mathbb{F}_q^*.$$

Let $\eta_q : \mathbb{F}_q^* \to \{\pm 1\}$ be the quadratic (multiplicative) character of \mathbb{F}_q . Namely, for $b \in \mathbb{F}_q^*$,

$$\eta_q(b) = \begin{cases} 1, & \text{if } b \text{ is a square in } \mathbb{F}_q^*, \\ -1, & \text{otherwise.} \end{cases}$$

Then $\eta_q(b) = (-1)^{\varphi_q(b)}$ where $\varphi_q(b) : \mathbb{F}_q^* \to \mathbb{F}_2$ is defined by

$$\varphi_q(b) = \begin{cases} 0, & \text{if } b \text{ is a square in } \mathbb{F}_q^*, \\ 1, & \text{otherwise.} \end{cases}$$

Namely, if $\mathbb{F}_q^* = \langle \theta \rangle$, then $\varphi_q(\theta^l) \equiv l \pmod{2} \ (1 \leq l \leq q-1)$.

A sufficient condition on set S has been given in [6] and [10] for C_{grs} and C_{egrs} being self-dual. From the proofs we can see that the sufficient condition is also necessary. Now we introduce the basic results given in [6] and [10].

Theorem 2.2. Let a_1, a_2, \ldots, a_n be distinct elements in \mathbb{F}_q , $\mathcal{S} = \{a_1, a_2, \ldots, a_n\}$.

- (1) ([6]) Suppose that n is even. There exists $v = (v_1, v_2, \dots, v_n) \in$ $(\mathbb{F}_{q}^{*})^{n}$ such that the (MDS) code $\mathcal{C}_{qrs}(\mathcal{S}, v, q)$ is self-dual if and only if all $\eta_q(\Delta_{\mathcal{S}}(a)) \ (a \in \mathcal{S}) \ are the same (which means that all \, \varphi_q(\Delta_{\mathcal{S}}(a)) \ (a \in \mathcal{S})$ S) are the same).
- (2) ([10]) Suppose that n is odd. There exists $v = (v_1, v_2, \dots, v_n) \in$ $(\mathbb{F}_q^*)^n$ such that the (MDS) code $\mathcal{C}_{eqrs}(\mathcal{S}, v, q)$ is self-dual code if and only if $\eta_q(-\Delta_{\mathcal{S}}(a)) = 1$ for all $a \in \mathcal{S}$ (which means that $\varphi_q(-\Delta_{\mathcal{S}}(a)) =$ 0 for all $a \in \mathcal{S}$).
- **Definition 2.3.** Let $\Sigma(q)$ be the set of all even number $n \geq 2$ such that there exists MDS self-dual code over \mathbb{F}_q with length n. Let $\Sigma(g,q)$ and $\Sigma(eg,q)$ be the set of all even number $n \geq 2$ such that there exists MDS self-dual code over \mathbb{F}_q with length n constructed by GRS code (Theorem 2.2 (1)) and EGRS code (Theorem 2.2 (2)) respectively. Namely,

$$\Sigma(g,q) = \left\{ n : \begin{array}{l} 2 \mid n \geq 2, \text{there exists a subset } \mathcal{S} \text{ of } \mathbb{F}_q, |\mathcal{S}| = n, \\ \text{such that all } \eta_q(\Delta_{\mathcal{S}}(a)) \ (a \in \mathcal{S}) \text{ are the same.} \end{array} \right\}.$$

$$\Sigma(eg,q) = \left\{ n : \begin{array}{l} 2 \mid n \geq 2, \text{there exists a subset} \quad \mathcal{S} \text{ of } \mathbb{F}_q, |\mathcal{S}| = n - 1, \\ \text{such that all } \eta_q(-\Delta_{\mathcal{S}}(a)) = 1 \text{ for all } a \in \mathcal{S}. \end{array} \right\}.$$

We have $\Sigma(q,q) \cup \Sigma(eq,q) \subseteq \Sigma(q)$.

2.2. Cyclotomic Numbers. A brief background on cyclotomic numbers is given in the following. For more details, the reader is referred to the book [9].

Let $q = p^m$ where p is an odd prime, $m \ge 1$. Let $q - 1 = ef, e \ge 1$ $2, \mathbb{F}_q^* = \langle \theta \rangle, \mathcal{D} = \langle \theta^e \rangle$. The cosets of the subgroup \mathcal{D} in \mathbb{F}_q^* are the following e-th cyclotomic classes

$$\mathcal{D}_{\lambda} = \mathcal{D}_{\lambda}^{(e)} = \theta^{\lambda} \mathcal{D} = \{ \theta^{\lambda + ej} : 0 \le j \le f - 1 \} \quad (0 \le \lambda \le e - 1).$$

Definition 2.4. For $0 \le i, j \le e-1$, the *e*-th cyclotomic numbers for $\mathbb{F}_q^* = \langle \theta \rangle$ are defined by

$$(i,j) = (i,j)_e = |(\mathcal{D}_i + 1) \cap \mathcal{D}_j| = \sharp \{x \in \mathcal{D}_i : x + 1 \in \mathcal{D}_j\}.$$

Lemma 2.5. Let $q = p^m$ where p is an odd prime, $m \ge 1, q - 1 = ef$ and $(i,j) = (i,j)_e$ $(0 \le i,j \le e-1)$ be the e-th cyclotomic numbers for $\mathbb{F}_q^* = \langle \theta \rangle.$

$$(1) (i,j) = (-i,j-i) = (pi,pj).$$

$$(1) (i, j) = (-i, j - i) = (pi, pj).$$

$$(2) (i, j) = \begin{cases} (j, i), & \text{if } 2 \mid f, \\ (j + \frac{e}{2}, i + \frac{e}{2}), & \text{if } 2 \nmid f. \end{cases}$$

(3)
$$\sum_{j=0}^{e-1} (i,j) = f - \theta_i$$
, where $\theta_i = \begin{cases} 1, & \text{if } 2 \mid f, i = 0 \text{ or } 2 \nmid f, i = \frac{e}{2}, \\ 0, & \text{otherwise.} \end{cases}$

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$$\sum_{i=0}^{e-1} (i,j) = f - \delta_{j,0}, \text{ where } \delta_{j,0} = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we are concerned with the cases of even number e. The values of e-th cyclotomic numbers for e=2 and 4 are listed as follows.

Lemma 2.6. ([9]) Let $q = p^m, p \ge 3, q - 1 = ef, (i, j) = (i, j)_e$ be the e-th cyclotomic numbers of \mathbb{F}_a^* .

(1) For e = 2,

(1.1) If $2 \mid f$, then $(0,0) = \frac{f}{2} - 1$, $(0,1) = (1,0) = (1,1) = \frac{f}{2}$; (1.2) If $2 \nmid f$, then $(0,1) = \frac{f+1}{2}$, $(0,0) = (1,0) = (1,1) = \frac{f-1}{2}$. (2) For e = 4, we have $q = s^2 + 4t^2$ where $s \in \mathbf{Z}$ is determined by $s \equiv 1 \pmod{4}$ and t is determined up to sign.

(2.1) If $2 \mid f$, the values of $(i, j) = (i, j)_4$ are listed in Table I where 16A = q - 11 - 6s, 16B = q - 3 + 2s + 8t, 16C = q - 3 + 2s, 16D = q - 3 + 2s, 16D = q - 3 + 2s, 16D = q - 3 + 2s + 8t, 16C = q - 3 + 2s + 16D =q-3+2s-8t, 16E=q+1-2s.

Tabl	e I	e =	4, 2 f		Tai	ble II	e =	$=4, 2 \nmid f$	
i j	0	1	2	3	v	\vee		2	
0	A	В	C E	\overline{D}	0	A	В	C	D
1	B	D	E	E	1	E	E	$C \\ D$	B
			C		2	A	E	A	E
3	D	E	E	B	3	E	D	B	E

(2.2) If $2 \nmid f$, the values of $(i, j) = (i, j)_4$ are listed in Table II where 16A = q - 7 + 2s, 16B = q + 1 + 2s - 8t, 16C = q + 1 - 6s, 16D =q + 1 + 2s + 8t, 16E = q - 3 - 2s.

3. Main Results

Let $q = p^m$ where p is an odd prime and $m \geq 1, \mathbb{F}_q^* = \langle \theta \rangle, q - 1 =$ $ef, 2 \mid e, \mathcal{D} = \langle \theta^e \rangle, \mathcal{D}_{\lambda} = \mathcal{D}_{\lambda}^{(e)} = \theta^{\lambda} \mathcal{D} \ (0 \leq \lambda \leq e - 1).$ For a subset I of $\mathbf{Z}_e = \{0, 1, \dots, e-1\}, |I| = l \ (1 \le l \le e).$ Let \mathcal{S} and $\widetilde{\mathcal{S}}$ be subsets of \mathbb{F}_q defined by

(3.1)
$$S = \bigcup_{\lambda \in I} \mathcal{D}_{\lambda}, \quad \widetilde{S} = S \bigcup \{0\},$$

then $|\mathcal{S}| = fl, |\widetilde{\mathcal{S}}| = fl + 1.$

The following Lemma follows from the aforementioned Theorem 2.2.

Lemma 3.1. (1) Assume that $2 \mid fl$. If $\varphi_q(\Delta_{\mathcal{S}}(a)) \in \mathbb{F}_2 = \{0, 1\}$ $(a \in \mathcal{S})$ are the same, then $fl \in \Sigma(g, q)$. If $\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a)) = \varphi_q(-1)$ for all $a \in \widetilde{\mathcal{S}}$, then $fl + 2 \in \Sigma(eg, q)$.

(2) Assume that $2 \nmid fl$. If $\varphi_q(\Delta_{\mathcal{S}}(a)) = \varphi_q(-1)$ for all $a \in \mathcal{S}$, then $fl + 1 \in \Sigma(eg, q)$. If $\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a))$ $(a \in \widetilde{\mathcal{S}})$ are the same, then $fl + 1 \in \Sigma(g, q)$.

Now we compute $\varphi_q(\Delta_{\mathcal{S}}(a))$ and $\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a))$ by using the *e*-th cyclotomic numbers $(i,j)=(i,j)_e$ on \mathbb{F}_q . For two subsets I,J of $\mathbf{Z}_e=\{0,1,\cdots,e-1\},|I|,|J|\geq l$. Denote

$$(I,J) = \sum_{\substack{i \in I \\ j \in J}} (i,j), \ (i,J) = (\{i\},J), (I,j) = (I,\{j\})$$

and

$$(I, \text{odd}) = \sum_{\substack{j=0\\2\nmid j}}^{e-1} (I, j), \quad (I, \text{even}) = \sum_{\substack{j=0\\2\mid j}}^{e-1} (I, j)$$

(odd, J) and (even, J) can be defined similarly.

Lemma 3.2. Let S and \widetilde{S} be subsets of \mathbb{F}_q defined by (3.1). Then for each $a \in \mathcal{D}_i, i \in I$,

$$\varphi_{q}(\Delta_{\mathcal{S}}(a)) \equiv (fl-1)(i+\frac{ef}{2}) + (odd, I-i) \pmod{2}
\varphi_{q}(\Delta_{\widetilde{\mathcal{S}}}(a)) \equiv \varphi_{q}(\Delta_{\mathcal{S}}(a)) + i \pmod{2}
\varphi_{q}(\Delta_{\widetilde{\mathcal{S}}}(0)) \equiv fl\frac{e}{2} + f|I_{odd}| \pmod{2}$$

 $where \ I_{odd} = \{i \in I: 2 \nmid i\}, I-i = \{j-i: j \in I\}.$

Proof. For each $a \in \mathcal{D}_i$, $i \in I$,

$$\Delta_{\mathcal{S}}(a) = \prod_{\substack{b \in \mathcal{S} \\ b \neq a}} (a - b) = \prod_{\lambda \in I} \prod_{\substack{b \in \mathcal{D}_{\lambda} \\ b \neq a}} (a - b) \quad (\text{let } b = ac)$$

$$= \prod_{\substack{\lambda \in I \\ c \in \mathcal{D}_{\lambda - i} \\ c \neq 1}} (a - ac) = (-a)^{fl-1} \prod_{\substack{\lambda \in I \\ c \in \mathcal{D}_{\lambda - i} \\ c \neq 1}} (c - 1).$$

Note $2 \mid e$, we know that for $\xi \in \mathcal{D}_{\lambda}$, $\varphi_q(\xi) \equiv \lambda \pmod{2}$ and $\varphi_q(-1) = \frac{ef}{2}$. Hence

$$\varphi_q(\Delta_{\mathcal{S}}(a)) \equiv (fl-1)(\frac{ef}{2}+i) + \prod_{\lambda \in I} \prod_{\substack{c \in \mathcal{D}_{\lambda-i} \\ c-1 \in \mathcal{D}_{\mu}, 2 \nmid \mu}} 1 \pmod{2}$$
$$\equiv (fl-1)(\frac{ef}{2}+i) + (\text{odd}, I-i) \pmod{2}.$$

On the other hand, from $\widetilde{S} = S \bigcup \{0\}$, we get $\Delta_{\widetilde{S}}(a) = \Delta_{S}(a)a$. Thus

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a)) \equiv \varphi_q(\Delta_{\mathcal{S}}(a)) + i \pmod{2}.$$

At last,
$$\Delta_{\widetilde{S}}(0) = \prod_{a \in S} (-a) = (-1)^{fl} \prod_{i \in I} \prod_{a \in \mathcal{D}_i} a$$
. Therefore

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(0)) \equiv fl \frac{ef}{2} + f|I_{\text{odd}}| \equiv fl \frac{e}{2} + f|I_{\text{odd}}| \text{ (mod 2)}.$$

The following theorem will play a central role in determining the existence of MDS self-dual codes.

Theorem 3.3. Let $q = p^m$ $(p \ge 3), q - 1 = ef, 2 \mid e, \mathbb{F}_q^* = \langle \theta \rangle, \mathcal{D}_{\lambda} = \theta^{\lambda} \langle \theta \rangle$ $(0 \le \lambda \le e - 1)$. Let I be a subset of $\mathbf{Z}_e = \{0, 1, \dots, e - 1\}, |I| = l, 1 \le l \le e, \mathcal{S} = \bigcup_{\lambda \in I} \mathcal{D}_{\lambda}, \widetilde{\mathcal{S}} = \mathcal{S} \cup \{0\}, \text{ so that } |\mathcal{S}| = fl \text{ and } |\widetilde{\mathcal{S}}| = fl + 1.$ We get

Case 1: $2 \mid f$.

(1.1) If $i + (odd, I - i) \pmod{2}$ are the same for all $i \in I$, then $fl \in \Sigma(g, q)$.

(1.2) If (odd, I - i) are even for all $i \in I$, then $fl + 2 \in \Sigma(eg, q)$. Case 2: $2 \nmid f$ and $2 \mid l$.

(2.1) If $i + (odd, I - i) \pmod{2}$ are the same for all $i \in I$, then $fl \in \Sigma(g,q)$.

(2.2) If $|I_{odd}| \equiv \frac{e}{2} \pmod{2}$ and $(odd, I-i) \equiv 0 \pmod{2}$ for all $i, \ then \ fl+2 \in \Sigma(eg,q)$.

Case 3: $2 \nmid fl$.

 $\begin{array}{c} (3.1) \ If \ (odd, I-i) \equiv \frac{e}{2} \ (\bmod \ 2) \ for \ all \ i \in I, \ then \ fl+1 \in \Sigma(eg,q). \\ (3.2) \ If \ i + (odd, I-i) \equiv \frac{e}{2} + |I_{odd}| \ (\bmod \ 2) \ for \ all \ i \in I, \ then \ fl+1 \in \Sigma(q,q). \end{array}$

Proof. Case 1: For $2 \mid f$. Then $|\mathcal{S}| = fl$ is even. By Lemma 3.2 we have, for $a \in \mathcal{D}_i, i \in I$,

$$\varphi_q(\Delta_{\mathcal{S}}(a)) \equiv i + (\text{odd}, I - i) \pmod{2}$$

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a)) \equiv (\text{odd}, I - i) \pmod{2}$$

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(0)) \equiv 0 \pmod{2}.$$

The conclusions (1.1) and (1.2) can be derived from Lemma 3.1 (1). Case 2: For $2 \nmid f$ and $2 \mid l$. Then $|\mathcal{S}| = fl$ is even and for $a \in \mathcal{D}_i, i \in I$,

$$\varphi_q(\Delta_{\mathcal{S}}(a)) \equiv i + \frac{e}{2} + (\text{odd}, I - i) \pmod{2}$$

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a)) \equiv \frac{e}{2} + (\text{odd}, I - i) \pmod{2}$$

$$\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(0)) \equiv |I_{\text{odd}}| \pmod{2}.$$

The conclusions (2.1) and (2.2) can be derived from Lemma 3.1 (1). Case 3: For $2 \nmid fl$, we have, for $a \in \mathcal{D}_i, i \in I$,

$$\varphi_q(\Delta_{\mathcal{S}}(a)) \equiv (\text{odd}, I - i) \pmod{2}
\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(a)) \equiv (\text{odd}, I - i) + i \pmod{2}
\varphi_q(\Delta_{\widetilde{\mathcal{S}}}(0)) \equiv \frac{e}{2} + |I_{\text{odd}}| \pmod{2}.$$

The conclusions (3.1) and (3.2) can be derived from Lemma 3.1 (2).

At the end of this section we show several general consequences of Theorem 3.3. For doing this we need to determine the parity of the number (odd, I) for certain subset I of $\{0, 1, \dots, e-1\}$.

Lemma 3.4. Let $q = p^m$ $(p \ge 3, m \ge 1), q - 1 = ef, 2 \mid e \ge 2, (i, j) = (i, j)_e$ $(i, j \in \mathbf{Z}_e = \mathbf{Z}/e\mathbf{Z})$ be the e-th cyclotomic numbers on \mathbb{F}_q . Then for $i \in \mathbf{Z}_e$,

- (1) $(odd, i) + (even, i) = f \delta_{i,0}$.
- (2) Assume that $2 \mid f$.
- If $2 \mid i$, then (odd, i) = (odd, -i), (even, i) = (even, -i).
- If $2 \nmid i$, then (odd, i) = (even, -i), (even, i) = (odd, -i). Particularly, if $e \equiv 2 \pmod{4}$, then $(odd, \frac{e}{2}) = (even, \frac{e}{2}) = \frac{f}{2}$.
 - (3) Assume that $2 \nmid f$.
 - $\textit{If } 2 \mid i + \tfrac{e}{2}, \textit{ then } (odd, i) = (odd, -i), (even, i) = (even, -i).$

If $2 \nmid i + \frac{e}{2}$, then (odd, i) = (even, -i). Particularly, if $e \equiv 2 \pmod{4}$, then $(odd, 0) = (even, 0) = \frac{f-1}{2}$.

Proof. (1) By Lemma 2.5 (3),
$$(\text{odd}, i) + (\text{even}, i) = \sum_{i=0}^{e-1} (j, i) = f - \delta_{i,0}$$
.

(2) Assume that $2 \mid f$. We have (i, j) = (j, i). Then (odd, i) = (i, odd), (even, i) = (i, even). From (i, j) = (-i, j - i) we get

Similarly, (even,
$$i$$
) =
$$\begin{cases} (\text{even}, -i), & \text{if } 2 \mid i, \\ (\text{odd}, -i), & \text{if } 2 \nmid i. \end{cases}$$
If $e \equiv 2 \pmod{4}$, then $\frac{e}{2}$ is odd and $(\text{odd}, \frac{e}{2}) = (\text{even}, -\frac{e}{2}) = (\text{even}, -\frac{e}{2})$

If $e \equiv 2 \pmod{4}$, then $\frac{e}{2}$ is odd and $(\text{odd}, \frac{e}{2}) = (\text{even}, -\frac{e}{2}) = (\text{even}, \frac{e}{2})$. But $(\text{odd}, \frac{e}{2}) + (\text{even}, \frac{e}{2}) = f$, we get $(\text{odd}, \frac{e}{2}) = (\text{even}, \frac{e}{2}) = \frac{f}{2}$. (3) Assume that $2 \nmid f$. From Lemma 2.5, we get

$$(j,i) = (i + \frac{e}{2}, j + \frac{e}{2}) = (-(i + \frac{e}{2}), j - i) = (j - (i + \frac{e}{2}), -i).$$

Therefore, if $2 \nmid i + \frac{e}{2}$, then

$$(\text{odd}, i) = \sum_{2 \nmid j} (j, i) = \sum_{2 \nmid j} (j - (i + \frac{e}{2}), -i) = (\text{even}, -i).$$

Similarly, if $2 \mid i + \frac{e}{2}$, then (odd, i) = (odd, -i) and (even, i) = (even, -i). If $e \equiv 2 \pmod{4}$, then $2 \nmid \frac{e}{2}$ and (odd, 0) = (even, 0). But (odd, 0) + (even, 0) = f - 1. Therefore $(\text{odd}, 0) = (\text{even}, 0) = \frac{f - 1}{2}$. \square

4. Examples

After above preparation, now we show several results on the length of MDS self-dual codes as applications of Theorem 3.3 and Lemma 3.2. It is known that if $q \equiv 3 \pmod{4}$, and $n \equiv 2 \pmod{4}$, then $n \notin \Sigma(q)$. Thus if $q \equiv 3 \pmod{4}$, we consider the case n = lf + a (a = 0, 1, 2) with $n \not\equiv 2 \pmod{4}$. Firstly, we consider the case l = |I| = 1 or 2.

Theorem 4.1. Let $q = p^m$ be a power of an odd prime $p, q-1 = ef, 2 \mid e \geq 2$.

- (1) If $2 \mid f$, then $f \in \Sigma(g,q)$. Moreover, if (odd,0) is even, then $f+2 \in \Sigma(eg,q)$. Particularly, if $e \equiv 2 \pmod{4}$ and $f \equiv 0 \pmod{4}$, then $f+2 \in \Sigma(eg,q)$.
- (2) If $2 \mid f$, then $2f + 2 \in \Sigma(eg, q)$. Moreover, if $4 \mid e$, then $2f \in \Sigma(eg, q)$.
- (3) If $2 \nmid f$. If $(odd, 0) \equiv \frac{e}{2} \pmod{2}$, then $f + 1 \in \Sigma(g, q)$ and $f + 1 \in \Sigma(eg, q)$. Particularly, if $e \equiv 2 \pmod{4}$ and $f \equiv 3 \pmod{4}$, then $f + 1 \in \Sigma(g, q) \cap \Sigma(eg, q)$.
- (4) If $2 \nmid f$ and there exists i such that $1 \equiv i \equiv \frac{e}{2} \pmod{2}$ and $(odd, 0) \equiv (odd, i) \pmod{2}$, then $2f + 2 \in \Sigma(eg, q)$.

Proof. (1) Suppose that $2 \mid f$. Take $I = \{0\}$, then l = |I| = 1 and (odd, I - 0) = (odd, 0 - 0) = (odd, 0). From Theorem 3.3 (1.1) and (1.2), we get $f \in \Sigma(g,q)$ and if $2 \mid (\text{odd}, 0)$, then $f + 2 \in \Sigma(eg,q)$. Moreover, by Lemma 3.4 (1) and (3) we know that

$$(\text{odd}, i) = \begin{cases} (\text{odd}, -i), & \text{if } 2 \mid i, \\ (\text{even}, -i) = f - (\text{odd}, -i) \equiv (\text{odd}, -i) \pmod{2}, & \text{if } 2 \nmid i. \end{cases}$$

Therefore

$$\sum_{i=0}^{e-1} (\text{odd}, i) = (\text{odd}, 0) + (\text{odd}, \frac{e}{2}) + \sum_{i=1}^{\frac{e}{2}-1} ((\text{odd}, i) + (\text{odd}, -i))$$

$$\equiv (\text{odd}, 0) + (\text{odd}, \frac{e}{2}) \text{ (mod 2)}.$$

On the other hand,

$$\sum_{i=0}^{e-1} (\text{odd}, i) = \sharp \{ x \in \mathcal{D}_1^{(2)} : x + 1 \neq 0 \} = \begin{cases} \frac{q-1}{2}, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q-3}{2}, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

If $e \equiv 2 \pmod{4}$ and $f \equiv 0 \pmod{4}$, then $q \equiv 1 \pmod{8}$ and $\sum_{i=0}^{e-1} (\text{odd}, i) = \frac{q-1}{2} \equiv 0 \pmod{2}$. Therefore $(\text{odd}, 0) \equiv (\text{odd}, \frac{e}{2}) \pmod{2}$

- 2). But from Lemma 3.4 (2), $(\text{odd}, \frac{e}{2}) = \frac{f}{2} \equiv 0 \pmod{2}$, we get 2 | (odd, 0) and $f + 2 \in \Sigma(eg, q)$.
 - (2) Assume that $2 \mid f$. By Lemma 3.4 (2), we have

$$(\mathrm{odd},i) = \left\{ \begin{array}{ll} (\mathrm{odd},-i), & \text{if } 2 \mid i, \\ (\mathrm{even},-i) = f - (\mathrm{odd},-i) \equiv (\mathrm{odd},-i) \pmod{2}, & \text{if } 2 \nmid i. \end{array} \right.$$

Therefore $(\text{odd}, i) \equiv (\text{odd}, -i) \pmod{2}$ for any i. Then we get

$$\sum_{i=0}^{e-1} (\text{odd}, i) = (\text{odd}, 0) + (\text{odd}, \frac{e}{2}) + \sum_{i=1}^{\frac{e}{2}-1} ((\text{odd}, i) + (\text{odd}, -i))$$

$$\equiv (\text{odd}, 0) + (\text{odd}, \frac{e}{2}) \text{ (mod 2)}.$$

But $\sum_{i=0}^{e-1} (\text{odd}, i) = f \equiv 0 \pmod{2}$. Therefore $(\text{odd}, 0) \equiv (\text{odd}, \frac{e}{2}) \pmod{2}$. Take $I = \{0, \frac{e}{2}\}$, then l = |I| = 2, and $(\text{odd}, I - 0) = (\text{odd}, I - \frac{e}{2}) = (\text{odd}, 0) + (\text{odd}, \frac{e}{2}) \equiv 0 \pmod{2}$. From Theorem 3.3 (1.2), we get $2f + 2 \in \Sigma(eg, q)$. Moreover, if $4 \mid e$, then $2f \in \Sigma(eg, q)$.

(3) Suppose that $2 \nmid f$, we also take $I = \{0\}, l = 1$. From Theorem 3.3 case 3, we know that if $(\text{odd}, 0) \equiv \frac{e}{2} \pmod{2}$, then $f + 1 \in \Sigma(g, q)$ and $f + 1 \in \Sigma(eg, q)$. Moreover, if $e \equiv 2 \pmod{4}$ and $f \equiv 3 \pmod{4}$,

then $(\text{odd}, 0) = \frac{f-1}{2} \equiv 1 \equiv \frac{e}{2} \pmod{2}$, we get $f + 1 \in \Sigma(g, q)$ and $f + 1 \in \Sigma(eg, q)$.

(4) Assume that $2 \nmid f$. By Lemma 3.4 (3), we have (odd, i) = (odd, -i) if $2 \mid i + \frac{e}{2}$. If there exists $i, 1 \equiv i \equiv \frac{e}{2} (\text{mod } 2)$ such that $(\text{odd}, 0) \equiv (\text{odd}, 2) (\text{mod } 2)$, we take $I = \{0, i\}$, then $|I_{\text{odd}}| = 1 \equiv \frac{e}{2} (\text{mod } 2)$ and $(\text{odd}, I - 0) = (\text{odd}, 0) + (\text{odd}, i) \equiv 0 \equiv (\text{odd}, 0) + (\text{odd}, -i) = (\text{odd}, I - i) (\text{mod } 2)$. By Theorem 3.3 (2.2), we get $2f + 2 \in \Sigma(eg, q)$.

Next we consider the semiprimitive case.

Definition 4.2. Let p be a prime, $p \nmid e \geq 2$. p is called semiprimitive module e if there exists a positive integer t such that $p^t \equiv -1 \pmod{e}$.

From now on, we take t to be the least positive integer such that $p^t \equiv -1 \pmod{e}$. Then the order of p module e is 2t.

Lemma 4.3. Let $2 \mid e \geq 4$, p be a semiprimitive prime module e and t be the least positive integer such that $p^t \equiv -1 \pmod{e}$. Let $r = p^m, q = r^2, m = ts, q - 1 = ef, R = r(-1)^s, \eta = \frac{R-1}{e}$. Then,

(1) $2 \mid f \text{ and } \eta \in \mathbb{Z}$;

(2) Let $(i,j) = (i,j)_e$ $(0 \le i,j \le e-1)$ be the cyclotomic numbers of order e on \mathbb{F}_q . Then (odd,0) is even and for $1 \le i \le e-1$,

$$(odd, i) = \begin{cases} \frac{R-1}{2} \pmod{2}, & \text{if } 2 \mid i, \\ \frac{R-1}{2} + \eta \pmod{2}, & \text{if } 2 \nmid i. \end{cases}$$

Proof. (1) $f = \frac{q-1}{e} = \frac{(r-1)(r+1)}{e}$ is even since $2 \nmid r$ and $r \equiv (-1)^s$ (mod e). Next, $R = p^{ts}(-1)^s \equiv (-1)^{s+s} \equiv 1 \pmod{e}$, we get $\eta = \frac{R-1}{e} \in \mathbb{Z}$.

(2) For the semiprimitive case, the cyclotomic numbers have been determined in ([10], Lemma 5) as follows

$$(0,0) = \eta^2 - (e-3)\eta - 1, (0,i) = (i,0) = (i,i) = \eta^2 + \eta \ (1 \le i \le e-1), (i,j) = \eta^2 \ (1 \le i \ne j \le e-1).$$

Then we get,

$$(\text{odd}, 0) = \sum_{\substack{i=0\\2\nmid i}}^{e-1} (i, 0) = \frac{e}{2} (\eta^2 + \eta) \equiv 0 \pmod{2},$$

and for $1 \le i \le e - 1$,

$$(\text{odd}, i) = \sum_{\substack{j=1\\2 \nmid i}}^{e-1} (j, i) = \begin{cases} \eta^2 \frac{e}{2} \equiv \eta \frac{e}{2} = \frac{R-1}{2} \pmod{2}, & \text{if } 2 \mid i, \\ (i, i) + (\frac{e}{2} - 1)\eta^2 \equiv \frac{R-1}{2} + \eta \pmod{2}, & \text{if } 2 \nmid i. \end{cases}$$

Theorem 4.4. Let p be a semiprimitive prime module e and t be the least positive integer such that $p^t \equiv -1 \pmod{e}$. Let $m = ts, r = p^m, q = r^2, q - 1 = ef, R = r(-1)^s, \eta = \frac{R-1}{e}$.

- (1) If $2 \mid \eta$, then $lf \in \Sigma(g,q)$ for all $1 \leq l \leq \frac{e}{2}$ and $lf + 2 \in \Sigma(eg,q)$ for all $1 \leq l \leq e$.
- (2) If $2 \nmid \eta$ and $4 \mid e$, then $lf \in \Sigma(g,q)$ for "all odd $l, 1 \leq l \leq e$ " and "all even $l, 2 \leq l \leq \frac{e}{2}$ ", and $lf + 2 \in \Sigma(eg,q)$ for "all even $2 \leq l \leq e$ " and "all odd $l, 1 \leq l \leq \frac{e}{2} 1$ ".
- (3) If $2 \nmid \eta$ and $e \equiv 2 \pmod{4}$, then $fl \in \Sigma(g,q)$ for "all odd $l, \leq l \leq e-1$ " and "all even $2 \leq l \leq \frac{e}{2}-1$ ", and $lf+2 \in \Sigma(eg,q)$ for "all even $l, 2 \leq l \leq e$ ".

Proof. Remark that f is even (Lemma 9 (1)).

- (1) If $2 \mid \eta = \frac{R-1}{e}$, then $2 \mid \frac{R-1}{2}$ and by Lemma 9, $(\text{odd}, i) \equiv 0 \pmod{2}$ for all $1 \leq i \leq e$. For any l, $1 \leq l \leq \frac{e}{2}$, we take a subset I of $2\mathbb{Z}_e = \{0, 2, 4, \dots, 2e-2\}$ with size |I| = l. For each $i \in I$, $i + (\text{odd}, I i) \equiv \sum_{j \in I} (\text{odd}, j i) \equiv 0 \pmod{2}$. By Theorem 3.3 (1.1)and (1.2), we get $fl \in \Sigma(g,q)$ and $fl + 2 \in \Sigma(eg,q)$ respectively.
- (2) If $2 \nmid \eta = \frac{R-1}{e}$ and $4 \mid e$, then $2 \mid \frac{R-1}{2}$ and for all $1 \leq i \leq e$, $(\text{odd}, i) \equiv i \pmod{2}$ (Lemma 9). Let $1 \leq l \leq e-1$ and I be a subset of $\{1, 2, \ldots, e\}$ with size |I| = l. Then for each $i \in I$,

$$i + (\text{odd}, I - i) = i + \sum_{j \in I} (\text{odd}, j - i) \equiv i + \sum_{j \in I} (j + i) \equiv (l + 1)i + \sum_{j \in I} j \pmod{2}.$$

If $2 \nmid l$, then $i + (\text{odd}, I - i) \equiv \sum_{j \in I} j \pmod{2}$ are the same for all $i \in I$. By Theorem 3.3 (1.1), we get $lf \in \Sigma(g, q)$. If $2 \mid l$, we also take $I \subseteq 2\mathbb{Z}_e, |I| = l, 2 \leq l \leq \frac{e}{2}$, then for all $i \in I$,

$$i + (\text{odd}, I - i) = i + \sum_{j \in I} j \equiv \sum_{j \in I} j \pmod{2}$$

are the same for all $i \in I$. By Theorem 3.3 (1,1), we get $lf \in \Sigma(g,q)$.

On the other hand, for each $I \subseteq \mathbb{Z}_e$, |I| = l, $(\text{odd}, I - i) \equiv li + \sum_{j \in I} j \pmod{2}$. If $2 \mid l, 2 \leq l \leq e$, it is easy to see that we have a subset I of \mathbb{Z}_e such that |I| = l and $\sum_{j \in I} j \equiv 0 \pmod{2}$. Therefore

 $(\text{odd}, I - i) \equiv 0 \pmod{2}$ for all $i \in I$. By Theorem 3.3 (1.2), we get $lf + 2 \in \Sigma(eg, q)$.

If $2 \nmid l$ and $1 \leq l \leq \frac{e}{2} - 1$, we take a subset I of $2\mathbb{Z}_e$ with size |I| = l. We also have $(\text{odd}, I - i) \equiv i + \sum_{j \in I} j \equiv 0 \pmod{2}$ for all $i \in I$. Then we get $lf + 2 \in \Sigma(eg, q)$ by Theorem 3.3 (1.2).

(3) If $2 \nmid \eta = \frac{R-1}{e}$ and $e \equiv 2 \pmod{4}$, then $2 \nmid \frac{R-1}{2}$. By Lemma 9, $2 \mid (\text{odd}, 0)$ and $(\text{odd}, i) \equiv i + 1 \pmod{2}$ for $1 \leq i \leq e - 1$. Let I be a

subset of Z_e with size $|I| = l, 1 \le l \le e$. Then

$$i + \sum_{j \in I} (\text{odd}, J - I) \equiv \left\{ \begin{array}{l} \sum\limits_{j \in I} 1 \ (\text{mod} \ \ 2), & \text{if} \ 2 \mid i, \\ 2 \nmid j & \\ 1 + \sum\limits_{j \in I} 1 \equiv 1 + l + \sum\limits_{j \in I} 1 \ (\text{mod} \ \ 2), & \text{if} \ 2 \nmid i. \end{array} \right.$$

By Theorem 3.3 (1.1), we get $lf \in \Sigma(g,q)$ for "odd $l, 1 \leq l \leq e$ ", and " even $l, 2 \le l \le \frac{e}{2} - 1$ ". On the other hand, for $i \in I$,

$$\sum_{i \in I} (\operatorname{odd}, I - i) \equiv \sum_{\substack{j \in I \\ j \neq i, 2 \nmid j - i}} 1 \equiv \left\{ \begin{array}{l} -1 + A \pmod{2}, & \text{if } 2 \mid i, \\ -1 + B \pmod{2}, & \text{if } 2 \nmid i. \end{array} \right.$$

where $A = \sharp \{j \in I : 2 \mid j\}, B = \sharp \{j \in I : 2 \nmid j\}, |A + B| = |I| = l$. Let $2 \mid l$ and $2 \leq l \leq e$. It is easy to find a subset I of \mathbb{Z}_e with size |I| = lsuch that both of A and B are odd. Then for each $i \in I$,

$$\sum_{i \in I} (\text{odd}, I - i) \equiv A - 1 \text{ or } B - 1 \equiv 0 \text{ (mod 2)}.$$

By Theorem 3.3 (1.2), $lf + 2 \in \Sigma(eg, q)$ for all even $l, 2 \le l \le e$.

- 5. Results for the cases e=2 and 4
- 5.1. When e=2. Let $q=p^n, p\geq 3, q-1=2f$. The cyclotomic numbers of order 2 are given in Lemma 2.6.

- **Theorem 5.1.** Let $q = p^n, p \ge 3, f = \frac{q-1}{2}$. (1) If $q \equiv 1 \pmod{4}$, then $2f + 2 \in \Sigma(eg, q)$. Moreover, if $q \equiv$ 1 (mod 8), then $f + 2 \in \Sigma(eg, q)$.
- (2) If $q \equiv 3 \pmod{4}$, then $2f + 2 \in \Sigma(eg, q)$. Moreover, if $q \equiv$ 7 (mod 8), then $f + 1 \in \Sigma(eg, q) \cap \Sigma(g, q)$.

Proof. (1) If $q \equiv 1 \pmod{4}$, then $2 \mid f$. Take $I = \{0, 1\}$,

$$(\text{odd}, I-0) = (\text{odd}, I-1) = (\text{odd}, 0) + (\text{odd}, 1) = (1, 0) + (1, 1) \equiv 0 \pmod{2},$$

from Theorem 3.3 (1.2), we get $2f + 2 \in \Sigma(eg, q)$. Moreover, if $q \equiv$ 1 (mod 8), we take $I = \{0\}$ or $\{1\}$,

$$(\text{odd}, I-0) = (\text{odd}, 0) = (1, 0) = \frac{f}{2} \text{ or } (\text{odd}, I-1) = (\text{odd}, 0) = (1, 0) = \frac{f}{2}$$

are even, so $f + 2 \in \Sigma(eg, q)$.

(2) If $q \equiv 3 \pmod{4}$, then $2 \nmid f$. Take $I = \{0, 1\}$, then $|I_{\text{odd}}| = 1 = \frac{e}{2}$ and $(\text{odd}, I - 0) = (\text{odd}, I - 1) = (1, 0) + (1, 1) = f - 1 \equiv 0 \pmod{2}$,

from Theorem 3.3 (2.2), we get $2f + 2 \in \Sigma(eg, q)$. Moreover, if $q \equiv 7 \pmod{8}$, we take $I = \{0\}$

$$(\text{odd}, I - 0) = (\text{odd}, 0) = (1, 0) = \frac{f - 1}{2} \equiv 1 \pmod{2},$$

from Theorem 3.3 (3.1), we get $f + 1 \in \Sigma(eg, q)$. If we take $I = \{1\}$,

$$1+(\text{odd}, I-1) = 1+(\text{odd}, 0) = 1+(1, 0) = \frac{f+1}{2} \equiv \frac{e}{2} + |I_{\text{odd}}| \pmod{2},$$
 from Theorem 3.3 (3.2), we get $f+1 \in \Sigma(g,q)$.

5.2. When e=4. The cyclotomic numbers of order 4 are given in Lemma 2.6. Now we get the following constructions of self-dual MDS codes by using cyclotomic classes of order four.

Theorem 5.2. Let $q = p^n \equiv 1 \pmod{4}, q - 1 = 4f$.

- (1) Assume that $p \equiv 1 \pmod{4}$.
- (1.1) If $2 \mid f \pmod{q} \equiv 1 \pmod{8}$, then $f, 2f \in \Sigma(g, q)$ and $f + 2, 2f + 2, 4f + 2 \in \Sigma(eg, q)$. Moreover, if $q \equiv 1 \pmod{16}$ and $t \equiv 2 \pmod{4}$, or $q \equiv 9 \pmod{16}$ and $4 \mid t$, then $3f \in \Sigma(g, q)$. If $q \equiv 1 \pmod{16}$ and $4 \mid t$, or $q \equiv 9 \pmod{16}$ and $t \equiv 2 \pmod{4}$, then $3f + 2 \in \Sigma(eg, q)$.
- (1.2) If $2 \nmid f \pmod{q} \equiv 5 \pmod{8}$, then $2f \in \Sigma(g,q)$ and $f+1,4f+2 \in \Sigma(eg,q)$.
 - (2) Assume that $p \equiv 3 \pmod{4}$. Then n = 2m is even.
- (2.1) If $p \equiv 7 \pmod{8}$ or $p \equiv 3 \pmod{8}$ and $2 \mid m$, then $f, 2f \in \Sigma(g,q)$ and $fl + 2 \in \Sigma(eg,q)$ for $1 \leq l \leq 4$.
- (2.2) If $p \equiv 3 \pmod{8}$ and $2 \nmid m$, then $fl \in \Sigma(g,q)$ for $1 \leq l \leq 3$ and $fl + 2 \in \Sigma(g,q)$ for l = 1, 2, 4.

Proof. (1.1) If $2 \mid f$, the conclusion $f, 2f \in \Sigma(g,q)$ can be derived from Theorem 3.3 (1.1) and Lemma 2.6 (2.1) by taking $I = \{0\}$ and $\{0,2\}$ respectively. The conclusion $f+2, 2f+2, 4f+2 \in \Sigma(eg,q)$ can be derived from Theorem 3.3 (1.2) and Lemma 2.6 (2.1) by taking $I = \{0\}, \{0,2\}$ and $\{0,1,2,3\}$ respectively.

Moreover, from $q \equiv 1 \pmod{8}$, $q = s^2 + 4t^2$ and $s \equiv 1 \pmod{4}$, we get $2 \mid t$. If $q \equiv 1 \pmod{16}$, $4 \mid t$ or $q \equiv 9 \pmod{16}$, $t \equiv 2 \pmod{4}$, then $8(\text{odd}, 1) = q - 1 - 4t \equiv 0 \pmod{16}$. By Lemma 2.6, $(\text{odd}, i) \equiv 0 \pmod{2}$ for all $0 \le i \le 3$. From Theorem 3.3 (1,1) and (1,2), we get $3f \in \Sigma(g,q)$, $3f + 2 \in \Sigma(eg,q)$ by taking $I = \{0,1,2\}$ respectively.

(1.2) If $2 \nmid f$, the conclusion $2f \in \Sigma(g,q)$ can be derived from Theorem 3.3 (II,1) and Lemma 2.6 (2.2), by taking $I = \{0,1\}$ or $\{0,3\}$ provided (odd, 1) = 0 or (odd, 3) = 0 respectively. The conclusion

- $4f + 2 \in \Sigma(eg, q)$ can be derived from Theorem 3.3 (II,2) and Lemma 2.6, by taking $I = \{0, 1, 2, 3\}$. The conclusion $f + 1 \in \Sigma(eg, q)$ can be derived from Theorem 3.3 by taking $I = \{0\}$.
- (2) Assume that $q \equiv 3 \equiv -1 \pmod{4}$. This is the semiprimitive case. Thus $q = r^2, r = p^m, 2 \mid f = \frac{q-1}{4}$ and $\eta = \frac{(-p)^m-1}{4}$. If $p \equiv 7 \pmod{8}$ or $p \equiv 3 \pmod{8}$ and $2 \mid m$, then $2 \mid \eta$. If $p \equiv 3 \pmod{8}$ and $2 \nmid m$, then $2 \nmid \eta$. The conclusion of (2.1) and (2.2) can be derived directly from Theorem 4.4.

Remark 5.3. We have examples satisfying the conditions provided in (1.1). Let $q = p^n, p \equiv 1 \pmod{4}$. For condition $q \equiv 1 \pmod{16}$ and $4 \mid t$, we have example $q = 113 = s^2 + 4t^2 = (-7)^2 + 4 \cdot 4^2$. For condition $q \equiv 9 \pmod{16}$ and $t \equiv 2 \pmod{4}$, we have examples $q = 25 = (-3)^2 + 4 \cdot 2^2$ and $q = 41 = 5^2 + 4 \cdot 2^2$.

Remark 5.4. One of our further work is consider the existence of MDS self-dual codes via cyclotomic numbers of order 6 and 8.

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