

# ON INTERSECTION DENSITY OF TRANSITIVE GROUPS OF DEGREE A PRODUCT OF TWO ODD PRIMES

Ademir Hujdurović<sup>a,b,1</sup>, Klavdija Kutnar<sup>a,b,2,\*</sup>, Bojan Kuzma<sup>a,b,c,3</sup>,  
Dragan Marušič<sup>a,b,c,4</sup>, Štefko Miklavič<sup>a,b,c,5</sup> and Marko Orel<sup>a,b,c,6</sup>

<sup>a</sup>University of Primorska, UP IAM, Muzejski trg 2, 6000 Koper, Slovenia

<sup>b</sup>University of Primorska, UP FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia

<sup>c</sup>IMFM, Jadranska 19, 1000 Ljubljana, Slovenia

## Abstract

Two elements  $g$  and  $h$  of a permutation group  $G$  acting on a set  $V$  are said to be *intersecting* if  $g(v) = h(v)$  for some  $v \in V$ . More generally, a subset  $\mathcal{F}$  of  $G$  is an *intersecting set* if every pair of elements of  $\mathcal{F}$  is intersecting. The *intersection density*  $\rho(G)$  of a transitive permutation group  $G$  is the maximum value of the quotient  $|\mathcal{F}|/|G_v|$  where  $G_v$  is a stabilizer of  $v \in V$  and  $\mathcal{F}$  runs over all intersecting sets in  $G$ . Intersection densities of transitive groups of degree  $pq$ , where  $p > q$  are odd primes, is considered. In particular, the conjecture that the intersection density of every such group is equal to 1 (posed in [*J. Combin. Theory, Ser. A* **180** (2021), 105390]) is disproved by constructing a family of imprimitive permutation groups of degree  $pq$  (with blocks of size  $q$ ), where  $p = (q^k - 1)/(q - 1)$ , whose intersection density is equal to  $q$ . The construction depends heavily on certain equidistant cyclic codes  $[p, k]_q$  over the field  $\mathbb{F}_q$  whose codewords have Hamming weight strictly smaller than  $p$ .

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## 1 Introductory remarks

Throughout this paper  $p$  and  $q$  will always denote prime numbers with  $p > q$ .

Let  $G \leq \text{Sym}(V)$  be a permutation group acting on a set  $V$ , where  $\text{Sym}(V)$  denotes the full symmetric group on  $V$ . Two elements  $g, h \in G$  are said to be *intersecting* if  $g(v) = h(v)$  for some  $v \in V$ . Furthermore, a subset  $\mathcal{F}$  of  $G$  is an *intersecting set* if every pair of elements of  $\mathcal{F}$  is intersecting. The *intersection density*  $\rho(\mathcal{F})$  of the intersecting set  $\mathcal{F}$  is defined to be the quotient

$$\rho(\mathcal{F}) = \frac{|\mathcal{F}|}{\max_{v \in V} |G_v|},$$

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\*Corresponding author e-mail: klavdija.kutnar@upr.si

where  $G_v$  is the point stabilizer of  $v \in V$ , and the *intersection density*  $\rho(G)$  (see [7]) of a group  $G$ , is the maximum value of  $\rho(\mathcal{F})$  where  $\mathcal{F}$  runs over all intersecting sets in  $G$ , that is,

$$\rho(G) = \max\{\rho(\mathcal{F}): \mathcal{F} \subseteq G, \mathcal{F} \text{ is intersecting}\} = \frac{\max\{|\mathcal{F}|: \mathcal{F} \subseteq G \text{ is intersecting}\}}{\max_{v \in V} |G_v|}.$$

Observe that every coset  $gG_v$ ,  $v \in V$  and  $g \in G$ , is an intersecting set, referred to as a *canonical* intersecting set. Clearly, in view of the above,  $\rho(G) \geq 1$ . In particular, for a transitive group  $G$  it follows that  $\rho(G) = 1$  if and only if the maximum cardinality of an intersecting set is  $|G|/|V|$ . Following [13] we define  $\mathcal{I}_n$  to be the set of all intersection densities of transitive permutation groups of degree  $n$ :

$$\mathcal{I}_n = \{\rho(G): G \text{ transitive of degree } n\},$$

and we let  $I(n)$  be the maximum value of  $\mathcal{I}_n$ . The *derangement graph*  $\Gamma_G = \text{Cay}(G, \mathcal{D})$  is a Cayley graph of  $G$  with the edge set consisting of all pairs  $(g, h) \in G \times G$  such that  $gh^{-1} \in \mathcal{D}$ , where  $\mathcal{D}$  is the set of all fixed-point-free elements (i.e. derangements) of  $G$ .

The following conjecture was posed in [13].

**Conjecture 1.1** [13, Conjecture 6.6] *Let  $G$  be a transitive permutation group of degree  $n$ . Then the following hold.*

- (i) *If  $n$  is even, but not a power of 2, then there is a transitive group  $H$  of degree  $n$  with  $\Gamma_H$  a complete multipartite graph with  $n/2$  parts.*
- (ii) *If  $n$  is a prime power, then  $I(n) = 1$ .*
- (iii) *If  $n = pq$  where  $p$  and  $q$  are odd primes, then  $I(n) = 1$ .*
- (iv) *If  $n = 2p$  where  $p$  is a prime, then  $I(n) = 2$ .*

Conjectures 1.1(ii) and (iv) were settled, respectively, in [6] and [17]. Furthermore in [6] it was shown that  $\mathcal{I}_{2p} = \{1, 2\}$  for every odd prime  $p$  and a complete characterization of groups of degree  $2p$  with intersection density 2 was also given there. In Proposition 2.1 we show that Conjecture 1.1(iii) is true for transitive groups of degree  $pq$ ,  $p > q$  odd primes, which contain a transitive subgroup with blocks of size  $p$ . Moreover, as a main result of this paper we give a construction of transitive groups of degree  $pq$  with blocks of size  $q$  and intersection density  $q$  (see Theorem 1.3 below). Consequently Conjecture 1.1(iii) is not true.

The above construction relies heavily on certain cyclic codes to which imprimitive permutation groups are associated in the following way. (See Section 3 for basic properties of cyclic codes needed in this construction.) Let  $C$  be a cyclic code of length  $m$  over  $\mathbb{F}_q$  and let  $V = \mathbb{Z}_q \times \mathbb{Z}_m$ . Let  $\alpha \in \text{Sym}(V)$  act according to the rule

$$\alpha: (i, j) \mapsto (i, j + 1) \text{ for } i \in \mathbb{Z}_q \text{ and } j \in \mathbb{Z}_m.$$

To each  $\mathbf{c} = (c_0, c_1, \dots, c_{m-1}) \in C$  a permutation  $\beta_{\mathbf{c}}$  acting according to the rule

$$\beta_{\mathbf{c}}: (i, j) \mapsto (i + c_j, j) \text{ for } i \in \mathbb{Z}_q \text{ and } j \in \mathbb{Z}_m$$

is assigned. Finally, we let  $G(C) \leq \text{Sym}(V)$  be the permutation group generated by  $\{\alpha\} \cup \{\beta_{\mathbf{c}} \mid \mathbf{c} \in C\}$ . Clearly,  $G(C)$  is an imprimitive permutation group with  $m$  blocks  $\{(i, j): i \in \mathbb{Z}_q\}$  of size  $q$ .

The following two theorems are the main results of this paper.

**Theorem 1.2** *Let  $q$  be a prime and  $m$  a positive integer. Let  $C$  be a nonzero cyclic code over  $\mathbb{F}_q$  of length  $m$  such that no codeword has maximal Hamming weight  $m$ . Then the corresponding permutation group  $G(C)$  of  $\mathbb{Z}_q \times \mathbb{Z}_m$  has intersection density equal to  $q$ .*

**Theorem 1.3** *Let  $p$  and  $q$  be odd primes such that  $p = \frac{q^k - 1}{q - 1}$  for some positive integer  $k$ . Then there exists an imprimitive group of degree  $pq$  with blocks of size  $q$  whose intersection density equals  $q$ .*

In Section 2 the structure of transitive permutation groups of degree  $pq$  is described, in Section 3 basic properties of cyclic codes are presented, and in Section 4 the proofs of Theorems 1.2 and 1.3 are given.

## 2 Hierarchy of transitive groups of degree $pq$

Let  $G$  be a transitive permutation group  $G$  acting on a set  $V$ . A partition  $\mathcal{B}$  of  $V$  is called  *$G$ -invariant* if the elements of  $G$  permute the parts, the so called *blocks* of  $\mathcal{B}$ , setwise. If the trivial partitions  $\{V\}$  and  $\{\{v\} : v \in V\}$  are the only  $G$ -invariant partitions of  $V$ , then  $G$  is *primitive*, and is *imprimitive* otherwise. In the latter case a corresponding nontrivial  $G$ -invariant partition will be referred to as a *complete imprimitivity block system* of  $G$ . We say that  $G$  is *doubly transitive* if given any two ordered pairs  $(u, v)$  and  $(u', v')$  of elements  $u, v, u', v' \in V$ , such that  $u \neq v$  and  $u' \neq v'$ , there exists an element  $g \in G$  such that  $g(u, v) = (u', v')$ . Note that a doubly transitive group is primitive. A primitive group which is not doubly transitive is called *simply primitive*.

A transitive group of degree  $pq$  falls into one of the following three classes: it either has blocks of size  $p$  or it has blocks of size  $q$  or it is a primitive group. (For a detailed description of these groups see [10, 11, 15, 16].) In the latter case the group is either doubly transitive in which case the intersection density is known to be equal to 1 (see [13, Lemma 2.1(3)]) or it is simply primitive. Using the fact that  $\rho(G) \leq \rho(H)$  for transitive groups  $H \leq G$  (see [13, Lemma 6.5]) the list of all simply primitive groups is further reduced to a shorter list of simply primitive groups containing no imprimitive subgroups (see [3, 11]). For each group on this list the corresponding intersection density will have to be computed. Coming back to imprimitive groups, the first of the above three classes is the easiest to deal with when considering intersection density. The following holds.

**Proposition 2.1** *Let  $G$  be a transitive group of degree  $pq$ ,  $p > q$  primes, containing an imprimitive subgroup  $H$  with  $q$  blocks of size  $p$ . Then  $\rho(G) = 1$ .*

PROOF. It may be seen that  $H$  contains a derangement of order  $p$ , in fact a semiregular element  $\alpha$  of order  $p$  such that the set of orbits of  $\langle \alpha \rangle$  forms an  $H$ -invariant partition  $\mathcal{B}$ , see [9]. Let  $\bar{H}$  be the permutation group induced by the action of  $H$  on  $\mathcal{B}$ . Then, by [6, Lemma 3.1],  $\rho(H) \leq \rho(\bar{H})$ . Since  $\bar{H}$  is a transitive group of prime degree we have  $\rho(\bar{H}) = 1$  by [6, Theorem 1.4], and so  $\rho(H) = 1$ . Moreover, by [13, Lemma 6.5],  $\rho(G) \leq \rho(H)$ , and the result follows. ■

This leaves us with imprimitive groups having blocks of size  $q$  and containing no transitive subgroups having blocks of size  $p$ . The case  $q = 2$  was settled in [6, 17]. As mentioned in Section 1 we will construct in Section 4 a family of imprimitive groups with blocks of size  $q$  having intersection density equal to  $q$ , thus disproving Conjecture 1.1(iii). In Example 2.2 below we give the smallest counter-example to this conjecture, a group of degree  $3 \cdot 11$  with intersection density 3. This group, however, is not part of the family of groups from Theorem 1.3 where the smallest

group occurs for  $q = 3$  and  $p = 13$ . In summary, in order to obtain a complete characterization of intersection densities of transitive groups of degree  $pq$ , simply primitive groups from the above mentioned list and imprimitive groups with blocks of size  $q$  will have to be addressed.

**Example 2.2** Let  $a, b_0 \in \text{Sym}(\mathbb{Z}_{33})$  be defined with  $a(i) = i + 3 \pmod{33}$  and  $b_0 = (0\ 1\ 2)$ . For  $k \in \{1, \dots, 10\}$  let  $b_k = b_0^{a^k} = (3k\ 3k + 1\ 3k + 2)$  and let  $b = b_0 \cdot b_2 \cdot b_3^2 \cdot b_4^2 \cdot b_5^2 \cdot b_6$ . Define  $G = \langle a, b \rangle \leq \text{Sym}(\mathbb{Z}_{33})$ . It can be verified (with MAGMA, for example) that  $G$  is a transitive group of order  $3^5 \cdot 11$  admitting blocks  $\{3k, 3k + 1, 3k + 2\}$ ,  $k \in \mathbb{Z}_{11}$ , of size 3. The kernel  $K$  of the action of  $G$  on these blocks is an elementary abelian group of order  $3^5$ , and contains no non-identity semiregular element. Hence  $K$  is an intersecting set of size  $3^5 = 3 \cdot |G_v|$ . This shows that  $\rho(G) \geq 3$ . Since  $G$  admits a semiregular subgroup  $\langle a \rangle$  with three orbits, by [6, Proposition 2.6] it follows that  $\rho(G) = 3$ .

### 3 Cyclic codes

Let  $m$  be a positive integer,  $r$  a power of a prime, and  $\mathbb{F}_r$  the finite field with  $r$  elements. The polynomial  $x^m - 1 \in \mathbb{F}_r[x]$  has no repeated factors (which are irreducible over  $\mathbb{F}_r$ ) if and only if  $r$  and  $m$  are relatively prime, i.e.  $\gcd(r, m) = 1$  (see [4, Exercise 201]), which we assume in this section.

Let  $\mathbb{F}_r^m$  be the  $m$ -dimensional vector space over  $\mathbb{F}_r$  formed by all row vectors  $(c_0, c_1, \dots, c_{m-1})$  with entries in  $\mathbb{F}_r$ . Let  $C$  be a linear  $[m, k]_r$  code, that is, a  $k$ -dimensional vector subspace in  $\mathbb{F}_r^m$ . A linear code  $C$  is *cyclic* if  $(c_0, c_1, \dots, c_{m-1}) \in C$  implies  $(c_{m-1}, c_0, \dots, c_{m-2}) \in C$ . The vector space  $\mathbb{F}_r^m$  can be identified with the principal ideal domain  $\mathbb{F}_r[x]/(x^m - 1)$ . Under this identification, cyclic codes correspond exactly to the ideals in  $\mathbb{F}_r[x]/(x^m - 1)$  (see [8, Theorem 9.36] or [4, Theorem 4.2.1]). The *generating polynomial*  $g(x)$  of a nonzero cyclic code  $C$  is the unique monic polynomial of the lowest degree in  $C$ . In this case  $C = \langle g(x) \rangle := \{a(x)g(x) : a(x) \in \mathbb{F}_r[x]\}$ , where the multiplication is done modulo  $x^m - 1$ . Moreover,  $g(x)$  divides the polynomial  $x^m - 1$  in  $\mathbb{F}_r[x]$ , and the dimension of the cyclic code  $C$  equals  $k = m - \deg g(x)$ . The polynomial  $h(x) = (x^m - 1)/g(x)$  is the *parity-check polynomial* of  $C$ . If  $g(x) = \sum_{i=0}^{m-k} g_i x^i$  where  $g_i \in \mathbb{F}_r$ , then  $C$ , viewed in  $\mathbb{F}_r^m$ , is spanned by  $k$  vectors

$$(g_0, g_1, \dots, g_{m-k}, 0, \dots, 0), (0, g_0, g_1, \dots, g_{m-k}, 0, \dots, 0), \dots, (0, \dots, 0, g_0, g_1, \dots, g_{m-k}).$$

Let

$$\Phi_m(x) = \prod_{d|m} (x^d - 1)^{\mu(m/d)} \tag{1}$$

be the  $m$ -th *cyclotomic polynomial*. Here  $\mu$  is the Möbius function

$$\mu(t) = \begin{cases} 1 & \text{if } t = 1, \\ 0 & \text{if a square of some prime divides } t, \\ (-1)^s & \text{if } t \text{ is a product of } s \text{ distinct primes.} \end{cases}$$

Then, by [18, Equation 9.20], we have

$$x^m - 1 = \prod_{d|m} \Phi_d(x). \tag{2}$$

By [18, Theorem 9.14], the function (1) is indeed a polynomial with integer coefficients. Hence, cyclotomic polynomials can be understood also as elements in  $\mathbb{F}_r[x]$ . Since  $\gcd(r, m) = 1$ , [18,

Theorem 9.16] implies that  $\Phi_m(x)$  is a product of  $\phi(m)/k$  distinct monic polynomials in  $\mathbb{F}_r[x]$  that are irreducible over  $\mathbb{F}_r$  and of degree  $k$ , which is the least positive integer such that  $r^k \equiv 1 \pmod{m}$ . Here,  $\phi$  is the Euler function. If  $h(x) \in \mathbb{F}_r[x]$  is one of the irreducible factors of  $\Phi_m(x)$  over  $\mathbb{F}_r$ , then, by (2), it divides the polynomial  $x^m - 1 \in \mathbb{F}$ . Hence,  $h(x)$  is the parity-check polynomial of the cyclic code  $\langle g(x) \rangle$ , where  $g(x) = (x^m - 1)/h(x)$ . The following result is proved in [12, Equation 2.10] (see also [14]).

**Lemma 3.1 ([12])** *Let  $\gcd(r, m) = 1$  and let  $h(x) \in \mathbb{F}_r[x]$  be a monic factor in  $\Phi_m(x)$ , which is irreducible over  $\mathbb{F}_r$  and of degree  $k$ . If  $\mathbf{c}$  is any nonzero codeword in the cyclic  $[m, k]_r$  code  $C$  with the parity-check polynomial  $h(x)$ , then the number  $Z(\mathbf{c})$  of zero entries in  $\mathbf{c}$  satisfies*

$$\left| Z(\mathbf{c}) - \frac{(r^{k-1} - 1)m}{r^k - 1} \right| \leq \left( 1 - \frac{1}{r} \right) \left( \frac{\gcd(m, r-1)}{r-1} - \frac{m}{r^k - 1} \right) r^{k/2}. \quad (3)$$

Recall that the value  $w(\mathbf{c}) := m - Z(\mathbf{c})$  is the *Hamming weight* of a codeword  $\mathbf{c}$  in a  $[m, k]_r$  code. If a code in Lemma 3.1 satisfies

$$\frac{r^k - 1}{r - 1} = \frac{m}{\gcd(m, r - 1)}, \quad (4)$$

then all its nonzero codewords have constant weight equal to

$$m - \frac{(r^{k-1} - 1)m}{r^k - 1} = m - (1 + r + r^2 + \cdots + r^{k-2})\gcd(m, r - 1). \quad (5)$$

In this case, the linearity of the code implies that the *Hamming distance*  $d(\mathbf{c}_1, \mathbf{c}_2) := w(\mathbf{c}_1 - \mathbf{c}_2)$  attains constant value (5) for all distinct codewords  $\mathbf{c}_1, \mathbf{c}_2 \in C$ , and the code is referred to as *equidistant*.

## 4 A family of groups with intersection density $q$

We start this section by proving Theorem 1.2. Recall from the introductory section that to every cyclic code  $C$  of length  $m$  over  $\mathbb{F}_q$  we can associate an imprimitive permutation group  $G(C)$  acting on  $\mathbb{Z}_q \times \mathbb{Z}_m$ .

**PROOF OF THEOREM 1.2.** Let  $C$  be a nonzero cyclic code of length  $m$  over  $\mathbb{F}_q$  such that no codeword has maximal Hamming weight  $m$ , and let  $G(C)$  be the permutation group associated with  $C$ . Let  $K$  be the subgroup of  $G(C)$  generated by  $\{\beta_{\mathbf{c}} \mid \mathbf{c} \in C\}$ . Observe that  $\beta : C \rightarrow K$  defined by  $\beta(\mathbf{c}) = \beta_{\mathbf{c}}$  is an isomorphism between the additive group of the code  $C$  and the group  $K$ . Therefore,  $K$  is an elementary abelian group of order  $q^k$ , where  $k$  is the dimension of  $C$ . Observe that  $K$  is normalized by  $\alpha$ , hence  $G(C) \cong K \rtimes \langle \alpha \rangle$ . It follows that  $|G(C)| = mq^k$ .

Since  $C$  is a nonzero cyclic code, it follows that for each  $j \in \mathbb{Z}_m$  there exists  $\mathbf{c} \in C$  with  $c_j \neq 0$ . Considering the action of  $\beta_{\mathbf{c}}$  it follows that  $K$  acts transitively on each of the sets  $\mathbb{Z}_q \times \{j\}$ , for each  $j \in \mathbb{Z}_m$ , and using the fact that  $\alpha$  permutes the sets  $\mathbb{Z}_q \times \{j\}$  it follows that  $G(C)$  acts transitively on  $\mathbb{Z}_q \times \mathbb{Z}_m$ . By the orbit-stabilizer theorem, it follows that the order of a point stabilizer in  $G(C)$  is  $\frac{mq^k}{mq} = q^{k-1}$ .

Let  $\mathbf{c} \in C$  and  $j \in \mathbb{Z}_m$  such that  $c_j = 0$ . Observe that  $\beta_{\mathbf{c}}$  fixes each element of the set  $\mathbb{Z}_q \times \{j\}$ . The assumption that each codeword of  $C$  has a zero entry implies that each element of  $K$  has a

fixed point. Since  $K$  is a subgroup of  $G(C)$  it follows that  $K$  is an intersecting set of  $G(C)$  of size  $q^k$ . We conclude that  $\rho(G) \geq q$ .

On the other hand, since  $G(C)$  admits a semiregular subgroup  $\langle \alpha \rangle$  with  $q$  orbits, by [6, Proposition 2.6] it follows that  $\rho(G(C)) \leq q$ , and so  $\rho(G(C)) = q$ . ■

Theorem 1.2 gives us a method of constructing transitive permutation groups with intersection density equal to  $q$ . However, the construction of cyclic codes of prime length  $p$  over the field  $\mathbb{F}_q$  with no codeword having the maximal Hamming weight  $p$  is an intriguing problem, which we address in the remainder of this section. The smallest example is an  $[11, 5]_3$  cyclic code, and the corresponding permutation group  $G(C)$  is given in Example 2.2. The following lemma will be needed.

**Lemma 4.1** *Let  $r$  be a power of a prime, and let  $k$  and  $m$  be positive integers satisfying (4). Then  $\gcd(m, r) = 1$  and  $k$  is the smallest positive integer such that  $r^k \equiv 1 \pmod{m}$ .*

PROOF. From (4) we deduce that

$$r^k - 1 = \frac{r - 1}{\gcd(m, r - 1)} m. \quad (6)$$

Let  $r$  be a power of a prime  $r_0$ . Suppose that  $\gcd(m, r) > 1$ . Then  $m \equiv 0 \pmod{r_0}$ . Since  $\frac{r-1}{\gcd(m, r-1)}$  is an integer, we deduce that the right-hand side of (6) vanishes modulo  $r_0$ , while the left-hand side equals  $-1$ , a contradiction. Hence,  $\gcd(m, r) = 1$ .

From (6) it is clear that  $r^k \equiv 1 \pmod{m}$ . Let  $i$  be any positive integer such that  $r^i \equiv 1 \pmod{m}$ . Then  $r^i - 1 = am$  and  $r - 1 = b \cdot \gcd(m, r - 1)$  for some integers  $1 \leq a$  and  $b \leq r - 1$ . Hence we deduce from (6) that

$$m \frac{r^k - 1}{r^i - 1} \leq am \frac{r^k - 1}{r^i - 1} = \frac{r - 1}{\gcd(m, r - 1)} m = bm \leq m(r - 1),$$

which yields  $r^i - 1 \geq 1 + r + r^2 + \dots + r^{k-1}$ . Hence,  $i \geq k$  as claimed. ■

The next result is deduced immediately from Lemmas 3.1 and 4.1.

**Corollary 4.2** *Let  $r$  be a power of a prime, and let  $k$  and  $m$  be positive integers satisfying (4). If  $h(x) \in \mathbb{F}_r[x]$  is a monic factor in  $\Phi_m(x)$ , which is irreducible over  $\mathbb{F}_r$ , then its degree is  $k$ , and the cyclic linear  $[m, k]_r$  code with the parity-check polynomial  $h(x)$  is equidistant and all its nonzero codewords have weight*

$$m \cdot \frac{r^k - r^{k-1}}{r^k - 1}.$$

Cyclic equidistant codes were characterized in [2]. Actually, they are just a bit more general than the codes from Corollary 4.2. (If you repeat a cyclic equidistant code you obtain a cyclic equidistant code.) For  $m$  a prime, Corollary 4.2 provides all cyclic equidistant codes. For a characterization of all (not necessarily cyclic) linear equidistant codes see [1].

In what follows we restrict the conditions on parameters  $m, r, k$  in Corollary 4.2. An (odd) prime  $p = m$  of the form

$$p = \frac{r^k - 1}{r - 1},$$

where  $r$  is a power of some prime, is said to be *projective* [5]. In this case it may be seen that  $k$  is necessarily a prime. Note that a projective prime with  $k = 2$  is necessarily a Fermat prime, and a projective prime with  $r = 2$  is a Mersenne prime (see [5]).

**Corollary 4.3** *Let  $r$  be a power of a prime, and let  $p = \frac{r^k-1}{r-1}$  be a projective prime. If  $h(x) \in \mathbb{F}_r[x]$  is an irreducible monic factor in  $\Phi_p(x)$  over  $\mathbb{F}_r$ , then its degree is  $k$  and the cyclic linear  $[p, k]_r$  code with the parity-check polynomial  $h(x)$  is equidistant and all its nonzero codewords have  $p - r^{k-1} > 0$  zero entries.*

PROOF. Since  $\frac{r^k-1}{r-1} = p = \frac{p}{\gcd(p, r-1)}$  the claim follows from Corollary 4.2. ■

**Example 4.4** *The Mersenne prime*

$$31 = \frac{2^5 - 1}{2 - 1} = \frac{5^3 - 1}{5 - 1}$$

*induces equidistant cyclic codes of the form  $[31, 5]_2$  and  $[31, 3]_5$ , where all nonzero codewords have 15 and 6 zero entries, respectively.*

We state a special case of Corollary 4.3, addressing the main goal of this paper, separately.

**Corollary 4.5** *Let  $p$  and  $q$  be odd primes such that  $p = \frac{q^k-1}{q-1}$  for some positive integer  $k$ . If  $h(x) \in \mathbb{F}_q[x]$  is an irreducible monic factor in  $\Phi_p(x)$  over  $\mathbb{F}_q$ , then its degree is  $k$  and the cyclic linear  $[p, k]_q$  code with the parity-check polynomial  $h(x)$  is equidistant and all its nonzero codewords have  $p - q^{k-1} > 0$  zero entries.*

Using Corollary 4.5 the proof of Theorem 1.3 is now straightforward.

PROOF OF THEOREM 1.3. Let  $C$  be a  $[p, k]_q$  cyclic code constructed in Corollary 4.5. Since  $C$  satisfies the assumptions of Theorem 1.2, it follows that the permutation group  $G(C)$  is a transitive permutation group of degree  $pq$  with the intersection density  $q$ . ■

Projective primes are related to the classification of finite simple groups. Based on heuristic arguments and computational evidence Jones and Zvonkin recently stated an interesting conjecture claiming that there are infinitely many projective primes [5, Conjecture 1.3]. They made an even stronger conjecture, claiming that for any fixed prime  $k \geq 3$ , there are infinitely many primes  $q$  such that

$$\frac{q^k - 1}{q - 1}$$

is a prime [5, Conjecture 6.5]. If this conjecture is true, then there are infinitely many pairs  $p, q$  of primes that satisfy the assumption in Theorem 1.3. In any case, the computational evidence in [5, Table 3] implies that there are plenty of such pairs.

**Remark 4.6** *Observe that if in Lemma 3.1 we have*

$$\frac{(r^{k-1} - 1)m}{r^k - 1} - \left(1 - \frac{1}{r}\right) \left(\frac{\gcd(m, r-1)}{r-1} - \frac{m}{r^k - 1}\right) r^{k/2} > 0$$

or equivalently

$$m > \gcd(m, r-1) \cdot (r^{k/2} + 1) \cdot \frac{r^{k/2-1}}{r^{k/2-1} + 1},$$

then  $Z(\mathbf{c}) > 0$ . In particular, this is true whenever  $m \geq \gcd(m, r-1) \cdot (r^{k/2} + 1)$ , which simplifies into  $m \geq r^{k/2} + 1$  whenever  $m$  is a prime.

So, let  $p$  and  $q$  be odd primes, let  $k$  be the smallest positive integer such that  $q^k = 1 \pmod{p}$ , and assume that  $p \geq q^{k/2} + 1$ . Then by the argument in the previous paragraph we again have that each codeword has some zero entries, leading to a construction of a transitive permutation group of degree  $pq$  and intersection density  $q$ .

For example, the parameters  $p = 757$ ,  $q = 3$ ,  $k = 9$  have these properties. Namely,  $q^k - 1 = 26 \cdot 757$ ,  $p \geq q^{k/2} + 1 \doteq 141.3$ , and it can be verified that  $q^i \neq 1 \pmod{p}$  for  $1 \leq i \leq 8$ .

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