# AVERAGES AND MAXIMAL AVERAGES OVER PRODUCT $j$-VARIETIES IN FINITE FIELDS 

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#### Abstract

We study both averaging and maximal averaging problems for Product $j$ varieties defined by $\Pi_{j}=\left\{x \in \mathbb{F}_{q}^{d}: \prod_{k=1}^{d} x_{k}=j\right\}$ for $j \in \mathbb{F}_{q}^{*}$, where $\mathbb{F}_{q}^{d}$ denotes a $d$-dimensional vector space over the finite field $\mathbb{F}_{q}$ with $q$ elements. We prove the sharp $L^{p} \rightarrow L^{r}$ boundedness of averaging operators associated to Product $j$-varieties. We also obtain the optimal $L^{p}$ estimate for a maximal averaging operator related to a family of Product $j$-varieties $\left\{\Pi_{j}\right\}_{j \in \mathbb{F}_{q}^{*}}$.


## 1. Introduction

In recent year, problems in Euclidean Harmonic analysis have been studied in the finite field setting, where the Euclidean structure is replaced by that of a vector space over a finite field. This approach may be efficient to relate analysis problems to well-studied problems in other areas such as the number theory and additive combinatorics. Moreover, problems in finite fields give us unique, interesting points as well as difficulties inherent in them.

In 1996, Wolff [21] suggested the finite field analogue of the Kakeya conjecture. In 2008, Dvir [3] solved this conjecture by using the polynomial method which is based on work in computer science. It has been applied to the Euclidean problems (for example, see [7, 5, 6]). In 2002, Mockenhaupt and Tao [18] initially posed and studied the finite field restriction problem for algebraic varieties. Much attention has been given to this problem, in part because there exist some different restriction phenomena between the Euclidean problem and its finite field analog. We refer readers to [4, 15, 17, 13, 16, 12, 10, 20, 11, 14] for background and recent development on the finite field restriction estimates for algebraic varieties. For the setting of rings of integers, see [8, 9].

More recently, Carbery, Stones, and Wright [2] introduced further harmonic analysis problems in the finite field setting. Among other things, they provided sharp results on

[^0]the finite field (maximal) averaging problems associated to certain $k$-dimensional varieties generated by vector-valued polynomials. Stones [19] addressed a sharp maximal theorem for dilation of quadratic surfaces in finite fields.

In this paper we will extend their work to more complicated varieties than quadratic surfaces. To precisely state our results, we need some notation and the definitions of the finite field averaging and maximal averaging problems. Let $\mathbb{F}_{q}^{d}$ be a $d$-dimensional vector space over a finite field with $q$ elements. Throughout this paper, we assume that $q$ is an odd prime power. Namely, the characteristic of $\mathbb{F}_{q}$ is strictly greater than two. We endow the space $\mathbb{F}_{q}^{d}$ with normalized counting measure, denoted by $d x$, which satisfies

$$
\int_{x \in \mathbb{F}_{q}^{d}} f(x) d x=q^{-d} \sum_{x \in \mathbb{F}_{q}^{d}} f(x),
$$

where $f$ is a complex valued function on $\mathbb{F}_{q}^{d}$. For $1 \leq s<\infty$, we define

$$
\|f\|_{L^{s\left(F_{q}^{d}\right)}}:=\left(q^{-d} \sum_{x \in \mathbb{F}_{q}^{d}}|f(x)|^{s}\right)^{1 / s}
$$

and $\|f\|_{L^{\infty}\left(\mathbb{F}_{q}^{d}\right)}:=\max _{x \in \mathbb{F}_{q}^{d}}|f(x)|$.
Let $V \subset \mathbb{F}_{q}^{d}$ be an algebraic variety, a set of common solutions to polynomial equations. Normalized surface measure on $V$, denoted by $d \sigma$, is associated to the variety $V$. Recall that the surface measure $d \sigma$ is defined by the relation

$$
\int_{\mathbb{F}_{q}^{d}} f(x) d \sigma(x)=\frac{1}{|V|} \sum_{x \in V} f(x)
$$

where $|V|$ denotes the cardinality of $V$.
The convolution function $f * g$ of functions $f, g$ on $\mathbb{F}_{q}^{d}$ is defined by

$$
f * g(x):=\frac{1}{q^{d}} \sum_{y \in \mathbb{F}_{q}^{d}} f(x-y) g(y) .
$$

Taking $g=d \sigma$, we see that

$$
f * d \sigma(x)=\frac{1}{|V|} \sum_{y \in V} f(x-y) .
$$

With the notation above, the averaging operator $A_{V}$ associated to $V$ is defined by

$$
A_{V} f(x):=f * d \sigma(x),
$$

where both $f$ and $A_{V} f$ are complex-valued functions defined on $\mathbb{F}_{q}^{d}$.
The following averaging problem for $V$ is first posed by Carbery-Stones-Wright [2].

Problem 1.1 (Averaging Problem). Let $d \sigma$ denote normalized surface measure on an algebraic variety $V$ in $\mathbb{F}_{q}^{d}$. Find all exponents $1 \leq p, r \leq \infty$ such that for some constant $C$ depending only on $p, r, d$, and $V$, the inequality

$$
\begin{equation*}
\|f * d \sigma\|_{L^{r}\left(\mathbb{F}_{q}^{d}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \tag{1.1}
\end{equation*}
$$

holds for all functions $f$ on $\mathbb{F}_{q}^{d}$.
The most important condition in the finite field averaging problem is that the operator norm is independent of the size of the underlying finite field $\mathbb{F}_{q}$. We will write $A_{V}(p \rightarrow$ $r) \lesssim 1$ if the averaging estimate (1.1) holds.

For each $1 \leq k \leq d-1$, Carbery-Stones-Wright [2] provided concrete $k$-dimensional surfaces in $\mathbb{F}_{q}^{d}$ for which they obtained the optimal result on the averaging problem. Indeed, for $1 \leq k \leq d-1$, they considered a variety $V_{k}$ as the image of the polynomial map $P_{k}: \mathbb{F}_{q}^{k} \rightarrow \mathbb{F}_{q}^{d}$ defined by

$$
\begin{equation*}
P_{k}(t)=\left(t_{1}, t_{2}, \ldots, t_{k}, t_{1}^{2}+\cdots+t_{k}^{2}, t_{1}^{3}+\cdots+t_{k}^{3}, \cdots, t_{1}^{d-k+1}+\cdots+t_{k}^{d-k+1}\right), \tag{1.2}
\end{equation*}
$$

and proved that $A_{V_{k}}(p \rightarrow r) \lesssim 1$ if and only if ( $1 / p, 1 / r$ ) is contained in the convex hull of points $(0,0),(0,1),(1.1)$, and $\left(\frac{d}{2 d-k}, \frac{d-k}{2 d-k}\right)$. We notice that the following Fourier decay estimate of the surface measure $d \sigma_{k}$ on $V_{k}$ was one of the most important ingredients in proving the optimal averaging estimate related to $V_{k}$ :

$$
\begin{equation*}
\max _{m \in \mathbb{F}_{q}^{d} \backslash\{(0, \ldots, 0)\}}\left|\left(d \sigma_{k}\right)^{\vee}(m)\right|:=\max _{m \in \mathbb{F}_{q}^{d} \backslash\{(0, \ldots, 0)\}}\left|\frac{1}{q^{k}} \sum_{x \in V_{k}} \chi(m \cdot x)\right| \leq C q^{-\frac{k}{2}} \tag{1.3}
\end{equation*}
$$

where $C$ is independent of $q$, and $\chi$ denotes a nontrivial additive character of $\mathbb{F}_{q}$.
In [2], Carbery-Stones-Wright also posed the maximal averaging problem for a family of algebraic varieties in $\mathbb{F}_{q}^{d}$. Let $\mathcal{A}$ be an indexing set. For each $\alpha \in \mathcal{A}$, let $d \sigma_{\alpha}$ denote normalized surface measure on an algebraic variety $V_{\alpha}$ in $\mathbb{F}_{q}^{d}$. Given any function $f$ : $\mathbb{F}_{q}^{d} \rightarrow \mathbb{C}$, the maximal averaging operator $M$ is defined by

$$
M f(x):=\sup _{\alpha \in \mathcal{A}}\left|f * d \sigma_{\alpha}(x)\right| \quad \text { for } x \in \mathbb{F}_{q}^{d}
$$

Problem 1.2 (Maximal Averaging Problem). Find all exponents $1 \leq p \leq \infty$ such that the inequality

$$
\|M f\|_{L^{p}\left(F_{q}^{d}\right)} \leq C\|f\|_{L^{p}\left(F_{q}^{d}\right)}
$$

holds for all complex-valued functions $f$ on $\mathbb{F}_{q}^{d}$, where the constant $C$ is independent of $q$.
Carbery-Stones-Wright [2] introduced a family of varieties in $\mathbb{F}_{q}^{d}$ for which they deduced the optimal result on the maximal averaging problem. More precisely, they first considered an indexing set $\mathcal{A}$ with $|\mathcal{A}|=q^{r}$ for some $0 \leq r \leq d-k$. For each $\alpha \in \mathcal{A}$, letting $M_{\alpha}$ be
an invertible $d \times d$ matrix over $\mathbb{F}_{q}$ and letting $b_{\alpha}$ be a vector in $\mathbb{F}_{q}^{d}$, they considered the following $k$-dimensional variety

$$
V_{k, \alpha}:=\left\{M_{\alpha} x+b_{\alpha} \in \mathbb{F}_{q}^{d}: x \in V_{k}\right\},
$$

where $V_{k}$ denotes the variety defined as in (1.2) and we identify a vector $x \in \mathbb{F}_{q}^{d}$ with a $d \times 1$ matrix. With the notation above, Carbery-Stones-Wright [2] proved that

$$
\left\|\sup _{\alpha \in \mathcal{A}}\left|f * d \sigma_{k, \alpha}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)}
$$

if and only if $r \leq k$ and $p \geq \frac{r+k}{k}$, where $d \sigma_{k, \alpha}$ denotes the normalized surface measure on the variety $V_{k, \alpha}$. Like the averaging problem for $V_{k}$, the Fourier decay estimate (1.3) was mainly used in deducing the optimal result on the maximal averaging problem.

The main purpose of this paper is to provide a complete solution of the averaging and maximal averaging problems related to certain varieties for which the Fourier decay estimate (1.3) is not satisfied. To this end, for each $j \in \mathbb{F}_{q}^{*}$, we consider an algebraic variety $\Pi_{j}$ in $\mathbb{F}_{q}^{d}$ defined by

$$
\Pi_{j}:=\left\{x=\left(x_{1}, x_{2}, \ldots, x_{d}\right) \in \mathbb{F}_{q}^{d}: \prod_{k=1}^{d} x_{k}=j\right\}
$$

We will call the variety $\Pi_{j}$ as Product $j$-variety. For each $j \in \mathbb{F}_{q}^{*}$, let $d \mu_{j}$ be normalized surface measure on Product $j$-variety $\Pi_{j}$. Unlike the Fourier decay estimate given in (1.3), the bound of $\left|\left(d \mu_{j}\right)^{\vee}(m)\right|$ becomes worse whenever we take any vector $m$ such that the number of zero components of $m=\left(m_{1}, m_{2}, \ldots, m_{d}\right)$ is large.

Definition 1.3. For each $m \in \mathbb{F}_{q}^{d}$, we denote by $\ell_{m}$ the number of zero components of $m$.

In fact, the inverse Fourier transform of the surface measure was explicitly computed as follows.

Lemma 1.4 ([1], Lemma 3.1). For each $j \in \mathbb{F}_{q}^{*}$, let $d \mu_{j}$ denote normalized surface measure on Product j-variety $\Pi_{j}$ in $\mathbb{F}_{q}^{d}$. Then we have

$$
\begin{equation*}
\left(d \mu_{j}\right)^{\vee}(m)=(-1)^{d-\ell_{m}}(q-1)^{-\left(d-\ell_{m}\right)} \quad \text { if } \quad 1 \leq \ell_{m} \leq d \tag{1.4}
\end{equation*}
$$

In addition, if $\ell_{m}=0$, then $\left|\left(d \mu_{j}\right)^{\vee}(m)\right| \lesssim q^{-\frac{(d-1)}{2}}$.

We may consider Product $j$-variety $\Pi_{j}$ in $\mathbb{F}_{q}^{d}$ as a $(d-1)$-dimensional surface since $\left|\Pi_{j}\right| \sim$ $q^{d-1}$. Notice that Lemma 1.4 implies that for every $j \neq 0$,

$$
\max _{\left.m \in \mathbb{F}_{q}^{d} \backslash(0, \ldots, 0)\right\}}\left|\left(d \mu_{j}\right)^{\vee}(m)\right| \sim q^{-1}
$$

Comparing this estimate with (1.3), we see that if $d>3$, then the Fourier decay on Product $j$-variety $\Pi_{j}$ is much worse than that on $V_{d-1}$. For this reason, the same argument used by Carbery-Stones-Wright [2] may not give us optimal results on both the averaging and maximal averaging problem related to Product $j$-varieties $\Pi_{j}$. However, our first result below indicates that even though the Fourier decay estimate on Product $j$-variety $\Pi_{j}$ is worse than that on the $V_{d-1}$, both varieties have the same mapping properties of the averaging operators.

Theorem 1.5. For each $j \in \mathbb{F}_{q}^{*}$, let $d \mu_{j}$ denote the normalized surface measure on Product $j$-variety $\Pi_{j}$. Then $A_{\Pi_{j}}(p \rightarrow r) \lesssim 1$ if and only if $(1 / p, 1 / r)$ lies on the convex hull of points $(0,0),(0,1),(1.1)$, and $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$.

Our next result is related to maximal averages associated to a family $\left\{\Pi_{j}\right\}_{j \in \mathbb{F}_{q}^{*}}$ of Product $j$-varieties $\Pi_{j}$.

Theorem 1.6. For each $j \in \mathbb{F}_{q}^{*}$, let $d \mu_{j}$ denote the normalized surface measure on Product $j$-variety $\Pi_{j}$. Then there is a constant $C$ independent of $q$ such that

$$
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)}
$$

if and only if $p \geq \frac{d}{d-1}$.

## 2. NECESSARY CONDITIONS

In this section, we prove the necessary conditions for the averaging and maximal averaging estimates given in Theorem 1.5 and Theorem 1.6, respectively. We begin by proving the following necessary conditions for the boundedness of the averaging operator $A_{\Pi_{j}}$ associated with Product $j$-variety $\Pi_{j}$.

Proposition 2.1. For each $j \in \mathbb{F}_{q}^{*}$, let $d \mu_{j}$ denote the normalized surface measure on Product $j$-variety $\Pi_{j}$. Suppose that the following inequality

$$
\begin{equation*}
\left\|f * d \mu_{j}\right\|_{L^{r}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \tag{2.1}
\end{equation*}
$$

holds for all functions $f$ on $\mathbb{F}_{q}^{d}$. Then $(1 / p, 1 / r)$ is contained in the convex hull of points $(0,0),(0,1),(1.1)$, and $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$.

Proof. We test the inequality (2.1) with $f=\delta_{\mathbf{0}}$, where $\delta_{\mathbf{0}}(x)=1$ if $x=(0, \ldots, 0)$, and 0 otherwise. Taking $f=\delta_{\mathbf{0}}$, we have

$$
\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)}=\left(\frac{1}{q^{d}} \sum_{x \in \mathbb{F}_{q}^{d}}\left|\delta_{\mathbf{0}}(x)\right|^{p}\right)^{\frac{1}{p}}=q^{-\frac{d}{p}}
$$

and

$$
\begin{aligned}
\left\|f * d \mu_{j}\right\|_{L^{r}\left(\mathbb{F}_{q}^{d}\right)} & =\left(\frac{1}{q^{d}} \sum_{x \in \mathbb{F}_{q}^{d}}\left|\left(\delta_{\mathbf{0}} * d \mu_{j}\right)(x)\right|^{r}\right)^{\frac{1}{r}} \\
& =\left(\frac{1}{q^{d}} \sum_{x \in \Pi_{j}}\left|\Pi_{j}\right|^{-r}\right)^{\frac{1}{r}} \sim q^{-\frac{d}{r}+\frac{(d-1)(1-r)}{r}} .
\end{aligned}
$$

Hence, invoking the assumption (2.1) we obtain a necessary condition:

$$
\begin{equation*}
\frac{1}{r} \geq \frac{d}{p}-d+1 \tag{2.2}
\end{equation*}
$$

Since the averaging operator is self-adjoint, we also have

$$
\frac{1}{p^{\prime}} \geq \frac{d}{r^{\prime}}-d+1
$$

which is equivalent to

$$
\begin{equation*}
\frac{1}{r} \geq \frac{1}{d p} \tag{2.3}
\end{equation*}
$$

Since $1 \leq p, r \leq \infty$, the proposition follows from (2.2) and (2.3).

We now state and prove the necessary conditions for the boundedness of the maximal averaging operator given in Theorem 1.6.

Proposition 2.2. Assume that the following maximal averaging estimate for a family of Product $j$-varieties $\Pi_{j}$ holds for all functions $f$ on $\mathbb{F}_{q}^{d}$ :

$$
\begin{equation*}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} . \tag{2.4}
\end{equation*}
$$

Then we have

$$
p \geq \frac{d}{d-1}
$$

Proof. As in the proof of Proposition 2.1, we test the inequality (2.4) with $f=\delta_{\mathbf{0}}$. Then it follows that

$$
\|f\|_{L^{p}\left(F_{q}^{d}\right)}=q^{-\frac{d}{p}} .
$$

On the other hand, we have

$$
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \geq\left(\frac{1}{q^{d}} \sum_{x \in \sum_{j \in \mathbb{F}_{q}^{*}} \Pi_{j}}\left(\sup _{j \in \mathbb{F}_{q}^{*}}\left|\left(\delta_{\mathbf{0}} * d \mu_{j}\right)(x)\right|\right)^{p}\right)^{\frac{1}{p}}
$$

Since $\left(\delta_{0} * d \mu_{j}\right)(x)=\frac{1}{\left|\Pi_{j}\right|} 1_{\Pi_{j}}(x) \sim q^{-(d-1)} 1_{\Pi_{j}}(x)$, we have

$$
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \gg q^{-\frac{d}{p}} q^{-d+1}\left(\sum_{x \in \bigcup_{j \in \mathbb{F}_{q}^{*}} \Pi_{j}}\left(\sup _{j \in \mathbb{F}_{q}^{*}} 1_{\Pi_{j}}(x)\right)^{p}\right)^{\frac{1}{p}} \sim q^{-\frac{d}{p}} q^{-d+1} q^{\frac{d}{p}}=q^{-d+1}
$$

Thus, the proposition follows from the hypothesis (2.4).

## 3. Sufficient conditions

Theorem 3.1. If $(1 / p, 1 / r)$ is contained in the convex hull of points $(0,0),(0,1),(1,1)$ and $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$, then the averaging inequality

$$
\begin{equation*}
\left\|f * d \mu_{j}\right\|_{L^{r}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \tag{3.1}
\end{equation*}
$$

holds for all functions $f$ on $\mathbb{F}_{q}^{d}$ and all $j \neq 0$.
Proof. Since $\left(f * d \mu_{j}\right)(x)=\left|\Pi_{j}\right|^{-1} \sum_{x \in \Pi_{j}} f(x)$ for all $x \in \mathbb{F}_{q}^{d}$, it is clear that the inequality (3.1) holds in the case when $(1 / p, 1 / r)=(0,0)$. By a direct computation, it is also true for $(1 / p, 1 / r)=(1,1)$. Thus, by using an interpolation theorem and nesting property of norms, we only need to verify the inequality (3.1) for $p=\frac{d+1}{d}$ and $r=d+1$. In other words, we aim to prove that the averaging estimate

$$
\begin{equation*}
\left\|f * d \mu_{j}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{\frac{d+1}{d}}\left(\mathbb{F}_{q}^{d}\right)} \tag{3.2}
\end{equation*}
$$

holds for all function $f$ on $\mathbb{F}_{q}^{d}$. For each $m \in \mathbb{F}_{q}^{d}$, let $\ell_{m}$ be the number of zero components of the vector $m$. Now, for each $k=0,1, \ldots, d$, we define

$$
N_{k}:=\left\{m \in \mathbb{F}_{q}^{d}: \ell_{m}=k\right\} .
$$

It is obvious that $\mathbb{F}_{q}^{d}=\bigcup_{k=0}^{d} N_{k}$. Since $\left(d \mu_{j}\right)^{\vee}=1_{N_{0}}\left(d \mu_{j}\right)^{\vee}+\sum_{k=1}^{d} 1_{N_{k}}\left(d \mu_{j}\right)^{\vee}$, we can decompose $d \mu_{j}$ as

$$
\begin{equation*}
d \mu_{j}=\widehat{1_{N_{0}}} * d \mu_{j}+\sum_{k=1}^{d} \widehat{1_{N_{k}}} * d \mu_{j} . \tag{3.3}
\end{equation*}
$$

By the definition of $N_{k}$ and the first part (1.4) of Lemma 1.4, we see that for $k=1,2, \ldots, d$,

$$
1_{N_{k}}\left(d \mu_{j}\right)^{\vee}=(-1)^{d-k}(q-1)^{-d+k} 1_{N_{k}}
$$

which in turn gives us

$$
\widehat{1_{N_{k}}} * d \mu_{j}=(-1)^{d-k}(q-1)^{-d+k} \widehat{1_{N_{k}}} .
$$

This can be combined with (3.3) to see that

$$
\begin{equation*}
d \mu_{j}=\widehat{1_{N_{0}}} * d \mu_{j}+\sum_{k=1}^{d}(-1)^{d-k}(q-1)^{-d+k} \widehat{1_{N_{k}}} . \tag{3.4}
\end{equation*}
$$

For each $j \neq 0$, let $\Omega_{j}=\widehat{1_{N_{0}}} * d \mu_{j}$. We have

$$
\left\|f * d \mu_{j}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} \leq\left\|f * \Omega_{j}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)}+\sum_{k=1}^{d}(q-1)^{-d+k}\left\|f * \widehat{1_{N_{k}}}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} .
$$

Hence, to prove (3.2), it will be enough to verify the following estimates:

$$
\begin{equation*}
\left\|f * \Omega_{j}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{\frac{d+1}{d}\left(F_{q}^{d}\right)}}, \tag{3.5}
\end{equation*}
$$

and for every $k=1,2, \ldots, d$,

$$
\begin{equation*}
(q-1)^{-d+k}\left\|f * \widehat{1_{N_{k}}}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{\frac{d+1}{d}\left(F_{q}^{d}\right)}} . \tag{3.6}
\end{equation*}
$$

3.1. proof of the inequality (3.5). We notice that the inequality (3.5) can be obtained by interpolating the following two estimates:

$$
\begin{gather*}
\left\|f * \Omega_{j}\right\|_{L^{2}\left(F_{q}^{d}\right)} \lesssim q^{-\frac{d-1}{2}}\|f\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)},  \tag{3.7}\\
\left\|f * \Omega_{j}\right\|_{L^{\infty}\left(\mathbb{F}_{q}^{d}\right)} \lesssim q\|f\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right.} . \tag{3.8}
\end{gather*}
$$

Thus, we only need to show that the inequalities (3.8) and (3.7) hold for any functions $f$ on $\mathbb{F}_{q}^{d}$. We can easily verify the inequality (3.7) by applying the Plancherel theorem and the second conclusion of Lemma 1.4 as follows:

$$
\begin{aligned}
\left\|f * \Omega_{j}\right\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)} & =\left\|f^{\vee} \Omega_{j}^{\vee}\right\|_{\ell^{2}\left(\mathbb{F}_{q}^{d}\right)} \\
& =\left(\sum_{m \in N_{0}}\left|f^{\vee}(m)\right|^{2}\left|\left(d \mu_{j}\right)^{\vee}(m)\right|^{2}\right)^{\frac{1}{2}} \\
& \lesssim q^{-\frac{d-1}{2}}\left(\sum_{m \in N_{0}}|\widehat{f}(m)|^{2}\right)^{\frac{1}{2}} \\
& \lesssim q^{-\frac{d-1}{2}}\|\widehat{f}\|_{\ell^{2}\left(\mathbb{F}_{q}^{d}\right)}=q^{-\frac{d-1}{2}}\|f\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)}
\end{aligned}
$$

To prove the inequality (3.8), we first notice by Young's inequality for convolution functions that

$$
\left\|f * \Omega_{j}\right\|_{L^{\infty}\left(\mathbb{F}_{q}^{d}\right)} \leq\left\|\widehat{1_{N_{0}}} * d \mu_{j}\right\|_{L^{\infty}\left(F_{q}^{d}\right)}\|f\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)} .
$$

Then the inequality (3.8) will be established by showing that the inequality

$$
\begin{equation*}
\max _{x \in \mathbb{F}_{q}^{d}}\left|\left(\widehat{1_{N_{0}}} * d \mu_{j}\right)(x)\right| \lesssim q \tag{3.9}
\end{equation*}
$$

holds for all $j \in \mathbb{F}_{q}^{*}$. To prove this inequality, we fix $x \in \mathbb{F}_{q}^{d}, j \in \mathbb{F}_{q}^{*}$, and observe that

$$
\begin{aligned}
& \left|\left(\widehat{1_{N_{0}}} * d \mu_{j}\right)(x)\right|=\left|\frac{1}{\left|\Pi_{j}\right|} \sum_{y \in \Pi_{j}} \widehat{1_{N_{0}}}(x-y)\right|=\left|\frac{1}{\left|\Pi_{j}\right|} \sum_{y \in \Pi_{j}} \sum_{m \in N_{0}} \chi(m \cdot(y-x))\right| \\
& \leq\left|\Pi_{j}\right|^{-1} \sum_{y \in \mathbb{F}_{q}^{d}}\left|\sum_{m \in N_{0}} \chi(m \cdot(y-x))\right|=\left|\Pi_{j}\right|^{-1} \sum_{z \in \mathbb{F}_{q}^{d}}\left|\sum_{m \in N_{0}} \chi(m \cdot z)\right| \\
& \sim q^{-d+1} \sum_{k=0}^{d} \sum_{y \in N_{k}}\left|\sum_{m_{1}, m_{2}, \ldots, m_{d} \in \mathbb{F}_{q}^{*}} \chi(m \cdot z)\right| .
\end{aligned}
$$

By using the orthogonality of $\chi$ and the definition of $N_{k}$, we conclude

$$
\left|\left(\widehat{1_{N_{0}}} * d \mu_{j}\right)(x)\right| \lesssim q^{-d+1} \sum_{k=0}^{d}\left|N_{k}\right|(q-1)^{k} \sim q,
$$

where the above similarity follows from the fact that $\left|N_{k}\right| \sim q^{d-k}$. This proves the inequality (3.9), as required. We have finished the proof of the inequality (3.5).
3.2. proof of the inequality (3.6). By Young's inequality for convolution functions, we have

$$
(q-1)^{-d+k}\left\|f * \widehat{1_{N_{k}}}\right\|_{L^{d+1}\left(\mathbb{F}_{q}^{d}\right)} \leq q^{-d+k}\|f\|_{L^{\frac{d+1}{d}\left(\mathbb{F}_{q}^{d}\right)}}\left\|\widehat{1_{N_{k}}}\right\|_{L^{\frac{d+1}{2}}\left(\mathbb{F}_{q}^{d}\right)} .
$$

Hence, to prove the inequality (3.6), it suffices to show that for each $k=1,2, \ldots, d$,

$$
\left\|\widehat{1_{N_{k}}}\right\|_{L^{\frac{d+1}{2}}\left(\mathfrak{F}_{q}^{d}\right)} \lesssim q^{d-k}
$$

We will prove this inequality separately in the cases of $d=2$ and $d \geq 3$.
Case 1: Let $d \geq 3$. Since $2 \leq(d+1) / 2<\infty$ for $d \geq 3$, we can invoke the Hausdorff-Young inequality to deduce the required estimate. More precisely, we have

$$
\left\|\widehat{1_{N_{k}}}\right\|_{L^{\frac{d+1}{2}}\left(\mathbb{F}_{q}^{d}\right)} \leq\left\|1_{N_{k}}\right\|_{\ell \frac{d+1}{d-1}\left(\mathbb{F}_{q}^{d}\right)}=\left|N_{k}\right|^{\frac{d-1}{d+1}} \sim q^{(d-k) \frac{d-1}{d+1}} \leq q^{d-k},
$$

as desired.
Case 2: Let $d=2$. We aim to prove that for $k=1,2$,

$$
\left\|\widehat{1_{N_{k}}}\right\|_{L^{\frac{3}{2}}\left(\mathbb{F}_{q}^{2}\right)} \lesssim q^{2-k}
$$

For $k=2$, it is clear that $N_{2}=\{(0,0)\}$. Hence, the above inequality follows by observing $\widehat{1_{N_{2}}}(x)=1$ for all $x \in \mathbb{F}_{q}^{2}$. To prove the above inequality for the case of $k=1$, we first notice
that

$$
N_{1}=\left(\mathbb{F}_{q}^{*} \times\{0\}\right) \cup\left(\{0\} \times \mathbb{F}_{q}^{*}\right)
$$

which implies that $\left|N_{1}\right|=2(q-1)$. For any $x \in \mathbb{F}_{q}^{2}$, we have

$$
\left|\widehat{N_{1}}(x)\right| \leq\left|N_{1}\right| \sim q .
$$

Therefore, it follows that

$$
\left\|\widehat{1_{N_{1}}}\right\|_{L^{\frac{3}{2}}\left(\mathbb{F}_{q}^{2}\right)} \leq \max _{x \in \mathbb{F}_{q}^{2}}\left|\widehat{1_{N_{1}}}(x)\right| \sim q .
$$

This completes the proof of the inequality (3.6).

Theorem 3.2. If $p \geq \frac{d}{d-1}$, then the following maximal averaging estimate for a family $\left\{\Pi_{j}\right\}_{j \in \mathbb{F}_{q}^{*}}$ of Product $j$-varieties $\Pi_{j}$ holds for all functions $f$ on $\mathbb{F}_{q}^{d}$ :

$$
\begin{equation*}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} \lesssim\|f\|_{L^{p}\left(\mathbb{F}_{q}^{d}\right)} . \tag{3.10}
\end{equation*}
$$

Proof. It is not hard to check that for every $x \in \mathbb{F}_{q}^{d}$,

$$
\sup _{j \in \mathbb{F}_{q}^{*}}\left|\left(f * d \mu_{j}\right)(x)\right| \leq \max _{y \in \mathbb{F}_{q}^{d}}|f(y)| .
$$

Hence, the inequality (3.10) is true for $p=\infty$. By interpolation theorem, it therefore suffices to prove the inequality (3.10) for $p=\frac{d}{d-1}$. Namely, our task is to verify the following estimate:

$$
\begin{equation*}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} \lesssim\|f\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} . \tag{3.11}
\end{equation*}
$$

As in (3.4), for each $j \in \mathbb{F}_{q}^{*}$, we can write

$$
d \mu_{j}=\Omega_{j}+\sum_{k=1}^{d}(-1)^{d-k}(q-1)^{-d+k} \widehat{1_{N_{k}}},
$$

where $\Omega_{j}:=\widehat{1_{N_{0}}} * d \mu_{j}$. Thus we have

$$
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * d \mu_{j}\right|\right\|_{L \frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)} \leq\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L \frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}+\sum_{k=1}^{d}(q-1)^{-d+k}\left\|f * \widehat{1_{N_{k}}}\right\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} .
$$

Therefore, to prove (3.11) (namely, to complete the proof of Theorem 3.2), it suffices to prove the following estimates:

$$
\begin{equation*}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} \lesssim\|f\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}}, \tag{3.12}
\end{equation*}
$$

and for every $k=1,2, \ldots, d$,

$$
\begin{equation*}
(q-1)^{-d+k}\left\|f * \widehat{1_{N_{k}}}\right\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} \lesssim\|f\|_{L^{\frac{d}{d-1}\left(\mathbb{F}_{q}^{d}\right)}} . \tag{3.13}
\end{equation*}
$$

First, let us verify the inequality (3.13). Notice by the Hőlder inequality that if $1 \leq t_{1} \leq$ $t_{2} \leq \infty$, then

$$
\|f\|_{L^{t_{1}\left(\mathbb{F}_{q}^{d}\right)}} \leq\|f\|_{L^{t_{2}\left(\mathbb{F}_{q}^{d}\right)}}
$$

By this nesting property of norms, it is not hard to see that the inequality (3.13) follows from the inequality (3.6).

Finally, we prove the inequality (3.12) which is a direct consequence from interpolating the following two estimates:

$$
\begin{gather*}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)} \lesssim q\|f\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)},  \tag{3.14}\\
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)} \lesssim q^{\frac{2-d}{2}}\|f\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)} . \tag{3.15}
\end{gather*}
$$

Hence, to finish the proof, it remains to prove the inequalities (3.14) and (3.15), which can be done by adapting an argument from [2]. The details are as follows. The inequality (3.14) follows, because we have

$$
\begin{aligned}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)} & \leq\left\||f| * \sup _{j \in \mathbb{F}_{q}^{*}}\left|\Omega_{j}\right|\right\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)} \\
& \leq\|f\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)}\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|\Omega_{j}\right|\right\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)} \lesssim q\|f\|_{L^{1}\left(\mathbb{F}_{q}^{d}\right)},
\end{aligned}
$$

where the last inequality $\lesssim$ is a direct consequence from the inequality (3.9).
For the inequality (3.15), we have

$$
\begin{aligned}
\left\|\sup _{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|\right\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)} & \leq\left\|\left(\sum_{j \in \mathbb{F}_{q}^{*}}\left|f * \Omega_{j}\right|^{2}\right)^{\frac{1}{2}}\right\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)}=\left(\sum_{j \in \mathbb{F}_{q}^{*}}\left\|f * \Omega_{j}\right\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)}^{2}\right)^{\frac{1}{2}} \\
& \leq\left(\sum_{j \in \mathbb{F}_{q}^{*}}\left(\max _{m \in \mathbb{F}_{q}^{d}}\left|\Omega_{j}^{\vee}(m)\right|^{2}\right)\left\|f^{\vee}\right\|_{\ell^{2}\left(\mathbb{F}_{q}^{d}\right)}^{2}\right)^{\frac{1}{2}} \\
& \leq\left(\max _{j \in \mathbb{F}_{q}^{*}, m \in \mathbb{F}_{q}^{d}}\left|\Omega_{j}^{\vee}(m)\right|\right) q^{\frac{1}{2}}\|f\|_{L^{2}\left(\mathbb{F}_{q}^{d}\right)} .
\end{aligned}
$$

Since $\Omega_{j}^{\vee}=1_{N_{0}}\left(d \mu_{j}\right)^{\vee}$, the second part of Lemma 1.4 implies that the maximum value above is dominated by $\sim q^{-\frac{d-1}{2}}$, and hence the inequality (3.15) is obtained, as desired. We have finished the proof of Theorem 1.6.

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