# Partial zeta functions, partial exponential sums, and $p$-adic estimates 

Noah Bertram, Xiantao Deng, C. Douglas Haessig, Yan Li

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#### Abstract

Partial zeta functions of algebraic varieties over finite fields generalize the classical zeta function by allowing each variable to be defined over a possibly different extension field of a fixed finite field. Due to this extra variation their rationality is surprising, and even simple examples are delicate to compute. For instance, we give a detailed description of the partial zeta function of an affine curve where the number of unit poles varies, a property different from classical zeta functions. On the other hand, they do retain some properties similar to the classical case. To this end, we give Chevalley-Warning type bounds for partial zeta functions and $L$-functions associated to partial exponential sums.


## 1 Introduction

Let $\mathbb{F}_{q}$ be the finite field with $q=p^{a}$ elements, $p$ a prime number. Fix $\mathbf{d}:=\left(d_{1}, \ldots, d_{n}\right) \in \mathbb{Z}_{\geq 1}^{n}$, and set $d:=$ $\operatorname{lcm}\left(d_{1}, \ldots, d_{n}\right)$. Throughout the paper, we will write $q_{i}:=q^{d_{i}}$. Let $X$ be an affine variety defined over $\mathbb{F}_{q^{d}}$, defined as the zero locus of $F_{1}, \ldots, F_{r} \in \mathbb{F}_{q^{d}}\left[x_{1}, \ldots, x_{n}\right]$. Associated to the sequence

$$
m \geq 1: \quad N_{m}(\mathbf{d}):=\#\left\{x:=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{F}_{q_{1}^{m}} \times \cdots \times \mathbb{F}_{q_{n}^{m}} \mid F_{1}(x)=\cdots=F_{r}(x)=0\right\}
$$

is the partial zeta function of $X$,

$$
Z\left(X / \mathbb{F}_{q}, \mathbf{d}, T\right):=\exp \left(\sum_{m \geq 1} N_{m}(\mathbf{d}) \frac{T^{m}}{m}\right) .
$$

Wan introduced partial zeta functions in [8] and proved their rationality in 9]:

$$
\begin{equation*}
Z\left(X / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{\prod_{i=1}^{R}\left(1-\alpha_{i} T\right)}{\prod_{j=1}^{S}\left(1-\beta_{j} T\right)} \in \mathbb{Q}(T) \tag{1}
\end{equation*}
$$

with $\alpha_{i}$ and $\beta_{j}$ Weil $q$-integers. Since each variable lies in a possibly different finite extension field of $\mathbb{F}_{q}$, even simple examples can be complicated, and very different from their classical counterpart. Further, their rationality is surprising, even in the most simple cases. This point is illustrated in Sections 2 and 3 where we take a detailed look at a couple of examples. In particular, the partial zeta function of the affine curve $\mathcal{C}_{n}$ defined by $y=x^{n}$ takes the form

$$
\begin{equation*}
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)}\left(\frac{\Phi_{k}(T)}{\Phi_{k}\left(q^{c} T\right)}\right)^{a_{k}} \tag{2}
\end{equation*}
$$

where $\Phi_{k}$ is the $k$-th cyclotomic polynomial, $\varphi$ is Euler's totient function, $c=\operatorname{gcd}\left(d_{1}, d_{2}\right)$, and

$$
a_{k}:=\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \zeta_{k}^{i} \in \mathbb{Z}
$$

where $\zeta_{k}$ is a primitive $k$-th root of unity and $M_{i}:=\left(q_{1}^{i}-1\right) /\left(q^{c i}-1\right)$. An interesting feature of this example is that, for fixed $n$ and $q$, the number of unit poles varies between zero and one as $d_{1}$ and $d_{2}$ vary. This does not occur for classical zeta functions, where affine curves never have unit poles.

We note that while Fu and Wan 3] have given a cohomological description of these zeta functions, rationality is still not immediate since their trace formula involves roots of unity, which translates into the zeta function having factors with roots of unity as exponents. Rationality is obtained by a further Galois theoretic argument. We wondered whether this Galois argument is an artifact of their method or intrinsic to these types of zeta functions, and indeed, it appears to be intrinsic. In the example of the affine curve $\mathcal{C}_{n}$, we examine its partial zeta function using an ad hoc method, and as you can see above, the exponents $a_{k}$ are exponential sums.

Little is known about the $p$-adic behavior of the zeros and poles of these types of zeta functions. A first step toward such an understanding is the following theorem, proven in Section 4 To state the result we need to define some notation. For a nonnegative integer $b$, write $b=a_{0}+a_{1} p+\cdots+a_{r} p^{r}$ with $0 \leq a_{i} \leq p-1$. Set $\sigma_{p}(b):=a_{0}+\cdots+a_{r}$. Extend this to $u=\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ by defining $\left|\sigma_{p}(u)\right|:=\sigma_{p}\left(u_{1}\right)+\cdots+\sigma_{p}\left(u_{n}\right)$. Using multi-index notation $x^{u}:=x_{1}^{u_{1}} \cdots x_{n}^{u_{n}}$, for a polynomial $F(x)=\sum a_{u} x^{u} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ define the $p$-weight $w_{p}(F):=\max _{u}\left|\sigma_{p}(u)\right|$.

Theorem 1.1. Let $X$ be an affine variety defined as the zero locus of $F_{1}, \ldots, F_{r} \in \mathbb{F}_{q^{d}}\left[x_{1}, \ldots, x_{n}\right]$. Then $p^{\omega} \mid N_{1}(\mathbf{d})$, where

$$
\omega:=\left\lceil a \cdot \frac{\left(d_{1}+\cdots+d_{n}\right)-d \sum_{i=1}^{r} w_{p}\left(F_{i}\right)}{\max _{i} w_{p}\left(F_{i}\right)}\right\rceil .
$$

A similar theorem was proven in [8, Theorem 1.4].
Corollary 1.2. $\operatorname{ord}_{q} \alpha_{i}$ and $\operatorname{ord}_{q} \beta_{j}$ from (11) are bounded below by

$$
\frac{\left(d_{1}+\cdots+d_{n}\right)-d \sum_{i=1}^{r} w_{p}\left(F_{i}\right)}{\max _{i} w_{p}\left(F_{i}\right)}
$$

We also prove an analogue of Theorem 1.1 for $L$-functions of partial exponential sums, which are defined as follows. Let $f(x) \in \mathbb{F}_{q^{d}}\left[x_{1}, \ldots, x_{n}\right], \psi$ a nontrivial additive character on $\mathbb{F}_{q^{d}}$, and $\chi_{1}, \ldots, \chi_{r}$ multiplicative characters on $\mathbb{F}_{q^{d_{1}}}^{\times}, \ldots, \mathbb{F}_{q^{d_{n}}}^{\times}$, respectively. Define the mixed partial character sum

$$
S(\chi, \mathbf{d}, f):=\sum \chi_{1}\left(x_{1}\right) \cdots \chi_{r}\left(x_{r}\right) \psi(f(x))
$$

where the sum runs over $x_{i} \in \mathbb{F}_{q^{d_{i}}}^{\times}$for $i=1, \ldots, r$ and $x_{i} \in \mathbb{F}_{q^{d_{i}}}$ for $i=r+1, \ldots, n$. To give a $p$-adic estimate for $S(\chi, \mathbf{d}, f)$, we embed it into the $p$-adic numbers.

Let $\mathbb{Q}_{p}$ be the $p$-adic numbers, and denote by $\mathbb{Q}_{q^{d}}$ the unramified extension of $\mathbb{Q}_{p}$ of degree ad (recall, $q=p^{a}$ ). We normalize the valuation such that $\operatorname{ord}_{q}(q)=1$. Denote by $T_{i}$ the set of solutions of $z^{q_{i}}=z$ in $\mathbb{Q}_{q^{d_{i}}} \subseteq \mathbb{Q}_{q^{d}}$, and set $T_{i}^{\times}:=T_{i} \backslash\{0\}$. Define

$$
\mathcal{T}:=T_{1}^{\times} \times \cdots \times T_{r}^{\times} \times T_{r+1} \times \cdots \times T_{n} .
$$

There exist $e_{i} \in\left\{1, \ldots, q_{i}-1\right\}$ for $i=1, \ldots, r$ such that

$$
S(\chi, \mathbf{d}, f)=\sum_{x \in \mathcal{T}} x_{1}^{e_{1}} \cdots x_{r}^{e_{r}} \psi(f(\bar{x})) \in \mathbb{Q}_{q^{d}}\left(\zeta_{p}\right)
$$

where $\bar{x}_{i}$ denotes the image of $x_{i}$ in the residue field $\mathbb{F}_{q^{d}} \subseteq \mathbb{F}_{q^{d}}$.
Theorem 1.3.

$$
\operatorname{ord}_{q} S(\chi, \mathbf{d}, f) \geq \frac{1}{w_{p}(f)}\left(\sum_{i=1}^{n} d_{i}-\frac{1}{a(p-1)} \sum_{i=1}^{r} \sigma_{p}\left(e_{i}\right)\right)
$$

When there is no twist (i.e. $r=0$ ) the estimate becomes

$$
\operatorname{ord}_{q} S(\mathbf{d}, f) \geq\left(d_{1}+\cdots+d_{n}\right) / w_{p}(f)
$$

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## 2 Example: Affine hyperplane

Let $\mathbf{d}:=\left(d_{1}, \ldots, d_{n}\right) \in \mathbb{Z}_{\geq 1}^{n}$. Observe that affine space $\mathbb{A}^{n}$ has the partial zeta function

$$
Z\left(\mathbb{A}^{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{1}{1-q^{d_{1}+\cdots+d_{n}} T}
$$

Next, consider the affine hyperplane $H$ defined by $a_{1} x_{1}+\cdots .+a_{n} x_{n}=0$, with $a_{i} \in \mathbb{F}_{q}^{*}$.
Theorem 2.1.

$$
\begin{equation*}
Z\left(H / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{1}{1-q^{e} T}, \quad \text { where } \quad e:=\sum_{k=2}^{n}(-1)^{k} \sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{i_{1}}, \ldots, d_{i_{k}}\right) \tag{3}
\end{equation*}
$$

The proof of the formula for $e$ requires the following lemma.

## Lemma 2.2.

1. $\mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)=\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{n}}}$.
2. The dimension of $\mathbb{F}_{q^{d_{1}}}+\cdots+\mathbb{F}_{q^{d_{n}}}$ as an $\mathbb{F}_{q}$-vector space is $\sum_{k=1}^{n}(-1)^{k+1} \sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{i_{1}}, \ldots, d_{i_{k}}\right)$.

Proof. For the first statement, we need only show

$$
\mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right) \subseteq \mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{n}}}
$$

since it is clear the right hand side is contained in the left. Set $m:=\operatorname{lcm}\left(d_{1}, \ldots, d_{n}\right)$. Assume for the moment that $\operatorname{ord}_{p} d_{1} \geq \operatorname{ord}_{p} d_{i}$ for every $i$, and thus $p \nmid m / d_{1}$. Write $\mathbb{F}_{q^{m}}^{*}=\langle\zeta\rangle=\left\{\zeta^{i} \mid 1 \leq i \leq q^{m}-1\right\}$, where $\zeta$ is a primitive $q^{m}-1$ root of unity. Observe that $\mathbb{F}_{q^{d_{i}}}^{*}=\left\langle\zeta^{a_{i}}\right\rangle$, where $a_{i}:=\left(q^{m}-1\right) /\left(q^{d_{i}}-1\right)$. Let $\eta \in \mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)$. Since
$\eta \in \mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}$ we may write $\eta=\sum_{i=2}^{n} b_{i} \zeta^{a_{i} t_{i}}$ for some $b_{i} \in \mathbb{F}_{p}$ and $t_{i} \in \mathbb{Z}$. Next, since $\eta \in \mathbb{F}_{q^{d_{1}}}$, then for all $j \in \mathbb{Z}$ we have

$$
\eta=\eta^{q^{j d_{1}}}=\sum_{i=2}^{n} b_{i} \zeta^{a_{i} t_{i} q^{j d_{1}}}
$$

Thus, setting $u:=m / d_{1}$ then $c \eta=\sum_{i=2}^{n} b_{i} \sum_{j=0}^{u-1} \zeta^{a_{i} t_{i} q^{j d_{1}}}$. Notice that

$$
\left(\sum_{j=0}^{u-1} \zeta^{a_{i} t_{i} q^{j d_{1}}}\right)^{q^{d_{1}}}=\sum_{j=0}^{u-1} \zeta^{a_{i} t_{i} q^{j d_{1}}}
$$

and so $\sum_{j=0}^{u-1} \zeta^{a_{i} t_{i} q^{j d_{1}}} \in \mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{i}}}$. Since $p \nmid u$, it follows that

$$
\eta=\sum_{i=1}^{n} \frac{b_{i}}{u} \sum_{j=0}^{c-1} \zeta^{a_{i} t_{i} q^{j d_{1}}} \in \mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}+\cdots .+\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{n}}}
$$

as desired. For this argument, we made the assumption that $\operatorname{ord}_{p} d_{1} \geq \operatorname{ord}_{p} d_{i}$ for every $i$. We may remove this restriction after proving the second statement of the lemma by induction as follows.

It is well-known that $\mathbb{F}_{q^{a}} \cap \mathbb{F}_{q^{b}}=\mathbb{F}_{q^{g c d}(a, b)}$, and so when $n=2$ we have

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{1}}}+\mathbb{F}_{q^{d_{2}}}\right) & =\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}}+\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{2}}}-\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}\right) \\
& \left.=\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}}+\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{2}}}-\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{\left.q^{\operatorname{gcd}\left(d_{1}, d_{2}\right)}\right)}\right) \\
& =d_{1}+d_{2}-\operatorname{gcd}\left(d_{1}, d_{2}\right) .
\end{aligned}
$$

Suppose now that $n>2$. Reorder the $d_{i}$ so that $\operatorname{ord}_{p} d_{1} \geq \operatorname{ord}_{p} d_{i}$. Set $r_{i}:=\operatorname{gcd}\left(d_{1}, d_{i}\right)$. Then

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{1}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right) & =\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}}+\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)-\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)\right) \\
& =\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}}+\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)-\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{r_{2}}}+\cdots+\mathbb{F}_{q^{r_{n}}}\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{d_{1}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right) & =d_{1}+\sum_{k=1}^{n-1}(-1)^{k+1} \sum_{2 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{i_{1}}, \ldots, d_{i_{k}}\right)-\sum_{k=1}^{n-1}(-1)^{k+1} \sum_{2 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(r_{i_{1}}, \ldots, r_{i_{k}}\right) \\
& =d_{1}+\sum_{k=1}^{n-1}(-1)^{k+1} \sum_{2 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{i_{1}}, \ldots, d_{i_{k}}\right)-\sum_{k=1}^{n-1}(-1)^{k+1} \sum_{2 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{1}, d_{i_{1}}, \ldots, d_{i_{k}}\right) \\
& =\sum_{k=1}^{n}(-1)^{k+1} \sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} \operatorname{gcd}\left(d_{i_{1}}, \ldots, d_{i_{k}}\right) .
\end{aligned}
$$

We may now remove the restriction $\operatorname{ord}_{p} d_{1} \geq \operatorname{ord}_{p} d_{i}$ as mentioned above. It is clear that

$$
\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{n}}} \subseteq \mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right)
$$

By a similar dimension calculation made above, we see that these two $\mathbb{F}_{q}$-vector spaces have the same dimension, and thus they are equal.

The formula for $e$ in (3) quickly follows from the lemma since

$$
\begin{aligned}
N_{1}(\mathbf{d}) & =\#\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{F}_{q^{d_{1}}} \times \cdots \times \mathbb{F}_{q^{d_{n}}} \mid a_{1} x_{1}+\cdots+a_{n} x_{n}=0\right\} \\
& =\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\cdots+\mathbb{F}_{q^{d_{n}}}\right) .
\end{aligned}
$$

In case it is of independent interest, we record here an alternative proof of Lemma 2.2 in the case $n=3$.
Alternate proof. Since $\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{2}}}+\mathbb{F}_{q^{d_{1}}} \cap \mathbb{F}_{q^{d_{3}}} \subseteq \mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\mathbb{F}_{q^{d_{3}}}\right)$, if we show the dimensions of these spaces are equal, then the spaces must be equal. Let $m:=\operatorname{lcm}\left(d_{1}, d_{2}, d_{3}\right)$. Let $\theta$ be a normal element of $\mathbb{F}_{q^{m}}$, and let $\sigma: x \mapsto x^{q}$ be the Frobenius morphism. Let $\alpha \in \mathbb{F}_{q^{d_{2}}}, \beta \in \mathbb{F}_{q^{d_{2}}}$, and $\gamma \in \mathbb{F}_{q^{d_{1}}}$ such that $\alpha+\beta=\gamma$. Using the normal basis,

$$
\alpha=\sum_{i=0}^{m-1} a_{i} \sigma^{i}(\theta) \in \mathbb{F}_{q^{d_{2}}} \Leftrightarrow \sigma^{d_{2}}(\alpha)=\alpha \Leftrightarrow a_{i}=a_{i+d_{2}} \text { for all } i .
$$

Thus, we may identify $\alpha$ with the periodic sequence $\left(a_{i}\right)_{i=0}^{\infty}$ of period $d_{2}$, and note its generating function has the form

$$
G_{\alpha}:=\sum_{i=0}^{\infty} a_{i} x^{i}=\frac{f_{\alpha}(x)}{1-x^{d_{2}}} \quad \text { with } \operatorname{deg} f_{\alpha}<d_{2}
$$

Similarly, the generating functions for $\beta$ and $\gamma$ satisfy

$$
G_{\beta}(x)=\frac{f_{\beta}(x)}{1-x^{d_{3}}}, \quad G_{\gamma}(x)=\frac{f_{\gamma}(x)}{1-x^{d_{3}}} \quad \text { with } \operatorname{deg} f_{\beta}<m_{3} \text { and } \operatorname{deg} f_{\gamma}<d_{1}
$$

Since $\alpha+\beta=\gamma$, we have

$$
\frac{f_{\alpha}(x)}{1-x^{d_{2}}}+\frac{f_{\beta}(x)}{1-x^{d_{3}}}=\frac{f_{\gamma}(x)}{1-x^{d_{1}}},
$$

or

$$
\begin{equation*}
\left(1-x^{d_{1}}\right)\left(1-x^{d_{3}}\right) f_{\alpha}(x)+\left(1-x^{d_{1}}\right)\left(1-x^{d_{2}}\right) f_{\beta}(x)=\left(1-x^{d_{2}}\right)\left(1-x^{d_{3}}\right) f_{\gamma}(x) . \tag{4}
\end{equation*}
$$

Set $r:=\operatorname{gcd}\left(d_{2}, d_{3}\right)$, and note that (4) shows

$$
\left(1-x^{d_{1}}\right) \mid\left(1-x^{d_{2}}\right)\left(1-x^{d_{3}}\right)\left(1-x^{r}\right)^{-1} f_{\gamma}(x)
$$

Thus, with

$$
h(x):=\operatorname{gcd}\left(\left(1-x^{d_{1}}\right),\left(1-x^{d_{2}}\right)\left(1-x^{d_{3}}\right)\left(1-x^{r}\right)^{-1}\right)
$$

we see that $\left(1-x^{d_{1}}\right) / h(x)$ divides $f_{\gamma}(x)$. That is, $f_{\gamma}(x)=g(x)\left(1-x^{d_{1}}\right) / h(x)$ for some $g$, and since $\operatorname{deg} f_{\gamma}<d_{1}$, we have $\operatorname{deg} g<\operatorname{deg} h$. Observe that this process is reversible: given any polynomial $g$ with $\operatorname{deg} g<\operatorname{deg} h$, if we set $f_{\gamma}(x)=g(x)\left(1-x^{d_{1}}\right) / h(x)$ then there exist polynomials $f_{\alpha}$ and $f_{\beta}$ of degrees $<d_{2}$ and $<d_{3}$, resp., satisfying (4). Since the set of such $g$ forms a vector space of dimension $\operatorname{deg} h$ over $\mathbb{F}_{q}$, we see that $\operatorname{dim}_{\mathbb{F}_{q}} \mathbb{F}_{q^{d_{1}}} \cap\left(\mathbb{F}_{q^{d_{2}}}+\mathbb{F}_{q^{d_{3}}}\right)=\operatorname{deg} h$. The result follows since

$$
\operatorname{deg} \operatorname{gcd}\left(\left(1-x^{d_{1}}\right),\left(1-x^{d_{2}}\right)\left(1-x^{d_{3}}\right)\left(1-x^{r}\right)^{-1}\right)=\operatorname{gcd}\left(d_{1}, d_{2}\right)+\operatorname{gcd}\left(d_{1}, d_{3}\right)-\operatorname{gcd}\left(d_{1}, d_{2}, d_{3}\right)
$$

## 3 Example: Affine curve $y=x^{n}$

In this section we will compute the partial zeta function of the affine curve $\mathcal{C}_{n}:=\left\{y=x^{n}\right\}$ in $\mathbb{A}^{2}$. Set $c:=\operatorname{gcd}\left(d_{1}, d_{2}\right)$, and denote by $\Phi_{k}$ the $k$-th cyclotomic polynomial over $\mathbb{Q}$. We will prove the following.
Theorem 3.1.

$$
\begin{equation*}
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)}\left(\frac{\Phi_{k}(T)}{\Phi_{k}\left(q^{c} T\right)}\right)^{a_{k}} \tag{5}
\end{equation*}
$$

where $a_{k}$ is an exponential sum defined by

$$
a_{k}:=\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \zeta_{k}^{i} \in \mathbb{Z}
$$

Here $\varphi$ is Euler's totient function, $\zeta_{k}$ is a primitive $k$-th root of unity, and $M_{i}:=\left(q_{1}^{i}-1\right) /\left(q^{c i}-1\right)$.
Also, note that the total degree of $Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)$ is bounded above by $1+2 n^{2}$, independent of $q$ and $\mathbf{d}$. This provides positive evidence for [3, Question 2.5].

Before proving the theorem, we give a few remarks. First, exponential sums involving the greatest common divisor are well known. For example, $\varphi(n)=\sum_{r=1}^{n} \operatorname{gcd}(n, r) \zeta_{n}^{-r}$. Inspired by the above, we wonder:
Conjecture 3.2. Let $n, a$, and $d$ be positive integers. Set $M_{i}:=\left(a^{d i}-1\right) /\left(a^{i}-1\right)$. We conjecture that the exponential $\operatorname{sum} \frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \zeta_{k}^{i}$ is a rational integer. (Is there a closed form expression?)

When $n$ is a prime number we give the following closed form for $a_{k}$. To state the result, define the Iverson bracket 1 [] for a logical proposition $\mathcal{P}$ by:

$$
[\mathcal{P}]:= \begin{cases}1 & \text { if } \mathcal{P} \text { is true } \\ 0 & \text { if } \mathcal{P} \text { is false }\end{cases}
$$

Theorem 3.3. Suppose $n$ is a prime number. If $n=p$, then $a_{k}=[k \mid 1]$. If $n \neq p$, then

$$
a_{k}=\frac{\operatorname{gcd}\left(n, d_{1} / c\right)-n}{t}[k \mid t]+\frac{\phi(n)}{t_{1}}\left[k \mid t_{1}\right]+[k \mid 1],
$$

where $t:=\operatorname{ord}_{n}^{*}\left(q^{c}\right)=$ the multiplicative order of $q^{c}$ modulo $n$, and $t_{1}:=\operatorname{ord}_{n}^{*}\left(q^{d_{1}}\right)$.
Proof. If $n=p$ then $\operatorname{gcd}\left(n, M_{i}\right)=1$, and so $a_{k}=[k \mid 1]$. Suppose now that $n$ is a prime number different from $p$. Observe that since $c \mid d_{1}$, we have $t_{1}=t / \operatorname{gcd}\left(t, d_{1} / c\right)$. Now, as $M_{i}=1+q^{c i}+q^{2 c i}+\cdots+q^{\left(d_{1}-c\right) i}$, we have

$$
\operatorname{gcd}\left(n, M_{i}\right)= \begin{cases}\operatorname{gcd}\left(n, d_{1} / c\right) & \text { if } t \mid i \\ \operatorname{gcd}\left(n, q^{d_{1} i}-1\right) & \text { otherwise }\end{cases}
$$

[^0]Moreover,

$$
\operatorname{gcd}\left(n, M_{i}\right)= \begin{cases}\operatorname{gcd}\left(n, d_{1} / c\right) & \text { if } t \mid i \\ n & \text { if } t_{1} \mid i \text { and } t \nmid i \\ 1 & \text { if } t_{1} \nmid i .\end{cases}
$$

Thus,

$$
\begin{equation*}
a_{k}=\frac{1}{\phi(n)}\left(\sum_{t \mid i} \operatorname{gcd}\left(n, d_{1} / c\right) \zeta_{k}^{i}+\sum_{t_{1} \mid i, t \nmid i} n \zeta_{k}^{i}+\sum_{t_{1} \nmid i} \zeta_{k}^{i}\right) \tag{6}
\end{equation*}
$$

For any divisor $t$ of $\phi(n)$ we have

$$
\begin{equation*}
\sum_{\substack{i=1 \\ t \mid i}}^{\phi(n)} \zeta_{k}^{i}=\sum_{j=1}^{\phi(n) / t} \zeta_{k}^{t j}=\frac{\phi(n)}{t}[k \mid t] \tag{7}
\end{equation*}
$$

Substituting (7) into (6), then

$$
\begin{aligned}
a_{k} & \left.=\frac{1}{\phi(n)}\left(\left.\operatorname{gcd}\left(n, d_{1} / c\right) \frac{\phi(n)}{t}[k \mid t]+p\left(\frac{\phi(n)}{t_{1}}\left[k \mid t_{1}\right]-\frac{\phi(n)}{t}[k \mid t]\right)+\phi(n)[k \mid 1]-\frac{\phi(n)}{t_{1}} k \right\rvert\, t_{1}\right]\right) \\
& \left.=\frac{\operatorname{gcd}\left(n, d_{1} / c\right)-n}{t} ; k \mid t\right]+\frac{\phi(n)}{t_{1}}\left[k \mid t_{1}\right]+[k \mid 1]
\end{aligned}
$$

as desired.
Next, let us illustrate Theorem 3.1 by explicitly computing the cases $n=2$ and 3 . In the quadratic case $n=2$,

$$
Z\left(\mathcal{C}_{2} / \mathbb{F}_{q}, \mathbf{d}, T\right)= \begin{cases}\frac{1}{1-q^{c} T} & \text { if } q \text { even; or } q \text { odd and } d_{1} / c \text { odd } \\ \frac{1-T}{\left(1-q^{c} T\right)^{2}} & \text { if } q \text { odd and } d_{1} / c \text { even. }\end{cases}
$$

For the cubic case $n=3$, we split into residue classes. For $q \equiv 0$ or 1 modulo 3 :

$$
Z\left(\mathcal{C}_{3} / \mathbb{F}_{q}, \mathbf{d}, T\right)= \begin{cases}\frac{1}{1-q^{c} T} & \text { if } q \equiv 0 \bmod 3 \\ & \text { if } q \equiv 1 \bmod 3 \text { and } \frac{d_{1}}{c} \not \equiv 0 \bmod 3 \\ \frac{(1-T)^{2}}{\left(1-q^{c} T\right)^{3}} & \text { if } q \equiv 1 \bmod 3 \text { and } \frac{d_{1}}{c} \equiv 0 \bmod 3\end{cases}
$$

and for $q \equiv 2 \bmod 3$ :

$$
Z\left(\mathcal{C}_{3} / \mathbb{F}_{q}, \mathbf{d}, T\right)= \begin{cases}\frac{1}{1-q^{c} T} & q \equiv 2 \bmod 3, c \text { even, } \frac{d_{1}}{c} \not \equiv 0 \bmod 3 ; \\ & q \equiv 2 \bmod 3, c \text { odd, } \frac{d_{1}}{c} \text { odd, } \frac{d_{1}}{c} \not \equiv 0 \bmod 3 ; \\ \frac{(1-T)^{2}}{\left(1-q^{c} T\right)^{3}} & q \equiv 2 \bmod 3, c \text { even, } \frac{d_{1}}{c} \equiv 0 \bmod 3 ; \\ & q \equiv 2 \bmod 3, c \text { odd, } \frac{d_{1}}{c} \text { even, } \frac{d_{1}}{c} \equiv 0 \bmod 3 ; \\ \frac{\left(1+q^{c} T\right)(1-T)}{\left(1-q^{c} T\right)^{2}(1+T)} & q \equiv 2 \bmod 3, \frac{d_{1}}{c} \text { even, } \frac{d_{1}}{c} \not \equiv 0 \bmod 3 \\ \frac{(1-T)(1+T)}{\left(1-q^{c} T\right)^{2}\left(1+q^{c} T\right)} & q \equiv 2 \bmod 3, c \text { odd, } \frac{d_{1}}{c} \equiv 0 \bmod 3\end{cases}
$$

We now move to the proof of Theorem 3.1 which will consist of the rest of this section.
Lemma 3.4. Let $\mathbf{d}=\left(d_{1}, d_{2}\right) \in \mathbb{Z}_{\geq 1}^{2}$. Set $c:=\operatorname{gcd}\left(d_{1}, d_{2}\right)$. For $\mathcal{C}_{n}$ the affine curve $y=x^{n}$ in $\mathbb{A}^{2}$,

$$
N_{m}(\mathbf{d})=\operatorname{gcd}\left(n, M_{m}\right)\left(q^{c m}-1\right)+1
$$

where $M_{m}:=\left(q_{1}^{m}-1\right) /\left(q^{c m}-1\right)=\sum_{i=0}^{\left(d_{1} / c\right)-1} q^{c m i}$.
Proof. The number of solutions to $y=x^{n}$ is the number of elements in $\mathbb{F}_{q_{1}^{m}}$ whose $n$-th power lies in $\mathbb{F}_{q_{2}^{m}}$. Let $\alpha$ be a generator of $\mathbb{F}_{q_{1}^{m}}^{\times}$. For notational convenience, for positive integers $i, j$ with $i$ dividing $j$, set

$$
[i, j]:=\{i, 2 i, 3 i, \ldots, j\} .
$$

Then the number of nonzero solutions to $y=x^{n}$ is

$$
\#\left\{k \in\left[1, q_{1}^{m}-1\right] \mid \alpha^{n k} \in \mathbb{F}_{q_{1}^{m}} \cap \mathbb{F}_{q_{2}^{m}}\right\}=\#\left\{k \in\left[1, q_{1}-1\right] \mid M_{m} \text { divides } n k\right\} .
$$

which follows since $M_{m}\left(q^{c m}-1\right)=q_{1}^{m}-1$ and $\alpha^{n k} \in \mathbb{F}_{q_{1}^{m}} \cap \mathbb{F}_{q_{2}^{m}}$ if and only if $\frac{q_{1}^{m}-1}{q^{c m}-1}$ divides $n k$. Therefore,

$$
\begin{aligned}
N_{m}(\mathbf{d}) & =\#\left\{k \in\left[1, q_{1}^{m}-1\right] \mid M_{m} \text { divides } n k\right\}+1 \\
& =\#\left\{k \in\left[n, n\left(q_{1}^{m}-1\right)\right] \mid M_{m} \text { divides } k\right\}+1 \\
& =\#\left\{k \in\left[1, n\left(q_{1}^{m}-1\right)\right] \mid M_{m} \text { and } n \text { divide } k\right\}+1 \\
& =\#\left\{k \in\left[M_{m}, n\left(q^{c m}-1\right) M_{m}\right] \mid n \text { divides } k\right\}+1 .
\end{aligned}
$$

Now any $M_{m} k \in\left[M_{m}, n\left(q^{c m}-1\right) M_{m}\right]$ is divisible by $n$ if and only if $n / \operatorname{gcd}\left(n, M_{m}\right)$ divides $k$. Thus

$$
\begin{aligned}
\left\{M_{m} k: k \in\left[1, n\left(q^{c m}-1\right)\right] \text { and } n \mid M k\right\} & =\left\{M_{m} k: k \in\left[1, n\left(q^{c m}-1\right)\right] \text { and } \frac{n}{\operatorname{gcd}\left(n, M_{m}\right)} \text { divides } k\right\} \\
& =\left\{\frac{n M_{m}}{\operatorname{gcd}\left(n, M_{m}\right)} k: k \in\left[1, \operatorname{gcd}\left(n, M_{m}\right)\left(q^{c m}-1\right)\right]\right\}
\end{aligned}
$$

which means that

$$
\begin{aligned}
N_{m}(\mathbf{d}) & =\#\left\{\frac{n M_{m}}{\operatorname{gcd}\left(n, M_{m}\right)} k: k \in\left[1, \operatorname{gcd}\left(n, M_{m}\right)\left(q^{c m}-1\right)\right]\right\}+1 \\
& =\operatorname{gcd}(n, M)\left(q^{c m}-1\right)+1
\end{aligned}
$$

Lemma 3.5. The sequence $\left\{\operatorname{gcd}\left(n, M_{m}\right)\right\}_{m=1}^{\infty}$ has period $\varphi(n)$.
Proof. By Euler's theorem, $M_{m} \equiv M_{m^{\prime}} \bmod n$ whenever $m \equiv m^{\prime} \bmod \varphi(n)$. We arrive at the statement by noting that $\operatorname{gcd}(n, b)=\operatorname{gcd}(n, b \bmod n)$, so in particular,

$$
\operatorname{gcd}\left(n, M_{m}\right)=\operatorname{gcd}\left(n, M_{m} \bmod n\right)=\operatorname{gcd}\left(n, M_{m^{\prime}} \bmod n\right)=\operatorname{gcd}\left(n, M_{m^{\prime}}\right)
$$

whenever $m \equiv m^{\prime} \bmod \varphi(n)$.
Applying Lemmas 3.4 and 3.5

$$
\begin{aligned}
& Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)=\exp \left(\sum_{i=1}^{\varphi(n)} \sum_{m \equiv i \bmod }^{m \geq 1} \varphi(n)\right. \\
&\left.\operatorname{mcd}\left(n, M_{i}\right)\left(q^{m c}-1\right) \frac{T^{m}}{m}+\frac{T^{m}}{m}\right) \\
&=\exp \left(\sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \sum_{\substack{m \equiv i \\
m \geq 1}} \frac{\left(q^{c} T\right)^{m}}{m}-\frac{T^{m}}{m}\right) /(1-T) .
\end{aligned}
$$

Now using the fact that

$$
\sum_{j=1}^{\varphi(n)} \zeta^{(m-i) j}= \begin{cases}\varphi(n) & \text { if } m \equiv i \bmod \varphi(n) \\ 0 & \text { otherwise }\end{cases}
$$

where $\zeta$ is a primitive $\varphi(n)$-th root of unity, observe that

$$
\sum_{\substack{m \equiv i \bmod \\ m \geq 1}} \frac{T^{m}}{m}=-\frac{1}{\varphi(n)} \sum_{j=1}^{\varphi(n)} \zeta^{-i j} \log \left(1-\zeta^{j} T\right)
$$

Set $\delta_{i}:=\operatorname{gcd}\left(n, M_{i}\right)$ and define the exponential sum

$$
S_{j}:=\sum_{i=1}^{\varphi(n)} \delta_{i} \zeta^{-i j}
$$

Then we may write

$$
\begin{align*}
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right) & =\exp \left(\sum_{i=1}^{\varphi(n)} \delta_{i} \sum_{m \equiv i \bmod } \varphi \frac{\left(q^{c} T\right)^{m}}{m \geq 1}-\frac{T^{m}}{m}\right) /(1-T) \\
& =\exp \left(\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \delta_{i} \sum_{j=1}^{\varphi(n)} \zeta^{-i j} \log \left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)\right) /(1-T) \\
& =\exp \left(\log \prod_{j=1}^{\varphi(n)}\left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{S_{j} / \varphi(n)}\right) /(1-T) \\
& =\frac{1}{1-T} \cdot \prod_{j=1}^{\varphi(n)}\left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{S_{j} / \varphi(n)} \tag{8}
\end{align*}
$$

At this point, it is not clear that $Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)$ is a rational function over $\mathbb{Q}$ due to the exponents $S_{j}$ being exponential sums, and unfortunately, we are unable to prove they are integers. However, Wan's rationality theorem [9] tells us it is a rational function over $\mathbb{Q}$, and thus since the reciprocal zeros and poles are distinct it must be the case that $S_{j}$ is an integer divisible by $\varphi(n)$.

Denote by $o(j)$ the additive order of $j$ modulo $\varphi(n)$. Then

$$
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)=\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)} \prod_{\substack{\leq j \leq \varphi(n) \\ o(j)=k}}\left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{S_{j} / \varphi(n)}
$$

Let $\sigma$ by an automorphism of $\mathbb{Q}(\zeta)(T)$ over $\mathbb{Q}(T)$, and denote by $j_{\sigma}$ the integer such that $1 \leq j_{\sigma} \leq \varphi(n)$ and $\sigma\left(\zeta^{j}\right)=\zeta^{j \sigma}$. Then

$$
\left.\begin{array}{rl}
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right) & =\sigma\left(Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)\right) \\
& =\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)} \prod_{\substack{1 \leq j \leq \varphi(n) \\
o(j)=k}}\left(\frac{1-\zeta^{j} \sigma}{1-\zeta^{j} \sigma} q^{c} T\right.
\end{array}\right)^{S_{j} / \varphi(n)} .
$$

Since there always exists an automorphism $\sigma$ for which $j_{\sigma}=j^{\prime}$ for any $j$ and $j^{\prime}$ having additive orders $k$ mod $\varphi(n)$, we see that $S_{j^{\prime}}=S_{j}$ for all $j, j^{\prime}$ with additive order $k \bmod \varphi(n)$, otherwise $\sigma\left(Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)\right)$ would have zeros and poles with multiplicities different from $Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right)$. We may now define

$$
a_{k}:=\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \zeta^{-i j}
$$

where $j$ is any positive integer with additive order $k$ modulo $\varphi(n)$; note that $\zeta_{k}:=\zeta^{j}$ is a primitive $k$-th root of unity, and so we may write

$$
\begin{equation*}
a_{k}:=\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \operatorname{gcd}\left(n, M_{i}\right) \zeta_{k}^{i} \tag{9}
\end{equation*}
$$

Finally,

$$
\begin{aligned}
Z\left(\mathcal{C}_{n} / \mathbb{F}_{q}, \mathbf{d}, T\right) & =\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)} \prod_{\substack{1 \leq j \leq \varphi(n) \\
o(j)=k}}\left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{S_{j} / \varphi(n)} \\
& =\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)} \prod_{\substack{\leq j \leq \varphi(n) \\
o(j)=k}}\left(\frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{a_{k}} \\
& =\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)}\left(\prod_{\substack{1 \leq j \leq \varphi(n) \\
o(j)=k}} \frac{1-\zeta^{j} T}{1-\zeta^{j} q^{c} T}\right)^{a_{k}} \\
& =\frac{1}{1-T} \cdot \prod_{k \mid \varphi(n)}\left(\frac{\Phi_{k}(T)}{\Phi_{k}\left(q^{c} T\right)}\right)^{a_{k}}
\end{aligned}
$$

where $\Phi_{k}$ is the $k$-th cyclotomic polynomial. This prove (5).

## 4 Partial zeta function

In this section, we prove Theorem 1.1 and Corollary 1.2 The proof is a slight generalization of the argument given in 6]; we provide the details for completeness. For each $i=1, \ldots, n$, let $\left\{\mu_{i j}\right\}_{j=1}^{a d_{i}}$ be a basis of $\mathbb{F}_{q^{d_{i}}} \subseteq \mathbb{F}_{q^{d}}$ over $\mathbb{F}_{p}$. Let $\left\{\mu_{j}\right\}_{j=1}^{a d}$ be a basis of $\mathbb{F}_{q^{d}}$ over $\mathbb{F}_{p}$. We now make a change of variable: set $x_{i}:=\sum_{j=1}^{a d_{i}} z_{i j} \mu_{i j}$. Observe that a monomial $x_{1}^{u_{1}} \cdots x_{n}^{u_{n}}$ transforms as follows. Writing each $u_{i}$ in base $p$, such as $u_{1}=a_{1} p^{\alpha_{1}}+\cdots$ and $u_{2}=a_{2} p^{\alpha_{2}}+\cdots$ with $0 \leq a_{i} \leq p-1$, we see that

$$
\begin{aligned}
x_{1}^{u_{1}} \cdots x_{n}^{u_{n}} & =\left(\sum_{j=1}^{a d_{1}} z_{1 j} \mu_{1 j}\right)^{u_{1}} \cdots\left(\sum_{j=1}^{a d_{n}} z_{n j} \mu_{n j}\right)^{u_{n}} \\
& =\left(\sum_{j=1}^{a d_{1}} z_{1 j} \mu_{1 j}^{p_{1}^{\alpha_{1}}}\right)^{a_{1}} \cdots\left(\sum_{j=1}^{a d_{n}} z_{n j} \mu_{n j}^{\alpha^{\alpha} n}\right)^{a_{n}} \cdots \\
& =\sum_{l=1}^{a d} G_{l}(z) \mu_{l}
\end{aligned}
$$

where each $G_{l}$ is a polynomial in the variables $z=\left(z_{i j}\right)$, defined over $\mathbb{F}_{p}$.
Performing this procedure for each $F_{i}$ creates a set of polynomials $\left\{G_{l}^{(i)}(z)\right\}$ for $1 \leq l \leq a d$ and $1 \leq i \leq r$ such that $\operatorname{deg} G_{l}^{(i)} \leq w_{p}\left(F_{i}\right)$ and with the property

$$
N_{1}(\mathbf{d})=\#\left\{z \in \mathbb{F}_{p}^{a\left(d_{1}+\cdots+d_{n}\right)} \mid G_{l}^{(i)}(z)=0 \text { for every } l \text { and } i\right\}
$$

That $p^{\omega} \mid N_{1}(\mathbf{d})$ now follows from the well-known theorem of Katz [4. This proves Theorem 1.1
With

$$
\rho:=\frac{\left(d_{1}+\cdots+d_{n}\right)-d \sum_{i=1}^{r} w_{p}\left(F_{i}\right)}{\max _{i} w_{p}\left(F_{i}\right)}
$$

observe that $\operatorname{ord}_{q} N_{m}(\mathbf{d}) \geq m \rho$ for every $m \geq 1$. Consequently, the argument of [2, p. 256] proves Corollary [1.2,

## 5 Partial mixed character sums

Fix positive integers $d_{1}, \ldots, d_{n} \in \mathbb{Z}_{\geq 1}$, and set $d:=\operatorname{lcm}\left(d_{1}, \ldots, d_{n}\right)$. Let $f(x)=\sum_{u \in U} a_{u} x^{u} \in \mathbb{F}_{q^{d}}\left[x_{1}, \ldots, x_{n}\right]$. Let $\psi$ be a nontrivial additive character on $\mathbb{F}_{q^{d}}$, and $\chi_{1}, \ldots, \chi_{r}$ multiplicative characters on $\mathbb{F}_{q^{d_{1}}}^{\times}, \ldots, \mathbb{F}_{q^{d_{n}}}^{\times}$, respectively, trivial or non-trivial. Define the mixed partial character sum

$$
S(\chi, \mathbf{d}, f):=\sum \chi_{1}\left(x_{1}\right) \cdots \chi_{r}\left(x_{r}\right) \psi(f(x))
$$

where the sum runs over $x_{i} \in \mathbb{F}_{q^{d_{i}}}^{\times}$for $i=1, \ldots, r$ and $x_{i} \in \mathbb{F}_{q^{d_{i}}}$ for $i=r+1, \ldots, n$. Our main theorem of this section provides an estimate for $\operatorname{ord}_{q} S(\chi, \mathbf{d}, f)$. We first embed $S(\chi, \mathbf{d}, f)$ into the $p$-adic numbers.

Let $\mathbb{Q}_{p}$ be the $p$-adic numbers, and denote by $\mathbb{Q}_{q^{d}}$ the unramified extension of $\mathbb{Q}_{p}$ of degree ad (recall, $q=p^{a}$ ). We normalize the valuation such that $\operatorname{ord}_{q}(q)=1$. Denote by $T_{i}$ the set of solutions of $z^{q_{i}}=z$ in $\mathbb{Q}_{q^{d_{i}}} \subseteq \mathbb{Q}_{q^{d}}$. Set

$$
\mathcal{T}:=T_{1}^{\times} \times \cdots \times T_{r}^{\times} \times T_{r+1} \times \cdots \times T_{n} .
$$

There exist $e_{i} \in\left\{1, \ldots, q_{i}-1\right\}$ for $i=1, \ldots, r$ such that

$$
S(\chi, \mathbf{d}, f)=\sum_{x \in \mathcal{T}} x_{1}^{e_{1}} \cdots x_{r}^{e_{r}} \psi(f(\bar{x})) \in \mathbb{Q}_{q^{d}}\left(\zeta_{p}\right)
$$

where $\bar{x}_{i}$ denotes the image of $x_{i}$ in the residue field $\mathbb{F}_{q^{d_{i}}} \subseteq \mathbb{F}_{q^{d}}$.
Theorem 5.1.

$$
\operatorname{ord}_{q} S(\chi, \mathbf{d}, f) \geq \frac{1}{w_{p}(f)}\left(\sum_{i=1}^{n} d_{i}-\frac{1}{a(p-1)} \sum_{i=1}^{r} \sigma_{p}\left(e_{i}\right)\right) .
$$

When there is no twist (ie. $r=0$ ), the estimate becomes

$$
\operatorname{ord}_{q} S(\mathbf{d}, f) \geq\left(d_{1}+\cdots+d_{n}\right) / w_{p}(f)
$$

Proof. Our proof follows that of [1] and [7, whose roots go back to at least [2]. Denote by $P$ the polynomial $P(t)=$ $\sum_{m=0}^{q^{d}-1} c_{m} t^{m}$ with the property that $P(z)=\psi(\bar{z})$ for every $\left(q^{d}-1\right)$-root of unity $z$ in $\mathbb{Q}_{q^{d}}$. Note that the coefficients satisfy

$$
c_{0}=1, \quad c_{q^{d}-1}=-q^{d} /\left(q^{d}-1\right), \quad \text { and } c_{m}=g_{m} /\left(q^{d}-1\right) \text { for } 1 \leq m \leq q^{d}-2,
$$

where $g_{m}$ is the Gauss sum

$$
g_{m}:=\sum_{z^{q^{d}-1}=1, z \in \mathbb{Q}_{q^{d}}} z^{-m} \psi(\bar{z})
$$

By a well-known result of Stickelberger,

$$
\begin{equation*}
\text { for } 0 \leq m \leq q^{d}-1: \quad \operatorname{ord}_{q} c_{m}=\frac{\sigma_{p}(m)}{a(p-1)} \tag{10}
\end{equation*}
$$

Recall, $f(x)=\sum_{u \in U} a_{u} x^{u} \in \mathbb{F}_{q^{d}}\left[x_{1}, \ldots, x_{n}\right]$. Let $\hat{a}_{u} \in \mathbb{Q}_{q^{d}}$ be the Teichmüller lift of $a_{u}$. Set $\mathbf{e}:=\left(e_{1}, \ldots, e_{r}, 0, \ldots, 0\right) \in$ $\mathbb{Z}_{\geq 0}^{n}$. Then

$$
\begin{aligned}
S(\chi, \mathbf{d}, f) & =\sum_{x \in \mathcal{T}} x^{\mathbf{e}} \psi(f(\bar{x})) \\
& =\sum_{x \in \mathcal{T}} x^{\mathbf{e}} \prod_{u \in U} \psi\left(a_{u} \bar{x}^{u}\right) \\
& =\sum_{x \in \mathcal{T}} x^{\mathbf{e}} \prod_{u \in U} P\left(\hat{a}_{u} x^{u}\right) \\
& =\sum_{x \in \mathcal{T}} x^{\mathbf{e}} \prod_{u \in U}\left(\sum_{m=0}^{q^{d}-1} c_{m} \hat{a}_{u}^{m} x^{m u}\right) .
\end{aligned}
$$

Denote by $\Phi$ the set of functions $\phi: U \rightarrow\left\{0,1, \ldots, q^{d}-1\right\}$. Then

$$
\begin{align*}
S(\chi, \mathbf{d}, f) & =\sum_{x \in \mathcal{T}} x^{\mathbf{e}} \sum_{\phi \in \Phi}\left(\prod_{u \in U} c_{\phi(u)} \hat{a}_{u}^{\phi(u)}\right) x^{\sum_{u \in U} \phi(u) u} \\
& =\sum_{\phi \in \Phi}\left(\prod_{u \in U} c_{\phi(u)} \hat{a}_{u}^{\phi(u)}\right)\left(\sum_{x \in \mathcal{T}} x^{\mathbf{e}+\sum_{u \in U} \phi(u) u}\right) . \tag{11}
\end{align*}
$$

The right most sum may be explicitly computed. In order to give its formula, we define the following. Denote by $m(\phi)$ the set of $i \in\{r+1, \ldots, n\}$ such that the $i$-th entry of $\sum_{u \in U} \phi(u) u$ is non-zero, and $(m(\phi))$ the complement of $m(\phi)$ in $\{r+1, \ldots, n\}$. Set

$$
q_{(m(\phi))}:=\prod_{i \notin m(\phi)} q_{i} \quad \text { and } \quad(q-1)_{m(\phi)}:=\prod_{i \in m(\phi)}\left(q_{i}-1\right) .
$$

Set $\left(\mathbf{q}_{*}-\mathbf{1}\right):=\left(q_{1}-1, \ldots, q_{n}-1\right)$. Also, define

$$
\left(\mathbf{q}_{*}-\mathbf{1}\right)\left(\mathbb{Z}_{>0}^{r} \times \mathbb{Z}_{\geq 0}^{n-r}\right):=\left(q_{1}-1\right) \mathbb{Z}_{>0} \times \cdots \times\left(q_{r}-1\right) \mathbb{Z}_{>0} \times\left(q_{r+1}-1\right) \mathbb{Z}_{\geq 0} \times \cdots \times\left(q_{n}-1\right) \mathbb{Z}_{\geq 0}
$$

Then, using the well-known identities

$$
j \in \mathbb{Z}: \quad \sum_{t \in T_{i}^{\times}} t^{j}= \begin{cases}q_{i}-1 & \text { if }\left(q_{i}-1\right) \mid j \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
j \in \mathbb{Z}_{\geq 0}: \quad \sum_{t \in T_{i}} t^{j}= \begin{cases}q_{i}-1 & \text { if }\left(q_{i}-1\right) \mid j \text { and } j \neq 0 \\ q_{i} & \text { if } j=0 \\ 0 & \text { if }\left(q_{i}-1\right) \nmid j,\end{cases}
$$

we have

$$
\begin{aligned}
& \sum_{x \in \mathcal{T}} x^{\mathbf{e}+\sum_{u \in U} \phi(u) u} \\
& \quad= \begin{cases}q_{(m(\phi))}(q-1)_{m(\phi)}\left(q_{1}-1\right) \cdots\left(q_{r}-1\right) & \text { if } \mathbf{e}+\sum_{u \in U} \phi(u) u \in\left(\mathbf{q}_{*}-\mathbf{1}\right)\left(\mathbb{Z}_{>0}^{r} \times \mathbb{Z}_{\geq 0}^{n-r}\right) \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

Denote by $\Phi_{0}$ the set of functions $\phi: U \rightarrow\left\{0,1, \ldots, q^{d}-1\right\}$ such that $\mathbf{e}+\sum_{u \in U} \phi(u) u \in\left(\mathbf{q}_{*}-\mathbf{1}\right)\left(\mathbb{Z}_{>0}^{r} \times \mathbb{Z}_{\geq 0}^{n-r}\right)$. It follows now by (11) that

$$
\operatorname{ord}_{q} S(\chi, \mathbf{d}, f) \geq \min _{\phi \in \Phi_{0}}\left\{\sum_{i \notin m(\phi)} d_{i}+\sum_{u \in U} \operatorname{ord}_{q} c_{\phi(u)}\right\} .
$$

Let $A \subseteq\{r+1, \ldots, n\}$, and denote by $\Phi_{0}^{(A)}$ the set of functions $\phi: U \rightarrow\left\{0,1, \ldots, q^{d}-1\right\}$ such that $\mathbf{e}+\sum_{u \in U} \phi(u) u \in$ $\left(\mathbf{q}_{*}-\mathbf{1}\right)\left(\mathbb{Z}_{>0}^{r} \times \mathbb{Z}_{\geq 0}^{n-r}\right)$ and the $i$-th entry of $\sum_{u \in U} \phi(u) u$ is non-zero if $i \in A$, and zero if $i \notin A$. Then

$$
\begin{aligned}
\operatorname{ord}_{q} S(\chi, \mathbf{d}, f) & \geq \min _{A \subseteq\{r+1, \ldots, n\}}\left\{\sum_{i \notin A} d_{i}+\min _{\phi \in \Phi_{0}^{(A)}}\left\{\sum_{u \in U} \operatorname{ord}_{q} c_{\phi(u)}\right\}\right\} \\
& \geq \min _{A \subseteq\{r+1, \ldots, n\}}\left\{\sum_{i \notin A} d_{i}+\frac{1}{a(p-1)} \min _{\phi \in \Phi_{0}^{(A)}}\left\{\sum_{u \in U} \sigma_{p}(\phi(u))\right\}\right\},
\end{aligned}
$$

where we used (10) for the second inequality.
For a vector $u=\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$, define $\sigma_{p}(u):=\left(\sigma_{p}\left(u_{1}\right), \ldots, \sigma_{p}\left(u_{n}\right)\right)$. Set $|u|:=u_{1}+\cdots+u_{n}$. Also, for $u, v \in \mathbb{Z}_{\geq 0}^{n}$, we write $u \geq v$ if $u_{i} \geq v_{i}$ for every $i$. Denote by $\mathbf{d}_{A}$ the vector in $\mathbb{Z}_{\geq 0}^{n}$ such that the $i$-th entry is 0 if $i \notin A$, and $d_{i}$ if $i \in A$. Last, we recall some properties of $\sigma_{p}$; see [7 Proposition 11]. For $a, b \in \mathbb{Z}_{\geq 0}$,

1. $\sigma_{p}(a+b) \leq \sigma_{p}(a)+\sigma_{p}(b)$;
2. $\sigma_{p}(a b) \leq \sigma_{p}(a) \sigma_{p}(b)$;
3. Let $q=p^{f}$. If $c$ is a positive multiple of $(q-1)$, then $\sigma_{p}(c) \geq \sigma_{p}(q-1)=f(p-1)$;

Using this, we see that for $\phi \in \Phi_{0}^{(A)}$,

$$
\sum_{u \in U} \sigma_{p}(\phi(u)) \sigma_{p}(u)+\sigma_{p}(\mathbf{e}) \geq a(p-1) \mathbf{d}_{A}
$$

Thus,

$$
\sum_{u \in U} \sigma_{p}(\phi(u))\left|\sigma_{p}(u)\right|+\left|\sigma_{p}(\mathbf{e})\right| \geq a(p-1)\left|\mathbf{d}_{A}\right|=a(p-1) \sum_{i \in A} d_{i}
$$

Since $w_{p}(f):=\max _{u \in U}\left|\sigma_{p}(u)\right|$,

$$
\sum_{u \in U} \sigma_{p}(\phi(u)) \geq \frac{1}{w_{p}(f)}\left(a(p-1) \sum_{i \in A} d_{i}-\left|\sigma_{p}(\mathbf{e})\right|\right)
$$

and so

$$
\begin{aligned}
\operatorname{ord}_{q} S(\chi, \mathbf{d}, f) & \geq \min _{A \subseteq\{r+1, \ldots, n\}}\left\{\sum_{i \notin A} d_{i}+\frac{1}{w_{p}(f)} \sum_{i \in A} d_{i}-\frac{\left|\sigma_{p}(\mathbf{e})\right|}{a(p-1) w_{p}(f)}\right\} \\
& \geq \frac{1}{w_{p}(f)}\left(\sum_{i=1}^{n} d_{i}-\frac{\left|\sigma_{p}(\mathbf{e})\right|}{a(p-1)}\right)
\end{aligned}
$$

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[^0]:    ${ }^{1}$ If you are unfamiliar with the Iverson bracket, take a look at Donald Knuth's article [5]

