# LINEAR FAMILIES OF SMOOTH HYPERSURFACES OVER FINITELY GENERATED FIELDS 

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#### Abstract

Let $K$ be a finitely generated field. We construct an $n$-dimensional linear system $\mathcal{L}$ of hypersurfaces of degree $d$ in $\mathbb{P}^{n}$ defined over $K$ such that each member of $\mathcal{L}$ defined over $K$ is smooth, under the hypothesis that the characteristic $p$ does not divide $\operatorname{gcd}(d, n+1)$ (in particular, there is no restriction when $K$ has characteristic 0 ). Moreover, we exhibit a counterexample when $p$ divides $\operatorname{gcd}(d, n+1)$.


## 1. Introduction

The study of hypersurfaces varying in a pencil, or more generally, in a linear system of arbitrary dimension, is an active research area. For instance, determining the number of reducible members in a pencil is already a challenging problem Ste89, Vis93, PY08. When the base field is a number field, the study of pencils has deep connections to Diophantine geometry; see, for example DGH21. Linear systems of hypersurfaces over finite fields have been studied by Ballico Bal07, Bal09.

Our primary goal in the present paper is to address the following question from a recent paper AG22] by the first two authors. While the version stated in AG22] was concerned with linear systems of hypersurfaces over finite fields, in this paper we will work over an arbitrary finitely generated field. Recall that a field $K$ is called finitely generated if it is generated by a finite number of elements as a field (or equivalently, as a field extension of its prime subfield).

Question 1. Let $K$ be a finitely generated field and $r \geqslant 1, n \geqslant 2, d \geqslant 2$ be integers. Do there exist $r+1$ linearly independent homogeneous polynomials $F_{0}, F_{1}, \ldots, F_{r} \in$ $K\left[x_{0}, \ldots, x_{n}\right]$ of degree $d$ such that the hypersurface

$$
X_{\left[a_{0}: a_{1}: \ldots: a_{r}\right]}=\left\{a_{0} F_{0}+a_{1} F_{1}+\ldots+a_{r} F_{r}=0\right\} \subset \mathbb{P}^{n}
$$

is smooth for every $\left[a_{0}: a_{1}: \ldots: a_{r}\right] \in \mathbb{P}^{r}(K)$ ?
Here, as usual, "smooth" means "smooth at every $\bar{K}$-point", not just at every $K$-point. Question 1 can be rephrased in geometric terms as follows. Consider the linear system $\mathcal{L}=\left\langle F_{0}, \ldots, F_{r}\right\rangle$ of (projective) dimension $r$ spanned by $F_{0}, \ldots, F_{r}$. We say that $\mathcal{L}$ is $K$-smooth if for every $\left[a_{0}: a_{1}: \ldots: a_{r}\right] \in \mathbb{P}^{r}(K)$, the hypersurface cut out by $a_{0} F_{0}+a_{1} F_{1}+\ldots+a_{r} F_{r}$ is smooth in $\mathbb{P}^{n}$. In other words, Question 1 asks for existence of a $K$-smooth linear system $\mathcal{L}$ in $\mathbb{P}^{n}$ of prescribed degree and dimension.

[^0]We show that, under a mild assumption on the characteristic, the maximum value of $r$ for which Question 1 has a positive answer is $r=n$.

Theorem 2. Let $K$ be an arbitrary field.
(1) If $r \geqslant n+1$, then there does not exist a $K$-smooth linear system of (projective) dimension $r$ (of any degree $d \geqslant 2$ ).
(2) Suppose $K$ is a finitely generated field of characteristic $p \geqslant 0$. If $r \leqslant n$ and $p \nmid \operatorname{gcd}(d, n+1)$, then there exist homogeneous polynomials $F_{0}, \ldots, F_{r}$ in $x_{0}, \ldots, x_{n}$ of degree $d$ such that $\mathcal{L}=\left\langle F_{0}, \ldots, F_{r}\right\rangle$ is a $K$-smooth linear system of (projective) dimension $r$.

Note that the assumption $p \nmid \operatorname{gcd}(d, p+1)$ on the characteristic of $K$ holds automatically when $\operatorname{char}(K)=0$. On the other hand, we will show in Section 5 that this assumption cannot be dropped in general. More precisely, we will show that no $n$-dimensional linear system of degree 2 hypersurfaces in $\mathbb{P}^{n}$ can be $K$ smooth in the case where $K$ is a field of characteristic 2 and $n \geqslant 1$ is an odd integer; see Theorem 6

The case where $r=1$, which corresponds to a pencil of hypersurfaces, is of particular interest. For any given $n$, the condition that $p \nmid \operatorname{gcd}(d, n+1)$ is satisfied for all but finitely many characteristics $p$. In particular, Theorem 2 tells us that for every value of $d \geqslant 1$ and every finitely generated field $K$ there exists

- a $K$-smooth pencil of degree $d$ in $\mathbb{P}^{2}$ if $\operatorname{char}(K) \neq 3$.
- a $K$-smooth pencil of degree $d$ in $\mathbb{P}^{3}$ if $\operatorname{char}(K) \neq 2$.
- a $K$-smooth pencil of degree $d$ in $\mathbb{P}^{4}$ if $\operatorname{char}(K) \neq 5$.
- a $K$-smooth pencil of degree $d$ in $\mathbb{P}^{5}$ if $\operatorname{char}(K) \neq 2,3$.

On the other hand, the main result of [AG22, Theorem 1.3] proves the existence of a $K$-smooth pencil $\mathcal{L}$ of degree $d$ hypersurfaces in $\mathbb{P}^{n}$ defined over the field $K=\mathbb{F}_{q}$ under a different hypothesis:

$$
q>\left(\frac{1+\sqrt{2}}{2}\right)^{2}\left((n+1)(d-1)^{n}\right)^{2}\left((n+1)(d-1)^{n}-1\right)^{2}\left((n+1)(d-1)^{n}-2\right)^{2}
$$

In particular, an $\mathbb{F}_{q}$-smooth pencil of degree $d$ hypersurfaces exists in any characteristic as long as $q$ is sufficiently large. It is reasonable to ask if smooth pencils of every degree exist over every finitely generated field.

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## 2. Proof of Theorem (2)

In this section $K$ will denote an arbitrary field. We will denote by $K\left[x_{0}, \ldots, x_{n}\right]_{d}$ the space of homogeneous polynomials of degree $d$ in $x_{0}, \ldots, x_{n}$ with coefficients in $K$. This is a $K$-vector space of dimension $N=\binom{n+d}{d}$. Points of the projective space $\mathbb{P}\left(K\left[x_{0}, \ldots, x_{n}\right]_{d}\right)$ are naturally identified with degree $d$ hypersurfaces in $\mathbb{P}^{n}$.

We proceed with the proof of part (11) of Theorem2, Assume the contrary: there exists a $K$-smooth linear system $\mathcal{L} \subset K\left[x_{0}, \ldots, x_{n}\right]_{d}$ of (affine) dimension $\geqslant n+2$.

Let $x_{0}^{d-1} K\left[x_{0}, \ldots, x_{n}\right]_{1}$ denote the $(n+1)$-dimensional $K$-vector space of degree $d$ forms divisible by $x_{0}^{d-1}$. Any such form can be written as $x_{0}^{d-1} l\left(x_{0}, \ldots, x_{n}\right)$, where $l \in K\left[x_{0}, \ldots, x_{n}\right]_{1}$. Consider the $K$-linear map

$$
\Psi: K\left[x_{0}, \ldots, x_{n}\right]_{d} \rightarrow x_{0}^{d-1} K\left[x_{0}, \ldots, x_{n}\right]_{1}
$$

which removes from $F \in \mathcal{L}(K)$ all monomials which are not multiples of $x_{0}^{d-1}$. In other words, for any non-negative integers $i_{0}, \ldots, i_{n}$ satisfying $i_{0}+\ldots+i_{n}=d$,

$$
\Psi\left(x_{0}^{i_{0}} x_{1}^{i_{1}} \ldots x_{n}^{i_{n}}\right)=\left\{\begin{array}{l}
x_{0}^{i_{0}} x_{1}^{i_{1}} \ldots x_{n}^{i_{n}}, \text { if } i_{0} \geqslant d-1, \text { and } \\
0, \text { otherwise } .
\end{array}\right.
$$

The kernel, $\operatorname{Ker}(\Psi)$, is precisely the set of polynomials $F \in K\left[x_{0}, \ldots, x_{n}\right]_{d}$ with the property that the associated hypersurface in $\mathbb{P}^{n}$ is singular at $P=[1: 0: \ldots: 0]$. Since the codimension of $\operatorname{Ker}(\Psi)$ in $K\left[x_{0}, \ldots, x_{n}\right]_{d}$ is at least $\operatorname{dim}\left(x_{0}^{d-1} K\left[x_{0}, \ldots, x_{n}\right]_{1}\right)=$ $n+1$ and $\operatorname{dim}(\mathcal{L}) \geqslant n+2$, we see that $\mathcal{L} \cap \operatorname{Ker}(\Psi)$ must contain a non-zero $K$-point of $\mathcal{L}$. In other words, $\mathcal{L}(K)$ contains a hypersurface which is singular at $P$. This shows that $\mathcal{L}$ cannot be $K$-smooth.

## 3. Proof of Theorem (2(2) in the case, where $K$ is a finite field

We begin by exhibiting two families of smooth hypersurfaces of degree $d \geqslant 2$ over an arbitrary field $K$ of characteristic $p \geqslant 0$.

Lemma 3. Suppose $p \nmid d$. Set $F=c_{0} x_{0}^{d}+c_{1} x_{1}^{d}+\ldots+c_{n} x_{n}^{d}$. If $c_{0}, c_{1}, \ldots, c_{n} \neq 0$, then $F$ cuts out a smooth hypersurface in $\mathbb{P}^{n}$.

Proof. This is clear from the Jacobian criterion: the equations

$$
\frac{\partial F}{\partial x_{i}}=d c_{i} x_{i}^{d-1}=0 \quad(i=0,1, \ldots, n)
$$

have no common solution in $\mathbb{P}^{n}$.
Lemma 4. Suppose $p \mid d$ but $p \nmid(n+1)$. Set $F=c_{0} x_{0}^{d-1} x_{1}+c_{1} x_{1}^{d-1} x_{2}+\ldots+$ $c_{n} x_{n}^{d-1} x_{0}$. If $c_{0}, c_{1}, \ldots, c_{n} \neq 0$, then $F$ cuts out a smooth hypersurface in $\mathbb{P}^{n}$.

Proof. Assume the contrary: the hypersurface cut out by $F$ in $\mathbb{P}^{n}$ is singular at some point $P=\left[u_{0}: u_{1}: \ldots: u_{n}\right] \in \mathbb{P}^{n}$. By symmetry we may assume without loss of generality that $u_{1} \neq 0$. Using the Jacobian criterion, and remembering that $p \mid d$, we obtain:

$$
\begin{equation*}
\frac{\partial F}{\partial x_{i}}(P)=c_{i-1} u_{i-1}^{d-1}-c_{i} u_{i}^{d-2} u_{i+1}=0 \tag{3.1}
\end{equation*}
$$

for each $0 \leqslant i \leqslant n$, where the subscripts are taken modulo $n+1$. Multiplying both sides of (3.1) by $u_{i}$, we obtain

$$
\begin{equation*}
c_{i-1} u_{i-1}^{d-1} u_{i}=c_{i} u_{i}^{d-1} u_{i+1} . \tag{3.2}
\end{equation*}
$$

Now recall that

$$
F(P)=c_{0} u_{0}^{d-1} u_{1}+c_{1} u_{1}^{d-1} u_{2}+\ldots+c_{n} u_{n}^{d-1} u_{0}=0 .
$$

By (3.2), the $n$ terms in this sum are all equal to each other. Hence,

$$
0=F(P)=\sum_{i=0}^{n} c_{i} u_{i}^{d-1} u_{i+1}=(n+1) c_{0} u_{0}^{d-1} u_{1}
$$

Since $p \nmid(n+1), c_{0} \neq 0$, and $u_{1} \neq 0$, we conclude that $u_{0}=0$.
We will divide the remainder of the proof into two cases, according to whether $d=2$ or $d \geqslant 3$. If $d \geqslant 3$, then (3.1) tells us that $u_{i}=0$ implies $u_{i-1}=0$ for any $i \in \mathbb{Z} /(n+1) \mathbb{Z}$. (Recall that the subscripts in (3.1) are viewed modulo $n+1$.) Using this implication recursively, starting from $u_{0}=0$, we see that $u_{0}=u_{n}=u_{n-1}=$ $\ldots=u_{1}=0$, a contradiction .

Now assume $d=2$. In this case (3.1) tells us that $u_{i-1}=0$ implies $u_{i+1}=0$ for any $i \in \mathbb{Z} /(n+1) \mathbb{Z}$. Since we know that $u_{0}=0$, this tells us that $u_{i}=0$ for every even $i$. Since $d=2$, the assumption that $p$ divides $d$ tells us that $p=2$ and the assumption that $p$ does not divide $n+1$ tells us that that $n=2 k$ is even. Thus, $2 k+2 \equiv 1$ modulo $n+1$ and hence, $0=u_{2 k+2}=u_{1}=0$, a contradiction.

We are now ready to prove Theorem (2(2) in the case, where $K=\mathbb{F}_{q}$ is a finite field. Since any $K$-linear subspace of a $K$-smooth linear system is again $K$-smooth, we may assume without loss of generality that $r=n$. Note also that $p \nmid \operatorname{gcd}(d, n+1)$ if and only if $p \nmid d$ or $p \nmid n+1$. Thus we may consider two cases.

Case 1: $p \nmid d$. We will explicitly construct a linear system $\mathcal{L}$ of dimension $r=n$ with the desired property. By the normal basis theorem, we can find an element $\alpha \in \mathbb{F}_{q^{n+1}}$ such that $\alpha, \alpha^{q}, \alpha^{q^{2}}, \ldots, \alpha^{q^{n}}$ form an $\mathbb{F}_{q^{-}}$basis for the $(n+1)$-dimensional vector space $\mathbb{F}_{q^{n+1}}$. Let

$$
\begin{aligned}
F_{0} & =\left(\alpha x_{0}+\alpha^{q} x_{1}+\alpha^{q^{2}} x_{2}+\ldots+\alpha^{q^{i}} x_{i}+\ldots+\alpha^{q^{n}} x_{n}\right)^{d}, \\
F_{1} & =\left(\alpha^{q} x_{0}+\alpha^{q^{2}} x_{1}+\alpha^{q^{3}} x_{2}+\ldots+\alpha^{q^{i+1}} x_{i}+\ldots+\alpha x_{n}\right)^{d}, \\
F_{2} & =\left(\alpha^{q^{2}} x_{0}+\alpha^{q^{3}} x_{1}+\alpha^{q^{4}} x_{2}+\ldots+\alpha^{q^{i+2}} x_{i}+\ldots+\alpha^{q} x_{n}\right)^{d}, \\
& \vdots \\
F_{n} & =\left(\alpha^{q^{n}} x_{0}+\alpha^{q} x_{1}+\alpha^{q^{2}} x_{2}+\ldots+\alpha^{q^{i+n}} x_{i}+\ldots+\alpha^{q^{n-1}} x_{n}\right)^{d} .
\end{aligned}
$$

Note that the polynomials $F_{i}$ are not defined over $\mathbb{F}_{q}$. However, the set $\left\{F_{0}, F_{1}, \ldots, F_{n}\right\}$ is invariant under the action of the $q$-th power Frobenius map. Thus, the linear system $\mathcal{L}=\left\langle F_{0}, \ldots, F_{n}\right\rangle$ is defined over $\mathbb{F}_{q}$, that is, one can find a set of new generators $G_{0}, G_{1}, \ldots, G_{n}$ for $\mathcal{L}$ where each $G_{i}$ is defined over $\mathbb{F}_{q}$.

We claim that $F_{0}, F_{1}, \ldots, F_{n}$ are linearly independent over $\overline{\mathbb{F}_{q}}$. To prove this claim, let

$$
\begin{equation*}
y_{j}=\alpha^{q^{j}} x_{0}+\alpha^{q^{j+1}} x_{1}+\alpha^{q^{j+2}} x_{2}+\ldots+\alpha^{q^{j+i}} x_{i}+\ldots+\alpha^{q^{j+n}} x_{n} \tag{3.3}
\end{equation*}
$$

for each $0 \leqslant j \leqslant n$, and observe that $F_{i}=y_{i}^{d}$. The linear map $x_{i} \mapsto y_{i}$ is a linear automorphism of $\mathbb{P}^{n}$. Indeed, the matrix of this linear transformation, known as a Moore matrix, is non-singular; see, e.g., Gos96, Corollary 1.3.4]. Thus, $y_{0}, \ldots, y_{n}$ are algebraically independent over $\mathbb{F}_{q}$ and hence, over $\overline{\mathbb{F}_{q}}$. Consequently, $F_{0}, F_{1}, \ldots, F_{n}$ are linearly independent over $\overline{\mathbb{F}_{q}}$. This proves the claim. In summary, $\mathcal{L}=\left\langle F_{0}, F_{1}, \ldots, F_{n}\right\rangle$ is a linear system of degree $d$ hypersurfaces in $\mathbb{P}^{n}$ defined over $\mathbb{F}_{q}$ of (projective) dimension $r=n$.

It remains to show that $\mathcal{L}$ is $\mathbb{F}_{q}$-smooth. Indeed, suppose

$$
\begin{equation*}
X=\left\{c_{0} F_{0}+c_{1} F_{1}+\ldots+c_{n} F_{n}=0\right\} \tag{3.4}
\end{equation*}
$$

is a singular hypersurface $X$ which belongs to $\mathcal{L}$, for some $c_{i} \in \overline{\mathbb{F}_{q}}$ where not all $c_{i}$ are zero. Our goal is to show that $X$ is not defined over $\mathbb{F}_{q}$. In the new coordinates $y_{i}$, we can express (3.4) as:

$$
X=\left\{c_{0} y_{0}^{d}+c_{1} y_{1}^{d}+\ldots+c_{n} y_{n}^{d}=0\right\}
$$

Since $X$ is singular, we can apply Lemma 3 to deduce that $c_{i}=0$ for some $i$. Without loss of generality, we may assume that $c_{0}=0$. By applying the Frobenius map, we see that $X$ is sent to:

$$
X^{\sigma}=\left\{c_{1}^{q} F_{2}+\ldots+c_{n}^{q} F_{0}=0\right\}
$$

We claim that $X$ and $X^{\sigma}$ are distinct. Indeed, their defining equations are not multiples of one another: otherwise, there would exist a nonzero constant $b \in \overline{\mathbb{F}_{q}}$ such that $c_{i}^{q}=b \cdot c_{i+1}$ for each $0 \leqslant i \leqslant n$ taken modulo $n+1$. As $c_{0}=0$, this would force $c_{i}=0$ for each $0 \leqslant i \leqslant n$, which is a contradiction. Thus, $X$ is not defined over $\mathbb{F}_{q}$, as desired. We conclude that the linear system $\mathcal{L}$ is $\mathbb{F}_{q}$-smooth.

Case 2: $p \mid d$ but $p \nmid(n+1)$. Define $y_{0}, \ldots, y_{n}$ by the formula (3.3), and set $F_{i}=y_{i}^{q} y_{i+1}$ for $0 \leqslant i \leqslant n-1$ and $F_{n}=y_{n}^{q} y_{0}$. Arguing as in Case 1, one readily checks that $\mathcal{L}=\left\langle F_{0}, F_{1}, \ldots, F_{n}\right\rangle$ is a linear subspace of (projective) dimension $n$ defined over $\mathbb{F}_{q}$. Moreover, the same argument as in Case 1, with Lemma 4 used in place of Lemma 3, shows that $\mathcal{L}$ is $\mathbb{F}_{q}$-smooth.

This completes the proof of Theorem (2(2) in the case, where $K=\mathbb{F}_{q}$ is a finite field.

## 4. Conclusion of the proof of Theorem [2](2)

Given a finitely generated field $K$, we define its dimension $\operatorname{dim}(K)$ to be the Krull dimension of any finitely generated $\mathbb{Z}$-algebra whose fraction field is $K$. In other words, $\operatorname{dim}(K)=\operatorname{trdeg}_{\mathbb{F}_{p}}(K)$ if $\operatorname{char}(K)=p>0$ and $\operatorname{dim}(K)=1+\operatorname{trdeg}_{\mathbb{Q}}(K)$ if $\operatorname{char}(K)=0$. In this section we will prove Theorem (2(2) over an arbitrary finitely generated field $K$ by induction on $\operatorname{dim}(K)$. The inductive step will be based on the following lemma.

Lemma 5. Let $R$ be discrete valuation ring with fraction field $K$ and residue field $L$, and let $F_{0}, \ldots, F_{r} \in L\left[x_{0}, \ldots, x_{n}\right]$ be linearly independent homogeneous polynomials of degree d. Denote their liftings to $R$ by $\overline{F_{0}}, \ldots, \overline{F_{r}} \in R\left[x_{0}, \ldots, x_{n}\right] \subset K\left[x_{0}, \ldots, x_{n}\right]$, respectively. If the linear system $\left\langle F_{0}, \ldots, F_{r}\right\rangle$ is L-smooth, then the linear system $\left\langle\overline{F_{0}}, \ldots, \overline{F_{r}}\right\rangle$ is $K$-smooth.

Proof. Let $\left(a_{0}, \ldots, a_{r}\right)$ be in $K^{r+1} \backslash\{(0, \ldots, 0)\}$. We will show that the hypersurface in $\mathbb{P}_{K}^{n}$ defined by the form $a_{0} \overline{F_{0}}+\ldots+a_{r} \overline{F_{r}}$ is smooth. By scaling the $a_{i}$, we may assume that $a_{i} \in R$ for all $i$ and $a_{i}$ is invertible in $R$ for at least one $i$. Consider the hypersurface $X \subset \mathbb{P}_{K}^{n}$ defined by $a_{0} \overline{F_{0}}+\cdots+a_{r} \overline{F_{r}}=0$. Then $X$ is flat over $\operatorname{Spec}(R)$ and its fiber over $\mathcal{L}$ is smooth by hypothesis. Since the smooth locus of the projection $X \rightarrow \operatorname{Spec}(R)$ is open in $X$, its complement must be empty. It follows that the fiber over the generic point of $\operatorname{Spec}(R)$ is smooth, as desired.

We are now ready to finish the proof of Theorem (2) by induction on the dimension of the finitely generated field $K$. If $\operatorname{dim}(K)=0$, then $K$ is a finite field. In this case Theorem 2(2) is proved in Section 3. If $\operatorname{dim}(K)>0$, then it is easy to see that $K$ admits a discrete valuation with finitely generated residue field $L$ such that $\operatorname{dim}(L)=\operatorname{dim}(K)-1$. Furthermore, if $\operatorname{char}(K)=0$, then this valuation can be chosen so that $\operatorname{char}(L)$ is positive and arbitrarily large. By applying Lemma 5 we can lift an $L$-smooth linear system of hypersurfaces in $\mathbb{P}^{n}$ to a $K$-smooth linear system of hypersurfaces in $\mathbb{P}^{n}$ of the same degree degree and the same dimension.

## 5. Quadrics in characteristic 2

In this section, we will show that the hypothesis $p \nmid \operatorname{gcd}(d, n+1)$ in our main theorem cannot be removed in general. We will focus on the case, where $p=d=2$ and $n$ is odd. Our goal is to prove the following result.

Theorem 6. Suppose $n$ is an odd positive integer, and $K$ be a field of characteristic 2 (not necessarily finitely generated). Then for any $d \geqslant 2$ there does not exist a linear system $\mathcal{L}=\left\langle F_{0}, \ldots, F_{n}\right\rangle \subset K\left[x_{0}, \ldots, x_{n}\right]_{2}$ of (projective) dimension $n$ over $K$ such that each $K$-member of $\mathcal{L}$ is a smooth quadric hypersurface in $\mathbb{P}^{n}$.

We begin with the following lemma.
Lemma 7. Let $K$ be a field of characteristic 2 and $n \geqslant 1$ be an odd integer. Consider a quadric hypersurface $X \subset \mathbb{P}^{n}$ cut out by

$$
F\left(x_{0}, \ldots, x_{n}\right)=x_{0}^{2}+G\left(x_{1}, x_{2}, \ldots, x_{n}\right)
$$

where $G \in K\left[x_{1}, \ldots, x_{n}\right]$ is a homogeneous polynomial of degree 2. Then $X$ is singular.

Proof. The Jacobian criterion gives rise to a homogeneous system

$$
\frac{\partial G}{\partial x_{1}}=\ldots=\frac{\partial G}{\partial x_{n}}=0
$$

of $n$ linear equations in $x_{1}, \ldots, x_{n}$. (Note $x_{0}$ never appears in this system.) We claim that this homogeneous linear system has a nontrivial solution. To prove the claim, it suffices to show that the matrix $M$ of this linear system is singular. Note that $M$ is the Hessian matrix of $G$ and hence, is symmetric. (Since $G$ is a quadratic polynomial, the entries of the Hessian matrix are constant.) Because we are in characteristic $2, M$ is also skew-symmetric. It remains to show that a skewsymmetric square $n \times n$ matrix $M$ over any commutative ring has zero determinant, when $n$ is odd.

Indeed, consider the universal skew-symmetric matrix $n \times n$ matrix $A$ over the polynomial ring $R=\mathbb{Z}\left[x_{i j} \mid 1 \leqslant i<j \leqslant n\right]$. By definition, the $(i, j)$-th entry of $A$ is $x_{i j}$ if $i<j, 0$ if $i=j$ and $-x_{i j}$ if $i>j$. Taking the determinant on both sides of $A^{T}=-A$, and remembering that $n$ is odd, we obtain $\operatorname{det}(A)=-\operatorname{det}(A)$ in $R$. Since $R$ is an integral domain of characteristic 0 , this implies that $\operatorname{det}(A)=0$. A simple specialization argument (specializing $x_{i j}$ to the $(i, j)$-th entry of $M$ ) now shows that $\operatorname{det}(M)=0$, as desired.

Thus, we have found $(0, \ldots, 0) \neq\left(t_{1}, \ldots, t_{n}\right) \in K^{n}$ such that for any point $P \in \mathbb{P}^{n}$ of the form $P=\left[t_{0}: \ldots: t_{n}\right]$, we have

$$
\begin{equation*}
\frac{\partial F}{\partial x_{0}}(P)=\ldots=\frac{\partial F}{\partial x_{n}}(P)=0 . \tag{5.1}
\end{equation*}
$$

Note that since $\operatorname{deg}(F)$ is even and we are in characteristic 2, conditions (5.1) do not guarantee that $F(P)=0$. On the other hand, the partial derivatives of $F\left(x_{0}, \ldots, x_{n}\right)$ depend only on $x_{1}, \ldots, x_{n}$ and not on $x_{0}$. We thus want to choose $t_{0}$ so that the resulting point $P=\left[t_{0}: \ldots: t_{n}\right]$ lies on the hypersurface $X$ cut out by $F$. To achieve this goal, we choose $t_{0} \in \bar{K}$ so that

$$
t_{0}^{2}=-G\left(t_{1}, t_{2}, \ldots, t_{n}\right)
$$

Then $P=\left[t_{0}: \ldots: t_{n}\right] \in \mathbb{P}^{n}(\bar{K})$ satisfies both (5.1) and $F(P)=0$. In other words, $X$ is singular at $P$.

Remark 8. If $K$ is a perfect field of characteristic 2, then the above construction gives rise to a singular point $P=\left[t_{0}: \ldots: t_{n}\right]$ of $X$ defined over $K$. Indeed, since $K$ is closed under taking square roots, we can always choose $t_{0} \in K$ in the last step.
Remark 9. The conclusion of Lemma 7 is false when $n=2 k$ is even. Indeed, the quadric hypersurface in $\mathbb{P}^{n}$ defined by the polynomial

$$
x_{0}^{2}+x_{1} x_{2}+x_{3} x_{4}+\ldots+x_{2 k-1} x_{2 k}=0
$$

is smooth.
We now proceed with a proof of Theorem6.
Proof of Theorem 6. Suppose, to the contrary, that $\mathcal{L}=\left\langle F_{0}, \ldots, F_{n}\right\rangle$ is a $K$-smooth linear system of quadric hypersurfaces of (projective) dimension $n$. Let $\mathcal{L}(K)$ denote the set of $K$-members of the system.

Consider the $K$-linear map

$$
\Psi: K\left[x_{0}, \ldots, x_{n}\right]_{2} \rightarrow x_{0} K\left[x_{0}, \ldots, x_{n}\right]_{1}
$$

introduced in Section 2 (with $d=2$ ). Recall that $x_{0} K\left[x_{0}, \ldots, x_{n}\right]_{1}$ denotes the $(n+1)$-dimensional $K$-vector space of quadratic forms in $x_{0}, \ldots, x_{n}$ divisible by $x_{0}$ and that $\Psi$ removes from $F \in K\left[x_{0}, \ldots, x_{n}\right]$ all monomials which are not multiples of $x_{0}$. When $d=2$, the map $\Psi$ is given by the simple formula

$$
(\Psi F)\left(x_{0}, \ldots, x_{n}\right)=F\left(x_{0}, x_{1}, \ldots, x_{n}\right)-F\left(0, x_{1}, \ldots, x_{n}\right)
$$

As we noted in Section 2, $F$ lies in the kernel of $\Psi$ if and only if the hypersurface in $\mathbb{P}^{n}$ cut out by $F$ is singular at the point $[1: 0: \ldots: 0]$. Since the linear system $\mathcal{L}$ is $K$-smooth, this tells us that the restricted map

$$
\Psi: \mathcal{L}(K) \rightarrow x_{0} K\left[x_{0}, \ldots, x_{n}\right]_{1}
$$

is injective. Since the vector spaces $\mathcal{L}(K)$ and $x_{0} K\left[x_{0}, \ldots, x_{n}\right]_{1}$ are of the same dimension $n+1$, we conclude that $\Psi$ must also be surjective. In particular, there exists some $F \in \mathcal{L}(K)$ whose image under $\Psi$ is $x_{0}^{2}$. In other words,

$$
F\left(x_{0}, \ldots, x_{n}\right)=x_{0}^{2}+G\left(x_{1}, \ldots, x_{n}\right)
$$

for some quadratic form $G$ in $x_{1}, \ldots, x_{n}$. By Lemma 7 , $F$ cuts out a singular quadric hypersurface. This contradicts the assumption that each $K$-member of $\mathcal{L}$
is smooth. We conclude that a $K$-smooth linear system $\mathcal{L}$ of quadric hypersurfaces in $\mathbb{P}^{n}$ of dimension $n$ does not exist.

We have shown that the hypothesis $p \nmid \operatorname{gcd}(d, n+1)$ of Theorem 2(2) cannot be removed in the case $p=2$. We do not know whether this assumption can be dropped for other primes $p$. We finish the paper with an example, which shows that it can be for one particular choice of $K, p, d$, and $n$.

Example 10. Set $d=3$ and $n=2$ and consider the following cubic homogeneous polynomials with coefficients in $K=\mathbb{F}_{3}$ :

$$
\begin{aligned}
& F_{0}=x^{3}+x^{2} y-x y^{2}+y^{3}+x^{2} z+x y z+y^{2} z-x z^{2}+z^{3} \\
& F_{1}=x^{3}+x^{2} y-x^{2} z-x y z+y^{2} z+z^{3} \\
& F_{2}=x^{3}-x^{2} y+x y^{2}+y^{3}+x^{2} z+x y z+y^{2} z-y z^{2}
\end{aligned}
$$

A computer calculation shows that $a F_{0}+b F_{1}+c F_{2}=0$ defines a smooth plane curve for each of the possible $3^{2}+3+1=13$ choices $[a: b: c] \in \mathbb{P}^{2}\left(\mathbb{F}_{3}\right)$. In other words, $\left\langle F_{0}, F_{1}, F_{2}\right\rangle$ is a $\mathbb{F}_{3}$-smooth linear system of (projective) dimension $n=2$. Thus, the conclusion of Theorem 2(2) holds in this example, even though $p$ divides $\operatorname{gcd}(d, n+1)$.

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