Constructions of cyclic codes and extended primitive cyclic codes with their applications

Ziling Heng, Xinran Wang^{*}, Xiaoru Li

School of Science, Chang'an University, Xi'an 710064, China

Abstract

Linear codes with a few weights have many nice applications including combinatorial design, distributed storage system, secret sharing schemes and so on. In this paper, we construct two families of linear codes with a few weights based on special polynomials over finite fields. The first family of linear codes are extended primitive cyclic codes which are affine-invariant. The second family of linear codes are reducible cyclic codes. The parameters of these codes and their duals are determined. As the first application, we prove that these two families of linear codes hold *t*-designs, where t = 2, 3. As the second application, the minimum localities of the codes are also determined and optimal locally recoverable codes are derived.

Keywords: Linear code, cyclic code, extended primitive cyclic code 2000 MSC: 94B05, 94A05

1. Introduction

Let \mathbb{F}_q be the finite field with q elements, where q is a power of a prime. Let $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. Let C be a non-empty set such that $C \subseteq \mathbb{F}_q^n$. If C is a k-dimensional linear subspace over \mathbb{F}_q , then C is called an [n,k,d] linear code over \mathbb{F}_q , where d denotes its minimum distance. In particular, if any codeword $(c_0, c_1, \dots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$, then C is called a cyclic code. The dual of an [n,k,d] linear code C is defined by

$$\mathcal{C}^{\perp} = \left\{ \mathbf{u} \in \mathbb{F}_{q}^{n} : \langle \mathbf{u}, \mathbf{c} \rangle = 0 \ \forall \ \mathbf{c} \in \mathcal{C} \right\},$$

where $\langle \mathbf{u}, \mathbf{c} \rangle$ denotes the standard inner product of \mathbf{u} and \mathbf{c} . It is obvious that \mathcal{C}^{\perp} is an [n, n-k] linear code. Let A_i denote the number of codewords with weight *i* in a linear code of length *n*, where $0 \le i \le n$. Then $A(z) = 1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$ is referred to as the weight enumerator of \mathcal{C} . The sequence $(1, A_1, \cdots, A_n)$ is called the weight distributions of \mathcal{C} . The weight enumerator can be used not only to characterize the error detection and correction capabilities of linear codes,

^{*}This research was supported in part by the National Natural Science Foundation of China under Grant 12271059, in part by the Young Talent Fund of University Association for Science and Technology in Shaanxi, China, under Grant 20200505, and in part by the Fundamental Research Funds for the Central Universities, CHD, under Grant 300102122202.

^{*}Corresponding author

Email addresses: zilingheng@chd.edu.cn (Ziling Heng), wangxr203@163.com (Xinran Wang*), lixiaoru@163.com (Xiaoru Li)

but also to calculate the error rate of error correction and detection. The weight enumerator of linear codes including cyclic codes has been studied in a large number of literatures in recent years [6, 11, 14, 15, 16, 17, 18, 19, 22, 29].

Let κ , *t* and *n* be positive integers with $1 \le t \le \kappa \le n$. Let \mathcal{P} be a set of *n* elements and \mathcal{B} be a set of κ -subsets of \mathcal{P} . The pair $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ is called a t- (n, κ, λ) design, or simply *t*-design, if each *t*-subset of \mathcal{P} is contained in precisely λ elements of \mathcal{B} . The elements of \mathcal{P} are called points and the elements of \mathcal{B} are referred to as blocks. A *t*-design without repeated blocks is said to be simple. A *t*-design is called a Steiner system if $\lambda = 1$ and $t \ge 2$, which is denoted by $S(t, \kappa, n)$. Some Steiner systems have been constructed in [4, 9, 23, 25, 26, 30].

Linear codes can be used to constructed *t*-designs. The well-known coding-theoretic construction is described below. Let $\mathcal{P} = \{1, 2, \dots, n\}$ be a set of coordinate positions of the codewords of linear code \mathcal{C} with length *n*. The support of a codeword $\mathbf{c} = \{c_1, c_2, \dots, c_n\}$ in \mathcal{C} is defined by $\operatorname{suppt}(\mathbf{c}) = \{1 \le i \le n : c_i \ne 0\}$. Let \mathcal{B}_{κ} denote the set of supports of all codewords with Hamming weight κ in \mathcal{C} . The pair $(\mathcal{P}, \mathcal{B}_{\kappa})$ may be a *t*- (n, κ, λ) design for some positive integer λ , which is referred to as a support design of \mathcal{C} . In other words, we say that the codewords with weight κ in \mathcal{C} support a *t*- (n, κ, λ) design. When the pair $(\mathcal{P}, \mathcal{B}_{\kappa})$ is a simple *t*- (n, κ, λ) design, we have the following relation:

$$|\mathcal{B}_{\kappa}| = \frac{1}{q-1} A_{\kappa}, \binom{n}{t} \lambda = \binom{\kappa}{t} \frac{1}{q-1} A_{\kappa}.$$
(1)

The following theorem developed by Assmus and Mattson gives a sufficient condition such that the pair $(\mathcal{P}, \mathcal{B}_{\kappa})$ defined in a linear code C is a *t*-design.

Theorem 1. [1] (Assmus-Mattson Theorem) Let C be an [n,k,d] code over \mathbb{F}_q , and let d^{\perp} denote the minimum distance of C^{\perp} . Let w be the largest integer satisfying $w \leq n$ and

$$w - \left\lfloor \frac{w+q-1}{q-2} \right\rfloor < d.$$

Define w^{\perp} analogously with d^{\perp} . Let (A_0, A_1, \dots, A_n) and $(A_0^{\perp}, A_1^{\perp}, \dots, A_n^{\perp})$ be the weight distributions of C and C^{\perp} , respectively. Let t be a positive integer with t < d such that there are at most $d^{\perp} - t$ weights of C in the sequence $(A_0, A_1, \dots, A_{n-t})$. Then

- 1. $(\mathcal{P}, \mathcal{B}_{\kappa})$ is a simple t-design provided that $A_{\kappa} \neq 0$ and $d \leq \kappa \leq w$;
- 2. $(\mathcal{P}, \mathcal{B}_{\kappa}^{\perp})$ is a simple t-design provided that $A_{\kappa}^{\perp} \neq 0$ and $d^{\perp} \leq \kappa \leq w^{\perp}$, where $\mathcal{B}_{\kappa}^{\perp}$ denotes the set of supports of all codewords of weight κ in \mathcal{C}^{\perp} .

The Assmus-Mattson Theorem is a powerful tool for construction *t*-design from linear codes [3, 4, 5, 6, 7, 8, 10, 20, 21, 26, 27, 28].

We can also use the automorphism group approach to obtain *t*-designs from linear codes. We review the automorphism group of linear codes for introducing this approach. The set of coordinate permutations that map a code C to itself forms a group denoted by PAut(C). PAut(C) is called the permutation automorphism group of C. If C is a code with length n, then PAut(C) is a subgroup of the symmetric group Sym(n). A monomial matrix over \mathbb{F}_q is a square matrix which has exactly one nonzero element of \mathbb{F}_q in each row and column. A monomial matrix M can be written either in the form PD or the form D'P with P a permutation matrix and D and D' being diagonal matrices. The set of monomial matrices that map C to itself forms a group denoted as MAut(C). MAut(C) is called the monomial automorphism group of C. It is obvious that PAut(C) \subseteq MAut(C). The automorphism group Aut(C) of C is a set of maps with form $M\sigma$ that map C to itself, where M is a monomial matrix and σ is a field automorphism. Then we have PAut(C) \subseteq MAut(C) \subseteq Aut(C). Note that PAut(C), MAut(C) and Aut(C) are the same in the binary case.

Clearly, every element in Aut(C) has the form $DP\sigma$, where D is a diagonal matrix, P is a permutation matrix and σ is an automorphism of \mathbb{F}_q . If for every pair of *t*-element ordered sets of coordinates, there exists an element $DP\sigma$ in Aut(C) such that its permutation part P sends the first set to the second set, then Aut(C) is called *t*-transitive. The following gives a sufficient condition for a linear code to hold *t*-designs.

Theorem 2. [13] Let C be a linear code of length n over \mathbb{F}_q . If Aut(C) is t-transitive, then the codewords of any weight $i \ge t$ of C hold a t-design.

The objective of this paper is to construct two families of linear codes with a few weights and study their applications. We first construct the linear codes based on special polynomials over finite fields. The first family of linear codes are extended primitive cyclic codes which are affine-invariant. The second family of linear codes are cyclic codes. The parameters of these codes and their duals are determined. As the first application, we prove that these two families of linear codes are also determined and optimal locally recoverable codes are derived.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary results on the number of zeros of some equations over finite fields and affine-invariant codes, which will be used in this paper. In Section 3, we construct a class of extended primitive cyclic codes by a special function and determine their parameters. We then derive some infinite families of 2-designs and 3-designs from these linear codes. In Section 4, we give another construction of linear codes, which are cyclic codes, and determine their parameters and weight distributions. It turns out that they hold 3-designs. In Section 5, we drive some optimal locally recoverable codes from these linear codes. In Section 6, we conclude the paper.

2. Preliminaries

In this section, we will present some preliminary results on the number of zeros of some equations over \mathbb{F}_q and affine-invariant codes.

2.1. The number of zeros of some equations over finite fields

Lemma 3. Let h and m be two integers with h < m and let $q = p^m$ with p a prime. Define a nonzero polynomial of the form

$$g(x) = \sum_{i=0}^{h} a_i x^{p^i}, \ a_i \in \mathbb{F}_q$$

Denote by N_g the number of zeros of g(x) in \mathbb{F}_q . Then $N_g \in \{1, p, p^2, p^3, \dots, p^h\}$.

Proof. It is obvious that $N_g \leq p^h$. Let G be the set of zeros of g(x) in \mathbb{F}_q . Then $G \neq \emptyset$ as $0 \in G$. It is easy to prove that (G, +) is a subgroup of \mathbb{F}_q . By Lagrange's Theorem, the order of G divides the order of \mathbb{F}_q . Then we have $N_g \in \{1, p, p^2, p^3, \dots, p^h\}$.

In the following, we let α be a generator of \mathbb{F}_q^* and give some examples to verify Lemma 3. **Example 4.** Let p = 2, m = 5 and $q = p^m$. Let

$$g_1(x) = \alpha^2 x + \alpha x^2 + \alpha^5 x^4.$$

Denote by N_{g_1} the number of zeros of $g_1(x)$ in \mathbb{F}_q . By Magma program, $N_{g_1} = 2$. Example 5. Let p = 2, m = 4 and $q = p^m$. Let

$$g_2(x) = \alpha^3 x + \alpha^5 x^2 + \alpha^8 x^4 + \alpha^7 x^8$$

Denote by N_{g_2} the number of zeros of $g_2(x)$ in \mathbb{F}_q . By Magma program, $N_{g_2} = 4$. Example 6. Let p = 3, m = 4 and $q = p^m$. Let

$$g_3(x) = \alpha^5 x + \alpha^9 x^3 + \alpha^{12} x^9 + \alpha^{11} x^{27}.$$

Denote by N_{g_3} the number of zeros of $g_3(x)$ in \mathbb{F}_q . By Magma program, $N_{g_3} = 9$. Example 7. Let p = 2, m = 4 and $q = p^m$. Let

$$g_4(x) = \alpha^{13}x + \alpha^7 x^2 + \alpha^{10} x^4 + \alpha x^8$$

Denote by N_{g_4} the number of zeros of $g_4(x)$ in \mathbb{F}_q . By Magma program, $N_{g_4} = 8$.

Example 8. Let p = 3, m = 3 and $q = p^m$. Let

$$g_5(x) = \alpha^{14}x + \alpha^{10}x^3 + \alpha^{24}x^9.$$

Denote by N_{g_5} the number of zeros of $g_5(x)$ in \mathbb{F}_q . By Magma program, $N_{g_5} = 9$.

Let *h* and *m* be positive integers with h < m and let $q = p^m$ with *p* a prime. Now we consider the zeros of the nonzero polynomial

$$f(x) = c + \sum_{i=0}^{h} a_i x^{p^i}, \ a_i, c \in \mathbb{F}_q,$$
 (2)

in \mathbb{F}_q . Let g(x) be the polynomial defined in Lemma 3. It is obvious that $f(x) = g(x) + c, c \in \mathbb{F}_q$.

Lemma 9. Let h and m be positive integers with h < m and let $q = p^m$ with p a prime. Denote by N_f the number of zeros of f(x) in \mathbb{F}_q . Then $N_f \in \{0, 1, p, p^2, p^3, \dots, p^h\}$.

Proof. If $a_i = 0$ for all $0 \le i \le h$ and $c \ne 0$, then $N_f = 0$. Now we assume that $(a_0, a_1, \dots, a_h) \ne (0, 0, \dots, 0)$. If f(x) has a zero u in \mathbb{F}_q , then f(u) = g(u) + c = 0 which implies c = -g(u). Thus f(x) = g(x) + c = g(x) - g(u) = g(x - u). This implies that $N_f = N_g$. By Lemma 3, we have $N_f \in \{1, p, p^2, p^3, \dots, p^h\}$. The desired conclusion follows.

Lemma 10. [25] Let $q = p^m$, where p is an odd prime, $m \ge 2$. Let $1 \le s \le m-1$, l = gcd(m,s). Let $U_{q+1} := \{x \in \mathbb{F}_{q^2} : x^{q+1} = 1\}$ and $f(x) = ax + bx^{p^s} + cx^{p^s+1} + u$, where $(a, b, c, u) \in \mathbb{F}_{q^2}^4 \setminus \{0, 0, 0, 0\}$. Then f(x) has 0, 1, 2 or $p^l + 1$ zeros in U_{q+1} .

2.2. Affine-invariant codes

In this subsection, we introduce affine-invariant codes.

We give the definition of primitive cyclic codes at first. A primitive cyclic code is a cyclic code of length $n = q^m - 1$ over \mathbb{F}_q , where *m* is a positive integer. Let R_n represent the quotient ring $\mathbb{F}_q[x]/(x^n - 1)$. Any primitive cyclic code *C* over \mathbb{F}_q is an ideal of R_n which is generated by a monic polynomial g(x) of the least degree over \mathbb{F}_q . We call this polynomial the generator polynomial of *C*. It can be represented as

$$g(x) = \prod_{t \in T} (x - \alpha^t),$$

where α is a generator of $\mathbb{F}_{q^m}^*$ and $T \subset \{0, 1, \dots, n-1\}$ is a union of some *q*-cyclotomic cosets modulo *n*.

We then introduce the extended primitive cyclic codes. Let *G* denote the generator matrix of a primitive cyclic code *C*. Define a matrix \overline{G} by adding a column to *G* such that the sum of the elements of each row of \overline{G} is 0. The matrix \overline{G} is the generator matrix of the extended code of *C*. The extended code of a primite cyclic code *C* is called an extended primitive cyclic code and denoted by \overline{C} .

Define the affine group $GA_1(\mathbb{F}_q)$ by the set of all permutations $\sigma_{u,v} : x \mapsto ux + v$ of \mathbb{F}_q , where $u \in \mathbb{F}_q^*, v \in \mathbb{F}_q$. An affine-invariant code is an extended primitive cyclic code \overline{C} such that $GA_1(\mathbb{F}_q) \subseteq$ PAut(\overline{C}). It is easy to prove that the group action of $GA_1(\mathbb{F}_q)$ on \mathbb{F}_q is doubly transitive, i.e. 2-transitive. Then by the Theorem 2, we have the following theorem.

Theorem 11. [13] For each *i* with $A_i \neq 0$ in an affine-invariant code \overline{C} , the supports of the codewords of weight *i* form a 2-design.

Theorem 11 is a very useful tool in constructing *t*-designs from extended primitive cyclic codes.

3. A family of extended primitive cyclic codes

In this section, let *h* and *m* be positive integers with h < m and $q = p^m$ with *p* a prime. For convenience, let dim(C) and d(C) respectively denote the dimension and minimum distance of a linear code C. Let α be a generator of \mathbb{F}_q^* and $\alpha_i := \alpha^i$ for $1 \le i \le q - 1$.

Define

$$D_{h} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_{1} & \alpha_{2} & \cdots & \alpha_{q-1} & 0 \\ \alpha_{1}^{p} & \alpha_{2}^{p} & \cdots & \alpha_{q-1}^{p} & 0 \\ \alpha_{1}^{p^{2}} & \alpha_{2}^{p^{2}} & \cdots & \alpha_{q-1}^{p^{2}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{1}^{p^{h}} & \alpha_{2}^{p^{h}} & \cdots & \alpha_{q-1}^{p^{h}} & 0 \end{bmatrix},$$
(3)

 D_h is an h+2 by q matrix over \mathbb{F}_q . Let C_{D_h} be the linear code over \mathbb{F}_q generated by D_h . Let D'_h be the h+2 by q-1 submatrix of D_h obtained by deleting the last column of D_h . Then the linear code $C_{D'_h}$ generated by D'_h is obvious a primitive cyclic code. It is easy to verify that C_{D_h} is the extended code of $C_{D'_h}$. Hence C_{D_h} is an extended primitive cyclic code.

In the following, we study the parameters of C_{D_h} and its dual $C_{D_h}^{\perp}$ and obtain *t*-designs from them.

Theorem 12. Let h and m be positive integers with h < m and let $q = 2^m$. Then C_{D_h} is a [q, h+2, d] linear code with at most h + 2 nonzero weights and $C_{D_h}^{\perp}$ is a [q, q - h - 2, 4] linear code over \mathbb{F}_q , where $d \in \{q - 2^h, q - 2^{h-1}, \dots, q - 2^j\}$ and j is the least integer such that $2^j \ge h + 1$. Moreover, C_{D_h} is affine-invariant and the supports of all codewords of any fixed nonzero weight in C_{D_h} form a 2-design. Besides, the minimum weight codewords of $C_{D_h}^{\perp}$ support a 3-(q, 4, 1) simple design, i.e. a Steiner system S(3, 4, q).

Proof. We prove that dim $(C_{D_h}) = h + 2$ at first. Let \mathbf{g}_i , $1 \le i \le h + 2$, represent the *i*-th row of D_h . Suppose that there are elements $r_i \in \mathbb{F}_q$, $1 \le i \le h + 2$, such that $\sum_{i=1}^{h+2} r_i \mathbf{g}_i = 0$. Then

$$\begin{cases} r_1 + r_2\alpha_1 + r_2\alpha_1^2 + \dots + r_{h+2}\alpha_1^{2^h} = 0, \\ r_1 + r_2\alpha_2 + r_2\alpha_2^2 + \dots + r_{h+2}\alpha_2^{2^h} = 0, \\ \vdots \\ r_1 + r_2\alpha_{q-1} + r_2\alpha_{q-1}^2 + \dots + r_{h+2}\alpha_{q-1}^{2^h} = 0, \\ r_1 = 0. \end{cases}$$

This implies that the polynomial $f(x) = r_1 + r_2 x + r_2 x^2 + \dots + r_{h+2} x^{2^h}$ has at least $q = 2^m$ solutions. Then f(x) must be a zero polynomial as h < m. In other words, we deduce that $r_i = 0, 1 \le i \le h+2$ and $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_{h+2}$ are linearly independent over \mathbb{F}_q . Thus dim $(\mathcal{C}_{D_h}) = h+2$.

We then prove that $C_{D_h}^{\perp}$ has parameters [q, q-h-2, 4]. Obviously, dim $(C_{D_h}^{\perp}) = q - (h+2) = q - h - 2$. Let x_1, x_2, x_3 be any three pairwise different elements in \mathbb{F}_q . Consider the following submatrix of D_h given by

$$M_1 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \\ x_1^{2^h} & x_2^{2^h} & x_3^{2^h} \end{bmatrix}.$$

Then we consider the following submatrix of M_1 given as

$$M_2 = \left[\begin{array}{rrrr} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{array} \right].$$

It is obvious that

$$|M_2| = \prod_{1 \leq i < j \leq 3} (x_j - x_i) \neq 0.$$

Then rank $(M_1) = 3$ and any 3 columns of D_h are linearly independent. This yields $d(\mathcal{C}_{D_h}^{\perp}) \ge 4$. Let x_1, x_2, x_3 be any three pairwise different elements in \mathbb{F}_q and $x_4 = x_1 + x_2 + x_3$. Consider the following submatrix M_3 of D_h given as

$$M_{3} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_{1} & x_{2} & x_{3} & x_{4} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1}^{2^{h}} & x_{2}^{2^{h}} & x_{3}^{2^{h}} & x_{4}^{2^{h}} \end{bmatrix}.$$

Let \mathbf{c}_i , $1 \leq i \leq 4$, represent the *i*-th column of M_3 . It is obvious that $\mathbf{c}_4 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$, i.e. there exist four columns of D_h which are linearly dependent. Hence, $d(\mathcal{C}_{D_h}^{\perp}) = 4$ and $\mathcal{C}_{D_h}^{\perp}$ is a [q, q - h - 2, 4] code.

Now we determine the parameters of C_{D_h} . By definition, we have

$$\mathcal{C}_{D_h} = \{\mathbf{c}_{c,a_0,a_1,\cdots,a_h} : c, a_0, a_1, \cdots, a_h \in \mathbb{F}_q\},\$$

where

$$\mathbf{c}_{c,a_0,a_1,\cdots,a_h} = \left(c + \sum_{i=0}^h a_i x^{2^i}\right)_{x \in \mathbb{F}_q}.$$

To determine the weight wt($\mathbf{c}_{c,a_0,a_1,\dots,a_h}$) of a nonzero codeword $\mathbf{c}_{c,a_0,a_1,\dots,a_h} \in C_{D_h}$, it is sufficient to determine the number of zeros of the equation

$$c + \sum_{i=0}^{h} a_i x^{2^i} = 0$$

in \mathbb{F}_q . By Lemma 9, the above equation has N_f zeros in \mathbb{F}_q , where $N_f \in \{0, 1, 2, 4, 8, \dots, 2^h\}$. Hence, wt($\mathbf{c}_{c,a_0,a_1,\dots,a_h}$) $\in \{q, q-1, q-2, q-4, q-8, \dots, q-2^h\}$. Then $q-2^h \leq d(\mathcal{C}_{D_h}) \leq q-h-1$ by the Singleton bound. We then derive that \mathcal{C}_{D_h} is a [q, h+2, d] code over \mathbb{F}_q , where $d \in \{q-2^h, q-2^{h-1}, \dots, q-2^j\}$, j is the least integer such that $2^j \geq h+1$.

In what follows, we prove that C_{D_h} is affine-invariant and holds 2-designs. Let

$$f(x) := c + \sum_{i=0}^{h} a_i x^{2^i}, \ c, a_0, a_1, \cdots, a_h \in \mathbb{F}_q.$$

For $u \in \mathbb{F}_q^*$, $v \in \mathbb{F}_q$. We have

$$f(ux+v) = c + \sum_{i=0}^{h} a_i (ux+v)^{2^i}$$
$$= c + \sum_{i=0}^{h} a_i v^{2^i} + \sum_{i=0}^{h} a_i u^{2^i} x^{2^i}$$

Let $\sigma_{(u,v)}(x) = ux + v \in GA_1(\mathbb{F}_q)$, where $u \in \mathbb{F}_q^*$ and $v \in \mathbb{F}_q$. Then we have

$$\sigma_{(u,v)}(\mathbf{c}_{c,a_0,a_1,\cdots,a_h}) = \mathbf{c}_{c',a'_0,a'_1,\cdots,a'_h} \in \mathcal{C}_{D_h},$$

where

$$c' = f(v), a'_i = a_i u^{2^i}, 0 \le i \le h.$$

Note that C_{D_h} is an extended primitive cyclic code by the definition in Subsection 2.2. Then by the discussion above and the definition of $PAut(C_{D_h})$, it is easy to deduce that $GA_1(\mathbb{F}_q) \subseteq PAut(C_{D_h})$. Thus, C_{D_h} is affine-invariant. By theorem 11, the supports of all codewords of any fixed nonzero weight in C_{D_h} form a 2-design.

Finally, we prove that the minimum weight codewords of $C_{D_h}^{\perp}$ support a 3-(q, 4, 1) simple design, i.e. a Steiner system S(3, 4, q). Taking any four different columns in D_h , we then obtain the matrix M_3 . We now prove that rank $(M_3) = 3$ if and only if $x_4 = x_1 + x_2 + x_3$. It is obvious that

rank(M_3) = 3 if $x_4 = x_1 + x_2 + x_3$. Conversely, let rank(M_3) = 3 and we suppose $x_4 \neq x_1 + x_2 + x_3$. Consider the following submatrix of M_3 given as

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{bmatrix}.$$

Note that

$$|M_4| = \prod_{1 \le i < j \le 4} (x_j - x_i)(x_1 + x_2 + x_3 + x_4) \neq 0$$

implying rank $(M_3) = 4$. This contradicts with rank $(M_3) = 3$. Therefore, rank $(M_3) = 3$ if and only if $x_4 = x_1 + x_2 + x_3$. Let $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ be any 4-subset of \mathbb{F}_q that satisfying $x_{i_4} = x_{i_1} + x_{i_2} + x_{i_3}$, where $1 \le i_j \le q$. Let $(r_{i_1}, r_{i_2}, r_{i_3}, r_{i_4})$ be a nonzero solution of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_{i_1} & x_{i_2} & x_{i_3} & x_{i_4} \\ x_{i_1}^2 & x_{i_2}^2 & x_{i_3}^2 & x_{i_4}^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_{i_1}^{2^h} & x_{i_2}^{2^h} & x_{i_3}^{2^h} & x_{i_4}^{2^h} \end{bmatrix} \begin{bmatrix} r_{i_1} \\ r_{i_2} \\ r_{i_3} \\ r_{i_4} \end{bmatrix} = \mathbf{0}.$$

Since rank $(M_3) = 3$, we have $r_{i_j} \neq 0$ for $1 \leq j \leq 4$. Let $\mathbf{c} = (c_1, c_2, \dots, c_q)$ be a codeword in $\mathcal{C}_{D_h}^{\perp}$, where $c_{i_j} = r_{i_j}$ and $c_v = 0$ for all $v \in \{1, 2, \dots, q\} \setminus \{i_1, i_2, i_3, i_4\}$. It is clear that wt(\mathbf{c}) = 4. Obviously, $\{a\mathbf{c} : a \in \mathbb{F}_q^*\}$ is a set of all codewords of weight 4 in $\mathcal{C}_{D_h}^{\perp}$ whose nonzero coordinates are in the set $\{i_1, i_2, i_3, i_4\}$. Therefore, every codeword of weight 4 and its nonzero multiples in $\mathcal{C}_{D_h}^{\perp}$ with nonzero coordinates $\{i_1, i_2, i_3, i_4\}$ must correspond to the set $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. For every three pairwise distinct elements $x_{i_1}, x_{i_2}, x_{i_3}$ in \mathbb{F}_q , the number of choices of x_{i_4} is equal to 1 and independent of $x_{i_2}, x_{i_3}, x_{i_4}$. We then deduce that the codewords of weight 4 in $\mathcal{C}_{D_h}^{\perp}$ support a 3-(q, 4, 1) design. By Equation (1), we have

$$A_4^{\perp} = \frac{q(q-1)^2(q-2)}{24}$$

The proof is completed.

In Theorem 12, the parameters of the 2-designs derived from C_{D_h} are not given. It is open to determine them. According to some examples confirmed by Magma program, we have the following conjecture.

Conjecture 13. The minimum weight codewords of C_{D_h} in Theorem 12 support 3-designs.

Theorem 14. Let p be an odd prime, h and m be positive integers with h < m and $q = p^m$. Then C_{D_h} is a [q, h+2, d] code with at most h+2 nonzero weights and $C_{D_h}^{\perp}$ is a [q, q-h-2, 3] code over \mathbb{F}_q , where $d \in \{q - p^h, q - p^{h-1}, \dots, q - p^j\}$, j is the least integer such that $p^j \ge h+1$. Moreover, C_{D_h} is affine-invariant and the supports of all codewords of any fixed nonzero weight in C_{D_h} form a 2-design. Besides, the minimum weight codewords of $C_{D_h}^{\perp}$ support a 2-(q, 3, p-2) simple design.

Proof. Similarly to Theorem 12, we can easily derive the paraments of C_{D_h} . The possible nonzero weights of C_{D_h} are $\{q - p^h, q - p^{h-1}, \dots, q - p^j\}$, where *j* is the least integer such that $p^j \ge h + 1$. Besides, we can also prove that C_{D_h} is affine-invariant. Thus the supports of all codewords of any fixed nonzero weight in C_{D_h} form a 2-design.

In the following, we prove $C_{D_h}^{\perp}$ has parameters [q, q-h-2, 3]. Obviously, dim $(C_{D_h}^{\perp}) = q - (h+2) = q - h - 2$. It is obvious that any two columns are linearly independent in D_h , which implies $d(C_{D_h}^{\perp}) \ge 3$. Let x_1 be an element in \mathbb{F}_q and $x_2 = ax_1, a \in \mathbb{F}_p^* \setminus \{1\}$. Consider the following submatrix of D_h given as

$$M_5 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & 0 \\ x_1^p & x_2^p & 0 \\ \vdots & \vdots & \vdots \\ x_1^{p^h} & x_2^{p^h} & 0 \end{bmatrix}.$$

Let \mathbf{c}_1 , \mathbf{c}_2 , \mathbf{c}_3 represent the first, second, third column of M_5 , respectively. It is easy to prove that $\mathbf{c}_2 = a\mathbf{c}_1 + (1-a)\mathbf{c}_3$. Hence, $d(\mathcal{C}_{D_h}^{\perp}) = 3$ and $\mathcal{C}_{D_h}^{\perp}$ has parameters [q, q-h-2, 3].

We now prove that the minimum weight codewords of $C_{D_h}^{\perp}$ support a 2-(q, 3, p-2) simple design. Taking any three columns in D_h , we obtain the submatrix M_6 , where x_1, x_2, x_3 are pairwise distinct elements in \mathbb{F}_q and

$$M_6 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^p & x_2^p & x_3^p \\ \vdots & \vdots & \vdots \\ x_1^{p^h} & x_2^{p^h} & x_3^{p^h} \end{bmatrix}.$$

Let \mathbf{g}_1 , \mathbf{g}_2 , \mathbf{g}_3 represent the first, second, third column of M_6 , respectively. We first prove that rank $(M_6) = 2$ if and only if $x_3 = ax_1 + (1-a)x_2, a \in \mathbb{F}_p \setminus \{0,1\}$. If $\mathbf{g}_3 = a\mathbf{g}_1 + (1-a)\mathbf{g}_2$ if $x_3 = ax_1 + (1-a)x_2, a \in \mathbb{F}_p \setminus \{0,1\}$, then rank $(M_6) = 2$. Conversely, we let rank $(M_6) = 2$ and assume that $\mathbf{g}_3 = a\mathbf{g}_1 + b\mathbf{g}_2, a, b \in \mathbb{F}_q \setminus \{0\}$. Then we have

$$\begin{cases}
 a + b = 1, \\
 ax_1 + bx_2 = x_3, \\
 ax_1^p + bx_2^p = x_3^p, \\
 \vdots \\
 ax_1^{p^h} + bx_2^{p^h} = x_3^{p^h}.
\end{cases}$$
(4)

By the first two equations in (4), we have b = 1 - a and $x_3 = ax_1 + (1 - a)x_2, a \in \mathbb{F}_q \setminus \{0, 1\}$. Then by System (4) we have

$$\begin{cases} ax_1^p + (1-a)x_2^p = (ax_1 + (1-a)x_2)^p, \\ ax_1^{p^2} + (1-a)x_2^{p^2} = (ax_1 + (1-a)x_2)^{p^2}, \\ \vdots \\ ax_1^{p^h} + (1-a)x_2^{p^h} = (ax_1 + (1-a)x_2)^{p^h}. \end{cases}$$
(5)

where $a \in \mathbb{F}_q \setminus \{0, 1\}$. The System (5) can be rewritten as

$$ax_1^{p^i} + (1-a)x_2^{p^i} = (ax_1 + (1-a)x_2)^{p^i}, \ 1 \le i \le h, \ a \in \mathbb{F}_q \setminus \{0,1\},\$$

which implies

$$a^{p^{i}}(x_{1}-x_{2})^{p^{i}}=a(x_{1}-x_{2})^{p^{i}},\ 1\leq i\leq h,\ a\in\mathbb{F}_{q}\setminus\{0,1\}.$$

Then $a^{p^i} = a$ for all $1 \le i \le h$. This implies

$$a \in \left(\bigcap_{i=1}^{h} \mathbb{F}_{p^{i}}\right) \setminus \{0,1\} = \mathbb{F}_{p} \setminus \{0,1\},\$$

and $x_3 = ax_1 + (1-a)x_2, a \in \mathbb{F}_p \setminus \{0, 1\}$. It is easy to prove that $x_3 \notin \{x_1, x_2\}$. Therefore, rank $(M_6) = 2$ if and only if $x_3 = ax_1 + (1-a)x_2, a \in \mathbb{F}_p \setminus \{0, 1\}$. If we fix x_1, x_2 , then different choices of *a* correspond to different x_3 . Then the total number of different choices of x_3 such that rank $(M_6) = 2$ is equal to p-2. Let x_{i_j} respectively denote the i_j -th column in D_h , where $1 \le i_j \le q$. Let $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ be any 3-subset of \mathbb{F}_q that satisfying $x_{i_3} = ax_{i_1} + (1-a)x_{i_2}, a \in \mathbb{F}_p \setminus \{0, 1\}$. Let $(r_{i_1}, r_{i_2}, r_{i_3})$ be a nonzero solution of

$$\begin{bmatrix} 1 & 1 & 1 \\ x_{i_1} & x_{i_2} & x_{i_3} \\ x_{i_1}^p & x_{i_2}^p & x_{i_3}^p \\ \vdots & \vdots & \vdots \\ x_{i_1}^{p^h} & x_{i_2}^{p^h} & x_{i_3}^{p^h} \end{bmatrix} \begin{bmatrix} r_{i_1} \\ r_{i_2} \\ r_{i_3} \end{bmatrix} = \mathbf{0}.$$

Since rank of the coefficient matrix equals 2, then all $r_{i_j} \neq 0$ for $1 \leq j \leq 3$. Let $\mathbf{c} = (c_1, c_2, \dots, c_q)$ be a codeword in $C_{D_h}^{\perp}$, where $c_{i_j} = r_{i_j}$ and $c_v = 0$ for all $v \in \{1, 2, \dots, q\} \setminus \{i_1, i_2, i_3\}$. It is clear that wt(\mathbf{c}) = 3. Obviously, $\{k\mathbf{c} : k \in \mathbb{F}_q^*\}$ is a set of all codewords of weight 3 in $C_{D_h}^{\perp}$ whose nonzero coordinates is $\{i_1, i_2, i_3\}$. Therefore, every codeword of weight 3 and its nonzero multiples in $C_{D_h}^{\perp}$ with nonzero coordinates $\{i_1, i_2, i_3\}$ must correspond to the set $\{x_{i_1}, x_{i_2}, x_{i_3}\}$. For every pair of distinct elements x_{i_1}, x_{i_2} in \mathbb{F}_q , the number of different choices of x_{i_3} is equal to p - 2. We then deduce that the codewords of weight 3 in $C_{D_h}^{\perp}$ support a 2-(q, 3, p - 2) design. By Equation (1), we have

$$A_3^{\perp} = \frac{q(q-1)^2(p-2)}{6}.$$

The proof is completed.

Remark 1. By Theorems 12 and 14, we have $q - p^h \leq d(C_{D_h}) \leq q - p^j$, where *j* is the least integer such that $p^j \geq h+1$. For h = 1,2,3,4, we compute the parameters of C_{D_h} and $C_{D_h}^{\perp}$ by magma in some cases. We list them in Table 1. These results show that the lower bound of $d(C_{D_h})$ is tight in these cases. It is open to determine the exact value of $d(C_{D_h})$ and the weight distribution of C_{D_h} for general *h*.

In the following subsections, we determine the parameters of C_{D_h} for some special h.

р	h	т	\mathcal{C}_{D_h}	$\mathcal{C}_{D_h}^{\perp}$
2	1	2	[4, 3, 2]	[4, 1, 4]
2	1	3	[8, 3, 6]	[8, 5, 4]
3	1	3	[27, 3, 24]	[27, 24, 3]
5	1	3	[125, 3, 120]	[125, 122, 3]
2	2	3	[8, 4, 4]	[8, 4, 4]
2	2	4	[16, 4, 12]	[16, 12, 4]
3	2	3	[27, 4, 18]	[27, 23, 3]
5	2	3	[125, 4, 100]	[125, 121, 3]
2	3	4	[16, 5, 8]	[16, 11, 4]
2	3	5	[32, 5, 24]	[32, 27, 4]
3	3	4	[81, 5, 54]	[81, 76, 3]
2	4	5	[32, 6, 16]	[32, 26, 4]
2	4	6	[64, 6, 48]	[64, 58, 4]
3	4	5	[243, 6, 162]	[243,237,3]

Table 1: The parameters of C_{D_h} and $C_{D_h}^{\perp}$ in Theorem 12, 14.

3.1. When h = 2

If p = h = 2, the weight distribution of C_{D_h} was studied in [26]. In this case, C_{D_h} is an NMDS code holding 3-designs.

Theorem 15. Let $q = 2^m$ with m > 2 and h = 2. Then C_{D_2} generated by the matrix G_{D_2} is an NMDS code with parameters [q, 4, q - 4] and weight enumerator

$$A(z) = 1 + \frac{q(q-1)^2(q-2)}{24}z^{q-4} + \frac{q(q-1)^2(q+4)}{4}z^{q-2} + \frac{q(q-1)(q^2+8)}{3}z^{q-1} + \frac{(q-1)(3q^3+3q^2-6q+8)}{8}z^q.$$

Moreover, the minimum weight codewords in C_{D_2} support a $3 \cdot (q, q - 4, \frac{(q-4)(q-5)(q-6)}{24})$ simple design and the minimum weight codewords in $C_{D_2}^{\perp}$ support a $3 \cdot (q, 4, 1)$ simple design, i.e., a Steiner system S(3, 4, q). Furthermore, the codewords of weight 5 in $C_{D_2}^{\perp}$ support a $3 \cdot (q, 5, \frac{(q-4)(q-8)}{2})$ simple design.

The weight enumerator of C_{D_h} if p > 2, h = 2 is determined in the following theorem.

Theorem 16. Let $q = p^m$ with p > 2, m > 2 and h = 2. Then C_{D_h} is a $[q, 4, q - p^2]$ code over \mathbb{F}_q with weight enumerator

$$\begin{split} A(z) &= 1 + \frac{q(q-p)(q-1)^2}{p^3(p-1)^2(p+1)} z^{q-p^2} + \frac{q(q-1)^2(p^2q+p^2-q-pq)}{p^2(p-1)^2} z^{q-p} + \\ & \frac{q(q-1)(p^3q^2+p^3q+p^3-2p^2q^2-p^2q-pq^2-2pq+3q^2)}{(p-1)^2(p+1)} z^{q-1} + \\ & \frac{(q-1)(p^3+p^2q^3-p^2q+pq^2-pq-q^3+q^2)}{p^3} z^q. \end{split}$$

Proof. By the proof Theorem 14, the possible nonzero weights of C_{D_h} are $q, q-1, q-p, q-p^2$. Denote by $w_1 = q, w_2 = q-1, w_3 = q-p, w_4 = q-p^2$. Let A_{w_i} represent the frequency of the weight $w_i, 1 \le i \le 4$. By the first five Pless Power Moments in [13], we have

$$\begin{cases} \sum_{i=1}^{4} A_{w_i} = q^4 - 1, \\ \sum_{i=1}^{4} w_i A_{w_i} = q^4 (q - 1), \\ \sum_{i=1}^{4} w_i^2 A_{w_i} = q^3 (q^2 - q + 1)(q - 1), \\ \sum_{i=1}^{4} w_i^3 A_{w_i} = q[q(q - 1)(q^4 - 2q^3 + 4q^2 - 4q + 2) - 6A_3^{\perp}], \end{cases}$$

where A_3^{\perp} is given in the proof of Theorem 14. Solving the above system of linear equations gives

$$\begin{cases} A_{q-p^2} = \frac{q(q-p)(q-1)^2}{p^3(p-1)^2(p+1)}, \\ A_{q-p} = \frac{q(q-1)^2(p^2q+p^2-q-pq)}{p^2(p-1)^2}, \\ A_{q-1} = \frac{q(q-1)(p^3q^2+p^3q+p^3-2p^2q^2-p^2q-pq^2-2pq+3q^2)}{(p-1)^2(p+1)}, \\ A_q = \frac{(q-1)(p^3+p^2q^3-p^2q+pq^2-pq-q^3+q^2)}{p^3}. \end{cases}$$

Then the weight enumerator of C_{D_h} follows.

3.2. When h = 3

Theorem 17. Let $q = p^m$ with p = 2, m > 3 and h = 3. Then C_{D_h} is a [q, 5, q-8] code over \mathbb{F}_q with weight enumerator

$$\begin{split} A(z) &= 1 + \frac{q(q-1)^2(q-2)(q-4)}{1344} z^{q-8} + \frac{q(q-1)^2(q-2)(3q+8)}{96} z^{q-4} + \\ & \frac{q(q-1)^2(7q^2+12q+32)}{24} z^{q-2} + \frac{2q(q-1)(3q^3+7q^2+32)}{21} z^{q-1} + \\ & \frac{(q-1)(25q^4+9q^3+22q^2-56q+64)}{64} z^q. \end{split}$$

Proof. By the proof of Theorem 12, the possible weight of C_{D_h} are q, q-1, q-2, q-4, q-8. Denote by $w_1 = q, w_2 = q-1, w_3 = q-2, w_4 = q-4, w_5 = q-8$. Let A_{w_i} represent the frequency of the weight $w_i, 1 \le i \le 5$. By the first five Pless Power Moments in [13], we have

$$\begin{cases} \sum_{i=1}^{5} A_{w_i} = q^4 - 1, \\ \sum_{i=1}^{5} w_i A_{w_i} = q^4 (q - 1), \\ \sum_{i=1}^{5} w_i^2 A_{w_i} = q^3 (q^2 - q + 1)(q - 1), \\ \sum_{i=1}^{5} w_i^3 A_{w_i} = q^2 (q - 1)^2 (q^4 - 2q^3 + 4q^2 - 4q + 2), \\ \sum_{i=1}^{5} w_i^4 A_{w_i} = q(q - 1)(q^6 - 3q^5 + 9q^4 - 17q^3 + 22q^2 - 17q + 6) + 24A_4^{\perp}, \end{cases}$$

where A_4^{\perp} is determined in the proof of Theorem 12. Solving the above system of linear equations yields

$$\left\{ \begin{array}{l} A_{q-8} = \frac{q(q-1)^2(q-2)(q-4)}{1344}, \\ A_{q-4} = \frac{q(q-1)^2(q-2)(3q+8)}{96}, \\ A_{q-2} = \frac{q(q-1)^2(7q^2+12q+32)}{96}, \\ A_{q-1} = \frac{2q(q-1)(3q^3+7q^2+32)}{21}, \\ A_{q} = \frac{(q-1)(25q^4+9q^3+22q^2-56q+64)}{64}, \end{array} \right.$$

Then the weight enumerator of C_{D_h} follows.

It is open to determine the weight enumerator of C_{D_h} when p > 2 and h = 3.

3.3. When h | *m*

Theorem 18. Let h and m be positive integers with h < m, $h \mid m$ and $q = p^m$ with p a prime. Let C_{D_h} be the linear code over \mathbb{F}_q generated by D_h . Then C_{D_h} has parameters $[q, h+2, q-p^h]$.

Proof. Consider the polynomial

$$f(x) = c + \sum_{i=0}^{h} a_i x^{p^i}, \ a_i, c \in \mathbb{F}_q.$$

Let $a_h = 1, a_0 = -1, c = 0, a_i = 0, 1 \le i \le h - 1$. Then $f(x) = x^{p^h} - x$. Let f(x) = 0, then we have $x^{p^h} = x$, which implies $x \in \mathbb{F}_{p^h} \subseteq \mathbb{F}_q$ as $h \mid m$. Thus, the number of zeros of f(x) in \mathbb{F}_q is equal to p^h . By the proofs of Theorems 12 and 14, C_{D_h} has parameters $[q, h+2, q-p^h]$ for any prime p. \Box

3.4. When h = m - 1

The trace function from \mathbb{F}_q onto \mathbb{F}_p is defined by

$$\operatorname{Tr}_{q/p}(x) = x + x^p + x^{p^2} + \dots + x^{p^{m-1}}.$$

Theorem 19. Let *m* be a positive integer and $q = p^m$ with *p* a prime. Let C_{D_h} be the linear code over \mathbb{F}_q generated by D_h and h = m - 1. Then C_{D_h} has parameters $[q, m + 1, q - p^{m-1}]$.

Proof. Let h = m - 1. Consider the polynomial

$$f(x) = c + \sum_{i=0}^{h} a_i x^{p^i}, \ a_i, c \in \mathbb{F}_q.$$

Let $c = 0, a_i = 1, 0 \le i \le h$. Then $f(x) = \sum_{i=0}^h x^{p^i} = \operatorname{Tr}_{q/p}(x)$. Let f(x) = 0. Then the number of zeros of f(x) in \mathbb{F}_q is equal to p^h . By the proofs of Theorems 12 and 14, the desired conclusion follows.

4. A family of cyclic codes with four weights

Let *q* be a power of an odd prime. Let $U_{q+1} := \{x \in \mathbb{F}_{q^2} : x^{q+1} = 1\}$. Let x_1, x_2, \dots, x_{q+1} denote all the elements of U_{q+1} . Let *G* be a generator matrix of the linear code C, where

$$G = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{q+1} \\ x_1^{p^s} & x_2^{p^s} & \cdots & x_{q+1}^{p^s} \\ x_1^{p^{s+1}} & x_2^{p^{s+1}} & \cdots & x_{q+1}^{p^{s+1}} \end{bmatrix}$$

In fact, C is a reducible cyclic code as U_{q+1} is a cyclic group.

Theorem 20. Let $q = p^m$, where p is an odd prime and $m \ge 2$. Let $1 \le s \le m-1$ and l = gcd(m,s). Then C is a $[q+1,4,q-p^l]$ cyclic code with weight enumerator

$$\begin{split} A(z) &= 1 + \frac{(q+1)q(q-1)^2(p^l-q+p^lq^2+2p^lq^4+q^2+2q^4)}{2(p^l+1)}z^{q+1} + \\ &\frac{(q+1)^2(q-1)(p^l-p^lq-q+p^lq^2-p^lq^3+p^lq^4+q^2)}{p^l}z^q + \\ &\frac{(q+1)^2q(q-1)(p^l-q+p^lq^2-q^2)}{2(p^l-1)}z^{q-1} + \frac{(q+1)^2q(q-1)^2}{p^l(p^{2l}-1)}z^{q-p^l}. \end{split}$$

Moreover, C^{\perp} has parameters [q+1,q-3,4]. The minimum weight codewords of C support a $3 \cdot (q+1,q-p^l,\frac{(q-p^l)(q-p^l-1)(q-p^l-2)}{p^l(p^{2l}-1)})$ simple design and the minimum weight codewords of C^{\perp} support a $3 \cdot (q+1,4,p^l-2)$ simple design. When p = 3, l = 1, the minimum weight codewords of C^{\perp} support a $3 \cdot (3^m+1,4,1)$ simple design, i.e. a Steiner system $S(3,4,3^m+1)$.

Proof. We first prove that $\dim(\mathcal{C}) = 4$. Let $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$ represent the first, second, third, forth row of *G*, respectively. Assume that $a\mathbf{g}_1 + b\mathbf{g}_2 + c\mathbf{g}_3 + u\mathbf{g}_4 = 0$, then we have

$$\begin{cases} a+bx_1+cx_1^{p^s}+ux_1^{p^s+1}=0,\\ \vdots\\ a+bx_{q+1}+cx_{q+1}^{p^s}+ux_{q+1}^{p^s+1}=0. \end{cases}$$

If $f(x) = a + bx + cx^{p^s} + ux^{p^{s+1}}$ is a nonzero polynomial, then it has at most $p^l + 1 \le p^{m-1} + 1$ solutions in U_{q+1} . By the above System of equations, we have a = b = c = d = 0 and dim $(\mathcal{C}) = 4$.

We then prove that C^{\perp} has parameters [q+1, q-3, 4]. It is obviously that $\dim(C^{\perp}) = q + 1 - 4 = q - 3$. We prove that $d(C^{\perp}) = 4$ in the following. Obviously, any two columns of *G* are \mathbb{F}_{q^2} -linearly independent. By the Singleton bound, we then have $3 \le d(C^{\perp}) \le 5$. Let x, y, z be three pairwise different elements in U_{q+1} . We consider the following submatrix given by

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^{p^s} & y^{p^s} & z^{p^s} \\ x^{p^s+1} & y^{p^s+1} & z^{p^s+1} \end{bmatrix}$$

If $\frac{z-x}{v-x} \notin \mathbb{F}_{p^s}^*$, we consider the submatrix D_1 of D, where

$$D_1 = \left[\begin{array}{rrr} 1 & 1 & 1 \\ x & y & z \\ x^{p^s} & y^{p^s} & z^{p^s} \end{array} \right].$$

Note that $|D_1| = (z-x)(y-x)^{p^s} - (y-x)(z-x)^{p^s} = 0$ if and only if $(\frac{z-x}{y-x})^{p^s-1} = 1$. Hence $|D_1| \neq 0$ if $\frac{z-x}{y-x} \notin \mathbb{F}_{p^s}^*$. If $\frac{z-x}{y-x} \in \mathbb{F}_{p^s}^*$, we consider the submatrix D_2 of D, where

$$D_2 = \begin{bmatrix} 1 & 1 & 1 \\ x^{p^s} & y^{p^s} & z^{p^s} \\ x^{p^s+1} & y^{p^s+1} & z^{p^s+1} \end{bmatrix}.$$

Suppose that $|D_2| = (y-x)(z-x)^{p^s}y^{p^s} - (z-x)(y-x)^{p^s}z^{p^s} = 0$. Then $(\frac{z-x}{y-x})^{p^s-1} = (\frac{z}{y})^{p^s}$. Since $\frac{z-x}{y-x} \in \mathbb{F}_{p^s}^*$, we have $(\frac{z}{y})^{p^s} = 1$. Then $\frac{z}{y} = 1$ as $gcd(p^s, q^2 - 1) = 1$. This contradicts with $y \neq z$. Hence, $|D_2| \neq 0$. We then deduce rank(D) = 3 and $4 \leq d(\mathcal{C}^{\perp}) \leq 5$. Now we prove that \mathcal{C} has four possible nonzero weights $w_1 = q + 1$, $w_2 = q$, $w_3 = q - 1$, $w_4 = q - p^l$. By definition,

$$\mathcal{C} = \{c_{a,b,c,u} = (a+bx+cx^{p^s}+ux^{p^s+1})_{x \in U_{q+1}} : a,b,c,u \in \mathbb{F}_{q^2}\}.$$

Note that the Hamming weight $wt(c_{a,b,c,u})$ of $c_{a,b,c,u}$ satisfies

$$\texttt{wt}(c_{a,b,c,u}) \in \{q+1,q,q-1,q-p^l\}$$

by Lemma 10. If $d(\mathcal{C}^{\perp}) = 5$, then \mathcal{C}^{\perp} is a MDS code with parameters [q+1, q-3, 5] and $d(\mathcal{C}) =$ q-2, which contradicts with wt $(c_{a,b,c,u}) \in \{q+1,q,q-1,q-p^l\}$. Therefore, $d(\mathcal{C}^{\perp}) = 4$ and \mathcal{C}^{\perp} is an AMDS code with parameters [q+1, q-3, 4].

Finally, we calculate the weight enumerator of C. Let $w_1 = q + 1, w_2 = q, w_3 = q - 1, w_4 = q + 1$ $q-p^{l}$. Let $A_{w_{i}}$ represent the frequency of the weight $w_{i}, 1 \leq i \leq 4$. Then by the first four Pless power moments in [13], we have

$$\begin{cases} \sum_{i=1}^{4} A_{w_i} = (q^2)^4 - 1, \\ \sum_{i=1}^{4} w_i A_{w_i} = (q^2)^8 (q^2 n - n), \\ \sum_{i=1}^{4} w_i^2 A_{w_i} = (q^2)^2 [(q^2 - 1)n(q^2 n - n + 1)], \\ \sum_{i=1}^{4} w_i^3 A_{w_i} = q^2 [(q^2 - 1)n(q^4 n^2 - 2q^2 n^2 + 3q^2 n - q^2 + n^2 - 3n)]. \end{cases}$$

Solving this system of linear equations yields

$$A_{w_1} = \frac{(q+1)q(q-1)^2(p^l-q+p^lq^2+2p^lq^4+q^2+2q^4)}{2(p^l+1)},$$

$$A_{w_2} = \frac{(q+1)^2(q-1)(p^l-p^lq-q+p^lq^2-p^lq^3+p^lq^4+q^2)}{p^l},$$

$$A_{w_3} = \frac{(q+1)^2q(q-1)(p^l-q+p^lq^2-q^2)}{2(p^l-1)},$$

$$A_{w_4} = \frac{(q+1)^2q(q-1)^2}{p^l(p^{2l}-1)}.$$

Then C has parameters $[q+1,4,q-p^l]$ and the weight enumerator of C follows. By the Pless power moments in [13], we have $A_4^{\perp} = \frac{(q+1)^2 q(q-1)^2 (p^l-2)}{24}$. It follows from Theorem 1 and Equation (1) that the minimum weight codewords of *C* support a $3 \cdot (q+1, q-p^l, \frac{(q-p^l)(q-p^l-1)(q-p^l-2)}{p^l(p^{2l}-1)})$ simple design and the minimum weight codewords of \mathcal{C}^{\perp} support a 3- $(q+1,4,p^l-2)$ simple design.

The proof is completed.

Example 21. Let p = 3, m = 2, s = 1. Then the linear code C is an NMDS code with parameters [10,4,6] and weight enumerator

$$A(z) = 1 + 2400z^{6} + 280800z^{8} + 4743200z^{9} + 38020320z^{10}.$$

The dual code C^{\perp} has parameters [10,6,4]. Besides, the codewords of weight 6 in C support a 3-(10,6,5) simple design and the codewords of weight 4 in C^{\perp} support a 3-(10,4,1) simple design, *i.e. a Steiner system* S(3, 4, 10).

Example 22. Let p = 5, m = 2, s = 1. Then the linear code *C* has parameters [26,4,20] and weight enumerator

 $A(z) = 1 + 81120z^{20} + 125736000z^{24} + 6095697504z^{25} + 146366376000z^{26}.$

The dual code C^{\perp} has parameters [26,22,4]. Besides, the minimum weight codewords of C support a 3-(26,20,57) simple design and the minimum weight codewords of C^{\perp} support a 3-(26,4,3) simple design.

5. Optimal locally recoverable codes

Let C be a linear code with parameters [n,k,d] over \mathbb{F}_q . For each positive integer n, let $[n] := \{0,1,\dots,n-1\}$. Then we use the the elements in [n] to index the coordinates of the codewords in C. For each $i \in [n]$, if there exist a subset $R_i \subseteq [n] \setminus i$ of size r and a function $f_i(x_1, x_2, \dots, x_r)$ on \mathbb{F}_q^r meeting $c_i = f_i(\mathbf{c}_{R_i})$ for any $\mathbf{c} = (c_0, \dots, c_{n-1}) \in C$, then C is referred to as an (n,k,d,q;r)-LRC, where \mathbf{c}_{R_i} is the projection of \mathbf{c} at R_i . The set R_i is known as the repair set of c_i and r is called the locality of C. If each f_i is a homogeneous function with degree 1, then C is called an (n,k,d,q;r)-LLRC (linearly local recoverable code) and has linear locality r. Obviously, each nontrivial linear code C has a minimum linear locality. The following lemma presents the relation between the minimum locality and the minimum linear locality of a nontrivial linear code.

Lemma 23 ([24]). *The minimum locality and minimum linear locality of a nontrivial linear code are equal.*

Besides, there exist some tradeoffs among the parameters of LRCs. For each (n,k,d,q;r)-LRC, the Singleton-like bound (see [2]) is given as

$$d \le n - k - \left\lceil \frac{k}{r} \right\rceil + 2. \tag{6}$$

LRCs achieving this bound are said to be distance-optimal. For any (n, k, d, q; r)-LRC, the Cadambe-Mazumdar bound (see [12]) is given by

$$k \le \min_{t \in \mathbb{Z}^+} [rt + k_{opt}^{(q)}(n - t(r+1), d)],$$
(7)

where $k_{opt}^{(q)}(n,d)$ represents the largest possible dimension of a linear code of length *n* and minimum distance *d* over \mathbb{F}_q , and \mathbb{Z}^+ represents the set of all positive integers. LRCs achieving the this bound are referred to as dimension-optimal ones.

The minimum locality of a nontrivial linear code C is given as follows.

Lemma 24. [24] Let C be a nontrivial linear code of length n and $d^{\perp} = d(C^{\perp})$. The minimum locality of C is $d^{\perp} - 1$ if $(\mathcal{P}(C^{\perp}), \mathcal{B}_{d^{\perp}}(C^{\perp}))$ is a 1- $(n, d^{\perp}, \lambda_1^{\perp})$ design with $\lambda_1^{\perp} \ge 1$.

Theorem 25. Let C_{D_h} be the code in Theorem 12 with p = 2. Then C_{D_h} is a

$$(q, h+2, d, q; 3)$$
-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q - h - 2, 4, q; d - 1)$$
 -LRC,

where $d \in \{q - 2^h, q - 2^{h-1}, \cdots, q - 2^j\}$ and j is the least integer such that $2^j \ge h + 1$.

Theorem 26. Let C_{D_h} be the code in Theorem 14 with p > 2. Then C_{D_h} is a

(q, h+2, d, q; 2)-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-h-2, 3, q; d-1)$$
-LRC

where $d \in \{q - p^h, q - p^{h-1}, \dots, q - p^j\}$, *j* is the least integer such that $p^j \ge h + 1$. *Proof.* The desired conclusion follows from Theorem 14 and Lemma 24.

Theorem 27. Let C_{D_2} be the code in Theorem 15 with p = 2, h = 2. Then C_{D_2} is a

(q, 4, q - 4, q; 3)-LRC

and $\mathcal{C}_{D_2}^{\perp}$ is a

$$(q, q-4, 4, q; q-5)$$
-LRC.

Besides, C_{D_2} and $C_{D_2}^{\perp}$ are both *d*-optimal and *k*-optimal.

Proof. By Theorem 15 and Lemma 24, the minimum localities of C_{D_2} and $C_{D_2}^{\perp}$ are $d(C_{D_2}^{\perp}) - 1 = 3$ and $d(C_{D_2}) - 1 = q - 5$, respectively. Then C_{D_2} is a

$$(q, 4, q - 4, q; 3)$$
-LRC

and $\mathcal{C}_{D_2}^{\perp}$ is a

$$(q, q-4, 4, q; q-5)$$
-LRC.

We then prove C_{D_2} is both *d*-optimal and *k*-optimal. By Equation (6),

$$q-4-\left\lceil\frac{4}{3}\right\rceil+2$$
$$q-4.$$

Hence, C_{D_2} is *d*-optimal. Let t = 1. Then

$$\min_{t \in \mathbb{Z}^+} \left[rt + k_{opt}^{(q)} \left(n - t(r+1), q - 4 \right) \right]$$

= $3 + k_{opt}^{(q)} \left(q - 4, q - 4 \right) = 4.$

Where the last equality holds due to $k_{opt}^{(q)}(q-4,q-4) = 1$ by the Singleton bound. By Equation (7), C_{D_2} is *k*-optimal. Similarly, we can prove $C_{D_2}^{\perp}$ is both *d*-optimal and *k*-optimal.

Similarly, we can easily prove the following four theorems.

Theorem 28. Let C_{D_h} be the code in Theorem 16 with h = 2, p > 2. Then C_{D_h} is a

$$(q, 4, q - p^2, q; 2)$$
 -LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-4, 3, q; q-p^2-1)$$
-LRC

Besides, $C_{D_h}^{\perp}$ is almost d-optimal.

Theorem 29. Let C_{D_h} be the code in Theorem 17, where h = 3. Then C_{D_h} is a

(q, 5, q-8, q; 3)-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-5, 4, q; q-9)$$
-LRC.

Besides, $C_{D_h}^{\perp}$ *is almost d-optimal.*

Theorem 30. Let C_{D_h} be the code in Theorem 18 with $h \mid m$.

1. If p = 2, then C_{D_h} is a

$$\left(q,h+2,q-2^{h},q;3\right)$$
-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-h-2, 4, q; q-2^{h}-1)$$
-LRC.

Besides, when h = 1 or 2, C_{D_h} and $C_{D_h}^{\perp}$ are both *d*-optimal and *k*-optimal. When h = 3, $C_{D_h}^{\perp}$ is almost *d*-optimal.

2. If p > 2, then C_{D_h} is a

$$\left(q,h+2,q-p^{h},q;2\right)$$
-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-h-2, 3, q; q-p^h-1)$$
-LRC.

Besides, when p = 3, h = 1, C_{D_h} is both *d*-optimal and *k*-optimal, and $C_{D_h}^{\perp}$ is *k*-optimal. When h = 1, $C_{D_h}^{\perp}$ is *d*-optimal. When h = 2, $C_{D_h}^{\perp}$ is almost *d*-optimal.

Theorem 31. Let C_{D_h} be the code in Theorem 19 with h = m - 1.

1. If p = 2, then C_{D_h} is a

$$(q, m+1, q-2^{m-1}, q; 3)$$
-LRC

and $\mathcal{C}_{D_{h}}^{\perp}$ is a

$$(q, q-m-1, 4, q; q-2^{m-1}-1)$$
-LRC.

Besides, when h = 1 or 2, C_{D_h} and $C_{D_h}^{\perp}$ are both *d*-optimal and *k*-optimal. When h = 3, $C_{D_h}^{\perp}$ is almost *d*-optimal.

2. If p > 2, then C_{D_h} is a

$$(q, m+1, q-p^{m+1}, q; 2)$$
-LRC

and $\mathcal{C}_{D_h}^{\perp}$ is a

$$(q, q-m-1, 3, q; q-p^{m-1}-1)$$
-LRC.

Besides, when p = 3, h = 1, C_{D_h} is both *d*-optimal and *k*-optimal, and $C_{D_h}^{\perp}$ is *k*-optimal. When h = 1, $C_{D_h}^{\perp}$ is *d*-optimal. When h = 2, $C_{D_h}^{\perp}$ is almost *d*-optimal.

The minimum locality of C in Theorem 20 is also studied in the following theorem.

Theorem 32. Let C be the code in Theorem 20. Then C is a

$$\left(q+1,4,q-p^l,q;3\right)$$
-LRC

and \mathcal{C}^{\perp} is a

$$(q+1, q-3, 4, q; q-p^l-1)$$
-LRC.

Besides, C is both d-optimal and k-optimal for p = 3, l = 1 and C^{\perp} is both d-optimal and k-optimal for all odd prime p and all l = gcd(m, s).

Proof. By Theorem 20 and Lemma 24, the minimum localities of C and C^{\perp} are $d(C^{\perp}) - 1 = 3$ and $d(C) - 1 = q - p^l - 1$, respectively. Then we directly derive that C is a

$$\left(q+1,4,q-p^l,q;3\right)$$
-LRC

and \mathcal{C}^{\perp} is a

$$(q+1, q-3, 4, q; q-p^l-1)$$
-LRC.

In the following, we prove C^{\perp} is both *d*-optimal and *k*-optimal.

$$q+1 - (q-3) - \left\lceil \frac{q-3}{q-p^l-1} \right\rceil + 2$$

= $6 - \left\lceil \frac{q-3}{q-p^l-1} \right\rceil = 4.$

where the last equality holds due to $q - p^l - 1 < q - 3 \le 2(q - p^l - 1)$. Hence, C^{\perp} is *d*-optimal by Equation (6). Let t = 1. Then

$$\min_{t \in \mathbb{Z}^+} \left[rt + k_{opt}^{(q)} \left(n - t(r+1), 4 \right) \right]$$

= $q - p^l - 1 + k_{opt}^{(q)} \left(p^l + 1, 4 \right) = q - 3.$

where the last equality holds due to $k_{opt}^{(q)}(p^l+1,4) = p^l - 2$ by the Singleton bound. Therefore, C^{\perp} is *k*-optimal by Equation (7).

Similarly, we can prove C is both d-optimal and k-optimal when p = 3, l = 1. The proof is completed.

6. Summary and concluding remarks

In this paper, we constructed a family of extended primitive cyclic codes and a family of reducible cyclic codes by special polynomials. The parameters of them and their duals were determined. It was shown that these codes have nice applications in combinatorial designs and locally recoverable codes. Besides, a conjecture was given in Conjecture 13 and an open problem was proposed in Remark 1. The reader is invited to solve them.

References

- C. Bracken, E. Byrne, N. Markin, G. McGuire, Determining the nonlinearity of a new family of APN functions, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes 4851 (2007) 72-79.
- [2] V. Cadambe, A. Mazumdar, An upper bound on the size of locally recoverable codes, Proc. IEEE Int. Symp. Network Coding (2013) 1-5.
- [3] C. Ding, Linear codes from some 2-designs, IEEE Trans. Inform. Theory 61 (6) (2015) 3265-3275.
- [4] C. Ding, C. Tang, Infinite families of near MDS codes holding t-designs, IEEE Trans. Inform. Theory 66 (9) (2020) 5419-5428.
- [5] C. Ding, C. Li, Infinite families of 2-designs and 3-designs from linear codes, Discrete Math. 340 (10) (2017) 2415-2431.
- [6] C. Ding, Infinite families of 3-designs from a type of five-weight code, Des. Codes Cryptogr. 86 (3) (2018) 703-719.
- [7] C. Ding, C. Tang, Combinatorial *t*-designs from special functions, Cryptogr. Commun. 12 (5) (2020) 1011-1033.
- [8] C. Ding, Designs from Linear Codes, World Scientific, Singapore, 2018.
- [9] C. Ding, An infinite family of Steiner systems from cyclic codes, J. Comb. Des. 26 (3) (2018) 127-144.
- [10] X. Du, R. Wang, C. Fan, Infinite families of 2-designs from a class of cyclic codes, J. Comb. Des. 28 (3) (2020) 157-170.
- [11] X. Du, X. Li, Y. WAN, A class of linear codes with three and five weights, Chin. J. Electron. 28 (3) (2019) 457-460.
- [12] P. Gopalan, C. Huang, H. Simitci, S. Yekhanin, On the locality of codeword symbols, IEEE Trans. Inform. Throry 58 (11) (2012) 6925-6934.
- [13] W. C. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, Cambridge, 2003.
- [14] Z. Heng, Projective linear codes from some almost difference sets, IEEE Trans. Inform. Theory, DOI: 10.1109/TIT.2022.3203380, 2022.
- [15] Z. Heng, F. Chen, C. Xie, D. Li, Constructions of projective linear codes by the intersection and difference of sets, Finite Fields Appli. 83 (2022) 102092.
- [16] Z. Heng, C. Li, X. Wang, Constructions of MDS, near MDS and almost MDS codes from cyclic subgroups of $\mathbb{F}_{q^2}^*$, IEEE Trans. Inform. Theory, DOI: 10.1109/TIT.2022.3194914, 2022.

- [17] C. Li, Q. Yue, F. Li, Hamming weights of the duals of cyclic codes with two zeros, IEEE Trans. Inform. Theory, 60(7) (2014) 3895-3902.
- [18] C. Li, Q. Yue, F. Li, Weight distributions of cyclic codes with respect to pairwise coprime order elements, Finite Fields Appl. 28 (2014) 94-114.
- [19] F. Li, Q. Yue, F. Liu, The weight distribution of a class of cyclic codes containing a subclass with optimal parameters, Finite Fields Appli. 45 (2017) 183-202.
- [20] C. Tang, Infinite families of 3-designs from APN functions, J. Comb. Des. 28 (2) (2020) 97-117.
- [21] C. Tang, C. Ding, An infinite family of linear codes supporting 4-designs, IEEE Trans. Inform. Theory 67 (1) (2020) 244-254.
- [22] C. Tang, C. Xiang, K. Feng, Linear codes with few weights from inhomogeneous quadratic functions, Des. Codes Cryptogr. 83 (3) (2017) 691-714.
- [23] C. Tang, C. Ding, M. Xiong, Steiner systems $S(2,4,\frac{3^m-1}{2})$ and 2-designs from ternary linear codes of length $\frac{3^m-1}{2}$, Des. Codes Cryptogr. 87 (2019) 2793-2811.
- [24] P. Tan, C. Fan, C. Ding, C. Tang, Z. Zhou, The minimum locality of linear codes, Des. Codes Cryptogr., https://doi.org/10.1007/s10623-022-01099-z, 2022.
- [25] X. Wang, C. Tang, C. Ding, Infinite families of cyclic and negacyclic codes supporting 3designs, arXiv: 2207.07262, 2022.
- [26] G. Xu, X. Cao, L. Qu, Infinite families of 3-designs and 2-designs from almost MDS codes, IEEE Trans. Inform. Theory, 68 (7) (2022) 4344-4353.
- [27] C. Xiang, X. Ling, Q Wang, Combinatorial *t*-designs from quadratic functions, Des. Codes Cryptogr. 88 (3) (2020) 553-565.
- [28] C. Xiang, Some t-designs from BCH codes, Cryptogr. Commun. 14 (3) (2022) 641-652.
- [29] C. Xiang, X. Wang, C. Tang, F. Fu, Two classes of linear codes and their weight distributions, Appl. Algebr. Eng. Comm. 29 (3) (2018) 209-225.
- [30] C. Xiang, C. Tang, Q. Liu, An infinite family of antiprimitive cyclic codes supporting Steiner systems S(3, 8, 7^m + 1). Des. Codes Cryptogr. 90 (2022) 1319-1333.