THE NUMBER OF RATIONAL POINTS OF A CLASS OF SUPERELLIPTIC CURVES

JOSÉ ALVES OLIVEIRA, DANIELA OLIVEIRA, AND F. E. BROCHERO MARTÍNEZ

ABSTRACT. In this paper, we study the number of \mathbb{F}_{q^n} -rational points on the affine curve $\mathcal{X}_{d,a,b}$ given by the equation

$$y^d = ax \operatorname{Tr}(x) + b,$$

where Tr denote the trace function from \mathbb{F}_{q^n} to \mathbb{F}_q and d is a positive integer. In particular, we present bounds for the number of \mathbb{F}_q -rational points on $\mathcal{X}_{d,a,b}$ and, for the cases where d satisfies a natural condition, explicit formulas for the number of rational points are obtained. Particularly, a complete characterization is given for the case d = 2. As a consequence of our results, we compute the number of elements α in \mathbb{F}_{q^n} such that α and $\operatorname{Tr}(\alpha)$ are quadratic residues in \mathbb{F}_{q^n} .

1. INTRODUCTION

Let \mathbb{F}_{q^n} be a finite field with q^n elements, where $q = p^s$ and p is an odd prime. Throughout the paper, $\operatorname{Tr}(x)$ denotes the trace function from \mathbb{F}_{q^n} into \mathbb{F}_q . The study of the number and existence of special elements in finite fields dates back to the 1980s. The famous *Primitive Normal Basis Theorem* was firstly proved by Lenstra and Schoof [11] in 1987.

More recently, sophisticated techniques have been created and employed in different problems regarding elements that satisfy special conditions (for example, see [1, 2, 3, 9]). The condition of having a fixed trace often appears in such articles because of its practical applications.

A problem that naturally arises is to find the number of elements $\alpha \in \mathbb{F}_{q^n}$ such that α and $\operatorname{Tr}(\alpha)$ are both quadratic residues in \mathbb{F}_{q^n} . This problem is easily solved in the case where n is even, since any element of \mathbb{F}_q is a quadratic residue in \mathbb{F}_{q^n} , so that, in particular, $\operatorname{Tr}(\alpha)$ does so. The problem gets more interesting for the case where n is odd. Heuristically, $q^n/4$ elements in \mathbb{F}_{q^n} must satisfy these two conditions, but there is no result in the literature addressing this problem. In both cases, such number of special elements is closely related to the number of \mathbb{F}_{q^n} -rational points on the affine curve $y^2 = x \operatorname{Tr}(x)$ (see the proof of Theorem 2.6). In this paper, we study a broader class of affine curves given by

$$\mathcal{X}_{d,a,b}: y^d = ax \mathrm{Tr}(x) + b,$$

where $a, b \in \mathbb{F}_{q^n}$ and d is a positive integer. The curve $\mathcal{X}_{d,a,b}$ belongs to a wide family of curves called *superelliptic* curves, whose points are given by the

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solutions of the equation

$$y^d = f(x). \tag{1}$$

There are few results in the literature regarding general superelliptic curves. For example, in [8] the authors give the distribution of points on smooth superelliptic curves over a finite field, when their degree goes to infinity. In [10] the authors describe the fluctuation in the number of points on a hyperelliptic curve, which is the especial class of superelliptic curves with d = 2.

In general, it is hard to compute the number of rational points on superelliptic curves, but this can be done in some cases, namely, for those that arise by choosing a suitable polynomial f in (1). For example, if $f(x) = -x^m + b$, the curve is the well-known Fermat curve, for which explicit formulas and bounds for the number of points are known ([13, 16]). The curve defined by Equation (1) over \mathbb{F}_{q^n} with $f(x) = x^q - x + b$ is known as Artin-Schreier curve. These special superellipic curves have also been well studied [4, 5, 15].

While these particular cases are well studied, a study of the number of rational points on $\mathcal{X}_{d,a,b}$ has not been provided. Our goal in this paper is to study the number of rational points on this curve, providing bounds for the number of rational points and explicit formulas for cases where $\mathcal{X}_{d,a,b}$ satisfy certain conditions.

In order to do that, we use the fact that the map $x \mapsto x \operatorname{Tr}(x)$ is a quadratic form over \mathbb{F}_{q^n} . Along the paper, we employ some classical results on quadratic forms over finite fields to provide an expression for the number of rational points on $\mathcal{X}_{d,a,b}$ in terms of Gauss sums (Proposition 4.5). Using this expression, we employ results on Gauss sums in order to obtain bounds for the number of rational points and, for suitable conditions on d, provide explicit formulas for this number. As a consequence of our results, we compute the number of elements α in \mathbb{F}_{q^n} such that both α and $\operatorname{Tr}(\alpha)$ are quadratic residues in \mathbb{F}_{q^n} (Theorem 2.6).

The paper is organized as follows. In Section 2, we present some remarks, comments and statements of our main results. Section 3 provides preliminary results on quadratic forms and Gauss sums that are used along the paper. In Section 4, we provide an expression for the number of rational points on the curve $\mathcal{X}_{d,a,b}$ in terms of Gauss sums. Bounds and explicit formulas for the number of rational points are presented in Section 5. In Section 6, we focus on the case $\operatorname{Tr}(b/a) \neq 0$.

2. Main results

In this section, we present our main results, along with a few comments. We observe that the curves $\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})$ and $\mathcal{X}_{\gcd(d,q^n-1),a,b}(\mathbb{F}_{q^n})$ have the same number of \mathbb{F}_{q^n} -rational points. Therefore, we may assume without loss of generality that d divides $q^n - 1$.

In [14], the authors give an improvement of the Hasse-Weil bound for curves with high genus. The curve $\mathcal{X}_{d,a,b}$ does not satisfy the conditions imposed in [14], but as a high genus curve, it is expected to it to be a curve whose number of points is far from Hasse-Weil's bound. Indeed, it turns out that Hasse-Weil can be significantly improved for $\mathcal{X}_{d,a,b}$. In Theorem 2.2, we provide sharp bounds for the number of rational points on such curves. In order to present this result, we introduce some notation. For a divisor d of $q^n - 1$, χ_d denotes a multiplicative character of $\mathbb{F}_{q^n}^*$ with order d. As usual, $\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})$ denotes the set of \mathbb{F}_{q^n} -rational points on $\mathcal{X}_{d,a,b}$.

Definition 2.1. Let
$$v = \gcd\left(d, \frac{q^n - 1}{q - 1}\right)$$
, $D = \frac{d}{v}$, $B = \operatorname{Tr}\left(\frac{b}{a}\right)$ and

$$\tau = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4}; \\ i, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Now we are able to present bounds for the number of \mathbb{F}_{q^n} -rational points on the curve $\mathcal{X}_{d,a,b}$

Theorem 2.2. The number of rational points on the curve $\mathcal{X}_{d,a,b}$ satisfies the following relations:

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| - q^n + (-1)^s q^{n-1} \sum_{\ell=1}^{d-1} \chi_d^\ell(b) \eqqcolon N_{d,a,b,n},$$
(2)

where

$$|N_{d,a,b,n}| \leq \begin{cases} (\frac{d}{D} - 1)(q - 1)q^{\frac{n}{2} - 1}, & \text{if } D \text{ is odd and } B = 0; \\ (q - 1)(\frac{d}{D}q^{\frac{n-1}{2}} + (\frac{d}{D} - 1)q^{\frac{n}{2} - 1}), & \text{if } D \text{ is even and } B = 0. \\ (d - 1)(q^{\frac{n}{2}} + q^{\frac{n-1}{2}}) & \text{if } B \neq 0. \end{cases}$$

We recall that the Hasse-Weil bound applied for $\mathcal{X}_{d,a,b}$ implies that

$$||\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| - q^n| \le 2gq^{\frac{n}{2}},$$

where the genus g can be computed via Riemann-Hurwitz's formula [7] and equals, at a minimum, the number

$$\frac{1}{2}(d-1)(q^{n-1}-1) - d + 1.$$

Note that Theorem 2.2 implies that

$$||\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| - q^n| \le (d-1)q^{n-1} + (q-1)(dq^{\frac{n-1}{2}} + (d-1)q^{\frac{n}{2}-1}),$$

which represents a significant improvement of Hasse-Weil's bound.

From now on, our main goal is to provide explicit formulas for the value $N = N_{d,a,b,n}$ defined in Theorem 2.2. In particular, the results obtained here include curves whose number of points attain the bounds provided in Theorem 2.2. Our starting point is the case d = 2, for which we present a simple expression for the number of affine rational points, that is given in the next theorem.

Theorem 2.3. The number of rational points on the curve $\mathcal{X}_{2,a,b}(\mathbb{F}_{q^n})$ is (1) If B = 0

$$|\mathcal{X}_{2,a,b}(\mathbb{F}_{q^n})| = q^n + q^{n-1}\chi_2(b) + N_1\tau^{ns}\chi_2(-a)(q-1),$$

where

$$N_1 = \begin{cases} q^{\frac{n-2}{2}}, & \text{if } n \text{ is even;} \\ \tau^s q^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

(2) If $B \neq 0$

$$|\mathcal{X}_{2,a,b}(\mathbb{F}_{q^n})| = q^n + q^{n-1}\chi_2(b) - N_2\tau^{ns}$$

where

$$N_{2} = \begin{cases} q^{\frac{n-2}{2}} [\chi_{2}(-a)\tau^{2s}q + \chi_{2}(-aB)], & \text{if } n \text{ is even;} \\ q^{\frac{n-1}{2}}\tau^{s}(\chi_{2}(-a) + \chi_{2}(-aB)), & \text{if } n \text{ is odd.} \end{cases}$$

This result allows us to compute the number of rational points in specific curves, as in the following examples.

Example 2.4. For the curve

$$\mathcal{X}_{2,-1,0}: y^2 = -x\mathrm{Tr}(x),$$

it follows that B = 0. Therefore, Theorem 2.3 provides the number of rational points for a pair (q, n):

q	n=2	n=3	n=4	n=5	n=6
3	$\tilde{\gamma}$	33	87	225	711
5	29	145	645	3225	15725
$\tilde{7}$	43	385	2443	16513	117355
9	89	801	6633	59697	532089
11	111	1441	14751	159841	1770351
25	649	16225	391225	9780625	254281250

Example 2.5. For the curve

$$\mathcal{X}_{2,-1,0}: y^2 = -x \operatorname{Tr}(x) + 1,$$

it follows that B = Tr(-1) = -n. Therefore, Theorem 2.3 provides the number of rational points for a pair (q, n):

q	n=2	n=3	n=4	n=5	n=6
3	10	42	96	385	954
5	34	150	720	3850	18600
$\tilde{\gamma}$	64	378	2688	19306	134162
9	80	882	7200	65448	591138
11	122	1452	15840	175692	1931402
25	625	16200	405600	10171250	253890000

One can check some values obtained in these two examples by using a computer program such as SageMath. For instance, in the case q = 25 and n = 6using a program in SageMath, spent 37h 34min. Nevertheless, the run time of the algorithm grows as q^n increases, making the computation unfeasible even for small values of q^n (such as $q^n > e^{20}$).

Back to the problem presented at the introduction, Theorem 2.3 is a key tool to compute the number of a special type of elements of \mathbb{F}_{q^n} , as we see in the proof of the following result.

Theorem 2.6. Let n be an odd positive integer. Then the number of elements $\alpha \in \mathbb{F}_{q^n}$ such that α and $\operatorname{Tr}(\alpha)$ are both quadratic residues in \mathbb{F}_{q^n} is

$$\frac{q^n + q^{n-1} + \tau^{s(n-1)}(q-1)q^{\frac{n-1}{2}} + 2}{4}.$$

In order to compute N, we present the following important definition, that is a generalization of a constant used by Wolfmann in [16] in the study of diagonal equations.

Definition 2.7. For *m* a divisor of $q^n - 1$, $a \in \mathbb{F}_{q^n}^*$ and $\epsilon \in \{1, -1\}$, we set

$$\theta_m(a,\epsilon) = \begin{cases} m-1, & \text{if } \chi_m(a) = \epsilon; \\ -1, & \text{otherwise.} \end{cases}$$

The following definition, that was introduced in the study of diagonal equations [13], will be useful in our results.

Definition 2.8. Let r be a positive integer. An integer d > 2 is (p, r)-admissible if $d \mid (p^r + 1)$ and there exists no r' < r such that $d \mid (p^{r'} + 1)$.

If d > 2 is (p, r)-admissible, then 2r is the multiplicative order of p module d. Since d divides $q^n - 1$ and $q = p^s$, the condition of (p, r)-admissibility on d implies that 2r divides ns and, in particular, that ns is an even number.

Definition 2.9. If d is a divisor of $q^n - 1$ and is (p, r)-admissible we define

$$\varepsilon = (-1)^{\frac{ns}{2r}}$$
 and $u = \frac{p^r + 1}{d}$.

We present now one of our main results, which provides a formula for the number of rational points on $\mathcal{X}_{d,a,b}$ in the case where B = 0 and some suitable conditions are required.

Theorem 2.10. Let d > 2 be an integer, $a, b \in \mathbb{F}_{q^n}$ such that B = 0 and $N = N_{d,a,b,n}$ defined as in Equation (2). The following holds:

(1) If v is (p, r)-admissible and D is odd, then

$$N = \begin{cases} 0, & \text{if } v = 1; \\ \tau^{sn} \chi_2(a)(q-1)q^{\frac{n-2}{2}}, & \text{if } v = 2; \\ \varepsilon \theta_v(-a, \varepsilon^u)(q-1)q^{\frac{n-2}{2}}, & \text{if } v > 2. \end{cases}$$

(2) If 2v is (p, r)-admissible and D is even. (a) If v = 1 then

$$N = (-1)^{s} \tau^{(n+1)s} \chi_2(a)(q-1)q^{\frac{n-1}{2}}.$$

(b) If
$$v = 2$$
 then

$$N = \left[(-1)^{s} \tau^{s} \sqrt{q} \varepsilon^{u \frac{D}{2} + 1} (\chi_{4}(-a) + \overline{\chi}_{4}(a)) + \tau^{ns} \chi_{2}(a) \right] (q - 1) q^{\frac{n-2}{2}}.$$
(c) If $v > 2$ then

$$N = \varepsilon \left[(-1)^{s} \tau^{s} \sqrt{q} \chi_{2v}(-a) \varepsilon^{\frac{uD}{2}} [1 + \theta_{v}(-a, 1)] + \theta_{v}(-a, 1) \right] (q - 1) q^{\frac{n-2}{2}}.$$

By employing this result, we can obtain a simple expression for the number of rational points in the case where a, b and d satisfy some restrictions.

Corollary 2.11. If d > 2 is a (p,r)-admissible divisor of $\frac{q^n-1}{q-1}$ and ns/2r is even, then

$$N_{d,-1,0,n} = (d-1)(q-1)q^{\frac{n-2}{2}}$$

Results similar to Theorem 2.10 can be obtained for the case $B \neq 0$, as we will see in Section 6.

3. Preliminary results

In this section, we provide some definitions and results that will be useful in the paper. Along the text, ψ and $\tilde{\psi}$ denote the canonicals additive characters of \mathbb{F}_{q^n} and \mathbb{F}_q , respectively, i.e.,

$$\psi(x) = \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_p}(x)}{p}\right) \quad \text{and} \quad \tilde{\psi}(x) = \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)}{p}\right)$$

We use χ_{q^n-1} to denote a fixed primitive multiplicative character of $\mathbb{F}_{q^n}^*$ and, for m a divisor of $q^n - 1$, χ_m denotes the multiplicative character of order m defined by $\chi_m = \chi_{q^n-1}^{(q^n-1)/m}$. The restriction of χ_m to \mathbb{F}_q^* is a multiplicative character of \mathbb{F}_q^* of order $M = \frac{m}{\gcd(m,(q^n-1)/(q-1))}$ and it will be denoted by η_M .

Definition 3.1. For multiplicative characters χ of $\mathbb{F}_{q^n}^*$ and η of \mathbb{F}_q^* , the Gauss sum of χ and η are the sums

$$G_n(\chi) = \sum_{x \in \mathbb{F}_{q^n}^*} \chi(x)\psi(x) \quad and \quad G_1(\eta) = \sum_{x \in \mathbb{F}_q^*} \eta(x)\tilde{\psi}(x),$$

respectively.

We present now some technical results that are used to compute the number of rational points on $\mathcal{X}_{d,a,b}$. Most of them are well-known and can be easily found in the literature.

Lemma 3.2 ([12, Theorem 5.4]). Let χ be a multiplicative character of \mathbb{F}_{q^n} . Then

$$\sum_{c \in \mathbb{F}_{q^n}} \chi(c) = \begin{cases} 0, & \text{if } \chi \text{ is nontrivial;} \\ q^n, & \text{if } \chi \text{ is trivial.} \end{cases}$$

Lemma 3.3. [12, Theorem 5.4] For $u \in \mathbb{F}_{q^n}$, we have that

$$\frac{1}{q^n} \sum_{c \in \mathbb{F}_{q^n}} \psi(uc) = \begin{cases} 0, & \text{if } u \neq 0; \\ 1, & \text{if } u = 0. \end{cases}$$

Lemma 3.4 ([12, Equation 5.4, p. 189]). Let d be a divisor of $q^n - 1$. If $c \in \mathbb{F}_{q^n}$, then

$$\sum_{j=0}^{d-1} \chi_d^j(c) = \begin{cases} 1, & \text{if } c = 0; \\ d, & \text{if } c \text{ is a } d\text{-power in } \mathbb{F}_{q^n}^*; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.5. [12, Theorems 5.11 and 5.12] Let χ_0 denote the trivial multiplicative character of $\mathbb{F}_{q^n}^*$. If $\chi \neq \chi_0$ is a multiplicative character of $\mathbb{F}_{q^n}^*$, then

(1) $G_n(\chi_0) = -1;$ (2) $|G_n(\chi)| = q^{n/2};$ (3) $G_n(\chi)G_n(\overline{\chi}) = \chi(-1)q^n.$

A Gauss sum is said to be *pure* if some positive integer power of it is real. The following result present necessary and sufficient conditions for a Gauss sum to be pure.

Theorem 3.6. [6, Theorem 1] Given a divisor d > 2 of $q^n - 1$ and a multiplicative character χ_d of $\mathbb{F}_{q^n}^*$ with order d, the following are equivalents:

- there exists r such that $d \mid (p^r + 1)$;
- $G_n(\chi_d^j)$ is pure for all $j \in \mathbb{Z}$;
- there exists a positive integer r such that $d \mid (p^r + 1), 2r \mid ns$ and

$$G_n(\chi_d^j) = -(-1)^{ns(uj+1)/2r} q^{n/2}$$

for all $j \not\equiv 0 \pmod{d}$, where $u = \frac{p^r + 1}{d}$.

Theorem 3.7. [12, Theorem 5.15] Let χ_2 be the quadratic character of $\mathbb{F}_{q^n}^*$. Then

$$G_n(\chi_2) = -(-1)^{sn} \tau^{sn} q^{n/2}.$$

Corollary 3.8. Let χ_2 be the quadratic character of $\mathbb{F}_{q^n}^*$. If ns is even, then

$$G_n(\chi_2) = -(-1)^{ns(u+1)/2} q^{n/2}$$

where $u = \frac{p+1}{2}$.

3.1. Quadratic forms. In order to determine the number of rational points on $\mathcal{X}_{d,a,b}$ we associate this curve to the quadratic form $\operatorname{Tr}(x\operatorname{Tr}(x))$. From this quadratic form, we provide its associate matrix and the dimension of its radical. In order to do that, we recall the following definitions.

Definition 3.9. For a quadratic form $\Phi : \mathbb{F}_{q^n} \to \mathbb{F}_q$, the symmetric bilinear form $\varphi : \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \to \mathbb{F}_q$ associated to Φ is

$$\varphi(\alpha,\beta) = \frac{1}{2} \left(\Phi(\alpha+\beta) - \Phi(\alpha) - \Phi(\beta) \right).$$

The radical of the symmetric bilinear form $\Phi: \mathbb{F}_{q^n} \to \mathbb{F}_q$ is the set

$$rad(\Phi) = \{ \alpha \in \mathbb{F}_{q^n} : \varphi(\alpha, \beta) = 0 \text{ for all } \beta \in \mathbb{F}_{q^n} \}.$$

If $rad(\Phi) = \{0\}$, then Φ is said to be non-degenerate.

Let $\mathcal{B} = \{v_1, \ldots, v\}$ be a basis of \mathbb{F}_{q^n} over \mathbb{F}_q . The $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} \Phi(v_i), & \text{if } i = j \\ \frac{1}{2}(\Phi(v_i + v_j) - \Phi(v_i) - \Phi(v_j)), & \text{if } i \neq j. \end{cases}$$

is the associated matrix of the quadratic form Φ in the basis \mathcal{B} . In particular, the dimension of rad (Φ) is equal to n - rank(A).

Let $\Phi : \mathbb{F}_q^m \to \mathbb{F}_q$ and $\Psi : \mathbb{F}_q^n \to \mathbb{F}_q$ be quadratic forms where $m \ge n$. Let U and V be associated matrix of Φ and Ψ , respectively. We say that Φ is equivalent to Ψ if there exists $M \in GL_m(\mathbb{F}_q)$ such that

$$M^{T}UM = \left(\begin{array}{c|c} V & 0\\ \hline 0 & 0 \end{array}\right) \in M_{m}(\mathbb{F}_{q}),$$

where $GL_m(\mathbb{F}_q)$ denotes the set of $m \times m$ invertible matrices over \mathbb{F}_q and $M_m(\mathbb{F}_q)$ denotes the set of $m \times m$ matrices over \mathbb{F}_q . Furthermore, Ψ is called a *reduced form* of Φ if $rad(\Psi) = \{0\}$.

The following theorem, which associate quadratic forms and characters sums, will be useful for our results and can be obtained from Theorems 6.26 and 6.27 of [12] by a straightforward computation.

Lemma 3.10. Let U be an $n \times n$ non null symmetric matrix over \mathbb{F}_q and $l = \operatorname{rank}(U)$. Then there exists $M \in GL_n(\mathbb{F}_q)$ such that $V = MUM^T$ is a diagonal matrix, i.e. $V = \operatorname{diag}(a_1, a_2, \ldots, a_l, 0, \ldots, 0)$ where $a_i \in \mathbb{F}_q^*$ for all $i = 1, \ldots, l$. For the quadratic form

$$F: \mathbb{F}_q^n \to \mathbb{F}_q, \quad F(X) = XUX^T \qquad X = (x_1, \dots, x_n) \in \mathbb{F}_q^n,$$

it follows that

$$\sum_{X \in \mathbb{F}_q^n} \tilde{\psi} \big(F(X) \big) = (-1)^{l(s+1)} \tau^{ls} \eta_2(\delta) q^{n-l/2},$$

where $\delta = a_1 \cdots a_l$.

We note that $\eta_2(\delta)$ independs of the system of coordinates chosen.

4. The number of rational points on the curve $\mathcal{X}_{d,a,b}$

For $a \in \mathbb{F}_{q^n}^*$, $b \in \mathbb{F}_{q^n}$ we compute the number of \mathbb{F}_{q^n} -rational points on the curve $\mathcal{X}_{d,a,b}$ by using well-known properties of character sums. For that, we have the following lemma.

Lemma 4.1. Let $a \in \mathbb{F}_{q^n}^*$, $b \in \mathbb{F}_{q^n}$ and d an integer that divides $q^n - 1$. The number of affine rational points on the curve $\mathcal{X}_{d,a,b}$ over \mathbb{F}_{q^n} is

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = q^n + \frac{1}{q^n} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \sum_{c \in \mathbb{F}_{q^n}^*} \psi(cb) \overline{\chi_d}^\ell(-c) \sum_{x \in \mathbb{F}_{q^n}} \psi(cax \operatorname{Tr}(x)).$$

Proof. It follows from Lemma 3.3 that

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = \frac{1}{q^n} \sum_{c \in \mathbb{F}_{q^n}} \sum_{x,y \in \mathbb{F}_{q^n}} \psi(c(ax \operatorname{Tr}(x) + b - y^d))$$

$$= q^n + \frac{1}{q^n} \sum_{c \in \mathbb{F}_{q^n}^*} \psi(cb) \sum_{x \in \mathbb{F}_{q^n}} \psi(cax \operatorname{Tr}(x)) \sum_{y \in \mathbb{F}_{q^n}} \psi(-cy^d).$$
(3)

Now, let $y^d = z$ and using Lemma 3.4 we obtain

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = q^n + \frac{1}{q^n} \sum_{c \in \mathbb{F}_{q^n}^*} \psi(cb) \sum_{x \in \mathbb{F}_{q^n}} \psi(cax \operatorname{Tr}(x)) \sum_{z \in \mathbb{F}_{q^n}} \psi(-cz) \left[1 + \dots + \chi_d^{d-1}(z)\right]$$

Making the change of variable w = -cz and using Lemma 3.3 we have that

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = q^n + \frac{1}{q^n} \sum_{c \in \mathbb{F}_{q^n}^*} \sum_{\ell=1}^{d-1} \psi(cb) \overline{\chi_d}^\ell(-c) \sum_{x \in \mathbb{F}_{q^n}} \psi(cax \operatorname{Tr}(x)) \sum_{w \in \mathbb{F}_{q^n}} \psi(w) \chi_d^\ell(w)$$

Therefore

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = q^n + \frac{1}{q^n} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \sum_{c \in \mathbb{F}_{q^n}^*} \psi(cb) \overline{\chi_d}^\ell(-c) \sum_{x \in \mathbb{F}_{q^n}} \psi(cax \operatorname{Tr}(x)).$$
(4)

In order to compute the value of the right-hand side sum of Equation (4), we will use the fact that $\operatorname{Tr}(cax\operatorname{Tr}(x))$ defines a quadratic form from \mathbb{F}_{q^n} into \mathbb{F}_q . For now on, let $Q_c(x)$ be a quadratic form of \mathbb{F}_{q^n} over \mathbb{F}_q defined by $Q_c(x) = \operatorname{Tr}(cx\operatorname{Tr}(x))$ and let $B_c(x, y)$ be the bilinear symmetric form associated to Q_c .

Proposition 4.2. For $c \in \mathbb{F}_{a^n}^*$, we have that

$$\dim_{\mathbb{F}_q} \left(rad(Q_c) \right) = \begin{cases} n-1 \ if \ c \in \mathbb{F}_q^*; \\ n-2 \ if \ c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q. \end{cases}$$

Proof. The dimension of the radical of the quadratic form Q_c is given by the dimension of the radical of the bilinear form $B_c(x, y)$, i.e., the dimension of the subspace generated by the elements $x \in \mathbb{F}_{q^n}$ such that $B_c(x, y) = 0$ for all $y \in \mathbb{F}_{q^n}$. We observe that

$$B_{c}(x,y) = \operatorname{Tr}(c(x+y)\operatorname{Tr}(x+y)) - \operatorname{Tr}(cx\operatorname{Tr}(x)) - \operatorname{Tr}(cy\operatorname{Tr}(y))$$

$$= \operatorname{Tr}(cx\operatorname{Tr}(y) + cy\operatorname{Tr}(x))$$

$$= \operatorname{Tr}(y)\operatorname{Tr}(cx) + \operatorname{Tr}(x)\operatorname{Tr}(cy)$$

$$= \operatorname{Tr}(y(\operatorname{Tr}(cx) + c\operatorname{Tr}(x))).$$
(5)

Then, $B_c(x, y) = 0$ for all $y \in \mathbb{F}_{q^n}$ if and only if $\operatorname{Tr}(cx) + c\operatorname{Tr}(x) = 0$. Therefore, are interested in computing the dimension of

$$V = \{ x \in \mathbb{F}_{q^n} : \operatorname{Tr}(cx) + c\operatorname{Tr}(x) = 0 \}.$$

We split the proof into two cases:

• For $c \in \mathbb{F}_q^*$, $\operatorname{Tr}(cx) + c\operatorname{Tr}(x) = c\operatorname{Tr}(x) + c\operatorname{Tr}(x) = 2c\operatorname{Tr}(x)$, that implies $|V| = |\{x \in \mathbb{F}_{q^n} : \operatorname{Tr}(x) = 0\}| = q^{n-1}$

and then $\dim(V) = n - 1$.

• For $c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$, if $x \in \mathbb{F}_{q^n}$ is such that $\operatorname{Tr}(x) \neq 0$, then $\frac{\operatorname{Tr}(cx)}{\operatorname{Tr}(x)} \in \mathbb{F}_q$ and $\frac{\operatorname{Tr}(cx)}{\operatorname{Tr}(x)} \neq -c$. Therefore for any element $x \in V$ we have that $\operatorname{Tr}(x) = 0$. It follows that $\operatorname{Tr}(cx) = -c\operatorname{Tr}(x) = 0$, then $V = V_1 \cap V_2$, where $V_1 = \{x \in \mathbb{F}_{q^n} \mid \operatorname{Tr}(x) = 0\}$ and $V_2 = \{x \in \mathbb{F}_{q^n} \mid \operatorname{Tr}(cx) = 0\}$. Since $c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ and $V_1 \neq V_2$, it is direct of the fact $\dim(V_1) = \dim(V_2) = n-1$ to verify that $\dim(V) = n-2$.

This completes the proof of our assertion.

Proposition 4.3. Let H be the symmetric matrix associated to Q_c and let δ be as defined in Lemma 3.10 for some basis of \mathbb{F}_{q^n} over \mathbb{F}_q . Then

$$\eta_2(\delta) = \begin{cases} \eta_2(c), & \text{if } c \in \mathbb{F}_q^*; \\ \eta_2(-1), & \text{if } c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q \end{cases}$$

Proof. Since $\eta_2(\delta)$ does not depend on the basis, we set a basis $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ of \mathbb{F}_{q^n} over \mathbb{F}_q such that

- $\alpha_0 = n^{-1}$ and $\operatorname{Tr}(\alpha_i) = 0$ for all $1 \le i \le n-1$ if $\operatorname{gcd}(n,q) = 1$;
- $\operatorname{Tr}(\alpha_0) = 1$, $\alpha_1 = 1$ and $\operatorname{Tr}(\alpha_0 \alpha_i) = \operatorname{Tr}(\alpha_i) = 0$ for all $2 \le i \le n-1$ in the case when $\operatorname{gcd}(q, n) \ne 1$.

For $0 \leq i \leq n-1$, let $x_i, y_i \in \mathbb{F}_q$ such that $y = \sum_{j=0}^{n-1} y_j \alpha_j$ and $x = \sum_{j=0}^{n-1} x_j \alpha_j$ and let us denote $X = (x_0, \dots, x_{n-1})$ and $Y = (y_0, \dots, y_{n-1})$. We recall that $Q_c(x) = XHX^T$ and

$$B_{c}(X,Y) = (X+Y)H(X+Y)^{T} - XHX^{T} - YHY^{T}$$

$$= XHY^{T} + YHX^{T}$$

$$= YH^{T}X^{T} + YHX^{T}$$

$$= Y(2H)X^{T}.$$

(6)

Therefore, we can determine δ from $B_c(x, y)$ by computing the determinant of the reduced matrix associated to 2H. In order to do this, we observe that $\text{Tr}(x) = x_0$ and

$$\operatorname{Tr}(cx) = \begin{cases} x_0 c_0 + \sum_{1 \le i, j \le n-1} x_i c_j \beta_{i,j}, & \text{if } p \nmid n \\ x_0 (c_1 + c_0 \beta_{0,0}) + c_0 x_1 + \sum_{2 \le i, j \le n-1} x_i c_j \beta_{i,j}, & \text{if } p \mid n, \end{cases}$$

where $\beta_{i,j} = \text{Tr}(\alpha_i \alpha_j)$. We obtain expressions for Tr(y) and Tr(cy) by a similar process. Therefore, it follows from (5) that

$$B_{c}(x,y) = \operatorname{Tr}(y)\operatorname{Tr}(cx) + \operatorname{Tr}(cy)\operatorname{Tr}(x)$$

$$= \begin{cases} 2x_{0}y_{0}c_{0} + \sum_{1 \le i,j \le n-1} c_{j}\beta_{i,j}(y_{0}x_{i} + x_{0}y_{i}), & \text{if } p \nmid n; \\ 2x_{0}y_{0}(c_{1} + c_{0}\beta_{0,0}) + c_{0}x_{1}y_{0} + c_{0}x_{0}y_{1} + \sum_{2 \le i,j \le n-1} c_{j}\beta_{i,j}(y_{0}x_{i} + x_{0}y_{i}), & \text{if } p \mid n. \end{cases}$$

We obtain

$$2H = \begin{pmatrix} 2c_0 & \sum_{j=1}^{n-1} \beta_{1,j}c_j & \cdots & \sum_{j=1}^{n-1} \beta_{n-1,j}c_j \\ \sum_{j=1}^{n-1} \beta_{1,j}c_j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n-1} \beta_{n-1,j}c_j & 0 & \cdots & 0 \end{pmatrix}$$

in the case when $p \nmid n$ and

$$2H = \begin{pmatrix} 2(c_1 + c_0\beta_{0,0}) & c_0 & \sum_{j=2}^{n-1}\beta_{2,j}c_j & \cdots & \sum_{j=2}^{n-1}\beta_{n-1,j}c_j \\ c_0 & 0 & 0 & \cdots & 0 \\ & \sum_{j=2}^{n-1}\beta_{2,j}c_j & 0 & 0 & \cdots & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & \sum_{j=1}^{n-1}\beta_{n-1,j}c_j & 0 & 0 & \cdots & 0 \end{pmatrix}$$

in the case when $p \mid n$.

In order to compute the reduced form of 2H, we observe that

- In the case when $c \in \mathbb{F}_q$ then $c_0 = c$ and $c_1 = c_2 = \cdots = c_{n-1} = 0$ if $p \nmid n$ and $c_1 = c$ and $c_0 = c_2 = \cdots = c_{n-1} = 0$ if $p \mid n$. Therefore the associated reduced matrix is (2c).
- In the case when $c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$, then Proposition 4.2 implies that either there exists $k \in \{1, \ldots, n-1\}$ such that $\sum_{j=1}^{n-1} \beta_{k,j} c_j \neq 0$ if $p \nmid n$ and $c_0 \neq 0$ or there exists $k \in \{2, \ldots, n-1\}$ such that $\sum_{j=2}^{n-1} \beta_{k,j} c_j \neq 0$ if $p \mid n$. Then straightforward manipulations of the lines and columns shows that 2H reduces to a matrix of the form

$$\begin{pmatrix} u & v \\ v & 0 \end{pmatrix},$$

where $v \neq 0$.

In sum, we have that the quadratic character of the determinant δ of the reduced matrix of H is given by

$$\eta_2(\delta) = \begin{cases} \eta_2(c), & \text{if } c \in \mathbb{F}_q^*; \\ \eta_2(-1), & \text{if } c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q. \end{cases}$$

This completes the proof.

Combining Lemma 3.10 and Proposition 4.2 and 4.3, we have the following result.

Theorem 4.4. For $c \in \mathbb{F}_{q^n}^*$, we have

$$\sum_{x \in \mathbb{F}_{q^n}} \psi(cx \operatorname{Tr}(x)) = \begin{cases} (-1)^{s+1} \eta_2(c) \tau^s q^{n-1/2}, & \text{if } c \in \mathbb{F}_q^*; \\ q^{n-1}, & \text{if } c \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q. \end{cases}$$

Theorem 4.4 allows us to express the number of rational points in terms of Gauss sums.

Proposition 4.5. Let v, D and B be as in Definition 2.1. Then

$$|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})| = q^n + \sum_{\ell=1}^{d-1} \chi_d^{\ell}(b) q^{n-1} + N,$$

where $N = N_{d,a,b,n}$ is given below.

(1) If D is odd and B = 0, then

$$N = -\frac{q-1}{q} \sum_{j=1}^{v-1} G_n(\chi_d^{jD}) \chi_d^{jD}(-a).$$

(2) If D is odd and $B \neq 0$, then

$$N = \frac{1}{q} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \Big[(-1)^{s+1} \chi_d^\ell(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_{2D}^{D-2\ell}) - \chi_d^\ell(-aB) G_1(\eta_{2D}^{-2\ell}) \Big]$$

(3) If D is even and B = 0, then

$$N = \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} \sum_{j=0}^{v-1} G_n(\chi_d^{jD+\frac{D}{2}}) \chi_d^{jD+\frac{D}{2}}(-a) - \sum_{j=1}^{v-1} G_n(\chi_d^{jD}) \chi_d^{jD}(-a) \right).$$

(4) If D is even and $B \neq 0$, then

$$N = \frac{1}{q} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \left[(-1)^{s+1} \chi_d^\ell(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_D^{\frac{D}{2}-\ell}) - \chi_d^\ell(-aB) G_1(\eta_D^{-\ell}) \right]$$

Proof. By Lemma 4.1 and Theorem 4.4, we have that $|\mathcal{X}_{d,a,b}(\mathbb{F}_{q^n})|$ is equal to

$$q^{n} + \frac{1}{q} \sum_{\ell=1}^{d-1} G_{n}(\chi_{d}^{\ell}) \left[\sum_{ca \in \mathbb{F}_{q^{n}} \setminus \mathbb{F}_{q}} \psi(cb) \overline{\chi_{d}}^{\ell}(-c) + \sum_{ca \in \mathbb{F}_{q}^{*}} \psi(cb) \overline{\chi_{d}}^{\ell}(-c) \left((-1)^{s+1} \eta_{2}(ac) \tau^{s} \sqrt{q}\right) \right].$$

That can be reewritten as

$$q^{n} + \frac{1}{q} \sum_{\ell=1}^{d-1} G_{n}(\chi_{d}^{\ell}) \left[\sum_{ca \in \mathbb{F}_{q^{n}}^{*}} \psi(cb) \overline{\chi_{d}}^{\ell}(-c) + \sum_{ca \in \mathbb{F}_{q}^{*}} \psi(cb) \overline{\chi_{d}}^{\ell}(-c) \left((-1)^{s+1} \eta_{2}(ac) \tau^{s} \sqrt{q} - 1 \right) \right]$$

$$\tag{7}$$

Let $T = \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \sum_{ca \in \mathbb{F}_{q^n}^*} \psi(cb) \overline{\chi_d}^\ell(-c)$. By Lemma 3.3, if b = 0, then

T = 0. Otherwise, Lemma 3.5 entails that

$$T = \sum_{\ell=1}^{d-1} \chi_d^{\ell}(-b) G_n(\chi_d^{\ell}) G_n(\overline{\chi_d}^{\ell}) = \sum_{\ell=1}^{d-1} \chi_d^{\ell}(b) q^n.$$
(8)

Now, we compute $S_{\ell} = \sum_{z \in \mathbb{F}_q^*} \psi(z_{\overline{a}}^{\underline{b}}) \overline{\chi_d}^{\ell} (\frac{-z}{a}) ((-1)^{s+1} \eta_2(z) \tau^s \sqrt{q} - 1).$ Assume that D is odd. We recall that η_D is the restriction of χ_d to \mathbb{F}_q^* and

that η_{2D} is such that $\eta_{2D}^2 = \eta_D$. Using this notation,

$$S_{\ell} = (-1)^{s+1} \chi_{d}^{\ell}(-a) \tau^{s} \sqrt{q} \sum_{z \in \mathbb{F}_{q}^{*}} \psi(z\frac{b}{a}) \eta_{2D}^{D-2\ell}(z) - \chi_{d}^{\ell}(-a) \sum_{z \in \mathbb{F}_{q}^{*}} \psi(z\frac{b}{a}) \eta_{2D}^{-2\ell}(z)$$
$$= (-1)^{s+1} \chi_{d}^{\ell}(-a) \tau^{s} \sqrt{q} \sum_{z \in \mathbb{F}_{q}^{*}} \tilde{\psi}(zB) \eta_{2D}^{D-2\ell}(z) - \chi_{d}^{\ell}(-a) \sum_{z \in \mathbb{F}_{q}^{*}} \tilde{\psi}(zB) \eta_{2D}^{-2\ell}(z).$$

We split the proof into two cases.

• If B = 0, then it follows from Lemma 3.3 that

$$S_{\ell} = \begin{cases} 0, & \text{if } D \nmid \ell; \\ -\chi_d^{\ell}(-a)(q-1), & \text{if } D \mid \ell. \end{cases}$$

• If $B \neq 0$, then

$$S_{\ell} = (-1)^{s+1} \chi_d^{\ell}(-a) \tau^s \sqrt{q} \eta_{2D}^{D-2\ell}(B^{-1}) G_1(\eta_{2D}^{D-2\ell}) - \chi_d^{\ell}(-a) \eta_{2D}^{-2\ell}(B^{-1}) G_1(\eta_{2D}^{-2\ell})$$

= $(-1)^{s+1} \chi_d^{\ell}(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_{2D}^{D-2\ell}) - \chi_d^{\ell}(-aB) G_1(\eta_{2D}^{-2\ell}).$

Our statement follows from the values of S_{ℓ} found and Equations (7) and (8). The case where D is even is obtained similarly.

5. Bounds and explict formulas for the number of rational POINTS ON $\mathcal{X}_{d.a.b}$

5.1. **Proof of Theorem 2.2.** The result follows by a direct employment of Proposition 4.5 and Lemma 3.5. Let us consider the case when D is odd.

• If B = 0, by Proposition 4.5 we have

$$|N_{d,a,b,n}| = \frac{1}{q} \left| -(q-1) \sum_{j=1}^{v-1} G_n(\chi_d^{jD}) \chi_d^{jD}(-a) \right|$$
$$\leq \frac{(q-1)}{q} \sum_{j=1}^{v-1} |G_n(\chi_d^{jD})|$$
$$= (q-1)q^{n/2-1} \left(\frac{d}{D} - 1\right).$$

• If
$$B \neq 0$$
, then

$$\begin{split} |N_{d,a,b,n}| &= \frac{1}{q} \left| \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \left[(-1)^{s+1} \chi_d^\ell (-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_{2D}^{D-2\ell}) - \chi_d^\ell (-aB) G_1(\eta_{2D}^{-2\ell}) \right] \\ &\leq \frac{1}{q} \sum_{\ell=1}^{d-1} |G_n(\chi_d^\ell)| \left[\sqrt{q} |G_1(\eta_{2D}^{D-2\ell})| + |G_1(\eta_{2D}^{-2\ell})| \right] \\ &\leq (d-1)(q^{n/2} + q^{(n-1)/2}). \end{split}$$

Those inequalities along with Proposition 4.5 assures us the result in the cases when D is odd. The case where D is even is obtained similarly.

5.2. Proof of Theorem 2.3. We have that

$$D = \frac{2}{v} = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

To compute $| \mathcal{X}_{2,a,b} |$ we need to determine N in Proposition 4.5. In the case when n is even we have D = 1, therefore

(1) If B = 0, then

$$N = \frac{1}{q} \left(-(q-1)G_n(\chi_2)\chi_2(-a) \right)$$

= $q^{\frac{n-2}{2}}(q-1)(-1)^{ns+2}\tau^{ns}\chi_2(-a)$
= $q^{\frac{n-2}{2}}(q-1)\tau^{ns}\chi_2(-a).$

(2) If $B \neq 0$:

$$N = \frac{1}{q} \left(G_n(\chi_2) \left[(-1)^{s+1} \chi_2(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_2) - \chi_2(-aB) G_1(\eta_2^2) \right] \right)$$

= $(-1)^{ns+1} \tau^{ns} q^{\frac{n-2}{2}} \left[(-1)^{s+1} \chi_2(-a) \tau^{2s} (-1)^{s+1} q + \chi_2(-aB) \right]$
= $-\tau^{ns} q^{\frac{n-2}{2}} \left[\chi_2(-a) \tau^{2s} q + \chi_2(-aB) \right].$

In the case when n odd, we have that D is even, then

(1) If B = 0:

$$N = \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} G_n(\chi_2) \chi_2(-a) \right)$$

= $\frac{q-1}{q} (-1)^{s+1} \tau^s (-1)^{ns-1} \tau^{ns} q^{\frac{n+1}{2}} \chi_2(-a)$
= $\tau^{(n+1)s} (q-1) \chi_2(-a) q^{\frac{n-1}{2}}.$

(2) If
$$B \neq 0$$
:

$$N = \frac{1}{q} G_n(\chi_2) \Big[(-1)^{s+1} \chi_2(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_2^0) - \chi_2(-aB) G_1(\eta_2) \Big]$$

$$= (-1)^{ns-1} \tau^{ns} q^{\frac{n-2}{2}} \Big[(-1)^{s+1} \chi_2(-a) \tau^s \sqrt{q} (-1) - \chi_2(-aB) (-1)^{s-1} \tau^s \sqrt{q} \Big]$$

$$= -\tau^{s(n+1)} q^{\frac{n-1}{2}} \Big[\chi_2(-a) + \chi_2(-aB) \Big].$$

Combined with Proposition 4.5, the theorem follows.

5.3. **Proof of Theorem 2.6.** Let M be the number of elements $\alpha \in \mathbb{F}_{q^n}$ such that both α and $\operatorname{Tr}(\alpha)$ are quadratic residues in \mathbb{F}_{q^n} . Let Λ_2 be the set of quadratic residues in $\mathbb{F}_{q^n}^*$ and $\Lambda = \mathbb{F}_{q^n}^* \setminus \Lambda_2$. We recall that, for $x \in \mathbb{F}_{q^n}$, $\chi_2(x) = 1$ if and only if $x \in \Lambda_2$. Then we have that

$$\left[1+\chi_2(x)\right]\left[1+\chi_2(\operatorname{Tr}(x))\right] = \begin{cases} 0, & \text{if either } x \in \Lambda \text{ or } \operatorname{Tr}(x) \in \Lambda; \\ 1, & \text{if } x = 0; \\ 2, & \text{if } x \in \Lambda_2 \text{ and } \operatorname{Tr}(x) = 0; \\ 4, & \text{if } x \in \Lambda_2 \text{ and } \operatorname{Tr}(x) \in \Lambda_2. \end{cases}$$

Therefore, Schur's orthogonality relations (Lemmas 3.3 and 3.4) imply that

$$M = \frac{1}{2} + \frac{1}{4} \sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x) \right] \left[1 + \chi_2(\operatorname{Tr}(x)) \right] + \frac{1}{4q} \sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x) \right] \sum_{c \in \mathbb{F}_q} \tilde{\psi}(c\operatorname{Tr}(x)).$$
(9)

We note that Lemma 3.2 implies that

$$\sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x) \right] \left[1 + \chi_2(\operatorname{Tr}(x)) \right] = \sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(\operatorname{Tr}(x)) + \chi_2(x\operatorname{Tr}(x)) \right].$$
(10)

Since n is odd, χ_2 is also the multiplicative character of order 2 over \mathbb{F}_q . Since $\operatorname{Tr}(x)$ is a linear transformation over \mathbb{F}_{q^n} whose image is \mathbb{F}_q , each element in \mathbb{F}_q has q^{n-1} elements in its preimage. Therefore,

$$\sum_{x \in \mathbb{F}_{q^n}} \chi_2(\mathrm{Tr}(x)) = q^{n-1} \sum_{y \in \mathbb{F}_q} \chi_2(y) = 0.$$
(11)

We note that

$$|\mathcal{X}_{2,1,0}(\mathbb{F}_{q^n})| = \sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x \operatorname{Tr}(x)) \right] = q^n + \sum_{x \in \mathbb{F}_{q^n}} \chi_2(x \operatorname{Tr}(x)).$$
(12)

From (10), (11) and (12), it follows that

$$\sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x) \right] \left[1 + \chi_2(\operatorname{Tr}(x)) \right] = |\mathcal{X}_{2,1,0}(\mathbb{F}_{q^n})|.$$
(13)

For the last sum in (9), Lemma 3.3 implies that

$$\sum_{x \in \mathbb{F}_{q^n}} \left[1 + \chi_2(x) \right] \sum_{c \in \mathbb{F}_q} \tilde{\psi}(c \operatorname{Tr}(x)) = q^{n-1} \cdot q + \sum_{c \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_{q^n}} \chi_2(x) \tilde{\psi}(c \operatorname{Tr}(x)).$$
(14)

The map $x \mapsto \operatorname{Tr}(x)$ is a linear map over \mathbb{F}_q so that $c\operatorname{Tr}(x) = \operatorname{Tr}(cx)$ for all $c \in \mathbb{F}_q$ and then $\tilde{\psi}(c\operatorname{Tr}(x) = \tilde{\psi}(\operatorname{Tr}(cx)) = \psi(cx)$. Therefore, by setting z = cx, we obtain that

$$\sum_{c \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_{q^n}} \chi_2(x) \tilde{\psi}(c \operatorname{Tr}(x)) = \sum_{c \in \mathbb{F}_q} \chi_2(c) \sum_{z \in \mathbb{F}_{q^n}} \chi_2(z) \psi(z)$$
$$= G_n(\chi_2) \sum_{c \in \mathbb{F}_q} \chi_2(c)$$
$$= 0,$$
(15)

where the last equality follows by Lemma 3.2. In sum, Equations (9), (13), (14) and (15) imply that

$$M = \frac{|\mathcal{X}_{2,1,0}(\mathbb{F}_{q^n})| + q^{n-1} + 2}{4}.$$

Now our result follows by Theorem 2.3.

5.4. Proof of Theorem 2.10.

(1) Assume that v is (p, r)-admissible and D is odd. Proposition 4.5 states that

$$N = -\frac{q-1}{q} \sum_{j=1}^{v-1} G_n(\chi_d^{jD}) \chi_d^{jD}(-a).$$

We recall that χ_d^D has order $\frac{d}{D} = v$. If v = 1, then N = 0. If v = 2, then Theorem 3.7 entails that

$$N = -(q-1)(-\tau^{sn}q^{\frac{n-2}{2}}\chi_2(-1)\chi_2(a))$$

= $(q-1)\tau^{sn}q^{\frac{n-2}{2}}\chi_2(a),$

since $\chi_2(-1) = \tau^{2ns} = 1$. If v > 2, then Theorem 3.6 implies that

$$N = -\frac{q-1}{q}(-\varepsilon)q^{\frac{n}{2}}\sum_{j=1}^{v-1} \left(\varepsilon^{uD}\chi^D_d(-a)\right)^j$$
$$= q^{\frac{n-2}{2}}(q-1)\varepsilon\theta_v(-a,\varepsilon^{uD}).$$

(2) Let us assume that 2v is (p, r)-admissible and D is even. By Proposition 4.5 we have

$$N = \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} \sum_{j=0}^{v-1} G_n(\chi_d^{jD+\frac{D}{2}}) \chi_d^{jD+\frac{D}{2}}(-a) - \sum_{j=1}^{v-1} G_n(\chi_d^{jD}) \chi_d^{jD}(-a) \right).$$

If v = 1, d = D and it follows that

$$N = \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} G_n(\chi_d^{\frac{d}{2}}) \chi_d^{\frac{d}{2}}(-a) \right)$$

= $\frac{q-1}{q} (-1)^{s+1} \tau^s \sqrt{q} G_n(\chi_2) \chi_2(-a)$
= $(-1)^{ns+s} (q-1) q^{\frac{n-1}{2}} \tau^{s(n+1)} \chi_2(a)$
= $(-1)^s (q-1) q^{\frac{n-1}{2}} \tau^{s(n+1)} \chi_2(a).$

If v = 2, we have d = 2D. By hypotheses, 4 is (p, r)-admissible, and $u = \frac{p^r + 1}{d} = \frac{p^r + 1}{4 \cdot D/2}$. Then

$$\begin{split} N &= \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} (G_n(\chi_d^{\frac{d}{4}}) \chi_d^{\frac{d}{4}}(-a) + G_n(\chi_d^{\frac{3d}{4}}) \chi_d^{\frac{3d}{4}}(-a)) - G_n(\chi_d^{\frac{d}{2}}) \chi_d^{\frac{d}{2}}(-a) \right) \\ &= \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} (G_n(\chi_4) \chi_4(-a) + G_n(\overline{\chi_4}) \overline{\chi_4}(-a)) - G_n(\chi_2) \chi_2(-a)) \right) \\ &= \frac{q-1}{q} \left((-1)^{s+1} \tau^s \sqrt{q} (-\varepsilon^{u\frac{D}{2}+1} q^{\frac{n}{2}} \chi_4(-a) + \chi_4(-1) (-\varepsilon^{u\frac{D}{2}+1} q^{\frac{n}{2}}) \overline{\chi_4}(-a)) + (-1)^{sn} \tau^{ns} q^{\frac{n}{2}} \chi_2(-a) \right) \\ &= q^{\frac{n-2}{2}} (q-1) \left((-1)^s \tau^s \sqrt{q} \varepsilon^{u\frac{D}{2}+1} (\chi_4(-a) + \overline{\chi_4}(a)) + \tau^{ns} \chi_2(a) \right). \end{split}$$

If v > 2, then Theorem 3.6 implies that

$$N = \frac{q-1}{q} \left((-1)^{s+1} \tau^s q^{\frac{n+1}{2}} \chi_{2v}(-a) (-\varepsilon^{1+\frac{uD}{2}}) \sum_{j=0}^{v-1} (\varepsilon^{2u} \chi_v(-a))^j - q^{\frac{n}{2}}(-\varepsilon) \sum_{j=1}^{v-1} (\varepsilon^{uD} \chi_v(-a))^j \right)$$
$$= \frac{q-1}{q} \left((-1)^s \tau^s q^{\frac{n+1}{2}} \chi_{2v}(-a) \varepsilon^{1+\frac{uD}{2}} [1 + \theta_v(-a, 1)] + q^{\frac{n}{2}} \varepsilon \theta_v(-a, 1) \right)$$
$$= (q-1) q^{\frac{n-2}{2}} \varepsilon \Big[(-1)^s \tau^s \sqrt{q} \chi_{2v}(-a) \varepsilon^{\frac{uD}{2}} [1 + \theta_v(-a, 1)] + \theta_v(-a, 1)] + \theta_v(-a, 1) \Big].$$

6. The case $\operatorname{Tr}(b/a) \neq 0$

Definition 6.1. If D is (p, r_0) -admissible, we define

$$\varepsilon_0 = (-1)^{\frac{s}{2r_0}}, \quad and \quad u_0 = \begin{cases} \frac{p^{r_0}+1}{2D}, & \text{if } D \text{ is odd;} \\ \frac{p^{r_0}+1}{D}, & \text{if } D \text{ is even} \end{cases}$$

Theorem 6.2. Let $a, b \in \mathbb{F}_{q^n}$ such that $B \neq 0$. If d is (p, r)-admissible, D is odd and (p, r_0) -admissible, then

$$N = N_1 q^{\frac{n}{2}} + N_2 q^{\frac{n-1}{2}} + N_3 q^{\frac{n-2}{2}},$$

where

- $N_1 = (-1)^{s+1} \varepsilon \varepsilon_0^{1+u_0 D} \tau^s \chi_2(B) \theta_d(-aB, \varepsilon^u);$ $N_2 = \varepsilon \varepsilon_0 \left(-\theta_d(-aB, \varepsilon^u) + \theta_v(-aB, \varepsilon^u)\right);$ $N_3 = -\varepsilon \theta_v(-aB, \varepsilon^u);$

Proof. By hypotheses d is (p, r)-admissible and D is odd. By Proposition 4.5,

$$N = \frac{1}{q} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \Big[(-1)^{s+1} \chi_d^\ell(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_{2D}^{D-2\ell}) - \chi_d^\ell(-aB) G_1(\eta_{2D}^{-2\ell}) \Big]$$

Since D is (p, r_0) -admissible and $p^{r_0} + 1$ is even, it follows that 2D divides $p^{r_0} + 1$. Therefore, Theorem 3.6 and Corollary 3.8 imply that

$$N = -q^{\frac{n-2}{2}} \sum_{\ell=1}^{d-1} \varepsilon^{u\ell+1} \chi_d^{\ell}(-aB) \left[(-1)^s \tau^s \chi_2(B) \varepsilon_0^{u_0(2\ell+D)+1} q + \varepsilon_0^{2u_0\ell+1} \sqrt{q} \right] + R$$

$$= -q^{\frac{n-1}{2}} \varepsilon \varepsilon_0 \Big((-1)^s \tau^s \chi_2(B) \varepsilon_0^{u_0 D} \sqrt{q} + 1 \Big) \sum_{\ell=1}^{d-1} \big(\chi_d(-aB) \varepsilon^u \varepsilon_0^{2u_0} \big)^{\ell} + R$$

$$= -q^{\frac{n-1}{2}} \varepsilon \varepsilon_0 \Big((-1)^s \tau^s \chi_2(B) \varepsilon_0^{u_0 D} \sqrt{q} + 1 \Big) \theta_d(-aB, \varepsilon^u) + R,$$

where R is the sum of the terms in the case when the multiplicative caracter is trivial, i.e.,

$$R = \frac{1}{q} \left(q^{\frac{n}{2}} \sum_{j=1}^{d/D-1} \varepsilon^{ujD+1} \chi_d^{jD}(-aB) \left[-1 + \sqrt{q} \varepsilon_0^{2u_0jD+1} \right] \right)$$
$$= -q^{\frac{n-2}{2}} \varepsilon \sum_{j=1}^{d/D-1} \left[\left(\varepsilon^{uD} \chi_d^D(-aB) \right)^j - \varepsilon_0 \left(\varepsilon^{uD} \varepsilon_0^{2u_0D} \chi_d^D(-aB) \right)^j \sqrt{q} \right]$$
$$= q^{\frac{n-2}{2}} \varepsilon (-1 + \varepsilon_0 \sqrt{q}) \theta_v (-aB, \varepsilon^u).$$

Rearranging the terms, we obtain the expression

$$N = N_1 q^{\frac{n}{2}} + N_2 q^{\frac{n-1}{2}} + N_3 q^{\frac{n-2}{2}},$$

proving the statement.

Theorem 6.3. Assume that $B \neq 0$ and d > 2. We denote $\alpha_1 = \theta_d(-aB, \varepsilon^u \varepsilon_0^{u_0})$ and $\alpha_2 = \theta_v(-aB, 1)$. If d is (p, r)-admissible, D is even and (p, r_0) -admissible, then

$$N = N_1 q^{\frac{n}{2}} + N_2 q^{\frac{n-1}{2}} + N_3 q^{\frac{n-2}{2}},$$

where

•
$$N_1 = (-1)^s \varepsilon \varepsilon_0 \tau^s \chi_2(B) \Big[-\varepsilon_0^{u_0 \frac{D}{2}} \alpha_1 + \varepsilon^{u \frac{D}{2}} \chi_{2v}(-aB)(1+\alpha_2) \Big],$$

• $N_2 = \varepsilon \varepsilon_0 (\alpha_2 - \alpha_1) - (-1)^s \varepsilon^{u \frac{D}{2} + 1} \tau^s \chi_{2v}(-aB) \chi_2(B)(1+\alpha_2),$

• $N_3 = -\varepsilon \alpha_2$,

Proof. By hypotheses d is (p, r)-admissible and D is even. By Proposition 4.5,

$$N = \frac{1}{q} \sum_{\ell=1}^{d-1} G_n(\chi_d^\ell) \Big[(-1)^{s+1} \chi_d^\ell(-aB) \chi_2(B) \tau^s \sqrt{q} G_1(\eta_D^{\frac{D}{2}-\ell}) - \chi_d^\ell(-aB) G_1(\eta_D^{-\ell}) \Big].$$

Therefore, since D is (p, r_0) admissible

$$\begin{split} N &= -q^{\frac{n-2}{2}} \sum_{\ell=1}^{d-1} \varepsilon^{u\ell+1} \chi_d^\ell (-aB) \Big[-(-1)^{s+1} \tau^s \chi_2(B) \varepsilon_0^{u_0(\ell+\frac{D}{2})+1} q + \varepsilon_0^{u_0\ell+1} \sqrt{q} \Big] + R \\ &= -q^{\frac{n-1}{2}} \varepsilon \varepsilon_0 \Big((-1)^s \tau^s \chi_2(B) \varepsilon_0^{u_0\frac{D}{2}} \sqrt{q} + 1 \Big) \sum_{\ell=1}^{d-1} \big(\chi_d(-aB) \varepsilon^u \varepsilon_0^{u_0} \big)^\ell + R, \\ &= -q^{\frac{n-1}{2}} \varepsilon \varepsilon_0 \Big((-1)^s \tau^s \chi_2(B) \varepsilon_0^{u_0\frac{D}{2}} \sqrt{q} + 1 \Big) \theta_d(-aB, \varepsilon^u \varepsilon_0^{u_0}) + R, \\ &\text{where } R = R_1 + R_2, \end{split}$$

$$R_{1} = -q^{\frac{n-2}{2}} \sum_{j=0}^{d/D-1} \varepsilon^{u\left(\frac{D}{2}(2j+1)\right)+1} \chi_{d}^{\frac{D}{2}(2j+1)} (-aB)(-1)^{s+1} \left[-\chi_{2}(B)\sqrt{q\tau^{s}} + \tau^{s}\chi_{2}(B)\varepsilon_{0}^{u_{0}\left(\frac{D}{2}(2j+1)+\frac{D}{2}\right)+1}q\right]$$

$$= (-1)^{s} q^{\frac{n-1}{2}} \varepsilon^{u\frac{D}{2}+1} \tau^{s} \chi_{2v}(-aB) \chi_{2}(B) \sum_{j=0}^{d/D-1} \left(\varepsilon^{uD} \chi_{v}(-aB) \right)^{j} (-1 + \sqrt{q} \varepsilon_{0}^{u_{0}Dj+u_{0}D+1})$$

$$= (-1)^{s} q^{\frac{n-1}{2}} \varepsilon^{u\frac{D}{2}+1} \tau^{s} \chi_{2v}(-aB) \chi_{2}(B) \sum_{j=0}^{d/D-1} \left[-\left(\varepsilon^{uD} \chi_{v}(-aB) \right)^{j} + \left(\varepsilon^{uD} \varepsilon_{0}^{u_{0}D} \chi_{v}(-aB) \right)^{j} \varepsilon_{0}^{u_{0}D+1} \sqrt{q} \right]$$

$$= (-1)^{s} q^{\frac{n-1}{2}} \varepsilon^{u\frac{D}{2}+1} \tau^{s} \chi_{2v}(-aB) \chi_{2}(B) \left[-\left(1 + \theta_{v}(-aB, \varepsilon^{uD}) \right) + \left(1 + \theta_{v}(-aB, \varepsilon^{uD} \varepsilon_{0}^{u_{0}D}) \right) \varepsilon_{0}^{u_{0}D+1} \sqrt{q} \right]$$

$$= (-1)^{s} q^{\frac{n-1}{2}} \varepsilon^{u\frac{D}{2}+1} \tau^{s} \chi_{2v}(-aB) \chi_{2}(B) \left[-\left(1 + \theta_{v}(-aB, \varepsilon^{uD}) \right) + \left(1 + \theta_{v}(-aB, \varepsilon^{uD} \varepsilon_{0}^{u_{0}D}) \right) \varepsilon_{0}^{u_{0}D+1} \sqrt{q} \right]$$
and

$$R_{2} = -q^{\frac{n-2}{2}} \sum_{j=1}^{d/D-1} \varepsilon^{ujD+1} \chi_{d}^{jD}(-aB) \left[1 - \sqrt{q} \varepsilon_{0}^{u_{0}jD+1}\right]$$
$$= -q^{\frac{n-2}{2}} \varepsilon \sum_{j=1}^{d/D-1} \left[\left(\varepsilon^{uD} \chi_{v}(-aB)\right)^{j} - \varepsilon_{0} \left(\varepsilon^{uD} \varepsilon_{0}^{u_{0}D} \chi_{v}(-aB)\right)^{j} \sqrt{q} \right]$$
$$= -q^{\frac{n-2}{2}} \varepsilon \left[\theta_{v}(-aB,1) - \varepsilon_{0} \theta_{v}(-aB,1) \sqrt{q} \right]$$
$$= -q^{\frac{n-2}{2}} \varepsilon \theta_{v}(-aB,1)(1 - \varepsilon_{0} \sqrt{q}).$$

Altogether, we have shown that

$$N = N_1 q^{\frac{n}{2}} + N_2 q^{\frac{n-1}{2}} + N_3 q^{\frac{n-2}{2}},$$

proving the statement.

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