# REMARKS ON CATALAN'S EQUATION OVER FUNCTION FIELDS 

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#### Abstract

Let $\ell$ be a prime number, $F$ be a global function field of characteristic $\ell$. Assume that there is a prime $P_{\infty}$ of degree 1. Let $\mathcal{O}_{F}$ be the ring of functions in $F$ with no poles outside of $\left\{P_{\infty}\right\}$. We study solutions to Catalan's equation $X^{m}-Y^{n}=1$ over $\mathcal{O}_{F}$ and show that under certain additional conditions, there are no non-constant solutions which lie in $\mathcal{O}_{F}$, when $m, n>1$.


## 1. Introduction

Let $m>1$ and $n>1$ be integers, and consider the diophantine equation

$$
X^{m}-Y^{n}=1
$$

The famous Catalan conjecture states that there are no non-trivial integer solutions to the above equation except when $m=2, n=3$ and $(X, Y)=( \pm 3,2)$. The celebrated result of Mihăilescu resolves this conjecture using techniques from the theory of cyclotomic fields (cf. [1]). Given the close analogy between number fields and function fields, it is of interest to study analogues of Catalan's conjecture in characteristic $\ell>0$. The field of rational numbers $\mathbb{Q}$ is the simplest number field to consider, and analogously, the most natural analogue is the field of rational functions $\mathbb{F}(T)$, where $T$ is a formal variable, and $\mathbb{F}$ is a finite field. The ring of integers $\mathbb{Z}$ is thus analogous to the ring of polynomial functions $\mathbb{F}[T]$, which shares similar properties to $\mathbb{Z}$. The reader is referred to $[2,3]$ for an introduction to the arithmetic of function fields, and further perspectives elaborating the close analogy between number fields and their counterparts in positive characteristic.

Let $\ell$ be a prime number and $F$ be a global function field of characteristic $\ell$. Denote by $\mathbb{F}_{\ell}$ the finite field with $\ell$ elements and set $\kappa$ to denote the algebraic closure of $\mathbb{F}_{\ell}$ in $F$. Note that $\kappa$ is a finite field (by assumption). Recall (from [2, Chapter 5]) that a prime in $F$ is defined to be the maximal ideal $v$ of a discrete valuation ring $R$ contained in $F$, with fraction field equal to $F$. The degree of $v$ is defined to be the dimension of $R / v$ over the field of constants $\kappa$. Each prime $v$ comes equipped with a valuation $\operatorname{ord}_{v}: F \rightarrow \mathbb{Z} \cup\{\infty\}$. Assume that there exists a prime $P_{\infty}$ of $F$ which has degree 1 , and let $\mathcal{O}_{F}$ be the ring of functions in $F$ with no poles outside $\left\{P_{\infty}\right\}$. The point $P_{\infty}$ is referred to as the point at infinity and $\mathcal{O}_{F}$ is the ring of integers of $F$. We say that a solution

[^0]$(X, Y) \in \mathcal{O}_{F}^{2}$ to $X^{m}-Y^{n}=1$ is constant if $X$ and $Y$ are both contained in $\kappa$, and non-constant otherwise.

Recall from loc. cit. that a divisor is a finite integral linear combination of primes of $F$. The principal divisor associated to $g \in F$ is denoted $\operatorname{div}(g)$, and two divisors $D_{1}$ and $D_{2}$ are said to be equivalent if $D_{1}-D_{2}$ is a principal divisor. The group of divisors classes of degree 0 is finite (cf. [2, Lemma 5.6]), and its cardinality is the class number of $F$, and this quantity is denoted by $h_{F}$. Given a prime number $p \neq \ell$, let $F\left(\mu_{p}\right)$ be the function field obtained by adjoining the $p$-th roots of unity $\mu_{p}$ to $F$. Note that $F\left(\mu_{p}\right)=\kappa^{\prime} \cdot F$, where $\kappa^{\prime}=\kappa\left(\mu_{p}\right)$. Thus, $F\left(\mu_{p}\right)$ is a constant field extension of $F$ in the sense of $[2$, Chapter 8].
Theorem 1.1. Let $F$ be a global function field of characteristic $\ell>0$. Let $p$ and $q$ be prime numbers and assume that all the following conditions are satisfied
(1) $p \neq \ell$ and $q \neq \ell$,
(2) if $p \neq q$, then either $q \nmid h_{F\left(\mu_{p}\right)}$ or $p \nmid h_{F\left(\mu_{q}\right)}$.
(3) if $q=2, p \neq 2$ and $q \mid h_{F\left(\mu_{p}\right)}$, then $p \nmid h_{F\left(\mu_{4}\right)}$.

Then, there are no non-constant solutions to $X^{p}-Y^{q}=1$ in $\mathcal{O}_{F}$. More generally, if $m>1$ and $n>1$ are integers such that $m$ is divisible by a prime $p$ and $n$ by a prime $q$ for which the above conditions are satisfied, then there are no non-constant solutions to $X^{m}-Y^{n}=1$ in $\mathcal{O}_{F}$.

The condition requiring that $p$ and $q$ are distinct from $\ell$ is necessary, since if $m=\ell$ for instance, it is easy to construct a large class of non-constant solutions if one of the primes is equal to $\ell$ (cf. Remark 2.3 for details).

We mention some related work of relevance. Silverman [4] considered a general class of equations of the form $a X^{m}+b Y^{n}=c$ over a general function field $K$, and proved that under some further conditions, there are only finitely many solutions when $a, b, c \in K^{*}$ are fixed. There is a mistake in the statement of Silverman's result, which has been corrected by Koymans [5]. The result of Koymans moreover applies to fields of larger dimension. The Catalan equation was studied by Nathanson [6] over $K[T]$ and $K(T)$ where $K$ is a field of positive characteristic. It is shown in loc. cit. that if $m>1$ and $n>1$ are coprime to $\ell$ then there are no solutions to Catalan's equation $X^{m}-Y^{n}=1$ that lie in $K[T]$ but not in $K$. Specializing to the case when $K$ is a finite field, one obtains the conclusion of Theorem 1.1 for the rational function field. This is because the class number of any rational function field is equal to 0 . Theorem 1.1 can thus be viewed as a generalization of Nathanson's result to general function fields $F$ with added stipulations on $(m, n)$.
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## 2. Proof of the main result

Recall that $F$ is a global function field of characteristic $\ell>0$ with field of constants $\kappa$. Let $\bar{\kappa}$ be the algebraic closure of $\kappa$ in a fixed algebraic closure of
$F$, and set $F^{\prime}$ to denote the composite $F \cdot \bar{\kappa}$. Also, denote by $A$ the composite $\mathcal{O}_{F} \cdot \bar{\kappa}$. The field $F^{\prime}$ is identified with the function field of a projective curve $\mathfrak{X}$ over $\bar{\kappa}$ and each point in $\mathfrak{X}(\bar{\kappa})$ corresponds to a valuation ring $R \subset F^{\prime}$ with residue field $\bar{\kappa}$ and fraction field $F^{\prime}$. The valuation ring associated to $w \in \mathfrak{X}(\bar{\kappa})$ is denoted $\mathcal{O}_{w}$, and we refer to $w$ as a prime of $F^{\prime}$. We say that $w$ divides (or lies above) a prime $v$ of $F$ if there is a natural inclusion of valuation rings $\mathcal{O}_{v} \hookrightarrow \mathcal{O}_{w}$ induced by the inclusion $F \hookrightarrow F^{\prime}$. Note that since $P_{\infty}$ has degree 1, it is totally inert in $F^{\prime}$. In particular, there is a single prime of $F^{\prime}$ that lies above $P_{\infty}$, which we identify with $P_{\infty}$. Given any prime $v$ of $F$, set $d_{v}$ to denote - $\operatorname{ord}_{v}$ and for any function $g \in F$, we refer to $d_{v}(g)$ as the order of the pole of $g$ at $v$. Given a prime $w$ of $F^{\prime}$ (i.e., point $\left.w \in \mathfrak{X}(\bar{\kappa})\right)$ and $g \in F^{\prime}$, denote by $d_{w}(g)$ the order of the pole of $g$ at $w$. We set $d: A \rightarrow \mathbb{Z}_{\geq 0}$ to denote $d_{P_{\infty}}$.
Lemma 2.1. Let $f, g \in A$ be non-zero. The following assertions hold.
(1) $d(g)=0$ if and only if $g$ is a constant function.
(2) We have that $d(f g)=d(f)+d(g)$.
(3) Suppose that $d(f)<d(g)$. Then, $d(g+f)=d(g)$.
(4) We have that $d(f) \geq 0$, and $d(f)>0$ if and only if $f$ is non-constant.

Proof. The proof of parts (1) to (3) are easy, hence, omitted. For part (4) note that a non-constant function $f \in A$ must have a pole at some point. By virtue of being contained in $A, f$ does not have poles outside $\left\{P_{\infty}\right\}$. Therefore, $f$ must have a pole at $P_{\infty}$, and thus, $d(f)>0$. On the other hand, if $f$ is constant, then $d(f)=0$. This proves part (4).
Lemma 2.2. Let $Y \in A$ and $c_{1}, c_{2} \in \bar{\kappa}$ be non-zero constants. If for some prime $p \neq \ell$ we have that

$$
\left(Y+c_{1}\right)^{p}-Y^{p}=c_{2},
$$

then $Y$ is a constant.
Proof. Setting $f(z):=\left(z+c_{1}\right)^{p}-z^{p}-c_{2}$, we find that $f(z)$ is a nonzero polynomial in $z$ with coefficients in $\bar{\kappa}$. Therefore, any solution $Y$ to the equation $f(Y)=0$ must also lie in $\bar{\kappa}$.
Proof of Theorem 1.1. First consider the case when $p=q$. Note that it is assumed that $p \neq \ell$. We show that there are no non-constant solutions to

$$
X^{p}-Y^{p}=1
$$

in $A$. Note that $(X-Y)$ divides $X^{p}-Y^{p}=1$, hence by Lemma 2.1,

$$
d(X-Y)=d(1)-d\left(X^{p-1}+X^{p-2} Y+\cdots+X Y^{p-2}+Y^{p-1}\right) \leq d(1)=0 .
$$

It follows from Lemma 2.1 part (1) that $(X-Y)$ is a constant $c \in \bar{\kappa}$. We thus deduced that

$$
\begin{equation*}
(Y+c)^{p}-Y^{p}=1 \tag{2.1}
\end{equation*}
$$

Lemma 2.2 implies that (2.1) has no non-constant solutions. Since $Y$ is a constant, it follows that $X$ is as well. If $X$ and $Y$ are in $\mathcal{O}_{F}$, it follows therefore that $X, Y \in \kappa$.

We assume therefore that $p$ and $q$ are distinct (and distinct from $\ell$ ). Note that there are further conditions on $p$ and $q$. First, we consider the case when $q \nmid$ $h_{F\left(\mu_{p}\right) \text {. All the variables introduced in the following argument will be contained }}$ in $F\left(\mu_{p}\right)$. Let $\zeta$ be a primitive $p$-th root of 1 . Since it is assumed that $p \neq \ell$, we note that $\zeta \neq 1$. In what follows we consider divisors over $F\left(\mu_{p}\right)$. Given a divisor $D=\sum_{v} n_{v} v$ involving primes $v$ of $F\left(\mu_{p}\right)$, the support consists of all primes $v$ such that the coefficient $n_{v}$ is not equal to 0 . Factor $X^{p}-1$ into linear factors to obtain the following equation

$$
\begin{equation*}
Y^{q}=\prod_{j=0}^{p-1}\left(X-\zeta^{j}\right) \tag{2.2}
\end{equation*}
$$

For $i \neq j$, note that $\left(X-\zeta^{i}\right)-\left(X-\zeta^{j}\right)=\zeta^{j}-\zeta^{i}$, which is a non-zero element of $\kappa\left(\mu_{p}\right)$. Hence, it follows that $\operatorname{div}\left(X-\zeta^{i}\right)$ and $\operatorname{div}\left(X-\zeta^{j}\right)$ have disjoint supports for $i \neq j$. From (2.2), we have the following relation between divisors that are formal linear combinations of primes in $F\left(\mu_{p}\right)$

$$
\sum_{j=0}^{p-1} \operatorname{div}\left(X-\zeta^{j}\right)=q \operatorname{div}(Y)
$$

The elements $\left(X-\zeta^{j}\right)$ are all contained in $F\left(\mu_{p}\right)$, while $Y$ is contained in $F$. Since the divisors $\operatorname{div}\left(X-\zeta^{j}\right)$ have disjoint supports for $i \neq j$, it follows that for each $i$, there is a divisor $D_{i}$ (involving linear combinations of primes in $F\left(\mu_{p}\right)$ ) such that $\operatorname{div}\left(X-\zeta^{i}\right)=q D_{i}$. Since $\operatorname{div}\left(X-\zeta^{i}\right)$ is a principal divisor, it has degree 0 , and hence $D_{i}$ does also have degree zero. Since $q \nmid h_{F\left(\mu_{p}\right)}$, there is no non-trivial $q$ torsion in the divisor class group. As a result, $D_{i}$ is a principal divisor $\operatorname{div}\left(\alpha_{i}\right)$, where $\alpha_{i} \in F\left(\mu_{p}\right)$. Thus, we have deduced that for all $i$,

$$
X-\zeta^{i}=u_{i} \alpha_{i}^{q}
$$

where $u_{i} \in F\left(\mu_{p}\right)$ is a non-zero function for which $\operatorname{div}\left(u_{i}\right)=0$. Therefore $u_{i}$ is a unit, and consequently, is contained in $\kappa\left(\mu_{p}\right)$. Recall that $p$ and $q$ are distinct, and we have shown that $u_{i} \in \bar{\kappa}$. It follows that $u_{i}$ is the $q$-th power of an element $v_{i} \in \bar{\kappa}^{\times}$. Replacing $\alpha_{i}$ with $v_{i} \alpha_{i}$, we write

$$
X-\zeta^{i}=\alpha_{i}^{q}
$$

where $\alpha_{i} \in\left(F^{\prime}\right)^{\times}$. Note that $\alpha_{i}$ is contained in $A$ since it has no poles outside $\left\{P_{\infty}\right\}$ (since $X-\zeta^{i}$ does not). We deduce that

$$
\begin{equation*}
\alpha_{0}^{q}-\alpha_{1}^{q}=(X-1)-(X-\zeta)=\zeta-1 \tag{2.3}
\end{equation*}
$$

It follows that $\alpha_{0}-\alpha_{1}$ divides $\zeta-1$, hence has no zeros or poles. As a result, $\alpha_{0}-\alpha_{1}$ is a constant $c \in \bar{\kappa}$. It is clear from (2.3) that $c$ is non-zero. Thus we find that

$$
\left(\alpha_{1}+c\right)^{q}-\alpha_{1}^{q}=\zeta-1 .
$$

Lemma 2.2 then implies that $\alpha_{1}$ and $\alpha_{0}$ are constants. We have thus shown that $X$, and hence $Y$ are both elements in $\bar{\kappa}$. Since $\kappa$ is the algebraic closure of $\mathbb{F}_{\ell}$ in $F$, and both $X$ and $Y$ are contained in $F$, it follows that $X, Y \in \kappa$.

It follows from the condition (2) of Theorem 1.1 that if $p \neq q$, then $q \nmid h_{F\left(\mu_{p}\right)}$ or $p \nmid h_{F\left(\mu_{q}\right)}$. We have shown that there are no non-constant solutions when
$p=q$, or when $q \nmid h_{F\left(\mu_{p}\right)}$. Throughout the rest of this proof, we shall therefore assume that $p \nmid h_{F\left(\mu_{q}\right)}$. If both $p$ and $q$ are odd, then we may replace $X$ with $-Y$ and $Y$ with $-X$ to obtain the equation $X^{q}-Y^{p}=1$, and thus the previous argument that gives the result applies in this case. We have therefore dealt with the case when both $p$ and $q$ are odd, and we are left to consider the case when $p \nmid h_{F\left(\mu_{q}\right)}$ and either $p$ or $q$ is 2 .

First consider the case when $p=2$. It has been shown that there no nonconstant solutions when $p=q$ and therefore $q$ must be odd. Moreover, as stated in the previous paragraph, we assume that $2 \nmid h_{F\left(\mu_{q}\right)}$. Then, we find that $X^{2}=Y^{q}+1=Y^{q}-(-1)^{q}=\prod_{j}\left(Y+\zeta^{j}\right)$, where $\zeta$ is a primitive $q$-th root of unity. For $i \neq j$, note that $\left(Y+\zeta^{i}\right)-\left(Y+\zeta^{j}\right)=\zeta^{i}-\zeta^{j}$ is a constant, and therefore, $\operatorname{div}\left(Y+\zeta^{i}\right)$ and $\operatorname{div}\left(Y+\zeta^{j}\right)$ have disjoint supports for $i \neq j$. We thus arrive at the equation

$$
\sum_{j=0}^{q-1} \operatorname{div}\left(Y+\zeta^{j}\right)=2 \operatorname{div}(X)
$$

The divisors $\operatorname{div}\left(Y+\zeta^{j}\right)$ have disjoint supports for $i \neq j$, and therefore, we may write $\operatorname{div}\left(Y+\zeta^{j}\right)=2 D_{j}$ for some divisors $D_{j}$ that are defined over $F\left(\mu_{q}\right)$. Recall that $2 \nmid h_{F\left(\mu_{q}\right)}$. An identical argument to the previous case implies that for all $j$, we have that

$$
Y+\zeta^{j}=\beta_{j}^{2}
$$

where $\beta_{j} \in A$. We deduce that

$$
\beta_{0}^{2}-\beta_{1}^{2}=(Y+1)-(Y+\zeta)=1-\zeta
$$

It follows that $\beta_{0}-\beta_{1}$ divides $1-\zeta$. Therefore, $\beta_{0}-\beta_{1}$ has no zeros or poles, and hence equals a constant $c \in \bar{\kappa}$. Thus we find that

$$
\left(\beta_{1}+c\right)^{2}-\beta_{1}^{2}=1-\zeta
$$

Lemma 2.2 then implies that $\beta_{1}=Y+\zeta$ and $\beta_{0}=Y+1$ are constants. From this, we deduce that both $X$ and $Y$ are constants.

Finally, assume that $p$ is odd, $q=2$. Note that the result has been proved when $q \nmid h_{F\left(\mu_{p}\right)}$. Therefore, we assume that $q \mid h_{F\left(\mu_{p}\right)}$. It follows from the condition (3) of Theorem 1.1 that $p \nmid h_{F\left(\mu_{4}\right)}$. We consider the equation $X^{p}=$ $Y^{2}+1=(Y+\eta)(Y-\eta)$, where $\eta^{2}=-1$. Note that $F\left(\mu_{4}\right)=F(\eta)$. Since $p$ does not divide the class number of $F(\eta)$, we find that $Y+\eta=\alpha_{0}^{p}$ and $Y-\eta=\alpha_{1}^{p}$, where $\alpha_{0}, \alpha_{1}$ are elements in $A$. Therefore, $2 \eta=\alpha_{0}^{p}-\alpha_{1}^{p}$. In particular, this implies that $\left(\alpha_{0}-\alpha_{1}\right)$ is a constant $c$. Since $\eta \neq 0$, it follows that $c \neq 0$. We have the following equation

$$
\left(\alpha_{1}+c\right)^{p}-\alpha_{1}^{p}=2 \eta .
$$

The result follows from Lemma 2.2.
Remark 2.3. At this point, it is pertinent to make a few remarks.

- The assumptions that $p$ and $q$ are not equal to $\ell$ are necessary. Indeed, suppose that $p=\ell$. Then, setting $X=1+z^{q}$ and $Y=z^{p}$ for any element $z \in \mathcal{O}_{F}$, one would obtain non-constant solutions.
- The methods introduced in this paper could potentially be applied to a more general class of diophantine equations, namely, equations of the form $X^{m}=f(Y)$, where $f(Y) \in \kappa[Y]$, where $\kappa$ is the field of constants of $F$.


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