# REMARKS ON CATALAN'S EQUATION OVER FUNCTION FIELDS

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ABSTRACT. Let  $\ell$  be a prime number, F be a global function field of characteristic  $\ell$ . Assume that there is a prime  $P_{\infty}$  of degree 1. Let  $\mathcal{O}_F$  be the ring of functions in F with no poles outside of  $\{P_{\infty}\}$ . We study solutions to Catalan's equation  $X^m - Y^n = 1$  over  $\mathcal{O}_F$  and show that under certain additional conditions, there are no non-constant solutions which lie in  $\mathcal{O}_F$ , when m, n > 1.

### 1. INTRODUCTION

Let m > 1 and n > 1 be integers, and consider the diophantine equation

$$X^m - Y^n = 1.$$

The famous Catalan conjecture states that there are no non-trivial integer solutions to the above equation except when m = 2, n = 3 and  $(X, Y) = (\pm 3, 2)$ . The celebrated result of Mihăilescu resolves this conjecture using techniques from the theory of cyclotomic fields (cf. [1]). Given the close analogy between number fields and function fields, it is of interest to study analogues of Catalan's conjecture in characteristic  $\ell > 0$ . The field of rational numbers  $\mathbb{Q}$  is the simplest number field to consider, and analogously, the most natural analogue is the field of rational functions  $\mathbb{F}(T)$ , where T is a formal variable, and  $\mathbb{F}$  is a finite field. The ring of integers  $\mathbb{Z}$  is thus analogous to the ring of polynomial functions  $\mathbb{F}[T]$ , which shares similar properties to  $\mathbb{Z}$ . The reader is referred to [2, 3] for an introduction to the arithmetic of function fields, and further perspectives elaborating the close analogy between number fields and their counterparts in positive characteristic.

Let  $\ell$  be a prime number and F be a global function field of characteristic  $\ell$ . Denote by  $\mathbb{F}_{\ell}$  the finite field with  $\ell$  elements and set  $\kappa$  to denote the algebraic closure of  $\mathbb{F}_{\ell}$  in F. Note that  $\kappa$  is a finite field (by assumption). Recall (from [2, Chapter 5]) that a *prime* in F is defined to be the maximal ideal v of a discrete valuation ring R contained in F, with fraction field equal to F. The *degree* of v is defined to be the dimension of R/v over the field of constants  $\kappa$ . Each prime v comes equipped with a valuation  $\operatorname{ord}_v : F \to \mathbb{Z} \cup \{\infty\}$ . Assume that there exists a prime  $P_{\infty}$  of F which has degree 1, and let  $\mathcal{O}_F$  be the ring of functions in F with no poles outside  $\{P_{\infty}\}$ . The point  $P_{\infty}$  is referred to as the *point at infinity* and  $\mathcal{O}_F$  is the *ring of integers* of F. We say that a solution

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 $(X, Y) \in \mathcal{O}_F^2$  to  $X^m - Y^n = 1$  is *constant* if X and Y are both contained in  $\kappa$ , and *non-constant* otherwise.

Recall from *loc. cit.* that a divisor is a finite integral linear combination of primes of F. The principal divisor associated to  $g \in F$  is denoted div(g), and two divisors  $D_1$  and  $D_2$  are said to be equivalent if  $D_1 - D_2$  is a principal divisor. The group of divisors classes of degree 0 is finite (cf. [2, Lemma 5.6]), and its cardinality is the *class number of* F, and this quantity is denoted by  $h_F$ . Given a prime number  $p \neq \ell$ , let  $F(\mu_p)$  be the function field obtained by adjoining the *p*-th roots of unity  $\mu_p$  to F. Note that  $F(\mu_p) = \kappa' \cdot F$ , where  $\kappa' = \kappa(\mu_p)$ . Thus,  $F(\mu_p)$  is a constant field extension of F in the sense of [2, Chapter 8].

**Theorem 1.1.** Let F be a global function field of characteristic  $\ell > 0$ . Let p and q be prime numbers and assume that all the following conditions are satisfied

(1)  $p \neq \ell$  and  $q \neq \ell$ ,

(2) if  $p \neq q$ , then either  $q \nmid h_{F(\mu_p)}$  or  $p \nmid h_{F(\mu_q)}$ .

(3) if q = 2,  $p \neq 2$  and  $q \mid h_{F(\mu_p)}$ , then  $p \nmid h_{F(\mu_4)}$ .

Then, there are no non-constant solutions to  $X^p - Y^q = 1$  in  $\mathcal{O}_F$ . More generally, if m > 1 and n > 1 are integers such that m is divisible by a prime p and n by a prime q for which the above conditions are satisfied, then there are no non-constant solutions to  $X^m - Y^n = 1$  in  $\mathcal{O}_F$ .

The condition requiring that p and q are distinct from  $\ell$  is necessary, since if  $m = \ell$  for instance, it is easy to construct a large class of non-constant solutions if one of the primes is equal to  $\ell$  (cf. Remark 2.3 for details).

We mention some related work of relevance. Silverman [4] considered a general class of equations of the form  $aX^m + bY^n = c$  over a general function field K, and proved that under some further conditions, there are only finitely many solutions when  $a, b, c \in K^*$  are fixed. There is a mistake in the statement of Silverman's result, which has been corrected by Koymans [5]. The result of Koymans moreover applies to fields of larger dimension. The Catalan equation was studied by Nathanson [6] over K[T] and K(T) where K is a field of positive characteristic. It is shown in *loc. cit.* that if m > 1 and n > 1 are coprime to  $\ell$ then there are no solutions to Catalan's equation  $X^m - Y^n = 1$  that lie in K[T]but not in K. Specializing to the case when K is a finite field, one obtains the conclusion of Theorem 1.1 for the rational function field. This is because the class number of any rational function field is equal to 0. Theorem 1.1 can thus be viewed as a generalization of Nathanson's result to general function fields Fwith added stipulations on (m, n).

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## 2. PROOF OF THE MAIN RESULT

Recall that F is a global function field of characteristic  $\ell > 0$  with field of constants  $\kappa$ . Let  $\bar{\kappa}$  be the algebraic closure of  $\kappa$  in a fixed algebraic closure of

F, and set F' to denote the composite  $F \cdot \bar{\kappa}$ . Also, denote by A the composite  $\mathcal{O}_F \cdot \bar{\kappa}$ . The field F' is identified with the function field of a projective curve  $\mathfrak{X}$  over  $\bar{\kappa}$  and each point in  $\mathfrak{X}(\bar{\kappa})$  corresponds to a valuation ring  $R \subset F'$  with residue field  $\bar{\kappa}$  and fraction field F'. The valuation ring associated to  $w \in \mathfrak{X}(\bar{\kappa})$  is denoted  $\mathcal{O}_w$ , and we refer to w as a prime of F'. We say that w divides (or lies above) a prime v of F if there is a natural inclusion of valuation rings  $\mathcal{O}_v \hookrightarrow \mathcal{O}_w$  induced by the inclusion  $F \hookrightarrow F'$ . Note that since  $P_\infty$  has degree 1, it is totally inert in F'. In particular, there is a single prime of F' that lies above  $P_\infty$ , which we identify with  $P_\infty$ . Given any prime v of F, set  $d_v$  to denote  $-\operatorname{ord}_v$  and for any function  $g \in F$ , we refer to  $d_v(g)$  as the order of the pole of g at v. Given a prime w of F' (i.e., point  $w \in \mathfrak{X}(\bar{\kappa})$ ) and  $g \in F'$ , denote by  $d_w(g)$  the order of the pole of g at w. We set  $d : A \to \mathbb{Z}_{>0}$  to denote  $d_{P_\infty}$ .

**Lemma 2.1.** Let  $f, g \in A$  be non-zero. The following assertions hold.

- (1) d(q) = 0 if and only if q is a constant function.
- (2) We have that d(fg) = d(f) + d(g).
- (3) Suppose that d(f) < d(g). Then, d(g+f) = d(g).
- (4) We have that  $d(f) \ge 0$ , and d(f) > 0 if and only if f is non-constant.

*Proof.* The proof of parts (1) to (3) are easy, hence, omitted. For part (4) note that a non-constant function  $f \in A$  must have a pole at some point. By virtue of being contained in A, f does not have poles outside  $\{P_{\infty}\}$ . Therefore, f must have a pole at  $P_{\infty}$ , and thus, d(f) > 0. On the other hand, if f is constant, then d(f) = 0. This proves part (4).

**Lemma 2.2.** Let  $Y \in A$  and  $c_1, c_2 \in \overline{\kappa}$  be non-zero constants. If for some prime  $p \neq \ell$  we have that

$$(Y + c_1)^p - Y^p = c_2,$$

then Y is a constant.

*Proof.* Setting  $f(z) := (z + c_1)^p - z^p - c_2$ , we find that f(z) is a nonzero polynomial in z with coefficients in  $\bar{\kappa}$ . Therefore, any solution Y to the equation f(Y) = 0 must also lie in  $\bar{\kappa}$ .

Proof of Theorem 1.1. First consider the case when p = q. Note that it is assumed that  $p \neq \ell$ . We show that there are no non-constant solutions to

$$X^p - Y^p = 1$$

in A. Note that (X - Y) divides  $X^p - Y^p = 1$ , hence by Lemma 2.1,

$$d(X - Y) = d(1) - d\left(X^{p-1} + X^{p-2}Y + \dots + XY^{p-2} + Y^{p-1}\right) \le d(1) = 0.$$

It follows from Lemma 2.1 part (1) that (X - Y) is a constant  $c \in \bar{\kappa}$ . We thus deduced that

(2.1) 
$$(Y+c)^p - Y^p = 1.$$

Lemma 2.2 implies that (2.1) has no non-constant solutions. Since Y is a constant, it follows that X is as well. If X and Y are in  $\mathcal{O}_F$ , it follows therefore that  $X, Y \in \kappa$ .

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We assume therefore that p and q are distinct (and distinct from  $\ell$ ). Note that there are further conditions on p and q. First, we consider the case when  $q \nmid h_{F(\mu_p)}$ . All the variables introduced in the following argument will be contained in  $F(\mu_p)$ . Let  $\zeta$  be a primitive p-th root of 1. Since it is assumed that  $p \neq \ell$ , we note that  $\zeta \neq 1$ . In what follows we consider divisors over  $F(\mu_p)$ . Given a divisor  $D = \sum_v n_v v$  involving primes v of  $F(\mu_p)$ , the support consists of all primes v such that the coefficient  $n_v$  is not equal to 0. Factor  $X^p - 1$  into linear factors to obtain the following equation

(2.2) 
$$Y^{q} = \prod_{j=0}^{p-1} (X - \zeta^{j}).$$

For  $i \neq j$ , note that  $(X - \zeta^i) - (X - \zeta^j) = \zeta^j - \zeta^i$ , which is a non-zero element of  $\kappa(\mu_p)$ . Hence, it follows that  $\operatorname{div}(X - \zeta^i)$  and  $\operatorname{div}(X - \zeta^j)$  have disjoint supports for  $i \neq j$ . From (2.2), we have the following relation between divisors that are formal linear combinations of primes in  $F(\mu_p)$ 

$$\sum_{j=0}^{p-1} \operatorname{div}(X - \zeta^j) = q \operatorname{div}(Y).$$

The elements  $(X - \zeta^j)$  are all contained in  $F(\mu_p)$ , while Y is contained in F. Since the divisors div $(X - \zeta^j)$  have disjoint supports for  $i \neq j$ , it follows that for each *i*, there is a divisor  $D_i$  (involving linear combinations of primes in  $F(\mu_p)$ ) such that div $(X - \zeta^i) = qD_i$ . Since div $(X - \zeta^i)$  is a principal divisor, it has degree 0, and hence  $D_i$  does also have degree zero. Since  $q \nmid h_{F(\mu_p)}$ , there is no non-trivial *q* torsion in the divisor class group. As a result,  $D_i$  is a principal divisor div $(\alpha_i)$ , where  $\alpha_i \in F(\mu_p)$ . Thus, we have deduced that for all *i*,

$$X - \zeta^i = u_i \alpha_i^q,$$

where  $u_i \in F(\mu_p)$  is a non-zero function for which  $\operatorname{div}(u_i) = 0$ . Therefore  $u_i$  is a unit, and consequently, is contained in  $\kappa(\mu_p)$ . Recall that p and q are distinct, and we have shown that  $u_i \in \bar{\kappa}$ . It follows that  $u_i$  is the q-th power of an element  $v_i \in \bar{\kappa}^{\times}$ . Replacing  $\alpha_i$  with  $v_i \alpha_i$ , we write

$$X - \zeta^i = \alpha_i^q$$

where  $\alpha_i \in (F')^{\times}$ . Note that  $\alpha_i$  is contained in A since it has no poles outside  $\{P_{\infty}\}$  (since  $X - \zeta^i$  does not). We deduce that

(2.3) 
$$\alpha_0^q - \alpha_1^q = (X - 1) - (X - \zeta) = \zeta - 1.$$

It follows that  $\alpha_0 - \alpha_1$  divides  $\zeta - 1$ , hence has no zeros or poles. As a result,  $\alpha_0 - \alpha_1$  is a constant  $c \in \bar{\kappa}$ . It is clear from (2.3) that c is non-zero. Thus we find that

$$(\alpha_1 + c)^q - \alpha_1^q = \zeta - 1.$$

Lemma 2.2 then implies that  $\alpha_1$  and  $\alpha_0$  are constants. We have thus shown that X, and hence Y are both elements in  $\bar{\kappa}$ . Since  $\kappa$  is the algebraic closure of  $\mathbb{F}_{\ell}$  in F, and both X and Y are contained in F, it follows that  $X, Y \in \kappa$ .

It follows from the condition (2) of Theorem 1.1 that if  $p \neq q$ , then  $q \nmid h_{F(\mu_p)}$ or  $p \nmid h_{F(\mu_q)}$ . We have shown that there are no non-constant solutions when p = q, or when  $q \nmid h_{F(\mu_p)}$ . Throughout the rest of this proof, we shall therefore assume that  $p \nmid h_{F(\mu_q)}$ . If both p and q are odd, then we may replace X with -Y and Y with -X to obtain the equation  $X^q - Y^p = 1$ , and thus the previous argument that gives the result applies in this case. We have therefore dealt with the case when both p and q are odd, and we are left to consider the case when  $p \nmid h_{F(\mu_q)}$  and either p or q is 2.

First consider the case when p = 2. It has been shown that there no nonconstant solutions when p = q and therefore q must be odd. Moreover, as stated in the previous paragraph, we assume that  $2 \nmid h_{F(\mu_q)}$ . Then, we find that  $X^2 = Y^q + 1 = Y^q - (-1)^q = \prod_j (Y + \zeta^j)$ , where  $\zeta$  is a primitive q-th root of unity. For  $i \neq j$ , note that  $(Y + \zeta^i) - (Y + \zeta^j) = \zeta^i - \zeta^j$  is a constant, and therefore,  $\operatorname{div}(Y + \zeta^i)$  and  $\operatorname{div}(Y + \zeta^j)$  have disjoint supports for  $i \neq j$ . We thus arrive at the equation

$$\sum_{j=0}^{q-1} \operatorname{div}(Y+\zeta^j) = 2\operatorname{div}(X).$$

The divisors  $\operatorname{div}(Y + \zeta^j)$  have disjoint supports for  $i \neq j$ , and therefore, we may write  $\operatorname{div}(Y + \zeta^j) = 2D_j$  for some divisors  $D_j$  that are defined over  $F(\mu_q)$ . Recall that  $2 \nmid h_{F(\mu_q)}$ . An identical argument to the previous case implies that for all j, we have that

$$Y + \zeta^j = \beta_i^2,$$

where  $\beta_j \in A$ . We deduce that

$$\beta_0^2 - \beta_1^2 = (Y+1) - (Y+\zeta) = 1 - \zeta.$$

It follows that  $\beta_0 - \beta_1$  divides  $1 - \zeta$ . Therefore,  $\beta_0 - \beta_1$  has no zeros or poles, and hence equals a constant  $c \in \bar{\kappa}$ . Thus we find that

$$(\beta_1 + c)^2 - \beta_1^2 = 1 - \zeta.$$

Lemma 2.2 then implies that  $\beta_1 = Y + \zeta$  and  $\beta_0 = Y + 1$  are constants. From this, we deduce that both X and Y are constants.

Finally, assume that p is odd, q = 2. Note that the result has been proved when  $q \nmid h_{F(\mu_p)}$ . Therefore, we assume that  $q \mid h_{F(\mu_p)}$ . It follows from the condition (3) of Theorem 1.1 that  $p \nmid h_{F(\mu_4)}$ . We consider the equation  $X^p =$  $Y^2 + 1 = (Y + \eta)(Y - \eta)$ , where  $\eta^2 = -1$ . Note that  $F(\mu_4) = F(\eta)$ . Since p does not divide the class number of  $F(\eta)$ , we find that  $Y + \eta = \alpha_0^p$  and  $Y - \eta = \alpha_1^p$ , where  $\alpha_0, \alpha_1$  are elements in A. Therefore,  $2\eta = \alpha_0^p - \alpha_1^p$ . In particular, this implies that  $(\alpha_0 - \alpha_1)$  is a constant c. Since  $\eta \neq 0$ , it follows that  $c \neq 0$ . We have the following equation

$$(\alpha_1 + c)^p - \alpha_1^p = 2\eta.$$

The result follows from Lemma 2.2.

*Remark* 2.3. At this point, it is pertinent to make a few remarks.

• The assumptions that p and q are not equal to  $\ell$  are necessary. Indeed, suppose that  $p = \ell$ . Then, setting  $X = 1 + z^q$  and  $Y = z^p$  for any element  $z \in \mathcal{O}_F$ , one would obtain non-constant solutions.

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• The methods introduced in this paper could potentially be applied to a more general class of diophantine equations, namely, equations of the form  $X^m = f(Y)$ , where  $f(Y) \in \kappa[Y]$ , where  $\kappa$  is the field of constants of F.

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