Girth of the algebraic bipartite graph D(k,q) *

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Abstract: For integer $k \geq 2$ and prime power q, the algebraic bipartite graph D(k,q) proposed by Lazebnik and Ustimenko (1995) is meaningful not only in extremal graph theory but also in coding theory and cryptography. This graph is q-regular, edge-transitive and of girth at least k + 4. Its exact girth g = g(D(k,q)) was conjectured in 1995 to be k+5 for odd k and $q \geq 4$. This conjecture was shown to be valid in 2016 when $\frac{k+5}{2}|_p(q-1)$, where p is the characteristic of \mathbb{F}_q and $m|_p n$ means that m divides $p^r n$ for some nonnegative integer r. In this paper, for $t \geq 1$ we prove that (a) g(D(4t+2,q)) = g(D(4t+1,q)); (b) g(D(4t+3,q)) = 4t+8 if g(D(2t,q)) = 2t+4; (c) g(D(8t,q)) = 8t+4 if g(D(4t-2,q)) = 4t+2; (d) $g(D(2^{s+2}(2t-1)-5,q)) = 2^{s+2}(2t-1)$ if $p \geq 3$, $(2t-1)|_p(q-1)$ and $2^s ||(q-1)$. A simple upper bound for the girth of D(k,q) is proposed in the end of this paper.

Keywords: Bipartite graph; Edge-transitive; Backtrackless walk; Girth; Homogeneous polynomial;

1 Introduction

The graphs considered in this paper are undirected, without loops and multiple edges. A graph G is said to be *edge-transitive* provided, for any two edges e_1 , e_2 of G, that there exists an automorphism ϕ of G such that ϕ maps the ends of e_1 into those of e_2 . A *backtrackless* (or *non-recurrent*) walk of length n means a sequence v_1, v_2, \ldots, v_n of vertices of G such that any two consecutive vertices are adjacent in G and $v_j \neq v_{j+2}$ for $j = 1, 2, \ldots, n-2$. A backtrackless walk v_1, v_2, \ldots, v_n is called a *backtrackless circuit* of length

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n further if *n* is greater than 2 and $v_3, v_4, \ldots, v_n, v_1, v_2$ is still a backtrackless walk. For any graph *G* which is not a tree, its girth, denoted by g(G), is equal to the length of the shortest backtrackless circuits in *G*.

Graphs with large girth and a high degree of symmetry have been applied to variant problems in extremal graph theory, finite geometry, coding theory, cryptography, communication networks and quantum computations (c.f. [1]–[21]). In this paper, we concetrate on the algebraic bipartite graph D(k,q), proposed by Lazebnik and Ustimenko in [3], which is edge-transitive and of girth at least k + 4, where $k \geq 2$ and $q = p^m$ is a power of prime p. The graph D(k,q) has been investigated quite well in literature (c.f. [3]–[20]). For the exact girth of D(k,q), the following conjecture was proposed in [5]:

Conjecture 1. D(k,q) has girth k + 5 for all odd k and all $q \ge 4$.

This conjecture was shown to be valid in [5] for the case that (k + 5)/2 is a factor of q - 1, in [19] for the case that (k + 5)/2 is a power of p, and in [20] for the case that (k + 5)/2 is a factor of q - 1 multiplied by a power of p, respectively. For a few small k's, the girth cycles (namely, the shortest backtrackless circuits) of D(k,q) are determined completely in [22].

In this paper, we will investigate the girth of D(k,q) further by means of a compact expression of some backtrackless walks of the bipartite graph $\Lambda_{k,q}$, which is defined as follows (c.f. [19], [20] and [22]). The left part of vertices of $\Lambda_{k,q}$, denoted by L_k , is the set of (k + 1)-dimensional vectors $[l] = (l_0, l_1, l_2, \ldots, l_k)$ over \mathbb{F}_q with $l_1 = l_2$. The right part of vertices of $\Lambda_{k,q}$, denoted by R_k , is the set of (k + 1)-dimensional vectors $\langle r \rangle =$ $(r_0, r_1, r_2, \ldots, r_k)$ over \mathbb{F}_q with $r_1 = 0$. Two vertices $(l_0, l_1, \ldots, l_k) \in L_k$ and $(r_0, r_1, \ldots, r_k) \in R_k$ are adjacent in $\Lambda_{k,q}$ if and only if, for $2 \leq i \leq k$,

$$l_i + r_i = \begin{cases} r_0 l_{i-2} & \text{if } i \equiv 2, 3 \mod 4, \\ l_0 r_{i-2} & \text{if } i \equiv 0, 1 \mod 4. \end{cases}$$
(1)

Since $\Lambda_{k,q}$ is isomorphic to D(k,q) [19], $\Lambda_{k,q}$ is also edge-transitive and of girth at least k+4. All the consequent arguments will be made on the graph $\Lambda_{k,q}$ instead of the original graph D(k,q).

This paper is arranged as follows. In Section 2 we introduce a class of homogeneous polynomials in several indeterminates and a compact expression for vertices over some backtrackless walks in $\Lambda_{k,q}$. An identity on such polynomials is shown in Section 3. By using of this identity, in Section 3 we show, for any $t \ge 1$, that each backtrackless circuit in $\Lambda_{4t+1,q}$ implies a backtrackless circuit in $\Lambda_{4t+2,q}$ of the same length, and $g(\Lambda_{4t+2,q}) = g(\Lambda_{4t+1,q})$ is then deduced. In Section 4, we construct some backtrackless circuits in $\Lambda_{4t+3,q}$ by using those in $\Lambda_{2t,q}$, and show for $n \geq 3$ that $g(\Lambda_{4t+3,q}) \leq 4n$ if $g(\Lambda_{2t,q}) \leq 2n$. A few results on the exact girth of $\Lambda_{k,q}$ are also given in this section. In Section 5, we deduce an upper bound for the girth of $\Lambda_{k,q}$ by combining a known result from [20] and the new results shown in Section 4. Some concluding remarks are given in Section 6.

2 Backtrackless Walks in $\Lambda_{k,q}$

At first, we introduce a class of homogeneous polynomials in several indeterminates which were defined in [19]. For indeterminates $\omega_1, \ldots, \omega_n$ whose values are usually limited to the set \mathbb{F}_q^* , let

$$\rho_0(\omega_1,\ldots,\omega_n)=\omega_1\cdots\omega_n$$

and, for $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, let $\rho_s(\omega_1, \ldots, \omega_n)$ denote the homogeneous polynomial of order n - 2s defined by

$$\rho_s(\omega_1,\ldots,\omega_n) = \sum_{1 \le i_1 < \cdots < i_s \le n-s} \frac{\prod_{j=1}^n \omega_j}{\prod_{j=1}^s \omega_{i_j+j-1} \omega_{i_j+j}},$$

where each term in the summation is a product of the remaining elements in the sequence $\omega_1, \ldots, \omega_n$ after deleting from it *s* disjoint pairs $\{\omega_i, \omega_{i+1}\}$ of consecutive elements. If n < 2s or s < 0, $\rho_s(\omega_1, \ldots, \omega_n)$ is defined as 0. For the null sequence η , we define $\rho_s(\eta)$ as

$$\rho_s(\eta) = \begin{cases} 1 & \text{if } s = 0\\ 0 & \text{if } s \neq 0 \end{cases}$$

One can show easily (c.f. [19], [20] and [22])

$$\rho_s(\omega_1,\ldots,\omega_n) = \rho_{s-1}(\omega_1,\ldots,\omega_{n-2}) + \omega_n \rho_s(\omega_1,\ldots,\omega_{n-1}), \qquad (2)$$

and, for $0 \leq j \leq n$,

$$\rho_{n-j}(\omega_1, \dots, \omega_{2n}) = \sum_{1 \le s_1 \le t_1 < s_2 \le t_2 < \dots < s_j \le t_j \le n} \prod_{k=1}^j \omega_{2s_k - 1} \omega_{2t_k}, \qquad (3)$$

$$\rho_{n-j}(\omega_1, \dots, \omega_{2n+1}) = \sum_{s=j}^n \rho_{s-j}(\omega_1, \dots, \omega_{2s})\omega_{2s+1}$$
$$= \sum_{1 \le s_0 \le t_1 < s_1 \le t_2 < s_2 \le \dots \le t_j < s_j \le n+1} \omega_{2s_0-1} \prod_{k=1}^j \omega_{2t_k} \omega_{2s_k-1}.$$
(4)

For $a, b \in \mathbb{F}_q^*$, let $\omega'_{2s-1} = a\omega_{2s-1}$ and $\omega'_{2s} = b\omega_{2s}$ for $s = 1, 2, \ldots$, then from (3) and (4) we see easily

$$\rho_{n-j}(\omega'_1,\ldots,\omega'_{2n}) = a^j b^j \rho_{n-j}(\omega_1,\ldots,\omega_{2n}), \tag{5}$$

$$\rho_{n-j}(\omega'_1, \dots, \omega'_{2n+1}) = a^{j+1} b^j \rho_{n-j}(\omega_1, \dots, \omega_{2n+1}).$$
(6)

Since the graph $\Lambda_{k,q}$ is edge-transitive, without loss of generality, we will concentrate on the backtrackless walks which are leading by the two vertices expressed by the all-zero vector. Let $\Gamma = [l^{(1)}]\langle r^{(1)}\rangle[l^{(2)}]\langle r^{(2)}\rangle\cdots$ be such a given backtrackless walk of $\Lambda_{k,q}$, where $[l^{(1)}] = (0, 0, \ldots, 0)$ and $\langle r^{(1)}\rangle = (0, 0, \ldots, 0)$. For $i \geq 1$, let x_i and y_i denote the first entries (or colors) of $[l^{(i)}]$ and $\langle r^{(i)}\rangle$, respectively, and write

$$u_i = x_{i+1} - x_i, \qquad v_i = y_{i+1} - y_i.$$
 (7)

Clearly, we have $u_i \neq 0$ and $v_i \neq 0$. As a refinement of a closed-form expression given in [19] for the backtrackless walks leading by $[l^{(1)}] = (0, 0, ..., 0)$, the following lemma was shown in [22].

Lemma 1. For any $i \ge 1$ and $j \ge 0$, let $l_j^{(i+1)}$ denote the (j+1)-th entries of $[l^{(i+1)}]$. Then, we have

$$l_{4j}^{(i+1)} = \rho_{i-j-1}(u_1, v_1, \dots, u_{i-1}, v_{i-1}, u_i),$$

$$(8)$$

$$l_{4j+1}^{(i+1)} = \rho_{i-j-2}(v_1, u_2, \dots, v_{i-1}, u_i), \tag{9}$$

$$l_{4j+2}^{(i+1)} = y_{i+1}l_{4j}^{(i+1)} - \rho_{i-j-1}(u_1, v_1, \dots, u_i, v_i),$$
(10)

$$l_{4j+3}^{(i+1)} = y_{i+1}l_{4j+1}^{(i+1)} - \rho_{i-j-2}(v_1, u_2 \dots, v_{i-1}, u_i, v_i).$$
(11)

This lemma shows a compact expression for the vertices on the walk Γ . For convenience, we say the walk Γ is of type $(u_1, v_1, u_2, v_2, \ldots)$. If the first 2i vertices in the walk Γ form a circuit in $\Lambda_{k,q}$ of length 2i, we also call it a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$.

We note that

$$\rho_{i-1}(v_1, u_2, \dots, v_{i-1}, u_i, v_i) = v_1 + \dots + v_i = y_{i+1} = y_1 = 0$$
(12)

is always a necessary condition for $\Lambda_{k,q}$ to have a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$.

3 Backtrackless Circuits in $\Lambda_{4t+2,q}$

In this section we show an identity on the homogeneous polynomials introduced in Section 2 at first. By using this identity, we show then that each backtrackless circuit in $\Lambda_{4t+1,q}$ ensures the existence of a backtrackless circuit of the same type in $\Lambda_{4t+2,q}$ and deduce $g(\Lambda_{4t+2,q}) = g(\Lambda_{4t+1,q})$ for any $t \ge 1$ in final.

Lemma 2. For any integers n, t with $n \ge 1$, let

$$\Delta_{2t-1}^{n} = \rho_{n-t}(v_1, u_2, \dots, v_{n-1}, u_n, v_n), \tag{13}$$

$$\nabla_{2t-1}^{n} = \rho_{n-t}(u_1, v_1, \dots, u_{n-1}, v_{n-1}, u_n), \tag{14}$$

$$\Delta_{2t}^n = \rho_{n-1-t}(v_1, u_2, \dots, v_{n-1}, u_n), \tag{15}$$

$$\nabla_{2t}^{n} = \rho_{n-t}(u_1, v_1, \dots, u_n, v_n).$$
(16)

Then, for $n \ge 1$ we have

$$\sum_{s} (-1)^{s} \nabla_{s}^{n} \Delta_{2j-s}^{n} = 0, \text{ for } j \ge 1.$$
(17)

Proof. From $\Delta_0^1 = \nabla_0^1 = 1$, $\Delta_1^1 = v_1$, $\nabla_1^1 = u_1$, $\nabla_2^1 = u_1v_1$, $\Delta_s^1 = \nabla_t^1 = 0$ for any $s \notin \{0,1\}$ and $t \notin \{0,1,2\}$, it can be checked easily that (17) is valid for n = 1.

To show that (17) is valid for n > 1, we note that for any integer t according to (2) we have

$$\begin{split} \Delta_{2t-1}^{n} = &\rho_{n-t-1}(v_{1}, u_{2}, \dots, v_{n-2}, u_{n-1}, v_{n-1}) \\ &+ v_{n}\rho_{n-t-1}(v_{1}, u_{2}, \dots, v_{n-2}, u_{n-1}) \\ &+ v_{n}u_{n}\rho_{n-t}(v_{1}, u_{2}, \dots, v_{n-2}, u_{n-1}, v_{n-1}) \\ &= &\Delta_{2t-1}^{n-1} + v_{n}\Delta_{2t-2}^{n-1} + v_{n}u_{n}\Delta_{2t-3}^{n-1}, \\ \nabla_{2t-1}^{n} = &\rho_{n-t-1}(u_{1}, v_{1}, \dots, u_{n-2}, v_{n-2}, u_{n-1}) \\ &+ u_{n}\rho_{n-t}(u_{1}, v_{1}, \dots, u_{n-1}, v_{n-1}) \\ &= &\nabla_{2t-1}^{n-1} + u_{n}\nabla_{2t-2}^{n-1}, \\ \Delta_{2t}^{n} = &\rho_{n-2-t}(v_{1}, u_{2}, \dots, v_{n-2}, u_{n-1}) \\ &+ u_{n}\rho_{n-1-t}(v_{1}, u_{2}, \dots, v_{n-2}, u_{n-1}, v_{n-1}) \\ &= &\Delta_{2t}^{n-1} + u_{n}\Delta_{2t-1}^{n-1}, \\ \nabla_{2t}^{n} = &\rho_{n-t-1}(u_{1}, v_{1}, \dots, u_{n-1}, v_{n-1}), \\ &+ v_{n}\rho_{n-t-1}(u_{1}, v_{1}, \dots, u_{n-2}, v_{n-2}, u_{n-1}) \\ &+ v_{n}u_{n}\rho_{n-t}(u_{1}, v_{1}, \dots, u_{n-1}, v_{n-1}) \\ &= &\nabla_{2t}^{n-1} + v_{n}\nabla_{2t-1}^{n-1} + v_{n}u_{n}\nabla_{2t-2}^{n-1}, \end{split}$$

then we get

$$\begin{split} &\sum_{s} (-1)^{s} \nabla_{s}^{n} \Delta_{2j-s}^{n} \\ &= \sum_{t} (\nabla_{2t}^{n-1} + v_{n} \nabla_{2t-1}^{n-1} + v_{n} u_{n} \nabla_{2t-2}^{n-1}) (\Delta_{2j-2t}^{n-1} + u_{n} \Delta_{2j-2t-1}^{n-1}) \\ &- \sum_{t} (\nabla_{2t-1}^{n-1} + u_{n} \nabla_{2t-2}^{n-1}) (\Delta_{2j-2t+1}^{n-1} + v_{n} \Delta_{2j-2t}^{n-1} + v_{n} u_{n} \Delta_{2j-2t-1}^{n-1}) \\ &= \sum_{t} (\nabla_{2t}^{n-1} \Delta_{2j-2t}^{n-1} - \nabla_{2t-1}^{n-1} \Delta_{2j-2t+1}^{n-1}) \\ &+ u_{n} \sum_{t} (\nabla_{2t}^{n-1} \Delta_{2j-2t-1}^{n-1} - \nabla_{2t-2}^{n-1} \Delta_{2j-2t+1}^{n-1}) \\ &= \sum_{s} (-1)^{s} \nabla_{s}^{n-1} \Delta_{2j-s}^{n-1}. \end{split}$$

Therefore, one can show easily by induction that (17) is valid for any positive integer n.

The following theorem is then a simple corollary of Lemmas 1 and 2.

Theorem 1. For $t \geq 1$, $\Lambda_{4t+2,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$ if and only if $\Lambda_{4t+1,q}$ has a backtrackless circuit of the same type. In particular, we have $g(\Lambda_{4t+2,q}) = g(\Lambda_{4t+1,q})$ for $t \geq 1$.

Proof. Assume that there is a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$ in $\Lambda_{4t+1,q}$, that is,

$$v_1 + \dots + v_i = y_{i+1} = y_1 = 0$$

and $l_k^{(i+1)} = 0$ for $0 \le k \le 4t + 1$. According to Lemma 1, by using the notations defined in Lemma 2 we have $\Delta_{2t+2}^i = 0$ and $\Delta_k^i = \nabla_k^i = 0$ for $1 \le k \le 2t + 1$. Therefore, from $\Delta_0^i = 1$ and Lemma 2 we see

$$l_{4t+2}^{(i+1)} = y_{i+1}l_{4t}^{(i+1)} - \nabla_{2t+2}^i = \sum_{s=0}^{2t+1} (-1)^s \nabla_s^i \Delta_{2t+2-s}^i = 0,$$

and then $\Lambda_{4t+2,q}$ also has a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$. On the other hand, we note that $\Lambda_{4t+2,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_i, v_i)$ implies naturally that $\Lambda_{4t+1,q}$ has a backtrackless circuit of the same type. The proof is complete.

4 Backtrackless Circuits in $\Lambda_{4s+3,q}$

All the arguments given in this section will be based on the existence of backtrackless circuits of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ with

$$v_{2j-1} = -v_{2j} = 1, \ j = 1, 2, \dots, n,$$
(18)

in the graph $\Lambda_{k,q}$. To show the existence of such circuits, we deduce some equalities on the homogeneous polynomials $\rho_s(\cdot,\ldots,\cdot)$ at first.

Lemma 3. For any integer t and tuple $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ over \mathbb{F}_q^* with (18), we have

$$\rho_{2n-2t}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n}) = (-1)^{t-1} \rho_{n-t}(u_1, \dots, u_{2n}) + (-1)^t \rho_{n-t-1}(u_2, \dots, u_{2n-1}),$$
(19)

$$\rho_{2n+1-2t}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n})$$

$$=(-1)^{t-1}\rho_{n-t}(u_2,\ldots,u_{2n})+(-1)^{t-1}\rho_{n-t}(u_1,\ldots,u_{2n-1}),$$
(20)

$$\rho_{2n-1-2t}(v_1, u_2, \dots, v_{2n-1}, u_{2n}) = (-1) \rho_{n-t-1}(u_2, \dots, u_{2n-1}), \tag{21}$$

$$\rho_{2n-2t}(v_1, u_2, \dots, v_{2n-1}, u_{2n}) = (-1) \quad \rho_{n-t}(u_2, \dots, u_{2n}), \tag{22}$$

$$\rho_{2n-2t}(u_1, v_1, \dots, u_{2n}, v_{2n}) = (-1)^t \rho_{n-t}(u_1, \dots, u_{2n}),$$
(23)

$$\rho_{2n+1-2t}(u_1, v_1, \dots, u_{2n}, v_{2n}) = (-1)^t \rho_{n-t}(u_2, \dots, u_{2n}), \tag{24}$$

$$\rho_{2n-2t}(v_1, u_2, \dots, v_{2n-1}, u_{2n}, v_{2n}) = (-1)^{\iota} \rho_{n-t}(u_2, \dots, u_{2n}),$$
(25)

$$\rho_{2n+1-2t}(v_1, u_2, \dots, v_{2n-1}, u_{2n}, v_{2n}) = 0.$$
(26)

Proof. We give a proof only for the equality (19). The others can be proved similarly.

It is obvious that (19) is valid if $t \leq 0$ or $t \geq n$. Since for $1 \leq s \leq n$ we

have $y_{2s-1} = 0$ and $y_{2s} = 1$, according to (3) and (4), for $1 \le t < n$, we have

$$\begin{split} \rho_{2n-2t}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n}) \\ &= \sum_{1 \le j_1 \le j_2 < j_3 \le \dots \le j_{4t-2} < j_{4t-1} \le 2n} u_{j_1} \prod_{s=1}^{2t-1} v_{j_{2s}} u_{j_{2s+1}} \\ &= \sum_{1 \le j_1 < j_3 < \dots < j_{4t-1} \le 2n} u_{j_1} \prod_{s=1}^{2t-1} u_{j_{2s+1}} \sum_{j_{2s-1} \le l < j_{2s+1}} v_{j_{2s}} \\ &= \sum_{1 \le j_1 < j_3 < \dots < j_{4t-1} \le 2n} u_{j_1} \prod_{s=1}^{2t-1} u_{j_{2s+1}} (y_{j_{2s+1}} - y_{j_{2s-1}}) \\ &= (-1)^{t-1} \sum_{1 \le i_1 \le i_2 < i_3 \le i_4 < \dots < i_{2t-1} \le i_{2t} \le n} \prod_{s=1}^t u_{2i_{2s-1}-1} u_{2i_{2s}} \\ &+ (-1)^t \sum_{1 \le i_1 < i_2 \le i_3 < i_4 \le \dots < i_{2t-1} < i_{2t} \le n} \prod_{s=1}^t u_{2i_{2s-1}} u_{2i_{2s-1}} \\ &= (-1)^{t-1} \rho_{n-t}(u_1, \dots, u_{2n}) + (-1)^t \rho_{n-t-1}(u_2, \dots, u_{2n-1}), \end{split}$$

i.e. (19) is valid for $1 \le t < n$.

The following lemma shows that, from any backtrackless circuit of length 2n in $\Lambda_{2s,q}$, one can construct a backtrackless circuit of length 4n in $\Lambda_{4s+3,q}$, for any $s \geq 1$ and $n \geq 3$.

Lemma 4. Assume that $s \geq 1$ and $n \geq 3$. The graph $\Lambda_{4s+3,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ with (18) if and only if $\Lambda_{2s,q}$ has a backtrackless circuit of length 2n.

Proof. Assume that $v_1, \ldots, v_{2n} \in \mathbb{F}_q^*$ satisfy (18). According to Lemma 1, the graph $\Lambda_{4s+3,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ if and only if

$$\begin{cases} \rho_{2n-j-2}(v_1, u_2, \dots, v_{2n-1}, u_{2n}, v_{2n}) = 0, \\ \rho_{2n-j-1}(u_1, v_1, \dots, u_{2n}, v_{2n}) = 0, \\ \rho_{2n-j-2}(v_1, u_2, \dots, v_{2n-1}, u_{2n}) = 0, \\ \rho_{2n-j-1}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n}) = 0, \end{cases}$$
 for $0 \le j \le s.$ (27)

If s = 2w is even, according to Lemma 3 we see that (27) is equivalent

to $\rho_{n-1}(u_1, \dots, u_{2n-1}) = \rho_{n-1}(u_2, \dots, u_{2n}) = 0$ and

$$\begin{cases} \rho_{n-j-1}(u_1, \dots, u_{2n-1}) = 0, \\ \rho_{n-j-1}(u_2, \dots, u_{2n}) = 0, \\ \rho_{n-j}(u_1, \dots, u_{2n}) = 0, \\ \rho_{n-j-1}(u_2, \dots, u_{2n-1}) = 0, \end{cases} \quad \text{for } 1 \le j \le w, \tag{28}$$

that is, the graph $\Lambda_{4w,q}$ has a backtrackless circuit of type (u_1, \ldots, u_{2n}) on account to Lemma 1.

If s = 2w - 1 is odd, according to Lemma 3 we see that (27) is equivalent to

$$\rho_{n-j}(u_1, \dots, u_{2n}) = 0,
\rho_{n-j-1}(u_2, \dots, u_{2n-1}) = 0,
\rho_{n-j}(u_1, \dots, u_{2n-1}) = 0,
\rho_{n-j}(u_2, \dots, u_{2n}) = 0,$$
for $1 \le j \le w.$ (29)

that is, the graph $\Lambda_{4w-2,q}$ has a backtrackless circuit of type (u_1, \ldots, u_{2n}) on account to Lemma 1.

Based on the existence of backtrackless circuits of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ with (18), on the girth of $\Lambda_{k,q}$ one can show the following theorem by using Lemmas 1, 3 and 4.

Theorem 2. Assume $n \ge 3$ and $s, w \ge 1$.

- 1. If $g(\Lambda_{2s,q}) \le 2n$, then $g(\Lambda_{4s+3,q}) \le 4n$.
- 2. If $g(\Lambda_{2n-4,q}) = 2n$, then $g(\Lambda_{4n-5,q}) = 4n$.
- 3. If $g(\Lambda_{4w-2,q}) = 4w + 2$, then $g(\Lambda_{8w,q}) = 8w + 4$.
- 4. If q is a power of 2 and $g(\Lambda_{4w,q}) \leq 2n$, then $g(\Lambda_{8w+4,q}) \leq 4n$.

Proof. The first statement follows simply from Lemma 4. Furthermore, the second statement follows immediately on account to $g(\Lambda_{4n-5,q}) \ge 4n$.

To show the third statement, we set n = 2w + 1 and assume that the graph $\Lambda_{4w-2,q}$ has a backtrackless circuit of type (u_1, \ldots, u_{2n}) . Then, according to Lemma 1 we have (29). If $\rho_{n-w-1}(u_2, \ldots, u_{2n}) = 0$, then $\Lambda_{4w-1,q}$ has a backtrackless circuit of type (u_1, \ldots, u_{2n}) , contradicts $g(\Lambda_{4w-1,q}) \geq$

4w + 4 > 2n. If $\rho_{n-w-1}(u_1, \ldots, u_{2n-1}) = 0$, then $\Lambda_{4w-1,q}$ has a backtrackless circuit of type (u_{2n}, \ldots, u_1) , contradicts $g(\Lambda_{4w-1,q}) \ge 4w + 4 > 2n$ too. Hence, we have $\rho_{n-w-1}(u_2, \ldots, u_{2n})\rho_{n-w-1}(u_1, \ldots, u_{2n-1}) \neq 0$. Let

$$\alpha = -\rho_{n-w-1}(u_2, \dots, u_{2n})/\rho_{n-w-1}(u_1, \dots, u_{2n-1}).$$

We multiply the entries with odd indices in the tuple (u_1, \ldots, u_{2n}) by α , and denote the resulting tuple by the same notation. Then, according to (5) and (6) one can check easily that the modified tuple (u_1, \ldots, u_{2n}) satisfies (29) and

$$\rho_{n-w-1}(u_2,\ldots,u_{2n}) + \rho_{n-w-1}(u_1,\ldots,u_{2n-1}) = 0.$$

Hence, according to Lemma 3 we have

$$\rho_{2n-2w-1}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n}) = 0$$

and (27) with s = 2w - 1, where the tuple (v_1, \ldots, v_{2n}) satisfies (18). Therefore, according to Lemma 1 we see $\Lambda_{8w,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ and thus we have $g(\Lambda_{8w,q}) \leq 4n = 8w + 4$. Hence, from $g(\Lambda_{8w,q}) \geq 8w + 4$ we see $g(\Lambda_{8w,q}) = 8w + 4$.

To show the last statement, we assume that q is a power of 2 and that the graph $\Lambda_{4w,q}$ has a backtrackless circuit of type (u_1, \ldots, u_{2n}) . Then, according to Lemma 1 we have

$$\rho_{n-1}(u_1,\ldots,u_{2n-1}) = \rho_{n-1}(u_2,\ldots,u_{2n}) = 0$$

and (28), that is, $\nabla_t^n = \Delta_t^n = 0$ holds for $1 \le t \le 2w + 1$ when we modify accordingly the definition of the notations ∇_t^n , Δ_t^n . Therefore, according to Lemma 2 and that the characteristic of \mathbb{F}_q is 2, we see

$$\rho_{n-w-2}(u_2, \dots, u_{2n-1}) - \rho_{n-w-1}(u_1, \dots, u_{2n})$$

= $\Delta_{2w+2}^n - \nabla_{2w+2}^n$
= $2\Delta_{2w+2}^n + \sum_{s=1}^{2w+1} (-1)^s \nabla_s^n \Delta_{2w+2-s}^n = 0.$

Hence, according to Lemma 3 we have

$$\rho_{2n-2w-2}(u_1, v_1, \dots, u_{2n-1}, v_{2n-1}, u_{2n}) = 0$$

and (27) with s = 2w, where the tuple (v_1, \ldots, v_{2n}) satisfies (18). Therefore, according to Lemma 1 we see $\Lambda_{8w+4,q}$ has a backtrackless circuit of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ and thus we have $g(\Lambda_{8w+4,q}) \leq 4n$.

- **Example 1.** For $k \ge 2$, the girth of $\Lambda_{k,2}$ has been determined [19]: $g(\Lambda_{k,2}) = 2^s$, where s is the integer with $2^{s-1} 4 < k \le 2^s 4$.
 - Suppose $q \geq 3$. According to Theorem 2, from $g(\Lambda_{2,q}) = 6$ [19] we see $g(\Lambda_{7,q}) = g(\Lambda_{8,q}) = 12$, $g(\Lambda_{18,q}) \leq g(\Lambda_{19,q}) = 24$, $g(\Lambda_{38,q}) \leq g(\Lambda_{39,q}) \leq 48$, $g(\Lambda_{78,q}) \leq g(\Lambda_{79,q}) \leq 96$ and $g(\Lambda_{159,q}) \leq 192$.
 - Suppose q > 3. According to Theorem 2, from $g(\Lambda_{4,q}) = 8$ [22] we see $g(\Lambda_{11,q}) = 16$. According to Theorem 1, from $g(\Lambda_{5,q}) = 10$ [22] we see $g(\Lambda_{6,q}) = 10$ and then, according to Theorem 2, we have $g(\Lambda_{15,q}) = g(\Lambda_{16,q}) = 20$ and $g(\Lambda_{35,q}) = 40$.

When the characteristic of \mathbb{F}_q is 2, one can deduce further the following corollary easily.

Corollary 1. Assume $g(\Lambda_{2s,q}) = 2s + 4$, where q is a power of 2 and $s \ge 1$. Then, for any $t \ge 1$ we have

$$g(\Lambda_{2^t(s+2)-4,q}) = g(\Lambda_{2^t(s+2)-5,q}) = 2^t(s+2),$$

where $\Lambda_{1,q}$ is defined as a graph isomorphic to $\Lambda_{2,q}$ for convenience.

Proof. From $g(\Lambda_{k,q}) \ge k+4$ and Theorem 2, we see easily that $g(\Lambda_{2^t(s+2)-4,q}) = 2^t(s+2)$ is valid for any $t \ge 1$. Furthermore, $g(\Lambda_{2^t(s+2)-5,q}) = 2^t(s+2)$ follows from $2^t(s+2) \le g(\Lambda_{2^t(s+2)-5,q}) \le g(\Lambda_{2^t(s+2)-4,q})$.

Example 2. Assume that $q \ge 4$ is a power of 2. According to Corollary 1, we have the following three statements.

- From $g(\Lambda_{2,q}) = 6$, we see $g(\Lambda_{2^t 3 4,q}) = g(\Lambda_{2^t 3 5,q}) = 2^t 3$ for $t \ge 1$.
- From $g(\Lambda_{4,q}) = 8$, we see $g(\Lambda_{2^{t+2}-4,q}) = g(\Lambda_{2^{t+2}-5,q}) = 2^{t+2}$ for $t \ge 1$.
- From $g(\Lambda_{6,q}) = 10$, we see $g(\Lambda_{2^t 5 4,q}) = g(\Lambda_{2^t 5 5,q}) = 2^t 5$ for $t \ge 1$.

For prime p, we write $m|_p n$ if $m|(np^r)$ for some $r \ge 0$. The following lemma is from [20].

Lemma 5. For $q = p^s$ and $t \ge 1$ with $(t+2)|_p(q-1)$,

$$g(\Lambda_{2t-1,q}) = g(\Lambda_{2t,q}) = 2t + 4.$$
(30)

The following theorem follows simply from Theorem 2 and Lemma 5.

Theorem 3. Assume that q is a power of odd prime p and s,t are positive integers with $(2t-1)|_p(q-1)$ and $2^s||(q-1)$. Then, Conjecture 1 is valid when $k = 2^{s+2}(2t-1) - 5$, i.e.

$$g(\Lambda_{2^{s+2}(2t-1)-5,q}) = 2^{s+2}(2t-1).$$
(31)

Proof. Clearly, we have $2^{s}(2t-1)|_{p}(q-1)$. Hence, from Lemma 5 we see

$$g(\Lambda_{2^{s+1}(2t-1)-5,q}) = g(\Lambda_{2^{s+1}(2t-1)-4,q}) = 2^{s+1}(2t-1),$$

therefore, from Theorem 2 we see (31).

We note that the result shown in this theorem is not included by Lemma 5. At the end of this section, we investigate the girth of $\Lambda_{k,3}$ for small k.

- **Example 3.** The positive integer t's satisfying $(t + 2)|_3(3 1)$ are 1,4,7,16,25,52,79,160,.... Then, according to Lemma 5 we have $g(\Lambda_{2,3}) = 6$, $g(\Lambda_{7,3}) = g(\Lambda_{8,3}) = 12$, $g(\Lambda_{13,3}) = g(\Lambda_{14,3}) = 18$, $g(\Lambda_{31,3}) = g(\Lambda_{32,3}) = 36$, $g(\Lambda_{49,3}) = g(\Lambda_{50,3}) = 54$, $g(\Lambda_{103,3}) =$ $g(\Lambda_{104,3}) = 108$, $g(\Lambda_{157,3}) = g(\Lambda_{158,3}) = 162$, $g(\Lambda_{319,3}) = g(\Lambda_{320,3}) =$ 324,
 - From $g(\Lambda_{32,3}) = 36$, according to Theorem 2 we see $g(\Lambda_{66,3}) \leq g(\Lambda_{67,3}) = 72$, $g(\Lambda_{134,3}) \leq g(\Lambda_{135,3}) \leq 144$ and $g(\Lambda_{271,3}) \leq 288$.
 - From $g(\Lambda_{104,3}) = 108$, according to Theorem 2 we see $g(\Lambda_{211,3}) = 216$.

The known results on the girth of $\Lambda_{k,3}$ for $2 \le k \le 320$ are summarized in the following table.

$\underline{\qquad}$										
2	3	4	5	6	7	8	13	14	19^{*}	
6	8	12	12	12	12	12	18	18	24	
31	32	39*	49	50	67*	79*	103	104	135^{*}	
36	36	≤ 48	54	54	72	≤ 96	108	108	≤ 144	
157	158	159*	211*	271*	319	320				
162	162	≤ 192	216	≤ 288	324	324				

Table 1: Girth of $\Lambda_{k,3}$ for $2 \le k \le 320$.

In this table, the mark * indicates the exact values or upper bounds of $g(\Lambda_{k,3})$ are obtained by the methods proposed in this paper. We note that the girth cycles of $\Lambda_{k,3}$ were determined in [22] for $3 \le k \le 8$. In particular, the results $g(\Lambda_{3,3}) = 8$ and $g(\Lambda_{k,3}) = 12$ for $4 \le k \le 8$ can be found therein.

5 Upper Bound of $g(\Lambda_{k,q})$

In this section, we manage to deduce an upper bound for the girth of $\Lambda_{k,q}$ for $q \geq 3$.

Assume that $q \geq 3$ is a given prime power and the number of positive factors of q-1 is n. Let k_1, k_2, \ldots be the odd integers in ascending order with $\frac{k_i+5}{2}|_p(q-1)$, where p is the characteristic of \mathbb{F}_q . Let i_0 be the integer with $k_{i_0} = 2q-5$.

Lemma 6. For any $i \ge i_0$

$$k_{i+n} = pk_i + 5p - 5. ag{32}$$

Proof. Suppose $q = p^m$. For $0 \le j \le m - 1$, let $d_{j,1}, \ldots, d_{j,t_j}$ denote the different factors of q - 1 with $p^j \le d_{j,t} < p^{j+1}$ for $t = 1, \ldots, t_j$. Then, we have $\sum_{j=0}^{m-1} t_j = n$ and for any $s \ge 0$ from $(p, d_{j,t}) = 1$ we see

$$\{k_{i_0+sn+i} | 0 \le i < n\} = \bigcup_{0 \le j < m} \{ 2d_{j,t} p^{s+m-j} - 5 | 1 \le t \le t_j \},\$$

which implies (32).

Let

$$T_q = \max_{i \ge i_0} \frac{k_{i+1} + 5}{k_i + 5}.$$
(33)

From Lemma 6 we see that T_q can also be given by

$$T_q = \max_{i_0 \le i < i_0 + n} \frac{k_{i+1} + 5}{k_i + 5}.$$
(34)

Clearly, $1 < T_q < p$, and for any $i \ge i_0$ we have $\frac{k_{i+1}+5}{2} \le T_q \frac{k_i+5}{2}$, i.e.

$$k_{i+1} \le T_q k_i + 5T_q - 5. \tag{35}$$

Example 4. If $q = 5^2$, then the positive factors of $5^2 - 1 = 24$ are 1, 2, 3, 4, 6, 8, 12, 24 and the positive integers t with $t|_524$ are

 $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 25, 30, 40, 50, 60, 75, 100, 120, 125, \ldots$

Hence, $T_q = \max\{6/5, 8/6, 10/8, 12/10, 15/12, 20/15, 24/20, 25/24\} = 4/3.$

From Lemma 5 we see $g(\Lambda_{k_i-1,q}) \leq g(\Lambda_{k_i,q}) = k_i + 5$, hence from Theorem 2 we see

$$g(\Lambda_{2(k_i-1)+2,q}) \le g(\Lambda_{2(k_i-1)+3,q}) \le 2(k_i+5)$$

and by induction we have

$$g(\Lambda_{2^s(k_i+1)-2,q}) \le g(\Lambda_{2^s(k_i+1)-1,q}) \le 2^s(k_i+5), \text{ for any } s \ge 0.$$
(36)

Theorem 4. Let q be a prime power.

1. If $T_q \leq 2$, then for $k \geq q$ we have

$$g(\Lambda_{k,q}) \le T_q(k+4). \tag{37}$$

2. If $T_q > 2$ and $k \ge \max\{q, 8T_q^2 - 10T_q - 3\}$, then we have

$$g(\Lambda_{k,q}) \le 2k + 4T_q + 1. \tag{38}$$

Proof. Without loss of generality, we assume $k_i < k < k_{i+1}$ for some $i \ge i_0$. If $T_q \le 2$, then from (35) and Lemma 5 we see

$$g(\Lambda_{k,q}) \le g(\Lambda_{k_{i+1},q}) = k_{i+1} + 5 \le T_q(k_i + 5) \le T_q(k+4),$$

i.e. (37) is valid.

Now we assume $T_q > 2$ and $k \ge 8T_q^2 - 10T_q - 3$. If $\frac{k_{i+1}}{2} < k \le k_{i+1}$, then from Lemma 5 we have

$$g(\Lambda_{k,q}) \le g(\Lambda_{k_{i+1},q}) = k_{i+1} + 5 \le 2k + 4.$$
(39)

If $k_i < k < \frac{k_{i+1}}{2}$, for the integer s with

$$2^{s}(k_{i}+1) \le k < 2^{s+1}(k_{i}+1) \tag{40}$$

from (35) we see

$$2^{s}(k_{i}+1) \le k \le \frac{k_{i+1}-1}{2} \le \frac{T_{q}k_{i}+5T_{q}-6}{2}$$

and then from (36) and (40) we have

$$g(\Lambda_{k,q}) \leq g(\Lambda_{2^{s+1}(k_i+1)-1,q}) \\ \leq 2^{s+1}(k_i+5) \\ \leq 2k+2^{s+3} \\ \leq 2k+4T_q + \frac{16T_q - 24}{k_i+1}.$$
(41)

From (35) and $k \ge 8T_q^2 - 10T_q - 3$ we see also

$$T_q k_i + 5T_q - 5 \ge k_{i+1} \ge 2k + 1 \ge 16T_q^2 - 20T_q - 5$$

and thus we have $(16T_q - 24)/(k_i + 1) \le 1$. Therefore, from $T_q > 1$, (39) and (41) we see that (38) is valid.

6 Concluding Remarks

Conjecture 1 was shown to be valid in [5] for the case $\frac{k+5}{2}|(q-1)$ based on the existence of a special automorphism of D(k,q), in [19] for the case $\frac{k+5}{2}$ is a power of p based on the existence of backtrackless circuit of type $(1, 1, \ldots, 1, 1)$, and in [19] for the case $\frac{k+5}{2}|_p(q-1)$ based on the existence of backtrackless circuit of type $(1, 1, b, b, \ldots, b^n, b^n)$, respectively, where pis the characteristic of \mathbb{F}_q . A few new results on the girth of D(k,q) are obtained in the present paper based on the existence of backtrackless circuit of type $(u_1, v_1, \ldots, u_{2n}, v_{2n})$ with (18). For example, Conjecture 1 is shown to be valid in Theorem 3 for a new class of infinite pairs (k,q): $p \ge 3$, $k = 2^{s+2}(2t-1)-5$ for positive integers s, t with $2^s ||(q-1)$ and $(2t-1)|_p(q-1)$. Almost the recent progresses made on the study of the girth of D(k,q) rely heavily on the computation of the homogeneous polynomial $\rho_s(\omega_1, \ldots, \omega_n)$. It is then of great interest to investigate the properties of these polynomials further in future.

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