# Girth of the algebraic bipartite graph $D(k, q)$ * 

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#### Abstract

For integer $k \geq 2$ and prime power $q$, the algebraic bipartite graph $D(k, q)$ proposed by Lazebnik and Ustimenko (1995) is meaningful not only in extremal graph theory but also in coding theory and cryptography. This graph is $q$-regular, edge-transitive and of girth at least $k+4$. Its exact girth $g=g(D(k, q))$ was conjectured in 1995 to be $k+5$ for odd $k$ and $q \geq 4$. This conjecture was shown to be valid in 2016 when $\left.\frac{k+5}{2}\right|_{p}(q-1)$, where $p$ is the characteristic of $\mathbb{F}_{q}$ and $\left.m\right|_{p} n$ means that $m$ divides $p^{r} n$ for some nonnegative integer $r$. In this paper, for $t \geq 1$ we prove that (a) $g(D(4 t+2, q))=g(D(4 t+1, q)) ;($ b) $g(D(4 t+3, q))=4 t+8$ if $g(D(2 t, q))=2 t+4$; (c) $g(D(8 t, q))=8 t+4$ if $g(D(4 t-2, q))=4 t+2$; (d) $g\left(D\left(2^{s+2}(2 t-1)-5, q\right)\right)=2^{s+2}(2 t-1)$ if $p \geq 3,\left.(2 t-1)\right|_{p}(q-1)$ and $2^{s} \|(q-1)$. A


 simple upper bound for the girth of $D(k, q)$ is proposed in the end of this paper.Keywords: Bipartite graph; Edge-transitive; Backtrackless walk; Girth; Homogeneous polynomial;

## 1 Introduction

The graphs considered in this paper are undirected, without loops and multiple edges. A graph $G$ is said to be edge-transitive provided, for any two edges $e_{1}, e_{2}$ of $G$, that there exists an automorphism $\phi$ of $G$ such that $\phi$ maps the ends of $e_{1}$ into those of $e_{2}$. A backtrackless (or non-recurrent) walk of length $n$ means a sequence $v_{1}, v_{2}, \ldots, v_{n}$ of vertices of $G$ such that any two consecutive vertices are adjacent in $G$ and $v_{j} \neq v_{j+2}$ for $j=1,2, \ldots, n-2$. A backtrackless walk $v_{1}, v_{2}, \ldots, v_{n}$ is called a backtrackless circuit of length

[^0]$n$ further if $n$ is greater than 2 and $v_{3}, v_{4}, \ldots, v_{n}, v_{1}, v_{2}$ is still a backtrackless walk. For any graph $G$ which is not a tree, its girth, denoted by $g(G)$, is equal to the length of the shortest backtrackless circuits in $G$.

Graphs with large girth and a high degree of symmetry have been applied to variant problems in extremal graph theory, finite geometry, coding theory, cryptography, communication networks and quantum computations (c.f. [1]-[21]). In this paper, we concetrate on the algebraic bipartite graph $D(k, q)$, proposed by Lazebnik and Ustimenko in [3, which is edge-transitive and of girth at least $k+4$, where $k \geq 2$ and $q=p^{m}$ is a power of prime $p$. The graph $D(k, q)$ has been investigated quite well in literature (c.f. 3][20]). For the exact girth of $D(k, q)$, the following conjecture was proposed in [5]:

Conjecture 1. $D(k, q)$ has girth $k+5$ for all odd $k$ and all $q \geq 4$.
This conjecture was shown to be valid in [5] for the case that $(k+5) / 2$ is a factor of $q-1$, in [19] for the case that $(k+5) / 2$ is a power of $p$, and in [20] for the case that $(k+5) / 2$ is a factor of $q-1$ multiplied by a power of $p$, respectively. For a few small $k$ 's, the girth cycles (namely, the shortest backtrackless circuits) of $D(k, q)$ are determined completely in [22.

In this paper, we will investigate the girth of $D(k, q)$ further by means of a compact expression of some backtrackless walks of the bipartite graph $\Lambda_{k, q}$, which is defined as follows (c.f. [19], [20] and [22]). The left part of vertices of $\Lambda_{k, q}$, denoted by $L_{k}$, is the set of $(k+1)$-dimensional vectors $[l]=\left(l_{0}, l_{1}, l_{2}, \ldots, l_{k}\right)$ over $\mathbb{F}_{q}$ with $l_{1}=l_{2}$. The right part of vertices of $\Lambda_{k, q}$, denoted by $R_{k}$, is the set of $(k+1)$-dimensional vectors $\langle r\rangle=$ $\left(r_{0}, r_{1}, r_{2}, \ldots, r_{k}\right)$ over $\mathbb{F}_{q}$ with $r_{1}=0$. Two vertices $\left(l_{0}, l_{1}, \ldots, l_{k}\right) \in L_{k}$ and $\left(r_{0}, r_{1}, \ldots, r_{k}\right) \in R_{k}$ are adjacent in $\Lambda_{k, q}$ if and only if, for $2 \leq i \leq k$,

$$
l_{i}+r_{i}= \begin{cases}r_{0} l_{i-2} & \text { if } i \equiv 2,3 \quad \bmod 4  \tag{1}\\ l_{0} r_{i-2} & \text { if } i \equiv 0,1 \quad \bmod 4\end{cases}
$$

Since $\Lambda_{k, q}$ is isomorphic to $D(k, q)$ [19], $\Lambda_{k, q}$ is also edge-transitive and of girth at least $k+4$. All the consequent arguments will be made on the graph $\Lambda_{k, q}$ instead of the original graph $D(k, q)$.

This paper is arranged as follows. In Section 2 we introduce a class of homogeneous polynomials in several indeterminates and a compact expression for vertices over some backtrackless walks in $\Lambda_{k, q}$. An identity on such polynomials is shown in Section 3. By using of this identity, in Section 3 we show, for any $t \geq 1$, that each backtrackless circuit in $\Lambda_{4 t+1, q}$ implies a backtrackless circuit in $\Lambda_{4 t+2, q}$ of the same length, and $g\left(\Lambda_{4 t+2, q}\right)=g\left(\Lambda_{4 t+1, q}\right)$
is then deduced. In Section 4, we construct some backtrackless circuits in $\Lambda_{4 t+3, q}$ by using those in $\Lambda_{2 t, q}$, and show for $n \geq 3$ that $g\left(\Lambda_{4 t+3, q}\right) \leq 4 n$ if $g\left(\Lambda_{2 t, q}\right) \leq 2 n$. A few results on the exact girth of $\Lambda_{k, q}$ are also given in this section. In Section 5, we deduce an upper bound for the girth of $\Lambda_{k, q}$ by combining a known result from [20] and the new results shown in Section 4. Some concluding remarks are given in Section 6.

## 2 Backtrackless Walks in $\Lambda_{k, q}$

At first, we introduce a class of homogeneous polynomials in several indeterminates which were defined in [19]. For indeterminates $\omega_{1}, \ldots, \omega_{n}$ whose values are usually limited to the set $\mathbb{F}_{q}^{*}$, let

$$
\rho_{0}\left(\omega_{1}, \ldots, \omega_{n}\right)=\omega_{1} \cdots \omega_{n}
$$

and, for $1 \leq s \leq\left\lfloor\frac{n}{2}\right\rfloor$, let $\rho_{s}\left(\omega_{1}, \ldots, \omega_{n}\right)$ denote the homogeneous polynomial of order $n-2 s$ defined by

$$
\rho_{s}\left(\omega_{1}, \ldots, \omega_{n}\right)=\sum_{1 \leq i_{1}<\cdots<i_{s} \leq n-s} \frac{\prod_{j=1}^{n} \omega_{j}}{\prod_{j=1}^{s} \omega_{i_{j}+j-1} \omega_{i_{j}+j}},
$$

where each term in the summation is a product of the remaining elements in the sequence $\omega_{1}, \ldots, \omega_{n}$ after deleting from it $s$ disjoint pairs $\left\{\omega_{i}, \omega_{i+1}\right\}$ of consecutive elements. If $n<2 s$ or $s<0, \rho_{s}\left(\omega_{1}, \ldots, \omega_{n}\right)$ is defined as 0 . For the null sequence $\eta$, we define $\rho_{s}(\eta)$ as

$$
\rho_{s}(\eta)= \begin{cases}1 & \text { if } s=0 \\ 0 & \text { if } s \neq 0\end{cases}
$$

One can show easily (c.f. [19], 20] and [22])

$$
\begin{equation*}
\rho_{s}\left(\omega_{1}, \ldots, \omega_{n}\right)=\rho_{s-1}\left(\omega_{1}, \ldots, \omega_{n-2}\right)+\omega_{n} \rho_{s}\left(\omega_{1}, \ldots, \omega_{n-1}\right) \tag{2}
\end{equation*}
$$

and, for $0 \leq j \leq n$,

$$
\begin{gather*}
\rho_{n-j}\left(\omega_{1}, \ldots, \omega_{2 n}\right)=\sum_{1 \leq s_{1} \leq t_{1}<s_{2} \leq t_{2}<\cdots<s_{j} \leq t_{j} \leq n} \prod_{k=1}^{j} \omega_{2 s_{k}-1} \omega_{2 t_{k}},  \tag{3}\\
\rho_{n-j}\left(\omega_{1}, \ldots, \omega_{2 n+1}\right)=\sum_{s=j}^{n} \rho_{s-j}\left(\omega_{1}, \ldots, \omega_{2 s}\right) \omega_{2 s+1} \\
=\sum_{1 \leq s_{0} \leq t_{1}<s_{1} \leq t_{2}<s_{2} \leq \cdots \leq t_{j}<s_{j} \leq n+1} \omega_{2 s_{0}-1} \prod_{k=1}^{j} \omega_{2 t_{k}} \omega_{2 s_{k}-1} . \tag{4}
\end{gather*}
$$

For $a, b \in \mathbb{F}_{q}^{*}$, let $\omega_{2 s-1}^{\prime}=a \omega_{2 s-1}$ and $\omega_{2 s}^{\prime}=b \omega_{2 s}$ for $s=1,2, \ldots$, then from (3) and (4) we see easily

$$
\begin{gather*}
\rho_{n-j}\left(\omega_{1}^{\prime}, \ldots, \omega_{2 n}^{\prime}\right)=a^{j} b^{j} \rho_{n-j}\left(\omega_{1}, \ldots, \omega_{2 n}\right),  \tag{5}\\
\rho_{n-j}\left(\omega_{1}^{\prime}, \ldots, \omega_{2 n+1}^{\prime}\right)=a^{j+1} b^{j} \rho_{n-j}\left(\omega_{1}, \ldots, \omega_{2 n+1}\right) . \tag{6}
\end{gather*}
$$

Since the graph $\Lambda_{k, q}$ is edge-transitive, without loss of generality, we will concentrate on the backtrackless walks which are leading by the two vertices expressed by the all-zero vector. Let $\Gamma=\left[l^{(1)}\right]\left\langle r^{(1)}\right\rangle\left[l^{(2)}\right]\left\langle r^{(2)}\right\rangle \ldots$ be such a given backtrackless walk of $\Lambda_{k, q}$, where $\left[l^{(1)}\right]=(0,0, \ldots, 0)$ and $\left\langle r^{(1)}\right\rangle=(0,0, \ldots, 0)$. For $i \geq 1$, let $x_{i}$ and $y_{i}$ denote the first entries (or colors) of $\left[l^{(i)}\right]$ and $\left\langle r^{(i)}\right\rangle$, respectively, and write

$$
\begin{equation*}
u_{i}=x_{i+1}-x_{i}, \quad v_{i}=y_{i+1}-y_{i} . \tag{7}
\end{equation*}
$$

Clearly, we have $u_{i} \neq 0$ and $v_{i} \neq 0$. As a refinement of a closed-form expression given in [19] for the backtrackless walks leading by $\left[l^{(1)}\right]=(0,0, \ldots, 0)$, the following lemma was shown in [22].

Lemma 1. For any $i \geq 1$ and $j \geq 0$, let $l_{j}^{(i+1)}$ denote the $(j+1)$-th entries of $\left[l^{(i+1)}\right]$. Then, we have

$$
\begin{align*}
& l_{4 j}^{(i+1)}=\rho_{i-j-1}\left(u_{1}, v_{1}, \ldots, u_{i-1}, v_{i-1}, u_{i}\right),  \tag{8}\\
& l_{4 j+1}^{(i+1)}=\rho_{i-j-2}\left(v_{1}, u_{2}, \ldots, v_{i-1}, u_{i}\right),  \tag{9}\\
& l_{4 j+2}^{(i+1)}=y_{i+1} l_{4 j}^{(i+1)}-\rho_{i-j-1}\left(u_{1}, v_{1}, \ldots, u_{i}, v_{i}\right),  \tag{10}\\
& l_{4 j+3}^{(i+1)}=y_{i+1} l_{4 j+1}^{(i+1)}-\rho_{i-j-2}\left(v_{1}, u_{2} \ldots, v_{i-1}, u_{i}, v_{i}\right) . \tag{11}
\end{align*}
$$

This lemma shows a compact expression for the vertices on the walk $\Gamma$. For convenience, we say the walk $\Gamma$ is of type $\left(u_{1}, v_{1}, u_{2}, v_{2}, \ldots\right)$. If the first $2 i$ vertices in the walk $\Gamma$ form a circuit in $\Lambda_{k, q}$ of length $2 i$, we also call it a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{i}, v_{i}\right)$.

We note that

$$
\begin{equation*}
\rho_{i-1}\left(v_{1}, u_{2} \ldots, v_{i-1}, u_{i}, v_{i}\right)=v_{1}+\cdots+v_{i}=y_{i+1}=y_{1}=0 \tag{12}
\end{equation*}
$$

is always a necessary condition for $\Lambda_{k, q}$ to have a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{i}, v_{i}\right)$.

## 3 Backtrackless Circuits in $\Lambda_{4 t+2, q}$

In this section we show an identity on the homogeneous polynomials introduced in Section 2 at first. By using this identity, we show then that each backtrackless circuit in $\Lambda_{4 t+1, q}$ ensures the existence of a backtrackless circuit of the same type in $\Lambda_{4 t+2, q}$ and deduce $g\left(\Lambda_{4 t+2, q}\right)=g\left(\Lambda_{4 t+1, q}\right)$ for any $t \geq 1$ in final.
Lemma 2. For any integers $n, t$ with $n \geq 1$, let

$$
\begin{align*}
\Delta_{2 t-1}^{n} & =\rho_{n-t}\left(v_{1}, u_{2}, \ldots, v_{n-1}, u_{n}, v_{n}\right)  \tag{13}\\
\nabla_{2 t-1}^{n} & =\rho_{n-t}\left(u_{1}, v_{1}, \ldots, u_{n-1}, v_{n-1}, u_{n}\right)  \tag{14}\\
\Delta_{2 t}^{n} & =\rho_{n-1-t}\left(v_{1}, u_{2}, \ldots, v_{n-1}, u_{n}\right)  \tag{15}\\
\nabla_{2 t}^{n} & =\rho_{n-t}\left(u_{1}, v_{1}, \ldots, u_{n}, v_{n}\right) . \tag{16}
\end{align*}
$$

Then, for $n \geq 1$ we have

$$
\begin{equation*}
\sum_{s}(-1)^{s} \nabla_{s}^{n} \Delta_{2 j-s}^{n}=0, \text { for } j \geq 1 . \tag{17}
\end{equation*}
$$

Proof. From $\Delta_{0}^{1}=\nabla_{0}^{1}=1, \Delta_{1}^{1}=v_{1}, \nabla_{1}^{1}=u_{1}, \nabla_{2}^{1}=u_{1} v_{1}, \Delta_{s}^{1}=\nabla_{t}^{1}=0$ for any $s \notin\{0,1\}$ and $t \notin\{0,1,2\}$, it can be checked easily that (17) is valid for $n=1$.

To show that (17) is valid for $n>1$, we note that for any integer $t$ according to (2) we have

$$
\begin{aligned}
\Delta_{2 t-1}^{n}= & \rho_{n-t-1}\left(v_{1}, u_{2}, \ldots, v_{n-2}, u_{n-1}, v_{n-1}\right) \\
& +v_{n} \rho_{n-t-1}\left(v_{1}, u_{2}, \ldots, v_{n-2}, u_{n-1}\right) \\
& +v_{n} u_{n} \rho_{n-t}\left(v_{1}, u_{2}, \ldots, v_{n-2}, u_{n-1}, v_{n-1}\right) \\
= & \Delta_{2 t-1}^{n-1}+v_{n} \Delta_{2 t-2}^{n-1}+v_{n} u_{n} \Delta_{2 t-3}^{n-1}, \\
\nabla_{2 t-1}^{n}= & \rho_{n-t-1}\left(u_{1}, v_{1}, \ldots, u_{n-2}, v_{n-2}, u_{n-1}\right) \\
& +u_{n} \rho_{n-t}\left(u_{1}, v_{1}, \ldots, u_{n-1}, v_{n-1}\right) \\
= & \nabla_{2 t-1}^{n-1}+u_{n} \nabla_{2 t-2}^{n-1}, \\
\Delta_{2 t}^{n}= & \rho_{n-2-t}\left(v_{1}, u_{2}, \ldots, v_{n-2}, u_{n-1}\right) \\
& +u_{n} \rho_{n-1-t}\left(v_{1}, u_{2}, \ldots, v_{n-2}, u_{n-1}, v_{n-1}\right) \\
= & \Delta_{2 t}^{n-1}+u_{n} \Delta_{2 t-1}^{n-1}, \\
\nabla_{2 t}^{n}= & \rho_{n-t-1}\left(u_{1}, v_{1}, \ldots, u_{n-1}, v_{n-1}\right), \\
& +v_{n} \rho_{n-t-1}\left(u_{1}, v_{1}, \ldots, u_{n-2}, v_{n-2}, u_{n-1}\right) \\
& +v_{n} u_{n} \rho_{n-t}\left(u_{1}, v_{1}, \ldots, u_{n-1}, v_{n-1}\right) \\
= & \nabla_{2 t}^{n-1}+v_{n} \nabla_{2 t-1}^{n-1}+v_{n} u_{n} \nabla_{2 t-2}^{n-1},
\end{aligned}
$$

then we get

$$
\begin{aligned}
& \sum_{s}(-1)^{s} \nabla_{s}^{n} \Delta_{2 j-s}^{n} \\
= & \sum_{t}\left(\nabla_{2 t}^{n-1}+v_{n} \nabla_{2 t-1}^{n-1}+v_{n} u_{n} \nabla_{2 t-2}^{n-1}\right)\left(\Delta_{2 j-2 t}^{n-1}+u_{n} \Delta_{2 j-2 t-1}^{n-1}\right) \\
& -\sum_{t}\left(\nabla_{2 t-1}^{n-1}+u_{n} \nabla_{2 t-2}^{n-1}\right)\left(\Delta_{2 j-2 t+1}^{n-1}+v_{n} \Delta_{2 j-2 t}^{n-1}+v_{n} u_{n} \Delta_{2 j-2 t-1}^{n-1}\right) \\
= & \sum_{t}\left(\nabla_{2 t}^{n-1} \Delta_{2 j-2 t}^{n-1}-\nabla_{2 t-1}^{n-1} \Delta_{2 j-2 t+1}^{n-1}\right) \\
& +u_{n} \sum_{t}\left(\nabla_{2 t}^{n-1} \Delta_{2 j-2 t-1}^{n-1}-\nabla_{2 t-2}^{n-1} \Delta_{2 j-2 t+1}^{n-1}\right) \\
= & \sum_{s}(-1)^{s} \nabla_{s}^{n-1} \Delta_{2 j-s}^{n-1} .
\end{aligned}
$$

Therefore, one can show easily by induction that (17) is valid for any positive integer $n$.

The following theorem is then a simple corollary of Lemmas 1 and 2,
Theorem 1. For $t \geq 1, \Lambda_{4 t+2, q}$ has a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots\right.$, $\left.u_{i}, v_{i}\right)$ if and only if $\Lambda_{4 t+1, q}$ has a backtrackless circuit of the same type. In particular, we have $g\left(\Lambda_{4 t+2, q}\right)=g\left(\Lambda_{4 t+1, q}\right)$ for $t \geq 1$.
Proof. Assume that there is a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{i}, v_{i}\right)$ in $\Lambda_{4 t+1, q}$, that is,

$$
v_{1}+\cdots+v_{i}=y_{i+1}=y_{1}=0
$$

and $l_{k}^{(i+1)}=0$ for $0 \leq k \leq 4 t+1$. According to Lemma (1) by using the notations defined in Lemma 2 we have $\Delta_{2 t+2}^{i}=0$ and $\Delta_{k}^{i}=\nabla_{k}^{i}=0$ for $1 \leq k \leq 2 t+1$. Therefore, from $\Delta_{0}^{i}=1$ and Lemma 2 we see

$$
l_{4 t+2}^{(i+1)}=y_{i+1} l_{4 t}^{(i+1)}-\nabla_{2 t+2}^{i}=\sum_{s=0}^{2 t+1}(-1)^{s} \nabla_{s}^{i} \Delta_{2 t+2-s}^{i}=0
$$

and then $\Lambda_{4 t+2, q}$ also has a backtrackless circuit of type ( $u_{1}, v_{1}, \ldots, u_{i}, v_{i}$ ). On the other hand, we note that $\Lambda_{4 t+2, q}$ has a backtrackless circuit of type ( $u_{1}, v_{1}, \ldots, u_{i}, v_{i}$ ) implies naturally that $\Lambda_{4 t+1, q}$ has a backtrackless circuit of the same type. The proof is complete.

## 4 Backtrackless Circuits in $\Lambda_{4 s+3, q}$

All the arguments given in this section will be based on the existence of backtrackless circuits of type ( $u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}$ ) with

$$
\begin{equation*}
v_{2 j-1}=-v_{2 j}=1, j=1,2, \ldots, n, \tag{18}
\end{equation*}
$$

in the graph $\Lambda_{k, q}$. To show the existence of such circuits, we deduce some equalities on the homogeneous polynomials $\rho_{s}(\cdot, \ldots, \cdot)$ at first.

Lemma 3. For any integer $t$ and tuple $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ over $\mathbb{F}_{q}^{*}$ with (18), we have

$$
\begin{align*}
& \rho_{2 n-2 t}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right) \\
= & (-1)^{t-1} \rho_{n-t}\left(u_{1}, \ldots, u_{2 n}\right)+(-1)^{t} \rho_{n-t-1}\left(u_{2}, \ldots, u_{2 n-1}\right),  \tag{19}\\
& \rho_{2 n+1-2 t}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right) \\
= & (-1)^{t-1} \rho_{n-t}\left(u_{2}, \ldots, u_{2 n}\right)+(-1)^{t-1} \rho_{n-t}\left(u_{1}, \ldots, u_{2 n-1}\right),  \tag{20}\\
& \rho_{2 n-1-2 t}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}\right)=(-1)^{t} \rho_{n-t-1}\left(u_{2}, \ldots, u_{2 n-1}\right),  \tag{21}\\
& \rho_{2 n-2 t}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}\right)=(-1)^{t-1} \rho_{n-t}\left(u_{2}, \ldots, u_{2 n}\right),  \tag{22}\\
& \rho_{2 n-2 t}\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)=(-1)^{t} \rho_{n-t}\left(u_{1}, \ldots, u_{2 n}\right)  \tag{23}\\
& \rho_{2 n+1-2 t}\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)=(-1)^{t} \rho_{n-t}\left(u_{2}, \ldots, u_{2 n}\right),  \tag{24}\\
& \rho_{2 n-2 t}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}, v_{2 n}\right)=(-1)^{t} \rho_{n-t}\left(u_{2}, \ldots, u_{2 n}\right),  \tag{25}\\
& \rho_{2 n+1-2 t}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}, v_{2 n}\right)=0 . \tag{26}
\end{align*}
$$

Proof. We give a proof only for the equality (19). The others can be proved similarly.

It is obvious that (19) is valid if $t \leq 0$ or $t \geq n$. Since for $1 \leq s \leq n$ we
have $y_{2 s-1}=0$ and $y_{2 s}=1$, according to (3) and (4), for $1 \leq t<n$, we have

$$
\begin{aligned}
& \rho_{2 n-2 t}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right) \\
&= \sum_{1 \leq j_{1} \leq j_{2}<j_{3} \leq \cdots \leq j_{4 t-2}<j_{4 t-1} \leq 2 n} u_{j_{1}} \prod_{s=1}^{2 t-1} v_{j_{2 s}} u_{j_{2 s+1}} \\
&= \sum_{1 \leq j_{1}<j_{3}<\cdots<j_{4 t-1} \leq 2 n} u_{j_{1}} \prod_{s=1}^{2 t-1} u_{j_{2 s+1}} \sum_{j_{2 s-1} \leq l<j_{2 s+1}} v_{j_{2 s}} \\
&= \sum_{1 \leq j_{1}<j_{3}<\cdots<j_{4 t-1} \leq 2 n} u_{j_{1}}^{2 t-1} \prod_{s=1}^{2 t} u_{j_{2 s+1}}\left(y_{j_{2 s+1}}-y_{j_{2 s-1}}\right) \\
&=(-1)^{t-1} \sum_{1 \leq i_{1} \leq i_{2}<i_{3} \leq i_{4}<\cdots<i_{2 t-1} \leq i_{2 t} \leq n} \prod_{s=1}^{t} u_{2 i_{2 s-1}-1} u_{2 i_{2 s}} \\
& \quad+(-1)^{t} \sum_{1 \leq i_{1}<i_{2} \leq i_{3}<i_{4} \leq \cdots \leq i_{2 t-1}<i_{2 t} \leq n} \prod_{s=1}^{t} u_{2 i_{2 s-1}} u_{2 i_{2 s}-1} \\
&=(-1)^{t-1} \rho_{n-t}\left(u_{1}, \ldots, u_{2 n}\right)+(-1)^{t} \rho_{n-t-1}\left(u_{2}, \ldots, u_{2 n-1}\right),
\end{aligned}
$$

i.e. (19) is valid for $1 \leq t<n$.

The following lemma shows that, from any backtrackless circuit of length $2 n$ in $\Lambda_{2 s, q}$, one can construct a backtrackless circuit of length $4 n$ in $\Lambda_{4 s+3, q}$, for any $s \geq 1$ and $n \geq 3$.

Lemma 4. Assume that $s \geq 1$ and $n \geq 3$. The graph $\Lambda_{4 s+3, q}$ has a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ with (18) if and only if $\Lambda_{2 s, q}$ has a backtrackless circuit of length $2 n$.

Proof. Assume that $v_{1}, \ldots, v_{2 n} \in \mathbb{F}_{q}^{*}$ satisfy (18). According to Lemma 1 , the graph $\Lambda_{4 s+3, q}$ has a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ if and only if

$$
\left\{\begin{array}{l}
\rho_{2 n-j-2}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}, v_{2 n}\right)=0  \tag{27}\\
\rho_{2 n-j-1}\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)=0, \\
\rho_{2 n-j-2}\left(v_{1}, u_{2}, \ldots, v_{2 n-1}, u_{2 n}\right)=0, \\
\rho_{2 n-j-1}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right)=0
\end{array} \quad \text { for } 0 \leq j \leq s\right.
$$

If $s=2 w$ is even, according to Lemma 3 we see that (27) is equivalent
to $\rho_{n-1}\left(u_{1}, \ldots, u_{2 n-1}\right)=\rho_{n-1}\left(u_{2}, \ldots, u_{2 n}\right)=0$ and

$$
\left\{\begin{array}{l}
\rho_{n-j-1}\left(u_{1}, \ldots, u_{2 n-1}\right)=0  \tag{28}\\
\rho_{n-j-1}\left(u_{2}, \ldots, u_{2 n}\right)=0, \\
\rho_{n-j}\left(u_{1}, \ldots, u_{2 n}\right)=0, \\
\rho_{n-j-1}\left(u_{2}, \ldots, u_{2 n-1}\right)=0
\end{array} \quad \text { for } 1 \leq j \leq w,\right.
$$

that is, the graph $\Lambda_{4 w, q}$ has a backtrackless circuit of type $\left(u_{1}, \ldots, u_{2 n}\right)$ on account to Lemma

If $s=2 w-1$ is odd, according to Lemma 3 we see that (27) is equivalent to

$$
\left\{\begin{array}{l}
\rho_{n-j}\left(u_{1}, \ldots, u_{2 n}\right)=0  \tag{29}\\
\rho_{n-j-1}\left(u_{2}, \ldots, u_{2 n-1}\right)=0, \\
\rho_{n-j}\left(u_{1}, \ldots, u_{2 n-1}\right)=0 \\
\rho_{n-j}\left(u_{2}, \ldots, u_{2 n}\right)=0
\end{array} \quad \text { for } 1 \leq j \leq w .\right.
$$

that is, the graph $\Lambda_{4 w-2, q}$ has a backtrackless circuit of type $\left(u_{1}, \ldots, u_{2 n}\right)$ on account to Lemma 1 .

Based on the existence of backtrackless circuits of type $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ with (18), on the girth of $\Lambda_{k, q}$ one can show the following theorem by using Lemmas 1, 3 and 4

Theorem 2. Assume $n \geq 3$ and $s, w \geq 1$.

1. If $g\left(\Lambda_{2 s, q}\right) \leq 2 n$, then $g\left(\Lambda_{4 s+3, q}\right) \leq 4 n$.
2. If $g\left(\Lambda_{2 n-4, q}\right)=2 n$, then $g\left(\Lambda_{4 n-5, q}\right)=4 n$.
3. If $g\left(\Lambda_{4 w-2, q}\right)=4 w+2$, then $g\left(\Lambda_{8 w, q}\right)=8 w+4$.
4. If $q$ is a power of 2 and $g\left(\Lambda_{4 w, q}\right) \leq 2 n$, then $g\left(\Lambda_{8 w+4, q}\right) \leq 4 n$.

Proof. The first statement follows simply from Lemma 4 . Furthermore, the second statement follows immediately on account to $g\left(\Lambda_{4 n-5, q}\right) \geq 4 n$.

To show the third statement, we set $n=2 w+1$ and assume that the graph $\Lambda_{4 w-2, q}$ has a backtrackless circuit of type $\left(u_{1}, \ldots, u_{2 n}\right)$. Then, according to Lemma 1 we have (29). If $\rho_{n-w-1}\left(u_{2}, \ldots, u_{2 n}\right)=0$, then $\Lambda_{4 w-1, q}$ has a backtrackless circuit of type $\left(u_{1}, \ldots, u_{2 n}\right)$, contradicts $g\left(\Lambda_{4 w-1, q}\right) \geq$
$4 w+4>2 n$. If $\rho_{n-w-1}\left(u_{1}, \ldots, u_{2 n-1}\right)=0$, then $\Lambda_{4 w-1, q}$ has a backtrackless circuit of type $\left(u_{2 n}, \ldots, u_{1}\right)$, contradicts $g\left(\Lambda_{4 w-1, q}\right) \geq 4 w+4>2 n$ too. Hence, we have $\rho_{n-w-1}\left(u_{2}, \ldots, u_{2 n}\right) \rho_{n-w-1}\left(u_{1}, \ldots, u_{2 n-1}\right) \neq 0$. Let

$$
\alpha=-\rho_{n-w-1}\left(u_{2}, \ldots, u_{2 n}\right) / \rho_{n-w-1}\left(u_{1}, \ldots, u_{2 n-1}\right) .
$$

We multiply the entries with odd indices in the tuple $\left(u_{1}, \ldots, u_{2 n}\right)$ by $\alpha$, and denote the resulting tuple by the same notation. Then, according to (5) and (6) one can check easily that the modified tuple $\left(u_{1}, \ldots, u_{2 n}\right)$ satisfies (29) and

$$
\rho_{n-w-1}\left(u_{2}, \ldots, u_{2 n}\right)+\rho_{n-w-1}\left(u_{1}, \ldots, u_{2 n-1}\right)=0 .
$$

Hence, according to Lemma 3 we have

$$
\rho_{2 n-2 w-1}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right)=0
$$

and (27) with $s=2 w-1$, where the tuple $\left(v_{1}, \ldots, v_{2 n}\right)$ satisfies (18). Therefore, according to Lemma we see $\Lambda_{8 w, q}$ has a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ and thus we have $g\left(\Lambda_{8 w, q}\right) \leq 4 n=8 w+4$. Hence, from $g\left(\Lambda_{8 w, q}\right) \geq 8 w+4$ we see $g\left(\Lambda_{8 w, q}\right)=8 w+4$.

To show the last statement, we assume that $q$ is a power of 2 and that the graph $\Lambda_{4 w, q}$ has a backtrackless circuit of type $\left(u_{1}, \ldots, u_{2 n}\right)$. Then, according to Lemma 1 we have

$$
\rho_{n-1}\left(u_{1}, \ldots, u_{2 n-1}\right)=\rho_{n-1}\left(u_{2}, \ldots, u_{2 n}\right)=0
$$

and (28), that is, $\nabla_{t}^{n}=\Delta_{t}^{n}=0$ holds for $1 \leq t \leq 2 w+1$ when we modify accordingly the definition of the notations $\nabla_{t}^{n}, \Delta_{t}^{n}$. Therefore, according to Lemma 2 and that the characteristic of $\mathbb{F}_{q}$ is 2 , we see

$$
\begin{aligned}
& \rho_{n-w-2}\left(u_{2}, \ldots, u_{2 n-1}\right)-\rho_{n-w-1}\left(u_{1}, \ldots, u_{2 n}\right) \\
= & \Delta_{2 w+2}^{n}-\nabla_{2 w+2}^{n} \\
= & 2 \Delta_{2 w+2}^{n}+\sum_{s=1}^{2 w+1}(-1)^{s} \nabla_{s}^{n} \Delta_{2 w+2-s}^{n}=0 .
\end{aligned}
$$

Hence, according to Lemma 3 we have

$$
\rho_{2 n-2 w-2}\left(u_{1}, v_{1}, \ldots, u_{2 n-1}, v_{2 n-1}, u_{2 n}\right)=0
$$

and (27) with $s=2 w$, where the tuple $\left(v_{1}, \ldots, v_{2 n}\right)$ satisfies (18). Therefore, according to Lemma 1 we see $\Lambda_{8 w+4, q}$ has a backtrackless circuit of type $\left(u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}\right)$ and thus we have $g\left(\Lambda_{8 w+4, q}\right) \leq 4 n$.

Example 1．－For $k \geq 2$ ，the girth of $\Lambda_{k, 2}$ has been determined［19］： $g\left(\Lambda_{k, 2}\right)=2^{s}$ ，where $s$ is the integer with $2^{s-1}-4<k \leq 2^{s}-4$.
－Suppose $q \geq 3$ ．According to Theorem 图 from $g\left(\Lambda_{2, q}\right)=6$［19］we see $g\left(\Lambda_{7, q}\right)=g\left(\Lambda_{8, q}\right)=12, g\left(\Lambda_{18, q}\right) \leq g\left(\Lambda_{19, q}\right)=24, g\left(\Lambda_{38, q}\right) \leq$ $g\left(\Lambda_{39, q}\right) \leq 48, g\left(\Lambda_{78, q}\right) \leq g\left(\Lambda_{79, q}\right) \leq 96$ and $g\left(\Lambda_{159, q}\right) \leq 192$ ．
－Suppose $q>3$ ．According to Theorem 圆，from $g\left(\Lambda_{4, q}\right)=8$［22］we see $g\left(\Lambda_{11, q}\right)=16$ ．According to Theorem 1，from $g\left(\Lambda_{5, q}\right)=10$［D2］we see $g\left(\Lambda_{6, q}\right)=10$ and then，according to Theorem 图，we have $g\left(\Lambda_{15, q}\right)=$ $g\left(\Lambda_{16, q}\right)=20$ and $g\left(\Lambda_{35, q}\right)=40$ ．

When the characteristic of $\mathbb{F}_{q}$ is 2 ，one can deduce further the following corollary easily．

Corollary 1．Assume $g\left(\Lambda_{2 s, q}\right)=2 s+4$ ，where $q$ is a power of 2 and $s \geq 1$ ． Then，for any $t \geq 1$ we have

$$
g\left(\Lambda_{2^{t}(s+2)-4, q}\right)=g\left(\Lambda_{2^{t}(s+2)-5, q}\right)=2^{t}(s+2),
$$

where $\Lambda_{1, q}$ is defined as a graph isomorphic to $\Lambda_{2, q}$ for convenience．
Proof．From $g\left(\Lambda_{k, q}\right) \geq k+4$ and Theorem2，we see easily that $g\left(\Lambda_{2^{t}(s+2)-4, q}\right)$ $=2^{t}(s+2)$ is valid for any $t \geq 1$ ．Furthermore，$g\left(\Lambda_{2^{t}(s+2)-5, q}\right)=2^{t}(s+2)$ follows from $2^{t}(s+2) \leq g\left(\Lambda_{2^{t}(s+2)-5, q}\right) \leq g\left(\Lambda_{2^{t}(s+2)-4, q}\right)$ ．

Example 2．Assume that $q \geq 4$ is a power of 2．According to Corollary $\mathbf{1}$ ， we have the following three statements．
－From $g\left(\Lambda_{2, q}\right)=6$ ，we see $g\left(\Lambda_{2^{t} 3-4, q}\right)=g\left(\Lambda_{2^{t} 3_{-5, q}}\right)=2^{t} 3$ for $t \geq 1$ ．
－From $g\left(\Lambda_{4, q}\right)=8$ ，we see $g\left(\Lambda_{2^{t+2}-4, q}\right)=g\left(\Lambda_{2^{t+2}-5, q}\right)=2^{t+2}$ for $t \geq 1$ ．
－From $g\left(\Lambda_{6, q}\right)=10$ ，we see $g\left(\Lambda_{2^{t} 5-4, q}\right)=g\left(\Lambda_{2^{t} 5-5, q}\right)=2^{t} 5$ for $t \geq 1$ ．
For prime $p$ ，we write $\left.m\right|_{p} n$ if $m \mid\left(n p^{r}\right)$ for some $r \geq 0$ ．The following lemma is from［20］．

Lemma 5．For $q=p^{s}$ and $t \geq 1$ with $\left.(t+2)\right|_{p}(q-1)$ ，

$$
\begin{equation*}
g\left(\Lambda_{2 t-1, q}\right)=g\left(\Lambda_{2 t, q}\right)=2 t+4 . \tag{30}
\end{equation*}
$$

The following theorem follows simply from Theorem 2 and Lemma 5

Theorem 3. Assume that $q$ is a power of odd prime $p$ and $s, t$ are positive integers with $\left.(2 t-1)\right|_{p}(q-1)$ and $2^{s} \|(q-1)$. Then, Conjecture 1 is valid when $k=2^{s+2}(2 t-1)-5$, i.e.

$$
\begin{equation*}
g\left(\Lambda_{2^{s+2}(2 t-1)-5, q}\right)=2^{s+2}(2 t-1) . \tag{31}
\end{equation*}
$$

Proof. Clearly, we have $\left.2^{s}(2 t-1)\right|_{p}(q-1)$. Hence, from Lemma 5 we see

$$
g\left(\Lambda_{2^{s+1}(2 t-1)-5, q}\right)=g\left(\Lambda_{2^{s+1}(2 t-1)-4, q}\right)=2^{s+1}(2 t-1),
$$

therefore, from Theorem 2 we see (31).
We note that the result shown in this theorem is not included by Lemma 5 . At the end of this section, we investigate the girth of $\Lambda_{k, 3}$ for small $k$.

Example 3. - The positive integer $t$ 's satisfying $\left.(t+2)\right|_{3}(3-1)$ are $1,4,7,16,25,52,79,160, \ldots$. Then, according to Lemma 5 we have $g\left(\Lambda_{2,3}\right)=6, g\left(\Lambda_{7,3}\right)=g\left(\Lambda_{8,3}\right)=12, g\left(\Lambda_{13,3}\right)=g\left(\Lambda_{14,3}\right)=18$, $g\left(\Lambda_{31,3}\right)=g\left(\Lambda_{32,3}\right)=36, g\left(\Lambda_{49,3}\right)=g\left(\Lambda_{50,3}\right)=54, g\left(\Lambda_{103,3}\right)=$ $g\left(\Lambda_{104,3}\right)=108, g\left(\Lambda_{157,3}\right)=g\left(\Lambda_{158,3}\right)=162, g\left(\Lambda_{319,3}\right)=g\left(\Lambda_{320,3}\right)=$ 324, ....

- From $g\left(\Lambda_{32,3}\right)=36$, according to Theorem圆 we see $g\left(\Lambda_{66,3}\right) \leq g\left(\Lambda_{67,3}\right)$ $=72, g\left(\Lambda_{134,3}\right) \leq g\left(\Lambda_{135,3}\right) \leq 144$ and $g\left(\Lambda_{271,3}\right) \leq 288$.
- From $g\left(\Lambda_{104,3}\right)=108$, according to Theorem 圆 we see $g\left(\Lambda_{211,3}\right)=216$.

The known results on the girth of $\Lambda_{k, 3}$ for $2 \leq k \leq 320$ are summarized in the following table.

Table 1: Girth of $\Lambda_{k, 3}$ for $2 \leq k \leq 320$.

| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 13 | 14 | $19^{*}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 8 | 12 | 12 | 12 | 12 | 12 | 18 | 18 | 24 |
| 31 | 32 | $39^{*}$ | 49 | 50 | $67^{*}$ | $79^{*}$ | 103 | 104 | $135^{*}$ |
| 36 | 36 | $\leq 48$ | 54 | 54 | 72 | $\leq 96$ | 108 | 108 | $\leq 144$ |
| 157 | 158 | $159^{*}$ | $211^{*}$ | $271^{*}$ | 319 | 320 |  |  |  |
| 162 | 162 | $\leq 192$ | 216 | $\leq 288$ | 324 | 324 |  |  |  |

In this table, the mark * indicates the exact values or upper bounds of $g\left(\Lambda_{k, 3}\right)$ are obtained by the methods proposed in this paper. We note that the girth cycles of $\Lambda_{k, 3}$ were determined in [22] for $3 \leq k \leq 8$. In particular, the results $g\left(\Lambda_{3,3}\right)=8$ and $g\left(\Lambda_{k, 3}\right)=12$ for $4 \leq k \leq 8$ can be found therein.

## 5 Upper Bound of $g\left(\Lambda_{k, q}\right)$

In this section, we manage to deduce an upper bound for the girth of $\Lambda_{k, q}$ for $q \geq 3$.

Assume that $q \geq 3$ is a given prime power and the number of positive factors of $q-1$ is $n$. Let $k_{1}, k_{2}, \ldots$ be the odd integers in ascending order with $\left.\frac{k_{i}+5}{2}\right|_{p}(q-1)$, where $p$ is the characteristic of $\mathbb{F}_{q}$. Let $i_{0}$ be the integer with $k_{i_{0}}=2 q-5$.

Lemma 6. For any $i \geq i_{0}$

$$
\begin{equation*}
k_{i+n}=p k_{i}+5 p-5 \tag{32}
\end{equation*}
$$

Proof. Suppose $q=p^{m}$. For $0 \leq j \leq m-1$, let $d_{j, 1}, \ldots, d_{j, t_{j}}$ denote the different factors of $q-1$ with $p^{j} \leq d_{j, t}<p^{j+1}$ for $t=1, \ldots, t_{j}$. Then, we have $\sum_{j=0}^{m-1} t_{j}=n$ and for any $s \geq 0$ from $\left(p, d_{j, t}\right)=1$ we see

$$
\left\{k_{i_{0}+s n+i} \mid 0 \leq i<n\right\}=\bigcup_{0 \leq j<m}\left\{2 d_{j, t} p^{s+m-j}-5 \mid 1 \leq t \leq t_{j}\right\}
$$

which implies (32).
Let

$$
\begin{equation*}
T_{q}=\max _{i \geq i_{0}} \frac{k_{i+1}+5}{k_{i}+5} \tag{33}
\end{equation*}
$$

From Lemma 6 we see that $T_{q}$ can also be given by

$$
\begin{equation*}
T_{q}=\max _{i_{0} \leq i<i_{0}+n} \frac{k_{i+1}+5}{k_{i}+5} \tag{34}
\end{equation*}
$$

Clearly, $1<T_{q}<p$, and for any $i \geq i_{0}$ we have $\frac{k_{i+1}+5}{2} \leq T_{q} \frac{k_{i}+5}{2}$, i.e.

$$
\begin{equation*}
k_{i+1} \leq T_{q} k_{i}+5 T_{q}-5 \tag{35}
\end{equation*}
$$

Example 4. If $q=5^{2}$, then the positive factors of $5^{2}-1=24$ are $1,2,3,4,6$, $8,12,24$ and the positive integers $t$ with $\left.t\right|_{5} 24$ are

$$
1,2,3,4,5,6,8,10,12,15,20,24,25,30,40,50,60,75,100,120,125, \ldots
$$

Hence, $T_{q}=\max \{6 / 5,8 / 6,10 / 8,12 / 10,15 / 12,20 / 15,24 / 20,25 / 24\}=4 / 3$.

From Lemma 5 we see $g\left(\Lambda_{k_{i}-1, q}\right) \leq g\left(\Lambda_{k_{i}, q}\right)=k_{i}+5$, hence from Theorem 2 we see

$$
g\left(\Lambda_{2\left(k_{i}-1\right)+2, q}\right) \leq g\left(\Lambda_{2\left(k_{i}-1\right)+3, q}\right) \leq 2\left(k_{i}+5\right)
$$

and by induction we have

$$
\begin{equation*}
g\left(\Lambda_{2^{s}\left(k_{i}+1\right)-2, q}\right) \leq g\left(\Lambda_{2^{s}\left(k_{i}+1\right)-1, q}\right) \leq 2^{s}\left(k_{i}+5\right), \text { for any } s \geq 0 . \tag{36}
\end{equation*}
$$

Theorem 4. Let $q$ be a prime power.

1. If $T_{q} \leq 2$, then for $k \geq q$ we have

$$
\begin{equation*}
g\left(\Lambda_{k, q}\right) \leq T_{q}(k+4) \tag{37}
\end{equation*}
$$

2. If $T_{q}>2$ and $k \geq \max \left\{q, 8 T_{q}^{2}-10 T_{q}-3\right\}$, then we have

$$
\begin{equation*}
g\left(\Lambda_{k, q}\right) \leq 2 k+4 T_{q}+1 . \tag{38}
\end{equation*}
$$

Proof. Without loss of generality, we assume $k_{i}<k<k_{i+1}$ for some $i \geq i_{0}$. If $T_{q} \leq 2$, then from (35) and Lemma 5 we see

$$
g\left(\Lambda_{k, q}\right) \leq g\left(\Lambda_{k_{i+1}, q}\right)=k_{i+1}+5 \leq T_{q}\left(k_{i}+5\right) \leq T_{q}(k+4),
$$

i.e. (37) is valid.

Now we assume $T_{q}>2$ and $k \geq 8 T_{q}^{2}-10 T_{q}-3$.
If $\frac{k_{i+1}}{2}<k \leq k_{i+1}$, then from Lemma 5 we have

$$
\begin{equation*}
g\left(\Lambda_{k, q}\right) \leq g\left(\Lambda_{k_{i+1}, q}\right)=k_{i+1}+5 \leq 2 k+4 . \tag{39}
\end{equation*}
$$

If $k_{i}<k<\frac{k_{i+1}}{2}$, for the integer $s$ with

$$
\begin{equation*}
2^{s}\left(k_{i}+1\right) \leq k<2^{s+1}\left(k_{i}+1\right) \tag{40}
\end{equation*}
$$

from (35) we see

$$
2^{s}\left(k_{i}+1\right) \leq k \leq \frac{k_{i+1}-1}{2} \leq \frac{T_{q} k_{i}+5 T_{q}-6}{2},
$$

and then from (36) and (40) we have

$$
\begin{align*}
g\left(\Lambda_{k, q}\right) & \leq g\left(\Lambda_{2^{s+1}\left(k_{i}+1\right)-1, q}\right) \\
& \leq 2^{s+1}\left(k_{i}+5\right) \\
& \leq 2 k+2^{s+3} \\
& \leq 2 k+4 T_{q}+\frac{16 T_{q}-24}{k_{i}+1} . \tag{41}
\end{align*}
$$

From (35) and $k \geq 8 T_{q}^{2}-10 T_{q}-3$ we see also

$$
T_{q} k_{i}+5 T_{q}-5 \geq k_{i+1} \geq 2 k+1 \geq 16 T_{q}^{2}-20 T_{q}-5
$$

and thus we have $\left(16 T_{q}-24\right) /\left(k_{i}+1\right) \leq 1$. Therefore, from $T_{q}>1$, (39) and (41) we see that (38) is valid.

## 6 Concluding Remarks

Conjecture 1 was shown to be valid in [5] for the case $\left.\frac{k+5}{2} \right\rvert\,(q-1)$ based on the existence of a special automorphism of $D(k, q)$, in [19] for the case $\frac{k+5}{2}$ is a power of $p$ based on the existence of backtrackless circuit of type $(1,1, \ldots, 1,1)$, and in 19 for the case $\left.\frac{k+5}{2}\right|_{p}(q-1)$ based on the existence of backtrackless circuit of type $\left(1,1, b, b, \ldots, b^{n}, b^{n}\right)$, respectively, where $p$ is the characteristic of $\mathbb{F}_{q}$. A few new results on the girth of $D(k, q)$ are obtained in the present paper based on the existence of backtrackless circuit of type ( $u_{1}, v_{1}, \ldots, u_{2 n}, v_{2 n}$ ) with (18). For example, Conjecture 1 is shown to be valid in Theorem 3 for a new class of infinite pairs $(k, q): p \geq 3, k=$ $2^{s+2}(2 t-1)-5$ for positive integers $s, t$ with $2^{s} \|(q-1)$ and $\left.(2 t-1)\right|_{p}(q-1)$. Almost the recent progresses made on the study of the girth of $D(k, q)$ rely heavily on the computation of the homogeneous polynomial $\rho_{s}\left(\omega_{1}, \ldots, \omega_{n}\right)$. It is then of great interest to investigate the properties of these polynomials further in future.

## References

[1] R. Wenger, Extremal graphs with no $C^{4} s, C^{6} s$, or $C^{10} s$, J. Combin. Theory Ser. B 52(1) (1991) 113-116.
[2] F. Lazebnik, V.A. Ustimenko, New examples of graphs without small cycles and of large size, European J. Combin. 14 (1993) 445-460.
[3] F. Lazebnik, V.A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, Discrete Appl. Math. 60(1995) 275-284.
[4] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, A new series of dense graphs of high girth, Bull. Amer. Math. Soc. 32(1) (1995) 73-79.
[5] Z. Füredi, F. Lazebnik, A. Seress, V.A. Ustimenko, A.J. Woldar, Graphs of pescribed girth and bi-degree, J. Combin. Theory, Ser. B 64(2) (1995) 228-239.
[6] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, A characterization of the components of the graphs $D(k, q)$, Discrete Math. 157 (1996) 271-283.
[7] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, New upper bounds on the order of cages, Electron. J. Combin. 14(R13) (1997) 1-11.
[8] F. Lazebnik, A.J. Woldar, General properties of families of graphs defined by some systems of equations, J. Graph Theory 38(2) (2001) 65-86.
[9] F. Lazebnik, R. Viglione, An infinite series of regular edge- but vertextransitive graphs, J. Graph Theory 41(4) (2002) 249-258.
[10] V.A. Ustimenko, A.J. Woldar, Extremal properties of regular and affine generalized m-gons as tactical configurations, European J. Combin. 24 (2003) 99-111.
[11] F. Lazebnik, R. Viglione, On the connectivity of certain graphs of high girth, Discrete Math. 277 (2004) 309-319.
[12] J. Kim, U. Peled, I. Perepelitsa, V. Pless, S. Friedland, Explicit construction of families of LDPC codes with no 4-cycles, IEEE Trans. Inform. Theory 50(10) (2004) 2378-2388.
[13] V. Dmytrenko, F. Lazebnik, J. Williford, On monomial graphs of girth eight, Finite Fields. Appl. 13 (2007) 828-842.
[14] V. Futorny, V. Ustimenko, On small world semiplanes with generalised Schubert cells, Acta Appl. Math. 98 (2007) 47-61.
[15] V.A. Ustimenko, On linguistic dynamical systems, families of graphs of large girth, and cryptography, J. Math. Sci. 140(3) (2007) 461-471.
[16] V.A. Ustimenko, On the homogeneous algebraic graphs of large girth and their applications, Linear Algebra Appl. 430(7) (2009) 1826-1837.
[17] T. Yan, Y. Tang, Constructions of LDPC codes based on polarity graphs with prescribed girth, in: Asia-Pacific Youth Conference on Communication, 2011APYCC, 2011, pp. 60C62.
[18] M. Polak, U. Romanczuk, V. Ustimenko, A. Wroblewska, On the applications of extremal graph theory to coding theory and cryptography, Electron. Notes Discrete Math. 43 (2013) 329-342.
[19] X. Cheng, W. Chen, Y. Tang, On the girth of the bipartite graph $D(k, q)$, Discrete Math. 335 (2014) 25-34.
[20] X. Cheng, W. Chen, Y. Tang, On the conjecture for the girth of the bipartite graph $D(k, q)$, Discrete Math. 335 (2016) 2384-2392.
[21] A. Dehghan, A. H. Banihashemi, Counting Short Cycles in Bipartite Graphs: A Fast Technique/Algorithm and a Hardness Result, IEEE Trans. Commun. 68(3) (2020) 1378-1390.
[22] M. Xu, X. Cheng, Y. Tang, On the girth cycles of the bipartite graph $D(k, q)$, submitted to Discrete Math. for publication, https://doi.org/10.48550/arXiv.2207.12752


[^0]:    *This work was supported by the National Natural Science Foundation of China (No. 61977056).
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