# ON THE DISTRIBUTION OF THE ENTRIES OF A FIXED-RANK RANDOM MATRIX OVER A FINITE FIELD 

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#### Abstract

Let $r>0$ be an integer, let $\mathbb{F}_{q}$ be a finite field of $q$ elements, and let $\mathcal{A}$ be a nonempty proper subset of $\mathbb{F}_{q}$. Moreover, let $\mathbf{M}$ be a random $m \times n$ rank- $r$ matrix over $\mathbb{F}_{q}$ taken with uniform distribution. We prove, in a precise sense, that, as $m, n \rightarrow+\infty$ and $r, q, \mathcal{A}$ are fixed, the number of entries of $\mathbf{M}$ that belong to $\mathcal{A}$ approaches a normal distribution.


## 1. Introduction

Let $\mathbb{F}_{q}$ be a finite field of $q$ element. For every matrix $\mathbf{M}$ over $\mathbb{F}_{q}$, let wt(M) be the weight of $\mathbb{F}_{q}$, that is, the number of nonzero entries of $\mathbf{M}$.

Migler, Morrison, and Ogle [3] proved a formula for the expected value of $\mathrm{wt}(\mathbf{M})$ when $\mathbf{M}$ is taken at random, with uniform distribution, from the set of $m \times n$ rank-r matrices over $\mathbb{F}_{q}$. Furthermore, they suggested that, as $m, n \rightarrow+\infty$ and $r, q$ are fixed, an appropriate scaling of $\mathrm{wt}(\mathbf{M})$ approaches a normal distribution. Sanna [6] proved this last claim for $q=2$ and assuming that $m / n$ converges to a positive real number.

For every $\mathcal{A} \subseteq \mathbb{F}_{q}$ and for every matrix $\mathbf{M}$ over $\mathbb{F}_{q}$, let $\operatorname{ct}_{\mathcal{A}}(\mathbf{M})$ be the number of entries of $\mathbf{M}$ that belong to $\mathcal{A}$. Moreover, put $\gamma_{\mathcal{A}}(q):=\sum_{a \in \mathcal{A}} \gamma_{a}(q)$, where $\gamma_{0}(q):=q^{-1}-1$ and $\gamma_{a}(q):=q^{-1}$ for each $a \in \mathbb{F}_{q}^{*}$, and let

$$
\begin{aligned}
\mu_{\mathcal{A}}(q, m, n) & :=\left(|\mathcal{A}| q^{-1}-\gamma_{\mathcal{A}}(q) q^{-r}\right) m n \\
\sigma_{\mathcal{A}}^{2}(q, m, n) & :=\gamma_{\mathcal{A}}(q)^{2} q^{-r}\left(1-q^{-r}\right)(m+n) m n
\end{aligned}
$$

for all integers $m, n>0$. Note that $\gamma_{\mathcal{A}}(q) \neq 0$ unless $\mathcal{A}=\varnothing$ or $\mathcal{A}=\mathbb{F}_{q}$.
Our result is the following.
Theorem 1.1. Fix an integer $r>0$ and a nonempty set $\mathcal{A} \subsetneq \mathbb{F}_{q}$. Let $\mathbf{M}$ be taken at random, with uniform distribution, from the set of $m \times n$ rank-r matrices over $\mathbb{F}_{q}$. Then, as $m, n \rightarrow+\infty$, we have that

$$
\begin{equation*}
\frac{\operatorname{ct}_{\mathcal{A}}(\mathbf{M})-\mu_{\mathcal{A}}(q, m, n)}{\sqrt{\sigma_{\mathcal{A}}^{2}(q, m, n)}} \tag{1}
\end{equation*}
$$

converges in distribution to a standard normal random variable.
Roughly speaking, Theorem 1.1 asserts that, as $m$ and $n$ both grow, $\operatorname{ct}_{\mathcal{A}}(\mathbf{M})$ approaches a normal random variable with expected value $\mu_{\mathcal{A}}(q, m, n)$ and variance $\sigma_{\mathcal{A}}^{2}(q, m, n)$. Note that, if the condition on the rank is dropped, that is, if $\mathbf{M}$ is taken at random with uniform distribution from the set of $m \times n$ matrices over $\mathbb{F}_{q}$, then an easy application of the central limit theorem yields that $\operatorname{ct}_{\mathcal{A}}(\mathbf{M})$ approaches a normal random variable with expected value $|\mathcal{A}| q^{-1} m n$ and variance $|\mathcal{A}| q^{-1}\left(1-|\mathcal{A}| q^{-1}\right) m n$.

Before we proceed, let us outline the main ideas of the proof of Theorem 1.1. First, using full-rank factorization and the well-known formula for the number of $m \times n$ rank- $r$ matrices over $\mathbb{F}_{q}$, it is shown that, for the sake of proving Theorem 1.1, we can assume that $\mathbf{M}=\mathbf{X Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are $m \times r$ and $r \times n$ independent random matrices taken with uniform distribution

[^0]from their respective spaces. Second, the event that the product of a row of $\mathbf{X}$ and a column of $\mathbf{Y}$ is equal to a prescribed element of $\mathbb{F}_{q}$ is handled via the Fourier transform of $\mathbb{F}_{q}$ respect to multiplicative characters. The use of multiplicative characters is necessary to conveniently "separate" the entries of $\mathbf{X}$ by the entries of $\mathbf{Y}$ in two factors of a product. However, it introduces some complications (essentially because the Fourier inversion formula holds only for functions $\mathbb{F}_{q}^{t} \rightarrow \mathbb{C}$ that are supported on $\left.\left(\mathbb{F}_{q}^{*}\right)^{t}\right)$, which are dealt with by a kind of Möbius transform. Finally, all of this makes possible to write (1) as a main term, which converges in distribution to a standard normal random variable, plus an error term, which is shown to be negligible.

It might be interesting to strenghten Theorem 1.1 by letting also $r$ goes to infinity, but in a way controlled by $m$ and $n$ (see Remark 5.1).

## 2. General notations and definitions

For every finite set $\mathcal{A}$, we let $|\mathcal{A}|$ be the number of elements of $\mathcal{A}$. For each statement $S$, we let $\mathbb{1}[S]$ be equal to 1 if $S$ is true, and to 0 if $S$ is false. For every event $E$, we let $\mathbb{P}[E]$ be the probability that $E$ occurs. For each real or complex random variable $X$, we write $\mathbb{E}[X]$ and $\mathbb{V}[X]$ for the expected value and the variance of $X$. For every sequence $\left(X_{n}\right)$ of random variables, we write $X_{n} \xrightarrow{d} X$ to denote that $\left(X_{n}\right)$ converges in distribution to $X$. For a complex random variable $Z=X+\boldsymbol{i} Y$, where $X$ and $Y$ are real random variables and $\boldsymbol{i}$ is the imaginary unity, the covariance matrix of $Z$ is the covariance matrix of the random vector $(X, Y)$. Also, we say that $Z$ is a complex normal random variable if the random vector $(X, Y)$ follows a bivariate normal distribution. For each integer $r>0$, we set $[r]:=\{1, \ldots, r\}$. We say that a function $f: \mathcal{X} \rightarrow \mathbb{C}$ is supported on a set $\mathcal{Y}$ if $f(x)=0$ for every $x \in \mathcal{X} \backslash \mathcal{Y}$. We adopt the usual convention that the empty sum and the empty product are equal to 0 and 1 , respectively.

## 3. Preliminaries on the Fourier transform

3.1. Characters of finite fields. We recall some basics facts about characters of finite fields (see, e.g., [2, Chapter 5, Section 1] and [4, Chapter 10, Section 1]).

Given a finite abelian group $G$, a character of $G$ is a group homomorphism $G \rightarrow \mathbb{C}^{*}$. The set of characters of $G$ is denoted by $\widehat{G}$ and is a finite abelian group respect to the pointwise product of functions. The identity of $\widehat{G}$ is the trivial character, which sends each element of $G$ to 1 , while the inverse of each $\chi \in \widehat{G}$ is the pointwise complex conjugation of $\chi$, which is denoted by $\bar{\chi}$.

The additive characters of $\mathbb{F}_{q}$ are the characters of $\mathbb{F}_{q}$ as an additive group. We let $\psi_{0}$ denote the trivial additive character of $\mathbb{F}_{q}$. The multiplicative characters of $\mathbb{F}_{q}$ are the characters of $\mathbb{F}_{q}^{*}$ as a multiplicative group. We let $\chi_{0}$ denote the trivial multiplicative character of $\mathbb{F}_{q}$. Moreover, we extend each multiplicative character $\chi$ of $\mathbb{F}_{q}$ to a function $\mathbb{F}_{q} \rightarrow \mathbb{C}$ by setting $\chi(0):=0$.

The additive and multiplicative characters of $\mathbb{F}_{q}$ satisfy the orthogonality relations:

$$
\begin{equation*}
\frac{1}{q} \sum_{\psi \in \widehat{\mathbb{F}_{q}}} \psi(a)=\mathbb{1}[a=0] \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{P}_{q}^{*}}} \chi(a)=\mathbb{1}[a=1] \tag{3}
\end{equation*}
$$

for every $a \in \mathbb{F}_{q}$, and

$$
\begin{equation*}
\frac{1}{q} \sum_{a \in \mathbb{F}_{q}} \psi(a)=\mathbb{1}\left[\psi=\psi_{0}\right] \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
\frac{1}{q-1} \sum_{a \in \mathbb{F}_{q}} \chi(a)=\mathbb{1}\left[\chi=\chi_{0}\right] \tag{5}
\end{equation*}
$$

for every $\psi \in \widehat{\mathbb{F}_{q}}$ and $\chi \in \widehat{\mathbb{F}_{q}^{*}}$.
For every function $f: \mathbb{F}_{q}^{t} \rightarrow \mathbb{C}$ that is supported on $\left(\mathbb{F}_{q}^{*}\right)^{t}$, the Fourier transform of $f$ is the function $\widehat{f}: \widehat{\mathbb{F}}_{q}^{t} \rightarrow \mathbb{C}$ defined by

$$
\begin{equation*}
\widehat{f}\left(\chi_{1}, \ldots, \chi_{t}\right):=\frac{1}{(q-1)^{t}} \sum_{a_{1}, \ldots, a_{t} \in \mathbb{F}_{q}} f\left(a_{1}, \ldots, a_{t}\right) \overline{\chi_{1}}\left(a_{1}\right) \cdots \overline{\chi_{t}}\left(a_{t}\right) \tag{6}
\end{equation*}
$$

for every $\chi_{1}, \ldots, \chi_{t} \in \widehat{\mathbb{F}_{q}^{*}}{ }^{1}$ From the orthogonality relation (3), it easily follows that

$$
\begin{equation*}
f\left(a_{1}, \ldots, a_{t}\right)=\sum_{\chi_{1}, \ldots, \chi_{t} \in \widehat{\mathbb{F}_{q}^{*}}} \widehat{f}\left(\chi_{1}, \cdots, \chi_{t}\right) \chi_{1}\left(a_{1}\right) \cdots \chi_{t}\left(a_{t}\right) \tag{7}
\end{equation*}
$$

for every $a_{1}, \ldots, a_{t} \in \mathbb{F}_{q}$, which is the Fourier inversion formula.
3.2. Möbius transform. We need a kind of Möbius transform and its corresponding inversion formula, which is essentially a consequence of the inclusion-exclusion principle (see, e.g., [7, Example 3.8.3]).

First, note that from the binomial theorem it easily follows that

$$
\begin{equation*}
\sum_{\mathcal{A} \subseteq \mathcal{B}}(-1)^{|\mathcal{A}|}=\mathbb{1}[\mathcal{B}=\varnothing], \tag{8}
\end{equation*}
$$

for every finite set $\mathcal{B}$.
Throughout the rest of Section 3, let $r>0$ be a fixed integer. For every $\mathcal{S} \subseteq[r]$, we write $a_{\mathcal{S}}$ to denote the $|\mathcal{S}|$-tuple $\left(a_{k_{1}}, \ldots, a_{k_{|\mathcal{S}|}}\right)$, where $k_{1}<\cdots<k_{|\mathcal{S}|}$ are the elements of $\mathcal{S}$. (If $\mathcal{S}$ is empty, then $a_{\mathcal{S}}$ is the empty tuple). Moreover, we write $a_{(\mathcal{S})}$ to denote the $r$-tuple ( $b_{1}, \ldots, b_{r}$ ), where $b_{k}:=0$ if $k \notin \mathcal{S}$, and $b_{k}:=a_{k}$ if $k \in \mathcal{S}$.

For every function $f: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$ and for every $\mathcal{S} \subseteq[r]$, we define the function $f_{\mathcal{S}}: \mathbb{F}_{q}^{|\mathcal{S}|} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
f_{\mathcal{S}}\left(a_{\mathcal{S}}\right):=\sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right), \tag{9}
\end{equation*}
$$

for every $a_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|}$.
Lemma 3.1. Let $f: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$. Then, for every $\mathcal{S} \subseteq[r]$, the function $f_{\mathcal{S}}$ is supported on $\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{S}|}$. Moreover, we have that

$$
\begin{equation*}
f\left(a_{1}, \ldots, a_{r}\right)=\sum_{\mathcal{S} \subseteq[r]} f_{\mathcal{S}}\left(a_{\mathcal{S}}\right) \tag{10}
\end{equation*}
$$

for every $a_{1}, \ldots, a_{r} \in \mathbb{F}_{q}$.
Proof. First, let us prove that for every $\mathcal{S} \subseteq[r]$ the function $f_{\mathcal{S}}$ is supported on $\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{S}|}$. Pick any $a_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|} \backslash\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{S}|}$. Hence, there exists $k_{0} \in \mathcal{S}$ such that $a_{k_{0}}=0$. Therefore, by (9) we have that

$$
\begin{aligned}
f_{\mathcal{S}}\left(a_{\mathcal{S}}\right) & =\sum_{\mathcal{T} \subseteq \mathcal{S} \backslash\left\{k_{0}\right\}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right)+\sum_{\left\{k_{0}\right\} \subseteq \mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right) \\
& =\sum_{\mathcal{T} \subseteq \mathcal{S} \backslash\left\{k_{0}\right\}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right)-\sum_{\mathcal{T}^{\prime} \subseteq \mathcal{S} \backslash\left\{k_{0}\right\}}(-1)^{\left|\mathcal{S} \backslash \mathcal{T}^{\prime}\right|} f\left(a_{\left(\mathcal{T}^{\prime}\right)}\right)=0,
\end{aligned}
$$

where we used the fact that each set $\mathcal{T}$ satisfying $\left\{k_{0}\right\} \subseteq \mathcal{T} \subseteq \mathcal{S}$ can be written in a unique way as $\mathcal{T}=\mathcal{T}^{\prime} \cup\left\{k_{0}\right\}$ with $\mathcal{T}^{\prime} \subseteq \mathcal{S} \backslash\left\{k_{0}\right\}$. The claim is proven.

Let us prove (10). From (9) and (8), we get that

$$
\begin{aligned}
\sum_{\mathcal{S} \subseteq[r]} f_{\mathcal{S}}\left(a_{\mathcal{S}}\right) & =\sum_{\mathcal{S} \subseteq[r]} \sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right)=\sum_{\mathcal{T} \subseteq[r]} f\left(a_{(\mathcal{T})}\right) \sum_{\mathcal{T} \subseteq \mathcal{S} \subseteq[r]}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} \\
& =\sum_{\mathcal{T} \subseteq[r]} f\left(a_{(\mathcal{T})}\right) \sum_{\mathcal{S}^{\prime} \subseteq[r] \backslash \mathcal{T}}(-1)^{\left|\mathcal{S}^{\prime}\right|}=f\left(a_{1}, \ldots, a_{r}\right),
\end{aligned}
$$

where we wrote $\mathcal{S}=\mathcal{S}^{\prime} \cup \mathcal{T}$. The proof is complete.

[^1]3.3. Möbius-Fourier inversion formula. We can combine the results of Sections 3.1 and 3.2 to obtain a Möbius-Fourier inversion formula.

Lemma 3.2. Let $f: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$. Then we have that

$$
f\left(a_{1}, \ldots, a_{r}\right)=\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{\mathcal { S }} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|} \widehat{f_{\mathcal{S}}}\left(\chi_{\mathcal{S}}\right) \prod_{k \in \mathcal{S}} \chi_{k}\left(a_{k}\right)
$$

for every $a_{1}, \ldots, a_{r} \in \mathbb{F}_{q}$.
Proof. The claim easily follows from the Fourier inversion formula (7) and Lemma 3.1.
For every function $f: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$ and for every $\mathcal{S} \subseteq[r]$, let $f_{(\mathcal{S})}: \mathbb{F}_{q}^{|\mathcal{S}|} \rightarrow \mathbb{C}$ be the function defined by $f_{(\mathcal{S})}\left(a_{\mathcal{S}}\right):=f\left(a_{(\mathcal{S})}\right)$ for each $a_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|}$.
Lemma 3.3. Let $f: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$ and $\mathcal{S} \subseteq[r]$. Then we have that

$$
\widehat{f_{\mathcal{S}}}\left(\chi_{\mathcal{S}}\right)=\sum_{\left\{\chi_{k} \neq \chi_{0}: k \in \mathcal{S}\right\} \subseteq \mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}| \widehat{f_{(\mathcal{T})}}\left(\chi_{\mathcal{T}}\right), ~, ~, ~ . ~}
$$

for every $\chi_{\mathcal{S}} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|$.
Proof. From (6) and (9), we get that

$$
\begin{align*}
& \widehat{f_{\mathcal{S}}}\left(\chi_{\mathcal{S}}\right)=\frac{1}{(q-1)^{|\mathcal{S}|}} \sum_{a_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|}} f_{\mathcal{S}}\left(a_{\mathcal{S}}\right) \prod_{k \in \mathcal{S}} \overline{\chi_{k}}\left(a_{k}\right)  \tag{11}\\
& =\frac{1}{(q-1)^{|\mathcal{S}|}} \sum_{a_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|}} \sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} f\left(a_{(\mathcal{T})}\right) \prod_{k \in \mathcal{S}} \overline{\chi_{k}}\left(a_{k}\right) \\
& =\sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|}\left(\frac{1}{(q-1)^{|\mathcal{T}|}} \sum_{a_{\mathcal{T} \in \mathbb{F}_{q}^{|\mathcal{T}|}}} f_{(\mathcal{T})}\left(a_{\mathcal{T}}\right) \prod_{k \in \mathcal{T}} \overline{\chi_{k}}\left(a_{k}\right)\right)
\end{align*}
$$

Furthermore, for every $\mathcal{U} \subseteq[r]$, we have that

$$
\begin{equation*}
\sum_{a_{U} \in \mathbb{F}_{q}^{|\mathcal{U}|}} \prod_{k \in \mathcal{U}} \overline{\chi_{k}}\left(a_{k}\right)=\prod_{k \in \mathcal{U}}\left(\sum_{a \in \mathbb{F}_{q}} \overline{\chi_{k}}(a)\right)=(q-1)^{|\mathcal{U}|} \mathbb{1}\left[\chi_{k}=\chi_{0} \text { for each } k \in \mathcal{U}\right] \tag{12}
\end{equation*}
$$

where we employed the orthogonality relation (5). At this point, the claim follows by combining (6), (11), and (12).
3.4. Some Fourier transforms. For every $a \in \mathbb{F}_{q}$, define the function $f^{(a)}: \mathbb{F}_{q}^{r} \rightarrow \mathbb{C}$ by

$$
f^{(a)}\left(a_{1}, \ldots, a_{r}\right)=\mathbb{1}\left[\sum_{k=1}^{r} a_{k}=a\right]
$$

for every $a_{1}, \ldots, a_{r} \in \mathbb{F}_{q}^{r}$.
Furthermore, let $\chi_{\mathbf{0}}$ denote a tuple $\left(\chi_{0}, \ldots, \chi_{0}\right)$, where the length will be always clear from the context.

Lemma 3.4. For every $a \in \mathbb{F}_{q}$ and $\mathcal{T} \subseteq[r]$, we have that

$$
\widehat{f_{(\mathcal{T})}^{(a)}}\left(\chi_{\mathbf{0}}\right)=\frac{1}{q}-\frac{\gamma_{a}(q)}{(1-q)^{|\mathcal{T}|}}
$$

Proof. This is essentially the evaluation of a generalized Jacobi sum of trivial characters, which is a well-known subject (see, e.g., [4, Theorem 6.1.35]), but we include the details for completeness.

First, from (4) it follows that
for every $\psi \in \widehat{\mathbb{F}_{q}}$. Then, from (6), (2), and (13), we get that

$$
\begin{aligned}
& \widehat{f_{(\mathcal{T})}^{(a)}}\left(\chi_{\mathbf{0}}\right)=\frac{1}{(q-1)^{|\mathcal{T}|}} \sum_{a_{\mathcal{T}} \in \mathbb{F}_{q}^{|\mathcal{T}|}} f_{(\mathcal{T})}^{(a)}\left(a_{\mathcal{T}}\right) \prod_{k \in \mathcal{T}} \overline{\chi_{0}}\left(a_{k}\right)=\frac{1}{(q-1)^{|\mathcal{T}|}} \sum_{a_{\mathcal{T}} \in\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{T}|}} f_{(\mathcal{T})}^{(a)}\left(a_{\mathcal{T}}\right) \\
& =\frac{1}{(q-1)^{|\mathcal{T}|}} \sum_{a \mathcal{T} \in\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{T}|}} \mathbb{1}\left[\sum_{k \in \mathcal{T}} a_{k}=a\right]=\frac{1}{(q-1)^{|\mathcal{T}|}} \sum_{a \mathcal{T} \in\left(\mathbb{F}_{q}^{*}\right)^{|\mathcal{T}|}} \frac{1}{q} \sum_{\psi \in \widehat{\mathbb{P}_{q}}} \psi\left(\sum_{k \in \mathcal{T}} a_{k}-a\right) \\
& =\frac{1}{q(q-1)^{|\mathcal{T}|}} \sum_{\psi \in \widehat{\mathbb{P}_{q}}} \sum_{a \mathcal{T} \in\left(\mathbb{F}_{q}^{*}\right)|\mathcal{T}|} \prod_{k \in \mathcal{T}} \psi\left(a_{k}\right) \bar{\psi}(a)=\frac{1}{q}+\frac{1}{q(1-q)^{|\mathcal{T}|}} \sum_{\psi \in \widehat{\mathbb{F}_{q} \backslash\left\{\psi_{0}\right\}}} \bar{\psi}(a) \\
& =\frac{1}{q}+\frac{1}{q(1-q)^{|\mathcal{T}|}}\left(\sum_{\psi \in \widehat{\mathbb{P}_{q}}} \bar{\psi}(a)-1\right)=\frac{1}{q}-\frac{\gamma_{a}(q)}{(1-q)^{|\mathcal{T}|}},
\end{aligned}
$$

since $\gamma_{a}(q)=q^{-1}-\mathbb{1}[a=0]$. The proof is complete.
Lemma 3.5. For every $a \in \mathbb{F}_{q}$ and $\mathcal{S} \subseteq[r]$, we have that

$$
\widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathbf{0}}\right)=\frac{\mathbb{1}[\mathcal{S}=\varnothing]}{q}-\gamma_{a}(q)\left(\frac{1}{q}-1\right)^{-|\mathcal{S}|}
$$

Proof. By Lemma 3.3 and Lemma 3.4, we have that

$$
\begin{aligned}
\widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathbf{0}}\right) & =\sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|} \widehat{f_{(\mathcal{T})}^{(a)}}\left(\chi_{\mathbf{0}}\right)=\sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|}\left(\frac{1}{q}-\frac{\gamma_{a}(q)}{\left.(1-q)^{\mid \mathcal{T \mathcal { T }}}\right)}\right. \\
& =\frac{1}{q} \sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|}-\gamma_{a}(q) \sum_{\mathcal{T} \subseteq \mathcal{S}}(-1)^{|\mathcal{S} \backslash \mathcal{T}|}(1-q)^{|\mathcal{T}|} \\
& =\frac{\mathbb{1}[\mathcal{S}=\varnothing]}{q}-\gamma_{a}(q)\left(\frac{1}{q}-1\right)^{-|\mathcal{S}|},
\end{aligned}
$$

where we used (8) and the more general fact that

$$
\sum_{\mathcal{A} \subseteq \mathcal{B}} s^{|\mathcal{B} \backslash \mathcal{A}|} t^{|\mathcal{A}|}=(s+t)^{|\mathcal{B}|}
$$

for every finite set $\mathcal{B}$ and for all real numbers $s$ and $t$.

## 4. Further preliminaries

For every field $\mathbb{K}$, let $\mathbb{K}^{m \times n}$ be the vector space of $m \times n$ matrices over $\mathbb{K}$, and let $\mathbb{K}^{m \times n, r}$ be the set of $m \times n$ rank- $r$ matrices over $\mathbb{K}$. The next lemma regards the full-rank factorization of matrices and it is well known (cf. [5, Theorem 2]).
Lemma 4.1. Let $\mathbb{K}$ be a field. For every $\mathbf{N} \in \mathbb{K}^{m \times n, r}$ there exist $\mathbf{X}_{0} \in \mathbb{K}^{m \times r, r}$ and $\mathbf{Y}_{0} \in \mathbb{K}^{r \times n, r}$ such that $\mathbf{N}=\mathbf{X}_{0} \mathbf{Y}_{0}$. Moreover, if $\mathbf{N}=\mathbf{X Y}$ for some $\mathbf{X} \in \mathbb{K}^{m \times r}$ and $\mathbf{Y} \in \mathbb{K}^{r \times n}$, then there exists $\mathbf{R} \in \mathbb{K}^{r \times r, r}$ such that $\mathbf{X}=\mathbf{X}_{0} \mathbf{R}$ and $\mathbf{Y}=\mathbf{R}^{-1} \mathbf{Y}_{0}$.

Proof. See, e.g., [6, Lemma 2.1]. There the second part of the lemma is stated with $\mathbf{X} \in \mathbb{K}^{m \times r, r}$ and $\mathbf{Y} \in \mathbb{K}^{r \times n, r}$ instead of $\mathbf{X} \in \mathbb{K}^{m \times r}$ and $\mathbf{Y} \in \mathbb{K}^{r \times n}$. However, if $\mathbf{X} \in \mathbb{K}^{m \times r}$ and $\mathbf{Y} \in \mathbb{K}^{r \times n}$ satisfy $\mathbf{X Y} \in \mathbb{K}^{m \times n, r}$, then $\mathbf{X} \in \mathbb{K}^{m \times r, r}$ and $\mathbf{Y} \in \mathbb{K}^{r \times n, r}$. Therefore, the two versions are equivalent.
Lemma 4.2. Let $\mathbf{M} \in \mathbb{F}_{q}^{m \times n, r}, \mathbf{X} \in \mathbb{F}_{q}^{m \times r}$, and $\mathbf{Y} \in \mathbb{F}_{q}^{r \times n}$ be independent random matrices uniformly distributed in their respective spaces. Then we have that

$$
\begin{equation*}
\sum_{\mathbf{N} \in \mathbb{F}_{q}^{m \times n}}|\mathbb{P}[\mathbf{X Y}=\mathbf{N}]-\mathbb{P}[\mathbf{M}=\mathbf{N}]| \rightarrow 0 \tag{14}
\end{equation*}
$$

as $m, n \rightarrow+\infty$ and $r$ is fixed.
Proof. It is well-known (see, e.g., [3, Formula 3]) that

$$
\begin{equation*}
\left|\mathbb{F}_{q}^{s \times t, r}\right|=\prod_{i=0}^{r-1} \frac{\left(q^{s}-q^{i}\right)\left(q^{t}-q^{i}\right)}{q^{r}-q^{i}} \tag{15}
\end{equation*}
$$

for all integers $s, t, r>0$ with $r \leq \min (s, t)$.
Furthermore, we have that

$$
\begin{equation*}
\frac{\prod_{i=0}^{r-1}\left(q^{m}-q^{i}\right)\left(q^{n}-q^{i}\right)}{q^{m r} \cdot q^{r n}}=\prod_{i=0}^{r-1} \frac{\left(q^{m}-q^{i}\right)\left(q^{n}-q^{i}\right)}{q^{m} \cdot q^{n}}=\prod_{i=0}^{r-1}\left(1-q^{i-m}\right)\left(1-q^{i-n}\right) \rightarrow 1 \tag{16}
\end{equation*}
$$

as $m, n \rightarrow+\infty$ and $r$ is fixed.
Let us split the sum in (14) into three sums $\Sigma_{(<)}, \Sigma_{(=)}, \Sigma_{(>)}$according to the rank of $\mathbf{N}$ being less than, equal to, or greater than $r$, respectively. We have to prove that, in the aforementioned limit, each of these sums goes to zero.

For every matrix $\mathbf{Z}$ over $\mathbb{F}_{q}$, let $\operatorname{rk}(\mathbf{Z})$ denote the rank of $\mathbf{Z}$. From (15) and (16), we get that

$$
\begin{aligned}
\Sigma_{(<)} & =\sum_{\substack{\mathbf{N} \in \mathbb{F}_{q}^{m \times n} \\
\operatorname{rk}(\mathbf{N})<r}} \mathbb{P}[\mathbf{X Y}=\mathbf{N}]=\mathbb{P}[\operatorname{rk}(\mathbf{X Y})<r]=1-\mathbb{P}\left[\mathbf{X} \in \mathbb{F}_{q}^{m \times r, r}\right] \mathbb{P}\left[\mathbf{Y} \in \mathbb{F}_{q}^{r \times n, r}\right] \\
& =1-\frac{\left|\mathbb{F}_{q}^{m \times r, r}\right|\left|\mathbb{F}_{q}^{r \times n, r}\right|}{\left|\mathbb{F}_{q}^{m \times r}\right|\left|\mathbb{F}_{q}^{r \times n}\right|}=1-\frac{\prod_{i=0}^{r-1}\left(q^{m}-q^{i}\right)\left(q^{n}-q^{i}\right)}{q^{m r} \cdot q^{r n}} \rightarrow 0,
\end{aligned}
$$

where we used the fact that $\operatorname{rk}(\mathbf{X Y}) \leq r$ with equality if and only if $\operatorname{rk}(\mathbf{X})=\operatorname{rk}(\mathbf{Y})=r$.
If $\mathbf{N} \in \mathbb{F}_{q}^{m \times n, r}$ then, by Lemma 4.1, there exist matrices $\mathbf{X}_{0} \in \mathbb{F}_{q}^{m \times r, r}$ and $\mathbf{Y}_{0} \in \mathbb{F}_{q}^{r \times n, r}$ such that $\mathbf{N}=\mathbf{X}_{0} \mathbf{Y}_{0}$. Moreover, again by Lemma 4.1, we have that $\mathbf{X Y}=\mathbf{N}$ if and only if there exists $\mathbf{R} \in \mathbb{F}_{q}^{r \times r, r}$ such that $\mathbf{X}=\mathbf{X}_{0} \mathbf{R}$ and $\mathbf{Y}=\mathbf{R}^{-1} \mathbf{Y}_{0}$. Consequently, we have that

$$
\mathbb{P}[\mathbf{X Y}=\mathbf{N}]=\sum_{\mathbf{R} \in \mathbb{F}_{q}^{r \times r, r}} \mathbb{P}\left[\mathbf{X}=\mathbf{X}_{0} \mathbf{R}\right] \mathbb{P}\left[\mathbf{Y}=\mathbf{R}^{-1} \mathbf{Y}_{0}\right]=\frac{\left|\mathbb{F}_{q}^{r \times r, r}\right|}{\left|\mathbb{F}_{q}^{m \times r}\right|\left|\mathbb{F}_{q}^{r \times n}\right|}
$$

Therefore, we get that

$$
\begin{aligned}
\Sigma_{(=)} & =\sum_{\mathbf{N} \in \mathbb{F}_{q}^{m \times n, r}}|\mathbb{P}[\mathbf{X Y}=\mathbf{N}]-\mathbb{P}[\mathbf{M}=\mathbf{N}]|=\sum_{\mathbf{N} \in \mathbb{F}_{q}^{m \times n, r}}\left|\frac{\left|\mathbb{F}_{q}^{r \times r, r}\right|}{\left|\mathbb{F}_{q}^{m \times r}\right|\left|\mathbb{F}_{q}^{r \times n}\right|}-\frac{1}{\left|\mathbb{F}_{q}^{m \times n, r}\right|}\right| \\
& =\left|\frac{\left|\mathbb{F}_{q}^{r \times r, r}\right|\left|\mathbb{F}_{q}^{m \times n, r}\right|}{\left|\mathbb{F}_{q}^{m \times r}\right|\left|\mathbb{F}_{q}^{r \times n}\right|}-1\right|=\left|\frac{\prod_{i=0}^{r-1}\left(q^{m}-q^{i}\right)\left(q^{n}-q^{i}\right)}{q^{m r} \cdot q^{r n}}-1\right| \rightarrow 0,
\end{aligned}
$$

where we employed (15) and (16).
Finally, since XY and $\mathbf{M}$ have rank not exceeding $r$, it follows that $\Sigma_{(>)}=0$. Thus all the three sums go to zero and the proof is complete.

The next result is a version of Slutsky's lemma (cf. [9, Lemma 2.8]).

Lemma 4.3. Let $\left(U_{n}\right)$ and $\left(V_{n}\right)$ be sequences of complex random variables such that $U_{n} \xrightarrow{d} U$ and $V_{n} \xrightarrow{d} c$ as $n \rightarrow+\infty$, where $U$ is a random variable and $c$ is a constant. Then we have that:
(i) $U_{n}+V_{n} \xrightarrow{d} U+c$; and
(ii) $U_{n} V_{n} \xrightarrow{d} U c$;
as $n \rightarrow+\infty$.
Proof. In [9, Lemma 2.8] the result is stated for real random variables. However, the proof can be easily adapted by identifying $\mathbb{C}$ with $\mathbb{R}^{2}$ and applying [9, Theorem 2.7] accordingly; noting that, with this identification, the addition and the multiplication of two complex numbers are continuous functions $\mathbb{R}^{2} \times \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$.

Lemma 4.4. Let $c_{1}, c_{2}$ be real numbers, and let $N_{1}, N_{2}$ be independent normal random variables of expected values $\mu_{1}, \mu_{2}$ and variances $\sigma_{1}^{2}, \sigma_{2}^{2}$, respectively. Then $c_{1} N_{1}+c_{2} N_{2}$ is a normal random variable of expected value $c_{1} \mu_{1}+c_{2} \mu_{2}$ and variance $c_{1}^{2} \sigma_{1}^{2}+c_{2}^{2} \sigma_{2}^{2}$.
Proof. This fact is well known (cf. [8, Exercise 2.1.9]).

## 5. Proof of Theorem 1.1

Let $m, n, r>0$ be integers with $r \leq \min (m, n)$. Let $\mathbf{X} \in \mathbb{F}_{q}^{m \times r}$ and $\mathbf{Y} \in \mathbb{F}_{q}^{r \times n}$ be independent random matrices taken with uniform distribution from their respective spaces.

For every $\mathcal{S} \subseteq[r]$ and $\chi_{\mathcal{S}} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|$, define the complex random variables

$$
\begin{equation*}
X_{\mathcal{S}, \chi}:=\sum_{i=1}^{m} \prod_{k \in \mathcal{S}} \chi_{k}\left(x_{i, k}\right) \quad \text { and } \quad Y_{\mathcal{S}, \chi}:=\sum_{j=1}^{n} \prod_{k \in \mathcal{S}} \chi_{k}\left(y_{k, j}\right) \tag{17}
\end{equation*}
$$

and also the real random variables

$$
Z:=\sum_{i=1}^{m} \prod_{k=1}^{r}\left(1-\chi_{0}\left(x_{i, k}\right)\right) \quad \text { and } \quad W:=\sum_{i=1}^{n} \prod_{k=1}^{r}\left(1-\chi_{0}\left(y_{k, j}\right)\right)
$$

where $x_{i, j}$ and $y_{i, j}$ denote the entries of $\mathbf{X}$ and $\mathbf{Y}$, respectively.
The next two lemmas provide the expected values of $X_{\mathcal{S}, \chi}$ and $Y_{\mathcal{S}, \chi}$, and the expected values and the variances of $Z$ and $W$.
Lemma 5.1. For all $\mathcal{S} \subseteq[r]$ and $\chi_{\mathcal{S}} \in{\widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|}$, we have that

$$
\mathbb{E}\left[X_{\mathcal{S}, \chi}\right]=C\left(\chi_{\mathcal{S}}\right)\left(1-\frac{1}{q}\right)^{|\mathcal{S}|} m \quad \text { and } \quad \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]=C\left(\chi_{\mathcal{S}}\right)\left(1-\frac{1}{q}\right)^{|\mathcal{S}|} n
$$

where

$$
C\left(\chi_{\mathcal{S}}\right):=\mathbb{1}\left[\chi_{k}=\chi_{0} \text { for each } k \in \mathcal{S}\right]
$$

Proof. Fix $\chi \in \widehat{\mathbb{F}_{q}^{*}}$ and let $c \in \mathbb{F}_{q}$ be taken at random with uniform distribution. From (5) it follows that

$$
\mathbb{E}[\chi(c)]=\frac{1}{q} \sum_{a \in \mathbb{F}_{q}} \chi(a)=\left(1-\frac{1}{q}\right) \mathbb{1}\left[\chi=\chi_{0}\right]
$$

Consequently, if $c_{\mathcal{S}} \in \mathbb{F}_{q}^{|\mathcal{S}|}$ is a random tuple taken with uniform distribution, then

$$
\mathbb{E}\left[\prod_{k \in \mathcal{S}} \chi_{k}\left(c_{k}\right)\right]=\prod_{k \in \mathcal{S}} \mathbb{E}\left[\chi_{k}\left(c_{k}\right)\right]=C\left(\chi_{\mathcal{S}}\right)\left(1-\frac{1}{q}\right)^{|\mathcal{S}|}
$$

At this point, the formulas for the expected values of $X_{\mathcal{S}, \chi}$ and $Y_{\mathcal{S}, \chi}$ follow by linearity.

Lemma 5.2. We have that

$$
\mathbb{E}[Z]=\frac{1}{q^{r}} m, \quad \mathbb{V}[Z]=\frac{1}{q^{r}}\left(1-\frac{1}{q^{r}}\right) m, \quad \mathbb{E}[W]=\frac{1}{q^{r}} n, \quad \mathbb{V}[W]=\frac{1}{q^{r}}\left(1-\frac{1}{q^{r}}\right) n .
$$

Proof. The claim follows easily by noticing that $Z$ and $W$ are binomial random variables of $m$ and $n$ trials, respectively, and probability of success equal to $q^{-r}$.

We can now prove a formula for $\operatorname{ct}_{\mathcal{A}}(\mathbf{X Y})$, for every $\mathcal{A} \subseteq \mathbb{F}_{q}$.
Lemma 5.3. For every $a \in \mathbb{F}_{q}$, we have that

$$
\begin{align*}
& \sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] X_{\mathcal{S}, \chi}=\frac{1}{q} m n-\gamma_{a}(q) n Z,  \tag{18}\\
& \sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{*}} \mathcal{S} \mid} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[X_{\mathcal{S}, \chi}\right] Y_{\mathcal{S}, \chi}=\frac{1}{q} m n-\gamma_{a}(q) m W,  \tag{19}\\
& \sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \in \widehat{\mathcal{S}} \mid \overrightarrow{\mathbb{F}_{q}^{*}}} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[X_{\mathcal{S}, \chi}\right] \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]=\left(\frac{1}{q}-\frac{\gamma_{a}(q)}{q^{r}}\right) m n . \tag{20}
\end{align*}
$$

Proof. From Lemma 5.1 and Lemma 3.5, it follows that

$$
\begin{aligned}
& \sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] X_{\mathcal{S}, \chi}=n \sum_{\mathcal{S} \subseteq[r]} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathbf{0}}\right)\left(1-\frac{1}{q}\right)^{|\mathcal{S}|} X_{\mathcal{S}, \chi_{0}} \\
& \quad= n \sum_{\mathcal{S} \subseteq[r]}\left(\frac{\mathbb{1}[\mathcal{S}=\varnothing]}{q}-\gamma_{a}(q)\left(\frac{1}{q}-1\right)^{-|\mathcal{S}|}\right)\left(1-\frac{1}{q}\right)^{|\mathcal{S}|} X_{\mathcal{S}, \chi_{0}} \\
& \quad=\frac{1}{q} m n-\gamma_{a}(q) n \sum_{\mathcal{S} \subseteq[r]}(-1)^{|\mathcal{S}|} X_{\mathcal{S}, \chi_{0}}
\end{aligned}
$$

since $X_{\varnothing, \chi_{0}}=m$. Furthermore, from (17), we have that

$$
\begin{aligned}
\sum_{\mathcal{S} \subseteq[r]}(-1)^{|\mathcal{S}|} X_{\mathcal{S}, \chi_{0}} & =\sum_{\mathcal{S} \subseteq[r]}(-1)^{|\mathcal{S}|} \sum_{i=1}^{m} \prod_{k \in \mathcal{S}} \chi_{0}\left(x_{i, k}\right)=\sum_{i=1}^{m} \sum_{\mathcal{S} \subseteq[r]} \prod_{k \in \mathcal{S}}\left(-\chi_{0}\left(x_{i, k}\right)\right) \\
& =\sum_{i=1}^{m} \prod_{k=1}^{r}\left(1-\chi_{0}\left(x_{i, k}\right)\right)=Z,
\end{aligned}
$$

and (18) follows. The proof of (19) proceeds similarly.
Finally, taking the expected value of both sides of (18), and employing Lemma 5.2, we obtain (20).
Lemma 5.4. For every $\mathcal{A} \subseteq \mathbb{F}_{q}$, we have that

$$
\begin{aligned}
\operatorname{ct}_{\mathcal{A}}(\mathbf{X Y})= & \mu_{\mathcal{A}}(q, m, n)+\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{\mathcal { S }} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|} \sum_{a \in \mathcal{A}} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right)\left(X_{\mathcal{S}, \chi}-\mathbb{E}\left[X_{\mathcal{S}, \chi}\right]\right)\left(Y_{\mathcal{S}, \chi}-\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]\right) \\
& -\gamma_{\mathcal{A}}(q) n(Z-\mathbb{E}[Z])-\gamma_{\mathcal{A}}(q) m(W-\mathbb{E}[W]) .
\end{aligned}
$$

Proof. Let $a \in \mathbb{F}_{q}$. From Lemma 3.2 and (17), we have that

$$
\begin{aligned}
\operatorname{ct}_{\{a\}}(\mathbf{X Y}) & =\sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{1}\left[\sum_{k=1}^{r} x_{i, k} y_{k, j}=a\right]=\sum_{i=1}^{m} \sum_{j=1}^{n} f^{(a)}\left(x_{i, 1} y_{1, j}, \ldots, x_{i, r} y_{r, j}\right) \\
& =\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{\mathrm{F}}}|\mathcal{S}|} \widehat{f_{\mathcal{S}}^{(a)}}(\chi \mathcal{S}) \prod_{k \in \mathcal{S}} \chi_{k}\left(x_{i, k} y_{k, j}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{|S|}}} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right)\left(\sum_{i=1}^{m} \prod_{k \in \mathcal{S}} \chi_{k}\left(x_{i, k}\right)\right)\left(\sum_{j=1}^{n} \prod_{k \in \mathcal{S}} \chi_{k}\left(y_{k, j}\right)\right) \\
& =\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{|S|}}} f_{\mathcal{S}}^{(a)}\left(\chi_{\mathcal{S}}\right) X_{\mathcal{S}, \chi} Y_{\mathcal{S}, \chi} .
\end{aligned}
$$

Then, from the identity

$$
\begin{aligned}
X_{\mathcal{S}, \chi} Y_{\mathcal{S}, \chi}=\left(X_{\mathcal{S}, \chi}\right. & \left.-\mathbb{E}\left[X_{\mathcal{S}, \chi}\right]\right)\left(Y_{\mathcal{S}, \chi}-\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]\right) \\
& +\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] X_{\mathcal{S}, \chi}+\mathbb{E}\left[X_{\mathcal{S}, \chi}\right] Y_{\mathcal{S}, \chi}-\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] \mathbb{E}\left[X_{\mathcal{S}, \chi}\right],
\end{aligned}
$$

we get that

$$
\begin{aligned}
& \operatorname{ct}_{\{a\}}(\mathbf{X Y})=\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{F}_{q}^{\mathcal{P}}} \mid} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right)\left(X_{\mathcal{S}, \chi}-\mathbb{E}\left[X_{\mathcal{S}, \chi}\right]\right)\left(Y_{\mathcal{S}, \chi}-\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]\right) \\
& +\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{\mathcal { S }} \in \widehat{\mathbb{F}_{q}^{*}}|\mathcal{S}|} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] X_{\mathcal{S}, \chi}+\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \in \widehat{\mathcal{\mathbb { F } _ { q } ^ { * }}|\mathcal{S}|}} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[X_{\mathcal{S}, \chi}\right] Y_{\mathcal{S}, \chi} \\
& -\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \in \widehat{\mathcal{\mathbb { F } _ { q } ^ { * }}|\mathcal{S}|}} \widehat{f_{\mathcal{S}}^{(a)}}\left(\chi_{\mathcal{S}}\right) \mathbb{E}\left[Y_{\mathcal{S}, \chi}\right] \mathbb{E}\left[X_{\mathcal{S}, \chi}\right] .
\end{aligned}
$$

At this point, the claim follows easily by applying Lemma 5.3 and Lemma 5.2, and by summing over all $a \in \mathcal{A}$.

Fix a nonempty $\mathcal{A} \subsetneq \mathbb{F}_{q}$ and, for the sake of brevity, let

$$
\widetilde{c t}_{\mathcal{A}}(\mathbf{N}):=\frac{\operatorname{ct}_{\mathcal{A}}(\mathbf{N})-\mu_{\mathcal{A}}(q, m, n)}{\sqrt{\sigma_{\mathcal{A}}^{2}(q, m, n)}}
$$

for every $\mathbf{N} \in \mathbb{F}_{q}^{m \times n}$. Moreover, hereafter, let $m, n \rightarrow+\infty$.
Note that each of the complex random variables $X_{\mathcal{S}, \chi}$ and $Y_{\mathcal{S}, \chi}$ is the sum of independent identically distributed random variables with finite covariance matrices. Therefore, by the Central Limit Theorem in $\mathbb{R}^{2}$ (see, e.g., [1, Theorem 3.9.6]), we have that $\left(X_{\mathcal{S}, \chi}-\mathbb{E}\left[X_{\mathcal{S}, \chi}\right]\right) / \sqrt{m}$ and $\left(Y_{\mathcal{S}, \chi}-\mathbb{E}\left[Y_{\mathcal{S}, \chi}\right]\right) / \sqrt{n}$ converge in distribution to some complex normal random variables, which we call $X_{\mathcal{S}, \chi}^{\prime}$ and $Y_{\mathcal{S}, \chi}^{\prime}$, respectively.

Similarly, each of the real random variables $Z$ and $W$ is the sum of independent identically distributed random variables. Hence, it follows from the Central Limit Theorem (in $\mathbb{R}$ ) that $(Z-\mathbb{E}[Z]) / \sqrt{\mathbb{V}[Z]}$ and $(W-\mathbb{E}[W]) / \sqrt{\mathbb{V}[W]}$ converge in distribution to standard normal random variables, which we call $Z^{\prime}$ and $W^{\prime}$, respectively.

From Lemma 5.4 and Lemma 5.2, it follows that

$$
\begin{equation*}
\widetilde{c t}_{\mathcal{A}}(\mathbf{X Y})=\sum_{\mathcal{S} \subseteq[r]} \sum_{\chi \mathcal{S} \in \widehat{\mathbb{P}_{q}^{*}} \mathcal{S} \mid} \frac{c_{\mathcal{A}, \mathcal{S}, \chi}(q)}{\sqrt{m+n}} X_{\mathcal{S}, \chi}^{\prime} Y_{\mathcal{S}, \chi}^{\prime}-\frac{Z^{\prime}}{\sqrt{1+m / n}}-\frac{W^{\prime}}{\sqrt{1+n / m}} \tag{21}
\end{equation*}
$$

where each $c_{\mathcal{A}, \mathcal{S}, \chi}(q)$ depends only on $\mathcal{A}, \mathcal{S}, \chi_{\mathcal{S}}, q, r$, and not on $m$ and $n$.
Since $X_{\mathcal{S}, \chi}^{\prime}$ and $Y_{\mathcal{S}, \chi}^{\prime}$ are independent, their product converges in distribution to $\widetilde{X}_{\mathcal{S}, \chi} \widetilde{Y}_{\mathcal{S}, \chi}$. Therefore, from Lemma 4.3(ii), we get that each term of the double sum in (21) converges in distribution to the constant 0 . Consequently, by Lemma 4.3(i), the double sum in (21) converges in distribution to the constant 0 .

Since $\widetilde{Z}$ and $\widetilde{W}$ are independent, from Lemma 4.4 it follows that

$$
U:=-\frac{\widetilde{Z}}{\sqrt{1+m / n}}-\frac{\widetilde{W}}{\sqrt{1+n / m}}
$$

is a standard normal random variable.
Moreover, from $Z^{\prime} \xrightarrow{d} \widetilde{Z}, W^{\prime} \xrightarrow{d} \widetilde{W}$, and the fact that $1 / \sqrt{1+m / n}$ and $1 / \sqrt{1+n / m}$ belong to $(0,1)$, we get easily that

$$
\frac{\widetilde{Z}-Z^{\prime}}{\sqrt{1+m / n}} \stackrel{d}{\rightarrow} 0 \quad \text { and } \quad \frac{\widetilde{W}-W^{\prime}}{\sqrt{1+n / m}} \xrightarrow{d} 0 .
$$

Therefore, Lemma 4.3(i) yields that

$$
-\frac{Z^{\prime}}{\sqrt{1+m / n}}-\frac{W^{\prime}}{\sqrt{1+n / m}}=U+\frac{\widetilde{Z}-Z^{\prime}}{\sqrt{1+m / n}}+\frac{\widetilde{W}-W^{\prime}}{\sqrt{1+n / m}} \xrightarrow{d} U .
$$

From a last application of Lemma 4.3(i) we get that $\widetilde{c t}_{\mathcal{A}}(\mathbf{X Y})$ converges in distribution to $U$.
Let $\mathbf{M}$ be a random matrix taken with uniform distribution from $\mathbb{F}_{q}^{m \times n, r}$. Thanks to Lemma 4.2, for every real number $t$, we have that

$$
\begin{aligned}
& \left|\mathbb{P}\left[\widetilde{c t}_{\mathcal{A}}(\mathbf{M}) \leq t\right]-\mathbb{P}\left[\widetilde{c t}_{\mathcal{A}}(\mathbf{X Y}) \leq t\right]\right|=\left|\sum_{\substack{\mathbf{N} \in \mathbb{F}_{q}^{m \times n} \\
\tilde{c t}_{\mathcal{A}}(\mathbf{N}) \leq t}}(\mathbb{P}[\mathbf{M}=\mathbf{N}]-\mathbb{P}[\mathbf{X Y}=\mathbf{N}])\right| \\
& \quad \leq \sum_{\mathbf{N} \in \mathbb{F}_{q}^{m \times n}}|\mathbb{P}[\mathbf{X Y}=\mathbf{N}]-\mathbb{P}[\mathbf{M}=\mathbf{N}]| \rightarrow 0 .
\end{aligned}
$$

Consequently, we get that $\widetilde{c t}_{\mathcal{A}}(\mathbf{M})$ and $\widetilde{c t}_{\mathcal{A}}(\mathbf{X Y})$ have the same limiting distribution (if it exists). Since we already proved that $\widetilde{c t}_{\mathcal{A}}(\mathbf{X Y})$ converges in distribution to a standard normal random variable, we get that $\widetilde{\mathrm{ct}_{\mathcal{A}}}(\mathbf{M})$ also converges in distribution to a standard normal random variable.

The proof of Theorem 1.1 is complete.
Remark 5.1. A crucial part of the proof is the fact that, since $r$ is fixed, the double sum in (21) has a fixed number of terms, and so it is possible to prove that it converges in distribution to the constant 0 without having to closely inspect its terms. If one let $r \rightarrow+\infty$, in a way controlled by $m$ and $n$, then it seems likely that understanding the behavior of $\widetilde{\mathrm{ct}_{\mathcal{A}}}(\mathbf{X Y})$ would require a more detailed study of the terms of the double sum in (21), since the number of such terms grows with $r$.

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[^1]:    ${ }^{1}$ We normalize the Fourier transform by the factor $(q-1)^{-t}$ because later this simplifies some formulas.

