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Additional Information

Stability analysis of polynomial fuzzy models via polynomial fuzzy Lyapunov functions

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Abstract

In this paper, stability of continuous-time polynomial fuzzy models by means of a polynomial generalization of fuzzy Lyapunov functions is studied. Fuzzy Lyapunov functions have been fruitfully used in literature for local analysis of Takagi-Sugeno models, a particular class of the polynomial fuzzy ones. Based on a recent Taylor-series approach which allows a polynomial fuzzy model to exactly represent a nonlinear model in a compact set of the state space, it is shown that a refinement of the polynomial Lyapunov function so as to make it share the fuzzy structure of the model proves advantageous. Conditions thus obtained are tested via SOS software.

Keywords: local stability, fuzzy modeling, fuzzy Lyapunov functions, polynomial fuzzy models, sum of squares.

1. INTRODUCTION

During the last twenty years, research on fuzzy models has evolved from a purely heuristic-based framework to a formal mathematical model-based one [Tanaka01]. This evolution has been based significantly on Takagi-Sugeno (TS) fuzzy models [Takagi85] since they allow exact representations of a given nonlinear model as a fuzzy system to be systematically obtained in a compact set of the state

variables [Taniguchi01]. A TS model is constructed as a nonlinear blending of linear models via membership functions (MFs) which hold the convex-sum property and capture the model nonlinearities through a technique known as the sector nonlinearity approach [Tanaka01]. The convex structure of a TS model enables Lyapunov-based stability analysis and controller design to be naturally applied. Quadratic Lyapunov functions have been extensively employed because they lead to conditions that can be easily cast as linear matrix inequalities (LMIs) [Tanaka01], which can be solved by convex optimization techniques from semi-definite programming [Boyd94]. Plenty of results have appeared under the TS-quadratic framework in the last twenty years proving their usefulness and applicability on traditional control tasks (see [Tanaka01, Sala05] and references therein).

Quadratic conditions are only sufficient for stability of TS models. Several directions have been explored to relax the inherent conservativeness of the quadratic approach, for instance: using a more general class of Lyapunov functions like the piecewise [Johansson99, Feng04] or the fuzzy ones [Tanaka03, Guerra04], handling in a less conservative way the membership-function information [Sala07, Sala08, Bernal09], or employing a class of models broader than the TS ones [Guerra07, Tanaka07a, Tanaka07b]. Among the latter direction, polynomial fuzzy (PF) models have established a new paradigm that overcomes many of the aforementioned problems of conservativeness since they are convex combinations of polynomial models instead of convex combinations of linear ones [Tanaka09a, Tanaka09b]. Moreover, conditions derived under this new framework can also be checked with semi-definite programming using Sum-of-Squares (SOS) tools [Prajna04a, Prajna04b].

This paper is based on two recent works: the first one [Sala09a, Sala09b] provides a systematic way of obtaining exact polynomial fuzzy representations of nonlinear models via a Taylor-series approach, thus generalizing sector nonlinearity approach; the second one [Guerra09, Bernal10] shows how to escape from the quadratic framework by combining local analysis and fuzzy Lyapunov functions for continuous-time TS models. Since local analysis can be easily included via Lagrange multipliers and the *Positivstellensatz*

argumentation in the polynomial framework [Prajna04a, Sala09b], the use of more general Lyapunov functions such as the polynomial fuzzy ones is investigated in this paper as a generalization of those employed in [Guerra09].

In summary, the objective of this paper is reducing the conservativeness of stability analysis of smooth nonlinear systems; this is achieved by generalizing previous results on local stability using non-quadratic Lyapunov functions to the polynomial-fuzzy case. Polynomial bounds on the partial derivatives of the membership functions, as well as information on the shape of the region of interest, will be used by means of *Positivstellensatz* multipliers.

The paper has the following structure: section II introduces notation, continuous-time PF models as well as polynomial fuzzy Lyapunov functions (PFLF) on which this paper is developed: a problem statement is made; section III develops the main result which combines PF models and PFLFs for local stability analysis; section IV provides some illustrative examples pointing out the advantages of using the proposed methodology; finally section V gathers some conclusions and discusses future work.

2. PROBLEM STATEMENT

Consider a nonlinear model $\dot{x}(t) = \mathbf{f}(x)$ having the origin as an equilibrium point, and assume that it can be expressed in the form:

$$\dot{x}(t) = \boldsymbol{\pi}\left(h^1(z_1(x)), \dots, h^\gamma(z_\gamma(x)), x(t)\right) \quad (1)$$

being $\boldsymbol{\pi}(\cdot): \mathbb{R}^{n+\gamma} \rightarrow \mathbb{R}^n$ a vector of polynomial functions, $x(t) \in \mathbb{R}^n$ the state vector, $z(x(t)) \in \mathbb{R}^\gamma$ another vector of polynomial functions of the state (denoted as the premise vector), and a set of functions $h^k(\cdot): \mathbb{R} \rightarrow \mathbb{R}$, $k \in \{1, \dots, \gamma\}$ representing possible *non-polynomial* nonlinearities in (1), such as trigonometric, exponential, etc., functions; nonlinearities $h^k(\cdot)$ are assumed bounded and smooth in a region of interest given by a compact set $\Omega \supset \mathbf{0}$. Any compact region of interest Ω can be included into a

semi-algebraic set with a piecewise polynomial boundary (for instance, a ball). This fact will be later used for SOS relaxations.

For instance, a model equation $\dot{x}_1 = \left(\sin(x_1^2 - x_2)\right)^2 x_2 + x_1$ can be expressed in the above form by considering $\pi(h, x_1, x_2) = h^2 x_2 + x_1$, $h(z) = \sin(z)$, and $z = x_1^2 - x_2$. As discussed below, if functions $h^k(z)$ are C^d they admit a representation as a fuzzy combination of polynomials of degree d , to be denoted as “polynomial fuzzy” model. The case $d=1$ amounts to the well-known Takagi-Sugeno models.

Once a nonlinear system in the above general form is assumed, fuzzy techniques will be used to analyze its stability. The first step is converting the system to a fuzzy model (a polynomial fuzzy one, in fact). In order to carry out such conversion, consider a particular non-polynomial nonlinearity $h(z)$ as those defined above (subscripts and arguments are omitted for simplicity). Employing the polynomial fuzzy modeling described in [Sala09a, Sala09b] (which is a generalization of sector nonlinearity in [Tanaka01]), this function can be rewritten as a convex sum of polynomials. Indeed, in order to do so, let us denote the d -th

degree Taylor approximation of $h(z)$ as $h_d(z) = \sum_{i=0}^{d-1} \frac{h^{(i)}(0)}{i!} z^i$, $d \in \mathbb{N}$, the residual term

$$T_d(z) = \frac{h(z) - h_d(z)}{z^d}, \text{ with } T_d(0) = \lim_{z \rightarrow 0} T_d(z), \text{ and the bounds } \overline{T}_d = \sup_{z \in \Omega} T_d(z), \underline{T}_d = \inf_{z \in \Omega} T_d(z), \text{ assuming}$$

the arbitrarily chosen degree d is low enough such that the required derivatives exist and $T_d(z)$ is continuous. This notation allows defining the pair of MFs:

$$w_0(z) = \frac{\overline{T}_d - T_d(z)}{\overline{T}_d - \underline{T}_d}, w_1(z) = 1 - w_0(z), w_0(z), w_1(z) \geq 0 \quad (2)$$

It is straightforward to see that the nonlinearity $h(z)$ can now be written as

¹ Expressions like $T_d(z)$ having a possible division by zero will appear in several expressions as a consequence of the Taylor-based modeling technique. In any case they will be defined at 0 as the limit of the expression in that point [Sala09a].

$$h(z) = w_0(z)q_0(z) + w_1(z)q_1(z) = \sum_{i=0}^1 w_i(z)q_i(z), \quad (3)$$

with two vertex polynomials of degree d given by: $q_0(z) = h_d(z) + \overline{T_d}z^d$ and $q_1(z) = h_d(z) + \underline{T_d}z^d$. For details, see [Sala09a, Sala09b]. On the sequel, arguments will be omitted when convenient for brevity, for instance, w_i will stand for $w_i(z)$. Basically, replacing (3) into the polynomial π in (1) will yield the overall fuzzy polynomial model. However, if the polynomial π is not linear in $h^k(\cdot)$, say it appears with degree d_k , it gives rise to multi-dimensional (nested) tensor-product convex sums. Indeed, in that case, every function $h^k(\cdot)$, $k \in \{1, \dots, \gamma\}$ can be written as the product of its d_k elementary convex sums of the form (3). Thus, expression (1) can be rewritten as the following PF model:

$$\dot{x}(t) = \pi \left(\left(\sum_{i_1=0}^1 \overline{w}_{i_1}^1 q_{i_1}^1 \right)^{d_1}, \left(\sum_{i_2=0}^1 \overline{w}_{i_2}^2 q_{i_2}^2 \right)^{d_2}, \dots, \left(\sum_{i_\gamma=0}^1 \overline{w}_{i_\gamma}^\gamma q_{i_\gamma}^\gamma \right)^{d_\gamma}, x \right) = \sum_{i_1=0}^1 \sum_{i_2=0}^1 \dots \sum_{i_p=0}^1 w_{i_1}^1 w_{i_2}^2 \dots w_{i_p}^p \mathbf{q}_{i_1 i_2 \dots i_p} = \sum_{\mathbf{i} \in \mathbf{I}_p} w_{\mathbf{i}} \mathbf{q}_{\mathbf{i}} \quad (4)$$

with :

- p being the sum of the degrees in $\pi(\cdot)$ of each of the γ nonlinearities in (1), i.e., $p = \sum_{j=1}^{\gamma} d_j$.
- $\mathbf{I}_p = \left\{ \mathbf{i} = (i_1, i_2, \dots, i_p) : i_j \in \{0, 1\}, j \in \{1, \dots, p\} \right\}$ is the set of all p -bit binary numbers, being its elements, \mathbf{i} , multidimensional index variables whose k -th bit is denoted as i_k .
- $w_{\mathbf{i}} = w_{i_1}^1 w_{i_2}^2 \dots w_{i_p}^p = \prod_{j=1}^p w_{i_j}^j(z_j)$ is a product of elementary MFs obtained from those describing each nonlinearity in (3),
- and $\mathbf{q}_{\mathbf{i}}(x)$ is a polynomial vector of the proper size.

Example 1: To illustrate the modeling process above, consider the model $\dot{x} = \sin^2(x) + e^{-x}x$. It will have a

polynomial model for $\sin(x) = \sum_{i_1=0}^1 \bar{w}_{i_1}^1 q_{i_1}^1$, and another one for $e^{-x} = \sum_{i_2=0}^1 w_{i_2}^2 q_{i_2}^2$, giving rise to an overall model

in the form: $\dot{x} = \sum_{i_1=0}^1 \sum_{i_2=0}^1 \sum_{i_3=0}^1 \bar{w}_{i_1}^1 w_{i_2}^2 \bar{w}_{i_3}^1 \mathbf{q}_{i_1 i_2 i_3}$, with $\mathbf{q}_{i_1 i_2 i_3} = q_{i_1}^1 q_{i_2}^1 + q_{i_2}^2 x$. Defining $w_i^3 = w_i^1 = \bar{w}_i^1$ yields an expression

in the form (4), i.e., a three-dimensional tensor product combination of vertex polynomials.

Recall that PF model (4) is equivalent to the original nonlinear model (1) in the compact set Ω of the state space including the origin; moreover, TS models are a subclass of the PF ones. A PF model is said to be of order d if the maximum order found in its Taylor approximations is d . This procedure generalizes those in [Sala07b, Bernal10] to the polynomial case.

Once a polynomial fuzzy model has been obtained, consider now the following polynomial-fuzzy Lyapunov function candidate:

$$V(x) = \sum_{i_1=0}^1 \sum_{i_2=0}^1 \cdots \sum_{i_p=0}^1 w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p p_{i_1 i_2 \dots i_p}(x) = \sum_{i \in \mathbf{I}_p} w_i p_i(x) \quad (5)$$

where $p_i(x) \in \mathbb{R}[x]$ are polynomials to be determined, and the MFs $w_{i_j}^j$ are those in the PF model (4). This function is a generalization of the fuzzy Lyapunov function in [Blanco01, Tanaka03] where $p_i(x)$ are restricted to be homogeneous quadratic polynomials in the state.

Asking this function to be a valid Lyapunov candidate means to ask $V(x)$ to be positive and radially unbounded; since $w_i \geq 0$, it is enough to guarantee $p_i(x) \geq 0$ to have $V(x) \geq 0$. As naturally follows from the polynomial nature of the PF model and the PFLF, positiveness will be tested by the sum-of-squares condition, i.e., $p_i(x)$ is SOS $\Rightarrow p_i(x) \geq 0$. Radial unboundedness is achieved by replacing zero in the right-hand side with an arbitrary radially-unbounded polynomial, such as $\varepsilon(x_1^2 + x_2^2)$, with $\varepsilon > 0$ an

arbitrary scalar. In the next section, a solution is proposed to the problem of deriving conditions to make (5) a valid PFLF for PF model (4) incorporating locality and membership-shape information (bounds on partial derivatives).

3. MAIN RESULT

Note that the time-derivative of $w_{\mathbf{i}}$ in (4) can be rewritten as follows [Guerra09, Bernal10]:

$$\dot{w}_{\mathbf{i}} = \frac{\partial w_{\mathbf{i}}}{\partial z} \dot{z} = \sum_{k=1}^p \frac{\partial w_{\mathbf{i}}}{\partial z_k} \dot{z}_k = \sum_{k=1}^p \frac{\partial}{\partial z_k} \left(\prod_{j=1}^p w_{i_j}^j(z_j) \right) \dot{z}_k = \sum_{k=1}^p \frac{\partial w_{i_k}^k}{\partial z_k} \left(\prod_{\substack{j=1 \\ j \neq k}}^p w_{i_j}^j(z_j) \right) \dot{z}_k ,$$

where the fact that each factor in $w_{\mathbf{i}}$ depends on only one premise variable has been used. Multiplying by

$$w_{i_k}^k + (1 - w_{i_k}^k) = 1 \text{ gives}$$

$$\dot{w}_{\mathbf{i}} = \sum_{k=1}^p \frac{\partial w_{i_k}^k}{\partial z_k} \left(w_{i_k}^k \prod_{\substack{j=1 \\ j \neq k}}^p w_{i_j}^j + (1 - w_{i_k}^k) \prod_{\substack{j=1 \\ j \neq k}}^p w_{i_j}^j \right) \dot{z}_k = \sum_{k=1}^p \frac{\partial w_{i_k}^k}{\partial z_k} (w_{\mathbf{i}} + w_{\bar{\mathbf{i}}(k)}) \dot{z}_k , \quad (6)$$

where $\bar{\mathbf{i}}(k)$ is defined as the p -bit binary index resulting from changing the k -th bit of \mathbf{i} to its complement. This form will allow convex expressions to be recovered on the Lyapunov method analysis.

Example 2: Consider $w_{\mathbf{i}} = w_{(1,0,1)} = w_1^1(z_1)w_0^2(z_2)w_1^3(z_3)$. To obtain expression (6) the expression

$$\dot{w}_{(1,0,1)} = \sum_{k=1}^3 \frac{\partial w_{(1,0,1)}}{\partial z_k} \dot{z}_k = \frac{\partial w_1^1}{\partial z_1} (w_0^2(z_2)w_1^3(z_3)) \dot{z}_1 + \frac{\partial w_0^2}{\partial z_2} (w_1^1(z_1)w_1^3(z_3)) \dot{z}_2 + \frac{\partial w_1^3}{\partial z_3} (w_1^1(z_1)w_0^2(z_2)) \dot{z}_3 \quad \text{must be}$$

written. Omitting arguments, the previous expression can be written as in (6) by multiplying each summand

by the proper term of the form $w_{i_k}^k + (1 - w_{i_k}^k) = 1$, i.e.:

$$\begin{aligned}
\dot{w}_{(1,0,1)} &= \frac{\partial w_1^1}{\partial z_1} (w_0^2 w_1^3) (w_0^1 + w_1^1) \dot{z}_1 + \frac{\partial w_0^2}{\partial z_2} (w_1^1 w_1^3) (w_0^2 + w_1^2) \dot{z}_2 + \frac{\partial w_1^3}{\partial z_3} (w_1^1 w_0^2) (w_0^3 + w_1^3) \dot{z}_3 \\
&= \frac{\partial w_1^1}{\partial z_1} (w_0^1 w_0^2 w_1^3 + w_1^1 w_0^2 w_1^3) \dot{z}_1 + \frac{\partial w_0^2}{\partial z_2} (w_1^1 w_0^2 w_1^3 + w_1^1 w_1^2 w_1^3) \dot{z}_2 + \frac{\partial w_1^3}{\partial z_3} (w_1^1 w_0^2 w_0^3 + w_1^1 w_0^2 w_1^3) \dot{z}_3 \\
&= \frac{\partial w_1^1}{\partial z_1} (w_{(0,0,1)} + w_{(1,0,1)}) \dot{z}_1 + \frac{\partial w_0^2}{\partial z_2} (w_{(1,0,1)} + w_{(1,1,1)}) \dot{z}_2 + \frac{\partial w_1^3}{\partial z_3} (w_{(1,0,0)} + w_{(1,0,1)}) \dot{z}_3.
\end{aligned}$$

Taking derivatives of the PFLF in (5) along the trajectories of PF model (4) and taking (6) into account gives

$$\dot{V}(x) = \sum_{\mathbf{i} \in \mathbf{I}_p} (w_{\mathbf{i}} \dot{p}_{\mathbf{i}} + \dot{w}_{\mathbf{i}} p_{\mathbf{i}}) = \sum_{\mathbf{i} \in \mathbf{I}_p} \left(w_{\mathbf{i}} \dot{p}_{\mathbf{i}} + \sum_{k=1}^p \frac{\partial w_{i_k}^k}{\partial z_k} (w_{\mathbf{i}} + w_{\bar{\mathbf{i}}(k)}) \dot{z}_k p_{\mathbf{i}} \right) = \sum_{\mathbf{i} \in \mathbf{I}_p} w_{\mathbf{i}} \left(\dot{p}_{\mathbf{i}} + \sum_{k=1}^p \frac{\partial w_0^k}{\partial z_k} \dot{z}_k (p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}) \right), \quad (7)$$

where the straightforward identity $\sum_{\mathbf{i} \in \mathbf{I}_p} w_{\bar{\mathbf{i}}(k)} p_{\mathbf{i}} = \sum_{\mathbf{i} \in \mathbf{I}_p} w_{\mathbf{i}} p_{\bar{\mathbf{i}}(k)}$ has been used to write the rightmost expression.

Example 3: Continuing with our previous example, note that according to (7), the polynomials $p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}$ sharing the same MF $w_{\mathbf{i}} = w_{(1,0,1)}$ are $p_{(1,0,1)} - p_{(0,0,1)}$ for $k=1$, $p_{(1,0,1)} - p_{(1,1,1)}$ for $k=2$, and $p_{(1,0,1)} - p_{(1,1,0)}$ for $k=3$. It is important to emphasize that should a stability problem have a non-fuzzy Lyapunov function solution, these terms will vanish since $\forall \mathbf{i}, \mathbf{j}, p_{\mathbf{i}} = p_{\mathbf{j}}$, thus proving the generalization ability behind the proposal in this paper.

Consider now expressions $\dot{z}_k = \left(\frac{\partial z_k}{\partial x} \right)^T \dot{x}$ and $\dot{p}_{\mathbf{i}} = \left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \dot{x}$ which are fuzzy polynomials (z_k and $p_{\mathbf{i}}$ are polynomials by assumption and \dot{x} is taken from its PF representation in (4)). The result of substituting them in (7) is:

$$\begin{aligned}
\dot{V}(x) &= \sum_{\mathbf{i} \in \mathbf{I}_p} w_{\mathbf{i}} \left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \sum_{\mathbf{l} \in \mathbf{I}_p} w_{\mathbf{l}} \mathbf{q}_{\mathbf{l}} + \sum_{k=1}^p \frac{\partial w_0^k}{\partial z_k} \left(\left(\frac{\partial z_k}{\partial x} \right)^T \sum_{\mathbf{l} \in \mathbf{I}_p} w_{\mathbf{l}} \mathbf{q}_{\mathbf{l}} \right) (p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}) \right) \\
&= \sum_{\mathbf{i} \in \mathbf{I}_p} \sum_{\mathbf{l} \in \mathbf{I}_p} w_{\mathbf{i}} w_{\mathbf{l}} \left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \sum_{k=1}^p \left(\left(\frac{\partial w_0^k}{\partial z_k} \cdot \frac{\partial z_k}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} \right) (p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}) \right)
\end{aligned} \quad (8)$$

All terms in the above expression are either MFs or polynomials, except possibly for $\frac{\partial w_0^k}{\partial z_k}$. The basic idea

is that, in the same way as the nonlinearities were fuzzified, $\frac{\partial w_0^k}{\partial z_k}$ can be recast again as a convex sum of polynomials, following the polynomial fuzzy modeling technique already described in (2) and (3) [Sala09a, Sala09b].

Example 4: Given a scalar nonlinearity $h(x) = \sin x$ in $\Omega = [-1, 1]$, it is easy to see that $h(x) = 0.8414w_0 - 0.8414(1-w_0)$ with $w_0 = 0.5942 \sin x + 0.5$, from which it follows that $dw_0/dx = 0.5942 \cos x$. These functions are all infinitely differentiable in the chosen region of interest $\Omega = [-1, 1]$. The latter one, dw_0/dx , can also be written as a convex sum of polynomials in Ω , for instance $dw_0/dx = 0.5942\mu_0 + 0.3211(1-\mu_0)$ with $\mu_0 = 2.1755 \cos x - 1.1755$. Actually, polynomials of degree zero have been chosen in this example, but the methodology applies to any arbitrarily chosen degree.

Since $\frac{\partial z_k}{\partial x} \in \square^{n \times 1}$ is assumed to be a polynomial vector, using a PF model of $\frac{\partial w_0^k}{\partial z_k}$, every expression

$\frac{\partial w_0^k}{\partial z_k} \cdot \frac{\partial z_k}{\partial x} \in \square^{n \times 1}$ in (8) can be written as

$$\frac{\partial w_0^k}{\partial z_k} \cdot \frac{\partial z_k}{\partial x} = \sum_{\mathbf{v}_k \in \mathbf{I}_{s_k}} \mu_{\mathbf{v}_k}^k(x) \mathbf{r}_{\mathbf{v}_k}^k(x), \quad k = 1, \dots, p, \quad (9)$$

with s_k being the number of possible non-polynomial nonlinearities in $\frac{\partial w_0^k}{\partial z_k}$, and $\mu_{\mathbf{v}_k}^k = \mu_{v_k^1}^k \cdots \mu_{v_k^{s_k}}^k$,

$\sum_{\mathbf{v}_k=0}^1 \mu_{\mathbf{v}_k}^k(\cdot) = 1$, $\mu_{\mathbf{v}_k}^k(\cdot) \geq 0$ being the MFs associated with each modelled nonlinearity, and $\mathbf{r}_{\mathbf{v}_k}^k(x) \in \square^{n \times 1}$

being the resulting polynomial vector.

Substituting (9) in (8) yields

$$\begin{aligned}\dot{V}(x) &= \sum_{\mathbf{i} \in \mathbf{I}_p} \sum_{\mathbf{l} \in \mathbf{I}_p} w_{\mathbf{i}} w_{\mathbf{l}} \left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \sum_{k=1}^p \left(\left(\sum_{\mathbf{v}_k \in \mathbf{I}_{s_k}} \mu_{\mathbf{v}_k}^k \mathbf{r}_{\mathbf{v}_k}^k \right)^T \mathbf{q}_{\mathbf{l}} \right) (p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}) \right) \\ &= \sum_{\mathbf{i} \in \mathbf{I}_p} \sum_{\mathbf{l} \in \mathbf{I}_p} \sum_{\mathbf{v}_1 \in \mathbf{I}_{s_1}} \dots \sum_{\mathbf{v}_p \in \mathbf{I}_{s_p}} w_{\mathbf{i}} w_{\mathbf{l}} \mu_{\mathbf{v}_1}^1 \dots \mu_{\mathbf{v}_p}^p \left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \sum_{k=1}^p \left(\left(\mathbf{r}_{\mathbf{v}_k}^k \right)^T \mathbf{q}_{\mathbf{l}} \right) (p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}) \right)\end{aligned}$$

Defining the polynomial vector $\hat{\mathbf{p}}_{\mathbf{i}} = \begin{bmatrix} p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(1)} \\ \vdots \\ p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(p)} \end{bmatrix} \in \mathbb{R}^{1 \times p}$, the polynomial matrix $\mathbf{R}_{\mathbf{v}} = \begin{bmatrix} \left(\mathbf{r}_{\mathbf{v}_1}^1 \right)^T \\ \vdots \\ \left(\mathbf{r}_{\mathbf{v}_p}^p \right)^T \end{bmatrix} \in \mathbb{R}^{p \times n}$, and

the multi-index $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_p)$, the previous expression can be rewritten as

$$\dot{V}(x) = \sum_{\mathbf{i} \in \mathbf{I}_p} \sum_{\mathbf{l} \in \mathbf{I}_p} \sum_{\mathbf{v} \in \mathbf{I}_{\sigma}} w_{\mathbf{i}} w_{\mathbf{l}} \mu_{\mathbf{v}} \left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \hat{\mathbf{p}}_{\mathbf{i}}^T \mathbf{R}_{\mathbf{v}} \mathbf{q}_{\mathbf{l}} \right) \quad (10)$$

with $\sigma = s_1 + \dots + s_p$.

The main result can now be stated:

Theorem 1: The PF model (4) with MF-derivatives as in (9) is asymptotically stable if there exist polynomials $p_{\mathbf{i}}(x) \in \mathbb{R}$, and non-negative, radially unbounded polynomials $\varepsilon_1(x), \varepsilon_2(x) > 0$ such that $p_{\mathbf{i}}(x) - \varepsilon_1(x)$ and $-\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \hat{\mathbf{p}}_{\mathbf{i}}^T \mathbf{R}_{\mathbf{v}} \mathbf{q}_{\mathbf{l}} \right) - \varepsilon_2(x)$ are SOS for all $\mathbf{i}, \mathbf{l} \in \mathbf{I}_p$, $\mathbf{v} \in \mathbf{I}_{\sigma}$ with $\hat{\mathbf{p}}_{\mathbf{i}}$ and $\mathbf{R}_{\mathbf{v}}$ defined as in (9)-(10).

Proof: It follows immediately from the fact that $p_{\mathbf{i}}(x) - \varepsilon_1(x)$ being SOS enforces the Lyapunov function candidate (5) to be non-negative and radially unbounded, whereas $-\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x} \right)^T \mathbf{q}_{\mathbf{l}} + \hat{\mathbf{p}}_{\mathbf{i}}^T \mathbf{R}_{\mathbf{v}} \mathbf{q}_{\mathbf{l}} \right) - \varepsilon_2(x)$ being SOS assures the time-derivative of the Lyapunov function to be strictly negative outside the origin, i.e., $\dot{V}(x) < 0$, as can be deduced from (10). \square

Remark: In order to reduce conservativeness of the above result, any relaxation scheme can be applied to the tensor-product double fuzzy summation in $w_1 w_1$ that appears in (10), for example, grouping those terms sharing the same factorization of $w_1 w_1$ [Tanaka01, Sala07, Sala07b].

3.1 LOCALITY ISSUES

As originally explained in [Parrilo03, Prajna04a] and illustrated in [Sala09b], the *Positivstellensatz* argumentation extends the use of Lagrange multipliers and S-procedure in the LMI framework to the polynomial-SOS case, thus permitting local information to be included as constraints in SOS conditions.

Assume that m known polynomial restrictions arranged as a vector $F = \{f_1(x), \dots, f_m(x)\} \geq 0$, $F(x) \in \mathbb{R}^m$ hold in Ω . Then, a sufficient condition for a polynomial $\pi(x)$ being positive in Ω , i.e., locally, is that

there exist multiplier SOS polynomials $q_i(x)$, $i = 1, \dots, n$, such that $\pi(x) - \sum_{i=1}^n q_i(x) \phi_i(x)$ is SOS, where

$\phi_i(x)$ are arbitrary fixed polynomials that are composed of products of those in F . *Positivstellensatz*

theorems allow for reaching a necessary and sufficient condition as the number of multipliers increase, but they are not constructive.

The previous reasoning as well as some practical considerations of polynomial order for SOS tests, leads to a procedure to include SOS restrictions into the local analysis. Briefly, it can be stated as follows, having two design parameters d_1 and d_2 :

1. Define a list of polynomial restrictions holding in the modelling area of the PF model

$F = \{f_1(x), \dots, f_m(x)\}$. Note that this non-unique list is naturally derived and a priori known from

the modelling region.

2. Construct polynomials $\phi_i(x)$ as all the product combinations of restrictions in F preserving the same sign up to a certain user-defined degree d_1 .
3. Set the degree of q_i such that $\pi(x) - \sum_{i=1}^n q_i(x)\phi_i(x)$ has a degree d_2 . Set the coefficients of q_i as free decision variables.

Then, we have the following theorem for local stability analysis:

Theorem 2: Assume that m restrictions $F = \{f_1(x), \dots, f_m(x)\} \geq 0$, $F(x) \in \mathbb{R}^m$ hold in Ω . The PF model (4) with MF-derivatives as in (9) is locally asymptotically stable in Ω if there exist polynomials $p_i(x) \in \mathbb{R}$, such that $p_i(x) - \sum_j u_j(x)\phi_j(x)$ and $-((\partial p_i / \partial x)^T \mathbf{q}_i + \hat{\mathbf{p}}_i^T \mathbf{R}_v \mathbf{q}_i) - \sum_k \tilde{u}_k(x)\phi_k(x)$ are SOS, with $u_j(x)$, $\tilde{u}_k(x)$ being SOS polynomials (multipliers) and $\phi_j(x)$ being arbitrary polynomials composed by the products of those in F , for all $\mathbf{i}, \mathbf{l} \in \mathbf{I}_p$, $\mathbf{v} \in \mathbf{I}_\sigma$ with $\hat{\mathbf{p}}_i$ and \mathbf{R}_v defined as in (9)-(10).

Proof: It follows immediately from the discussion above.

These sufficient conditions are less conservative than those without local restrictions.

The higher d_1 and d_2 are chosen the higher the number of decision variables and the lower the conservativeness.

3.2 NUMERICAL ISSUES

Polynomial-programming techniques, even if convex for a *fixed* degree of the polynomials, are computationally hard in the fuzzy-control context. The basic drawbacks are: (a) a high-degree Taylor series is needed to approximate the nonlinearities in a large domain; (b) the number of rules is two to the power of

the number of nonlinearities and the degree of them in π (however, this is also a drawback in classical TS modelling); (c) as polynomials diverge wildly, sometimes the obtained results are worse than ordinary TS ones unless *Positivstellensatz* multipliers are used; (d) the number of decision variables increases heavily as d_1 and d_2 in Section 3.1 increase.

Hence, practical engineering applications of SOS techniques in high-order nonlinear systems may have severe limitations with current software but, nevertheless, they have theoretical interest and Takagi-Sugeno results are the particular case of degree-1. In the authors' opinion, broadly speaking, TS models may be the best option regarding the trade-off "quality of results / computational resources needed" in high-order control applications. However, in case TS fails, conceiving an application in which SOS techniques might be helpful by increasing the precision of the representation of a couple of nonlinearities to second or third degree is always reasonable.

4. EXAMPLES

Example 5: Consider the following nonlinear model proposed in [Tanaka03, Tanaka09b]:

$$\dot{x}(t) = \begin{bmatrix} -\frac{7}{2}x_1 - 4x_2 - \frac{3}{2}x_1 \sin x_1 \\ \frac{19}{2}x_1 - 2x_2 - \frac{21}{2}x_1 \sin x_1 \end{bmatrix}. \quad (11)$$

The stability properties of the previous model in $\Omega = \{|x_i| \leq 1\}$ will be investigated. To do so, the nonlinearity $\sin x_1$ is written as a convex sum of polynomials following the techniques described above with $z_1 = x_1$, leading to the following PF model structure:

$$\dot{x}(t) = \sum_{i \in \mathbf{I}_1} w_i \mathbf{q}_i = w_0^1 \begin{bmatrix} -\frac{7}{2}x_1 - 4x_2 - \frac{3}{2}x_1 q_0^1(x) \\ \frac{19}{2}x_1 - 2x_2 - \frac{21}{2}x_1 q_0^1(x) \end{bmatrix} + w_1^1 \begin{bmatrix} -\frac{7}{2}x_1 - 4x_2 - \frac{3}{2}x_1 q_1^1(x) \\ \frac{19}{2}x_1 - 2x_2 - \frac{21}{2}x_1 q_1^1(x) \end{bmatrix}. \quad (12)$$

where $q_0^1(x)$, $q_1^1(x)$ are polynomials of certain degree, and $w_0^1(x)$, $w_1^1(x)$ are the corresponding MFs.

Consider a 0-degree PF model: in this case, $q_0^1(x)=0.8414$ and $q_1^1(x)=-0.8414$ are, plainly, constants

while $w_0^1=0.5942\sin x_1+0.5$ and $w_1^1=1-w_0^1$ are the corresponding MFs. As discussed in example 4, for

expression $\frac{\partial w_0^1}{\partial x_1}=0.5942\cos x_1$, consider a 0-degree modeling as in (9), i.e., bounds $r_0^1(x)=0.5942$,

$r_1^1(x)=0.3211$, and MFs $\mu_0^1=2.1755\cos x_1-1.1755$, $\mu_1^1=1-\mu_0^1$. Theorem 1 is now used to analyze

stability for a degree-2 PFLF candidate of the form $V(x)=w_0^1p_1(x)+w_1^1p_2(x)$.

When no Lagrange multipliers are used (global analysis) the SOS problem is unfeasible. In order to make local analysis as pointed out in Section 3.1, a simple list of 1- and 2-degree polynomial restrictions have

been made: $F=\{x_1^2-\bar{x}^2<0, x_2^2-\bar{x}^2<0, \bar{x}^2-x_1<0, \bar{x}^2-x_2<0, -(x_1+\bar{x}^2)<0, -(x_2+\bar{x}^2)<0\}$. Then,

double products of constraints in the list have been used to construct a second order polynomial Lagrange multiplier multiplied by the following constraints (valid in Ω with $\bar{x}=1$):

$$\begin{aligned} x_1^2-\bar{x}^2<0, \quad x_2^2-\bar{x}^2<0, \quad -(x_1^2-\bar{x}^2)(x_2^2-\bar{x}^2)<0, \quad -(x_1^2-\bar{x}^2)(x_1-\bar{x}^2)<0, \\ (x_1^2-\bar{x}^2)(x_1+\bar{x}^2)<0, \quad -(x_2^2-\bar{x}^2)(x_2-\bar{x}^2)<0, \quad (x_2^2-\bar{x}^2)(x_2+\bar{x}^2)<0. \end{aligned} \quad (13)$$

Via SOSTools, conditions in Theorem 1 are then satisfied for $p_1(x)=3.9106x_1^2+1.863x_1x_2+4.1858x_2^2$ and

$p_2(x)=10.692x_1^2+1.2375x_1x_2+2.569x_2^2$. In Fig. 2 some level curves of this PFLF are displayed in dashed-

lines; the outermost Lyapunov level is in bold-dashed. Some trajectories in solid lines are also included.

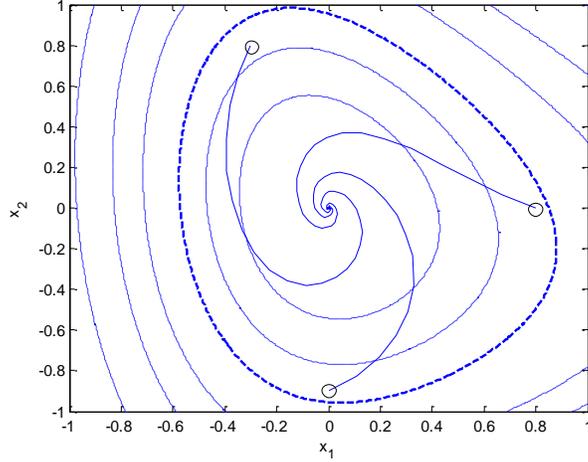


Figure 1: Lyapunov levels for the 0-degree PFLF in Example 2

Now consider a 3rd-degree PF model in (12) with polynomials $q_0^l(x) = x_1 - 0.1585x_1^3$,

$q_1^l(x) = x_1 - 0.1667x_1^3$, and MFs $w_0^l = 122.9 \frac{(\sin x_1 - x_1)}{x_1^3} + 20.48$, $w_1^l = -19.48 - 122.9 \frac{(\sin x_1 - x_1)}{x_1^3}$. It can be

checked that

$$\frac{\partial w_0^l}{\partial x_1} = 122.9 \left(\frac{x_1^3 (\cos x_1 - 1) - 3x_1^2 (\sin x_1 - x_1)}{x_1^6} \right) = 122.9 \left(\frac{\cos x_1}{x_1^3} - \frac{3 \sin x_1}{x_1^4} + \frac{2}{x_1^3} \right), \quad (14)$$

which can be written as follows from the Taylor-series representation of its components

$$\begin{aligned} \frac{\partial w_0^l}{\partial x_1} &= 122.9 \left(\frac{1}{x_1^3} \left(1 - \frac{x_1^2}{2!} + \frac{x_1^4}{4!} - \dots \right) - \frac{3}{x_1^4} \left(x - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} - \dots \right) + \frac{2}{x_1^3} \right) \\ &= 122.9 \left(\frac{1}{x_1^3} \left(\frac{x_1^4}{4!} - \dots \right) - \frac{3}{x_1^4} \left(\frac{x_1^5}{5!} - \dots \right) \right) = 122.9 \left(\left(\frac{1}{4!} - \frac{3}{5!} \right) x_1 - \left(\frac{1}{6!} - \frac{3}{7!} \right) x_1^3 + \dots \right) \end{aligned} \quad (15)$$

thus proving that it can be defined in 0 as the limit of (14) and it is therefore a smooth function.

Then, since $z_1 = x_1$ the following 3rd-degree Taylor-based PF model in $x_1 \in [-1, 1]$ arises:

$$\frac{\partial w_0^l}{\partial z_1} \cdot \frac{\partial z_1}{\partial x} = \sum_{v_1=0}^1 \mu_{v_1}^l(x) \mathbf{r}_{v_1}^l(x) = \mu_0^l \begin{bmatrix} 2.0483x_1 - 0.09555x_1^3 \\ 0 \end{bmatrix} + \mu_1^l \begin{bmatrix} 2.0483x_1 - 0.0975x_1^3 \\ 0 \end{bmatrix}$$

with $T(x) = \frac{122.9}{x_1^3} \left(\frac{\cos x_1}{x_1^3} - \frac{3 \sin x_1}{x_1^4} + \frac{2}{x_1^3} - 0.0167x_1 \right)$, $\mu_0^1 = \frac{T(x) + 0.0975}{-0.09555 + 0.0975}$, $\mu_1^1 = 1 - \mu_0^1$. Recall that

according to definitions (9)-(10), in this example $\mathbf{v} = \mathbf{v}_1 \in \mathbf{I}_1$, so matrix $\mathbf{R}_v = (r_{v_i}^1)^T \in \mathbb{R}^{2 \times 1}$. The example is now analyzed via Theorem 1.

Via SOSTools polynomials $p_1(x) = p_2(x) = 8.4852x_1^2 + 0.23829x_1x_2 + 2.8658x_2^2$ are found satisfying conditions in Theorem 1 under the aforementioned constraints. Note that the corresponding Lyapunov function has lost its fuzzy structure since $p_1(x) = p_2(x)$, i.e., $V(x) = w_0^1 p(x) + (1 - w_0^1) p(x) = p(x)$, a solution which is not ruled out by conditions in Theorem 1.

Discussion: Independently of their degree, PF models obtained by the aforementioned methodology are all exact representations of nonlinearities associated to a nonlinear model or the MFs' derivatives. Then, a natural question arises: what is the difference between lower or higher degrees in PF modeling? The answer originates from the previous example: as the PF model degree increases the vertex polynomials converge to the Taylor series under mild assumptions; then, MFs yield their modeling influence only to the corresponding polynomials terms of higher degree. Therefore, the fuzzy character of the PF model becomes less significant for higher degree models. As a consequence of this phenomenon, in the previous example an ordinary quadratic polynomial Lyapunov function could not be found when the PF model was highly fuzzy (degree zero approximations): a non-quadratic PFLF has been found instead. On the other hand, when the PF model degree was increased the family of models thus represented seems to have been reduced in such a way that an ordinary quadratic Lyapunov function was found, thus having no need of the fuzzy structure for it.

Example 6: Consider the following nonlinear model:

$$\dot{x}(t) = \begin{bmatrix} -0.2363x_1^2 + 0.0985x_1 \left((0.1x_1)^2 + x_1 (0.1x_2)^2 \right) - 0.9x_2 \\ \sinh x_1 - 2x_1 - 0.7097x_2 + 0.3427x_2 \left((0.1x_1)^2 + (0.1x_2)^2 \right) \end{bmatrix}, \quad (16)$$

which has a stable focus at the origin and an unstable limit cycle; it is therefore not globally stable.

For different values of $\bar{x} > 0$, let $\Omega = \{|x_i| \leq \bar{x}\}$ be a square region of interest in which a decreasing Lyapunov function is to be found. Simulation shows that $\bar{x} = 4.15$ is the maximum admissible value for \bar{x} for the whole Ω to be in the basin of attraction.

1st- and 3rd-degree PF models of (16) have been obtained depending on whether 1st- or 3rd-degree polynomials were used for bounding $\sinh x_1$. The MFs' derivatives corresponding to these PF models have been also bounded by 1st- and 3rd-degree polynomials with an analogous methodology. Then, under 2nd-order Lagrange multipliers with constraints (13), Theorem 1 has been used to search the maximum $\bar{x} > 0$ for which stability can be proved for each combination of the previous cases.

The test is first run for quadratic non-fuzzy polynomial Lyapunov functions of the form $V(x) = p(x)$, where of course the time-derivatives of the MFs play no role (conditions in Theorem 1 have $\hat{\mathbf{p}}_i = \mathbf{0}$); these results are then compared with those obtained with a 2nd-order fuzzy polynomial function $V(x) = \sum_{i \in I_p} w_i(x) p_i(x)$. The results are shown in Table 1.

$\Omega = \{ x_i \leq \bar{x}\}$	$\deg(\mathbf{q}_i) = 1$	$\deg(\mathbf{q}_i) = 3$
Non-fuzzy PLF	$\bar{x} = 2.1094$	$\bar{x} = 2.6406$
PFLF, $\deg(\mathbf{r}_{v_k}^k) = 1$	$\bar{x} = 2.500$	$\bar{x} = 2.6875$
PFLF, $\deg(\mathbf{r}_{v_k}^k) = 3$	$\bar{x} = 2.5313$	$\bar{x} = 2.7344$

Table 1: Comparing polynomial Lyapunov functions versus polynomial fuzzy Lyapunov functions in Example 3: maximum size of a square region of interest where a decreasing LF is feasible. In all cases, the degree of the candidate Lyapunov function was fixed to 4.

As expected, better approximations on the PF model and/or the MFs' derivatives lead to better results. On the other hand, given a particular PF model, PFLFs clearly improve over non-fuzzy ones.

5. CONCLUSIONS AND PERSPECTIVES

A new methodology for analyzing the stability of continuous-time nonlinear models in the polynomial fuzzy form has been presented. It combines recent advances on Taylor-based fuzzy polynomial models and local stability via fuzzy polynomial Lyapunov functions, exploiting both polynomial bounds on the model's non-polynomial nonlinearities and, also, polynomial bounds on the partial derivatives of the membership functions.

The examples in the paper illustrate that fuzzy-polynomial Lyapunov functions prove useful in performing better than the unstructured polynomial Lyapunov functions, getting larger estimates of the region of attraction. Research on control design under this technique is under course. However, as in other polynomial-based approaches to control in literature, computational cost increases heavily as system order, polynomial degrees and number of rules in the fuzzy model increase.

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