Construction of fuzzy automata from fuzzy regular expressions $\overset{\circ}{\sim}$

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Abstract

Li and Pedrycz [Y. M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids, Fuzzy Sets and Systems 156 (2005) 68–92] have proved fundamental results that provide different equivalent ways to represent fuzzy languages with membership values in a lattice-ordered monoid, and generalize the well-known results of the classical theory of formal languages. In particular, they have shown that a fuzzy language over an integral lattice-ordered monoid can be represented by a fuzzy regular expression if and only if it can be recognized by a fuzzy finite automaton. However, they did not give any effective method for constructing an equivalent fuzzy finite automaton from a given fuzzy regular expression. In this paper we provide such an effective method.

Transforming scalars appearing in a fuzzy regular expression α into letters of the new extended alphabet, we convert the fuzzy regular expression α to an ordinary regular expression α_R . Then, starting from an arbitrary nondeterministic finite automaton \mathscr{A} that recognizes the language $||\alpha_R||$ represented by the regular expression α_R , we construct fuzzy finite automata \mathscr{A}_{α} and $\mathscr{A}_{\alpha}^{\mathrm{r}}$ with the same or even less number of states than the automaton \mathscr{A} , which recognize the fuzzy language $||\alpha||$ represented by the fuzzy regular expression α . The starting nondeterministic finite automaton \mathscr{A} can be obtained from α_R using any of the well-known constructions for converting regular expressions to nondeterministic finite automata, such as Glushkov-McNaughton-Yamada's position automaton, Brzozowski's derivative automaton, Antimirov's partial derivative automaton, or Ilie-Yu's follow automaton.

Keywords: Fuzzy automata; fuzzy regular expressions, nondeterministic automata; regular expressions, position automata; state reduction; right invariant equivalences; lattice-ordered monoids;

1. Introduction

Study of fuzzy automata and languages was initiated in 1960s by Santos [73–75], Wee [81], Wee and Fu [82], and Lee and Zadeh [50]. From late 1960s until early 2000s mainly fuzzy automata and languages with membership values in the Gödel structure have been considered (see for example [28, 31, 62]). The idea of studying fuzzy automata with membership values in some structured abstract set comes back to W. Wechler [80], and in recent years researcher's attention has been aimed mostly to fuzzy automata with membership values in complete residuated lattices, lattice-ordered moinoids, and other kinds of lattices. Fuzzy automata taking membership values in a complete residuated lattice were first studied in [68, 69], where some basic concepts have been discussed, and later, extensive research of these fuzzy automata has been carried out in [70, 71, 83–87]. From a different point of view, fuzzy automata with membership values in a complete residuated in [23, 24, 35–37, 39, 78]. Fuzzy automata over other types of lattices were the subject of [27, 47, 48, 54, 56, 63–66], and automata which generalize fuzzy automata over any type of lattices, as well as weighted automata over semirings, have been studied recently in [18, 26, 46]. It is worth noting that fuzzy automata and languages are widely used in lexical analysis, description of natural and

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programming languages, learning systems, control systems, neural networks, clinical monitoring, pattern recognition, databases, discrete event systems, and many other areas.

Li and Pedrycz [55] have proved fundamental results that provide different equivalent ways to represent fuzzy languages with membership values in a lattice-ordered monoid, e.g., by fuzzy finite automata, crispdeterministic fuzzy finite automata, fuzzy regular expressions, and fuzzy regular grammars. These results generalize the well-known results of the classical theory of formal languages. In particular, they have shown that a fuzzy language over an integral lattice-ordered monoid can be represented by a fuzzy regular expression if and only if it can be recognized by a fuzzy finite automaton. However, Li and Pedrycz did not give any effective method for constructing an equivalent fuzzy finite automaton from a given fuzzy regular expression. The purpose of the present paper is to provide such an effective method.

Our basic idea is to convert a fuzzy regular expression α into an ordinary regular expression α_R , transforming scalars appearing in the fuzzy regular expression α into letters of the new extended alphabet. Then, starting from an arbitrary nondeterministic finite automaton \mathscr{A} that recognizes the language $||\alpha_{R}||$ represented by the regular expression α_R , we construct a fuzzy finite automaton \mathscr{A}_α with the same number of states as the automaton \mathscr{A} , which recognizes the fuzzy language $\|\alpha\|$ represented by the fuzzy regular expression α . Moreover, we construct a reduced version \mathscr{A}^{r}_{α} of the fuzzy automaton \mathscr{A}_{α} , a fuzzy finite automaton which also recognizes the fuzzy language $||\alpha||$ and can have even smaller number of states than \mathscr{A}_{α} . The method is generic, which means that it can be used in combination with any method for constructing a nondeterministic finite automaton from an ordinary regular expression. In the past, many different techniques for constructing nondeterministic finite automata from regular expressions have been proposed. Besides Thompson's construction [79], which build nondeterministic finite automata with ε -transitions, other well-known constructions build nondeterministic finite automata without ε -transitions. The best known and most used such constructions are the position automaton, discovered independently by Glushkov [30] and McNaughton and Yamada [61], Brzozowski's derivative automaton [7], Antimirov's partial derivative automaton [2], and Ilie and Yu's follow automaton [40-43]. Each of these constructions can serve as a basis for the construction of our fuzzy finite automata. More information on the algorithms for building small nondeterministic finite automata from regular expressions can be found in [42].

It should be noted that the same idea of treating scalars appearing in a fuzzy regular expression as the letters of a new extended alphabet, and then treating a fuzzy regular expression as an ordinary regular expression over a larger alphabet, has been recently used by Kuske [49] in the context of weighted regular expressions and weighted finite automata over semirings. However, there are some significant differences between his and our approach. First, Kuske considered only weighted regular expressions that define proper power series, i.e., power series with zero as the coefficient of the empty word. In terms of the theory of fuzzy languages, these are fuzzy languages which (absolutely) do not contain the empty word. There is one even more important difference. In the mentioned paper [49], Kuske gave a new proof of the famous Schützenberger's theorem [29, 72] which asserts that the behaviors of weighted finite automata over an arbitrary semiring are precisely the rational formal power series, i.e., formal power series defined by weighted regular expressions. In his proof, Kuske first converts a weighted regular expression *E* to a regular expression *E'*, then he starts from an arbitrary deterministic finite automaton that recognizes the language defined by E', and from this automaton he constructs a weighted finite automaton whose behavior is the formal power series defined by E. However, the number of states of deterministic finite automata obtained from regular expressions can be exponentially larger than the lengths of the corresponding regular expressions. For this reason, regular expressions are more often converted to nondeterministic finite automata, and the above mentioned constructions outputs nondeterministic finite automata whose number of states is equal to the length of the regular expression plus one, or even less than that number. In addition, our constructions output fuzzy finite automata with the same or even smaller number of states than the original nondeterministic finite automaton.

As we have said, the size of an automaton obtained from a regular expression plays a very important role, and for that reason regular expressions are mostly converted to nondeterministic finite automata. On the other hand, for practical applications deterministic finite automata are usually needed, but determinization of a nondeterministic finite automaton can cause an exponential blow up in the number of states. That is why the number of states of a nondeterministic finite automaton has to be reduced prior to determinization. As the minimization of nondeterministic finite automata is computationally hard, we must be satisfied with the methods for reducing the number of states that do not necessarily give a minimal automaton, but rather provide a reasonably small automaton that can be effectively computed. Such reduction methods have been recently investigated in [9, 11, 41–45], in the context of nondeterministic finite automata, and in [23, 24, 78], in the context of fuzzy finite automata (see also [19, 21, 22]). Key role in the state reduction of nondeterministic finite automata play right and left invariant equivalences, which have been generalized in the fuzzy framework as right and left invariant fuzzy equivalences (cf. [23, 24, 78]). It is worth noting that right and left invariant (fuzzy) equivalences are also known as forward and backward bisimulation (fuzzy) equivalences (cf. [19, 21, 22]). In particular, it has been proved in [14–16, 40–43] that both the partial derivative automaton and the follow automaton are factor automata of the position automaton with respect to certain right invariant equivalences. State reduction of fuzzy finite automata by means of right invariant fuzzy and crisp equivalences will be also considered in this paper. Let us also note that the above mentioned determinization problem has been recently investigated in the fuzzy framework in [4, 18, 36, 46, 55].

Our main results are the following. We start from a given fuzzy regular expression α over an alphabet Xand a lattice-ordered monoid $\mathscr{L} = (L, \land, \lor, \otimes, 0, 1, e)$, and we define an ordinary regular expression α_R over a new alphabet $X \cup Y$, where Y consists of the letters associated with different scalars appearing in α . The mapping φ_{α} of $X \cup Y$ to L, which maps all letters from X to e, and letters from Y to related scalars appearing in α , can be extended in a natural way to a homomorphism φ_{α}^* of the free monoid $(X \cup Y)^*$ to the monoid (L, \otimes, e) . In the case when \mathscr{L} is an integral lattice-ordered monoid, using this homomorphism we establish a relationship between the fuzzy language $||\alpha||$ represented by α and the language $||\alpha_R||$ represented by α_R (cf. Theorem 3.6), and starting from any nondeterministic finite automaton \mathscr{A} that recognizes the language $||\alpha_R||$ we define the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α , and we prove that \mathscr{A}_{α} recognizes the fuzzy language $||\alpha||$ represented by the fuzzy regular expression α (cf. Theorem 3.7).

However, the aforementioned definition of the fuzzy automaton \mathscr{A}_{α} is not sufficiently constructive, because the computing of the fuzzy transition relation and the fuzzy set of terminal states of \mathscr{A}_{α} requires the computing of minimal words in certain infinite languages with respect to the embedding order, which might be a problem. We solve this problem introducing a reflexive and transitive fuzzy relation $R_{\mathscr{A}}$ on the set of states of the starting nondeterministic finite automaton \mathscr{A} , which can be effectively computed as the *n*-th power of an easily computable fuzzy relation, where *n* is the number of states of \mathscr{A} . We express the fuzzy relation $R_{\mathscr{A}}$ in terms of the homomorphism φ_{α}^* and the transition relation of \mathscr{A} (cf. Theorem 4.3), and then we express the fuzzy transition relation and the fuzzy set of terminal states of \mathscr{A}_{α} in terms of the fuzzy relation of \mathscr{A} , and the set of terminal states of \mathscr{A}_{α} in terms of the fuzzy relation of \mathscr{A} , and the set of terminal states of \mathscr{A}_{α} in terms of the fuzzy relation of \mathscr{A} , and the set of terminal states of \mathscr{A}_{α} in terms of the fuzzy relation $R_{\mathscr{A}}$, the transition relation of \mathscr{A} , and the set of terminal states of \mathscr{A}_{α} in terms of the fuzzy relation \mathscr{A}_{α} associated with \mathscr{A} and the fuzzy regular expression α .

Using the fuzzy relation $\mathcal{R}_{\mathscr{A}}$ we also construct a version \mathscr{A}_{α}^{r} of the fuzzy finite automaton \mathscr{A}_{α} which can have even smaller number of states than the fuzzy automaton \mathscr{A}_{α} and the automaton \mathscr{A} , and recognizes the same fuzzy language $||\alpha||$ (cf. Theorem 5.1). We show by an example that the number of states of \mathscr{A}_{α}^{r} can be strictly smaller than the number of states of \mathscr{A} and \mathscr{A}_{α} . We also discuss the state reduction of the fuzzy automaton \mathscr{A}_{α} by means of right invariant crisp equivalences, and we show that even if the starting automaton \mathscr{A} is a minimal deterministic automaton, the number of states of the fuzzy automaton \mathscr{A}_{α} could be reduced. Finally, we describe certain properties of fuzzy automata obtained from the position and the follow automaton.

The structure of the paper is as follows. In Section 2 we recall some basic definitions and results concerning fuzzy sets and relations over lattice ordered monoids, nondeterministic and fuzzy automata, and regular and fuzzy regular expressions. In Section 3 we give the basic construction of a fuzzy finite automaton \mathcal{A}_{α} associated with a fuzzy regular expression α and a nondeterministic finite automaton \mathcal{A} recognizing the language $||\alpha_R||$. Section 4 addresses the issue of the effective construction of the fuzzy automaton \mathcal{A}_{α} , and in Section 5 we deal with the version of this construction that gives a fuzzy automaton with a reduced number of states with respect to the original construction. Finally, in Section 6 we discuss the problem of the reduction of the number of states of fuzzy finite automata constructed from fuzzy regular expressions.

2. Preliminaries

In this section we recall some basic definitions and results concerning fuzzy sets and relations over lattice ordered monoids, nondeterministic and fuzzy automata, and regular and fuzzy regular expressions.

2.1. Lattice-ordered monoids

A *lattice-ordered monoid* or an ℓ -monoid [52, 53, 55, 77] is an algebra $\mathscr{L} = (L, \land, \lor, \otimes, 0, 1, e)$ such that

- (L1) $(L, \land, \lor, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
- (L2) (L, \otimes, e) is a monoid with the unit e,
- (L3) $x \otimes 0 = 0 \otimes x = 0$, for every $x \in L$,
- (L4) $x \otimes (y \vee z) = x \otimes y \vee x \otimes z$, $(x \vee y) \otimes z = x \otimes z \vee y \otimes z$, for all $x, y, z \in L$.

The operation \otimes is called the *multiplication*. In addition, if (*L*, \wedge , \vee , 0, 1) is a complete lattice and satisfies the following infinite distributive laws

$$x \otimes \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (x \otimes x_i), \qquad \left(\bigvee_{i \in I} x_i\right) \otimes x = \bigvee_{i \in I} (x_i \otimes x), \tag{1}$$

then \mathscr{L} is called a *quantale*. In the general case, in an ℓ -monoid $\mathscr{L} = (L, \land, \lor, \otimes, 0, 1, e)$ the greatest element 1 of the lattice $(L, \land, \lor, 0, 1)$ and the unit element *e* of the monoid (L, \otimes, e) are different. If 1 and *e* coincide, then \mathscr{L} is called an *integral* ℓ -monoid.

It can be easily verified that with respect to \leq , the multiplication \otimes in an ℓ -monoid is isotone in both arguments, i.e., for all $x, y, z \in L$ we have

$$x \leq y \text{ implies } x \otimes z \leq y \otimes z \text{ and } z \otimes x \leq z \otimes y.$$
(2)

An integral quantale with commutative multiplication is known as a *complete residuated lattice* (cf. [5, 6]). The most studied and applied kinds of complete residuated lattices, with the support [0, 1], $x \land y = \min(x, y)$ and $x \lor y = \max(x, y)$, are the *Lukasiewicz structure*, with the multiplication defined by $x \otimes y = \max(x+y-1, 0)$, the *Goguen* or *product structure*, with $x \otimes y = x \cdot y$, and the *Gödel structure*, with $x \otimes y = \min(x, y)$. The fourth important type of complete residuated lattices is the two-element Boolean algebra of classical logic with the support {0, 1}, called the *Boolean structure*.

In the further text, if not noted otherwise, \mathscr{L} will be an ℓ -monoid. A *fuzzy subset* of a set *A* is defined as any mapping from *A* into *L*. Ordinary crisp subsets of *A* are considered as fuzzy subsets of *A* taking membership values in the set $\{0, e\} \subseteq L$. Let *f* and *g* be two fuzzy subsets of *A*. The *equality* of *f* and *g* is defined as the usual equality of functions, i.e., f = g if and only if f(x) = g(x), for every $x \in A$. The *inclusion* $f \leq g$ is also defined pointwise: $f \leq g$ if and only if $f(x) \leq g(x)$, for every $x \in A$. Endowed with this partial order the set L^A of all fuzzy subsets of *A* forms the distributive lattice, in which the meet (intersection) $f \wedge g$ and the join (union) $f \vee g$ of any fuzzy subsets f, g of *A* are also fuzzy subsets of *A* over \mathscr{L} defined by

$$(f \wedge g)(x) = f(x) \wedge g(x), \quad (f \vee g)(x) = f(x) \vee g(x). \tag{3}$$

for each $x \in L$. The *crisp part* of a fuzzy subset $f \in L^A$ is a crisp subset $\widehat{f} = \{a \in A \mid f(a) = e\}$ of A. We will also consider \widehat{f} as a mapping $\widehat{f} : A \to L$ defined by $\widehat{f}(a) = e$, if f(a) = e, and $\widehat{f}(a) = 0$, otherwise.

A *fuzzy relation* on *A* is any fuzzy subset of $A \times A$. The equality, inclusion and ordering of fuzzy relations are defined as for fuzzy sets. For fuzzy relations *R* and *S* on a set *A*, their *composition* $R \circ S$ is a fuzzy relation on *A* defined by

$$(R \circ S)(a, b) = \bigvee_{c \in A} R(a, c) \otimes S(c, b), \tag{4}$$

for all $a, b \in A$, and for a fuzzy subset f of A and a fuzzy relation R on A, the *compositions* $f \circ R$ and $R \circ f$ are fuzzy subsets of A defined, for any $a \in A$, by

$$(f \circ R)(a) = \bigvee_{b \in A} f(b) \otimes R(b, a), \quad (R \circ f)(a) = \bigvee_{b \in A} R(a, b) \otimes f(b).$$
(5)

For fuzzy subsets *f* and *g* we write

$$f \circ g = \bigvee_{a \in A} f(a) \otimes g(a).$$
(6)

It is well known that the composition of fuzzy relations is associative. Moreover

$$(f \circ R) \circ S = f \circ (R \circ S), \qquad (R \circ S) \circ f = R \circ (S \circ f), \qquad (f \circ R) \circ g = f \circ (R \circ g), \tag{7}$$

for all fuzzy subsets f and g of A, and fuzzy relations R and S on A. If A is a finite set with n elements, then R and S can be treated as $n \times n$ matrices over \mathcal{L} , and $R \circ S$ is their matrix product, whereas $f \circ R$ can be treated as the product of the $1 \times n$ matrix f and the $n \times n$ matrix R, and $R \circ f$ as the product of the $n \times n$ matrix R and the $n \times 1$ matrix f^t (the transpose of f).

For a finite set *A* and an fuzzy relation *R* on *A*, a fuzzy relation R^n is defined inductively as follows: R^0 is the crisp equality on *A*, and $R^{n+1} = R^n \circ R$, for $n \in \mathbb{N} \cup \{0\}$.

A fuzzy relation *R* on *A* is said to be

- (R) *reflexive* if R(a, a) = e, for every $a \in A$;
- (S) symmetric if R(a, b) = R(b, a), for all $a, b \in A$;
- (T) *transitive* if for all $a, b, c \in A$ we have $R(a, b) \otimes R(b, c) \leq R(a, c)$.

It is easy to check that a reflexive fuzzy relation *R* is transitive if and only if $R^2 = R$, and then $R^n = R$, for every $n \in \mathbb{N}$. A reflexive, symmetric and transitive fuzzy relation is called a *fuzzy equivalence*. For a fuzzy equivalence *E* on *A* and $a \in A$ we define a fuzzy subset E_a of *A* by $E_a(x) = E(a, x)$, for every $x \in A$. We call E_a the *equivalence class* of *E* determined by *a*. The set $A/E = \{E_a | a \in A\}$ is called the *factor set* of *A* with respect to *E* (cf. [5, 6, 20]). We use the same notation for crisp equivalences, i.e., for an equivalence π on *A*, the related factor set is denoted by A/π , the equivalence class of an element $a \in A$ is denoted by π_a . A fuzzy equivalence *E* on a set *A* is called a *fuzzy equality* if for all $x, y \in A$, E(x, y) = e implies x = y. In other words, *E* is a fuzzy equality if and only if its crisp part \widehat{E} is a crisp equality.

2.2. Fuzzy regular expressions

Let *X* be a non-empty set, which is called an *alphabet* and whose elements are called *letters*, and let X^* be the free monoid over *X*, i.e., the set of all finite sequences of letters from *X*, including the empty sequence, equipped with the concatenation operation. Elements of X^* are called *words*, and the empty sequence is denoted by ε and called the *empty word*.

A *fuzzy language* in X^* is defined as any fuzzy subset of X^* . A *language* in X^* is a fuzzy language in X^* taking membership values in the set $\{0, e\}$. For a fuzzy language f and a scalar $\lambda \in L$, the *scalar multiplication* $\lambda \otimes f$ is a fuzzy language in X^* defined by

$$(\lambda \otimes f)(u) = \lambda \otimes f(u),$$

for any $u \in X^*$. The *union* (*join*) $f \lor g$ of fuzzy languages f and g is defined as the union of fuzzy subsets f and g. The *concatenation* (*product*) fg of fuzzy languages f and g is defined by

$$(fg)(u) = \bigvee_{u=vw} f(v) \otimes g(w).$$

The concatenation of fuzzy languages is an associative operation, and for $n \in \mathbb{N}$, the *n*-th power of a fuzzy language *f* is defined inductively by $f^0 = f_{\varepsilon}$, where f_{ε} is a characteristic function of the empty word ε , i.e.,

$$f_{\varepsilon}(u) = \begin{cases} e & \text{if } u = \varepsilon \\ 0 & \text{otherwise}' \end{cases}$$
(8)

and $f^{n+1} = f^n f$, for each $n \in \mathbb{N} \cup \{0\}$. The *Kleene closure* of a fuzzy language *f*, denoted by f^* , is defined by

$$f = \bigvee_{n \in \mathbb{N} \cup \{0\}} f^n.$$

Recall the following result proved in [55].

Proposition 2.1. If \mathscr{L} is an integral ℓ -monoid, then for any fuzzy language f, the Kleene closure is well defined.

The family \mathcal{LR} of *fuzzy regular expressions* over a finite alphabet *X* is defined inductively in the following way (cf. [52, 55]):

- (i) $\emptyset \in \mathscr{LR}$;
- (ii) $\varepsilon \in \mathcal{LR}$;
- (iii) $x \in \mathcal{LR}$, for all $x \in X$;
- (iv) $(\lambda \alpha) \in \mathscr{LR}$, for all $\lambda \in L$ and $\alpha \in \mathscr{LR}$ (scalar multiplication);
- (v) $(\alpha_1 + \alpha_2) \in \mathcal{LR}$, for all $\alpha_1, \alpha_2 \in \mathcal{LR}$ (addition);
- (vi) $(\alpha_1 \alpha_2) \in \mathscr{LR}$, for all $\alpha_1, \alpha_2 \in \mathscr{LR}$ (concatenation);
- (vii) $(\alpha^*) \in \mathcal{LR}$, for all $\alpha \in \mathcal{LR}$ (star operation);

(viii) There are no other fuzzy regular expressions than those given in steps (i)-(viii).

In order to avoid parentheses it is assumed that the star operation has the highest priority, then concatenation and then addition. For any fuzzy regular expression $\alpha \in \mathcal{LR}$, the *fuzzy language* $||\alpha||$ *determined by* α is defined inductively as follows (cf. [52, 55]):

- (i) $\|\emptyset\|(u) = 0$, for every $u \in X^*$,
- (ii) For $\alpha \in X \cup \{\varepsilon\}$, $\|\alpha\| = f_{\alpha}$, where f_{α} is the characteristic function of α defined by

$$f_{\alpha}(u) = \begin{cases} e & \text{if } u = \alpha \\ 0 & \text{otherwise} \end{cases}$$

(iii) $\|\lambda \alpha\| = \lambda \otimes \|\alpha\|$ for all $\lambda \in L$ and $\alpha \in \mathcal{LR}$;

- (iv) $\|(\alpha_1 + \alpha_2)\| = \|\alpha_1\| \vee \|\alpha_2\|$, for all $\alpha_1, \alpha_2 \in \mathcal{LR}$;
- (v) $\|(\alpha_1\alpha_2)\| = \|\alpha_1\| \|\alpha_2\|$, for all $\alpha_1, \alpha_2 \in \mathcal{LR}$;
- (v) $\|\alpha^*\| = \|\alpha\|^*$, for all $\alpha \in \mathcal{LR}$.

For a fuzzy regular expression α over *X*, the *length* of α , denoted by $|\alpha|_X$, is the number of occurrences of letters from *X* in α .

A fuzzy regular expression α which does not contain any occurrence of an element of *L* is called a *regular expression* over an alphabet *X*. In other words, regular expressions are those fuzzy regular expressions that are obtained without using any scalar mutiplication. Note that the fuzzy language $||\alpha||$ defined by a regular expression α takes membership values in the set {0, *e*}, and thus, it can be considered as an ordinary subset of *X*^{*}.

For the free monoid X^* we set $X^+ = X^* \setminus \{\varepsilon\}$. The *length* of a word $u \in X^*$, in notation |u|, is the number of appearances of letters from X in u. The embedding order relation \leq_{em} is defined on X^* by

$$u \leqslant_{em} v \Leftrightarrow u = u_1 u_2 \cdots u_n \text{ and } v = v_0 u_1 v_1 u_2 \cdots v_{n-1} u_n v_n, \tag{9}$$

where $n \in \mathbb{N}$ and $u, v, u_1, u_2, ..., u_n, v_0, v_1, ..., v_n \in X^*$.

Proposition 2.2. ([32, 33]) For any alphabet X, \leq_{em} is a partial order on X^* . Any set of pairwise incomparable words in the partially ordered set (X^* , \leq_{em}) is finite.

Consequently, for any $U \subseteq X^*$ *, the set* M(U) *of all minimal words from* U *with respect to* \leq_{em} *is finite.*

Throughout the paper, the set of all minimal words from $U \subseteq X^*$ with respect to the embedding order \leq_{em} will be denoted by M(U), as in the previous proposition.

2.3. Fuzzy automata

Let \mathscr{L} be an ℓ -monoid. A *fuzzy automaton* (over \mathscr{L}) is defined as a five-tuple $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, where A and X are non-empty sets, called respectively the *set of states* and the *input alphabet*, $\delta^A : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called the *fuzzy transition relation*, $\sigma^A \in L^A$ is the fuzzy set of *initial states*, and $\tau^A \in L^A$ is the fuzzy set of *terminal states*. We will assume that the input alphabet X is always finite. A fuzzy automaton whose set of states is finite is called a *fuzzy finite automaton*. Since all fuzzy automata considered in this paper will be finite, we will speak simply *fuzzy automaton* instead of fuzzy finite automaton. Cardinality of a fuzzy automaton \mathscr{A} , in notation $|\mathscr{A}|$, is defined as the cardinality |A| of its set of states A.

The fuzzy transition relation δ^A can be extended up to a mapping $\delta^A_* : A \times X^* \times A \to L$ in the following way: If $a, b \in A$ and $\varepsilon \in X^*$ is the empty word, then

$$\delta_*^A(a,\varepsilon,b) = \begin{cases} e & \text{if } a = b \\ 0 & \text{otherwise}' \end{cases}$$
(10)

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^A_*(a, ux, b) = \bigvee_{c \in A} \delta^A_*(a, u, c) \otimes \delta^A(c, x, b)$$
(11)

Without danger of confusion we shall write just δ^A instead of δ^A_* .

By (L4) and Theorem 3.1 in [55] we have that

$$\delta^{A}(a, uv, b) = \bigvee_{c \in A} \delta^{A}(a, u, c) \otimes \delta^{A}(c, v, b),$$
(12)

for all $a, b \in A$ and $u, v \in X^*$.

For any $u \in X^*$ we define a fuzzy relation $\delta_u^A \in L^{A \times A}$, called the *fuzzy transition relation* determined by u, by $\delta_u^A(a, b) = \delta^A(a, u, b)$, for all $a, b \in A$. Then for all $u, v \in X^*$, the equality (12) can be written as $\delta_{uv}^A = \delta_u^A \circ \delta_v^A$.

A fuzzy language recognized by a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, denoted by $L(\mathscr{A})$, is a fuzzy language in X* defined by

$$L(\mathscr{A})(u) = \bigvee_{a,b\in A} \sigma^{A}(a) \otimes \delta^{A}(a,u,b) \otimes \tau^{A}(b),$$
(13)

or equivalently,

$$L(\mathscr{A})(u) = \sigma^{A} \circ \delta_{u}^{A} \circ \tau^{A} = \sigma^{A} \circ \delta_{x_{1}}^{A} \circ \delta_{x_{2}}^{A} \circ \cdots \circ \delta_{x_{n}}^{A} \circ \tau^{A},$$
(14)

for any $u = x_1 x_2 ... x_n \in X^*$ with $x_1, x_2, ..., x_n \in X$.

In particular, if $\mathscr{A} = (A, X, \delta^A, a_0, \tau^A)$ is a fuzzy automaton having a single crisp initial state a_0 , then the fuzzy language $L(\mathscr{A})$ recognized by \mathscr{A} is given by

$$L(\mathscr{A})(u) = \bigvee_{a \in A} \delta^A(a_0, u, a) \otimes \tau^A(a).$$
(15)

or equivalently,

$$L(\mathscr{A})(u) = (\delta_u^A \circ \tau^A)(a_0) = (\delta_{x_1}^A \circ \delta_{x_2}^A \circ \dots \circ \delta_{x_n}^A \circ \tau^A)(a_0),$$
(16)

for any $u = x_1 x_2 ... x_n \in X^*$ with $x_1, x_2, ..., x_n \in X$.

In the further text, ordinary nondeterministic automata will be considered as fuzzy automata. Namely, by a nondeterministic automaton we mean a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ such that δ_x^A is a fuzzy relation taking values in the set $\{0, e\}$, for each $x \in X$, and σ^A and τ^A are fuzzy sets also taking values in $\{0, e\}$. In this case, the fuzzy language recognized by \mathscr{A} is a crisp language, and it is exactly the language recognized by a nondeterministic automaton in the sense of the well-known definition from the classical theory of non-deterministic automata.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton and let *E* be a fuzzy equivalence on *A*. Without any restriction on the fuzzy equivalence *E*, we define a fuzzy transition relation $\delta^{A/E} : A/E \times X \times A/E \to L$ by

$$\delta^{A/E}(E_a, x, E_b) = \bigvee_{a', b' \in A} E(a, a') \otimes \delta(a', x, b') \otimes E(b', b) = (E \circ \delta_x \circ E)(a, b) = E_a \circ \delta_x \circ E_b, \tag{17}$$

and fuzzy sets $\sigma^{A/E} \in L^{A/E}$ and $\tau^E \in L^{A/E}$ of initial and terminal states by

$$\sigma^{A/E}(E_a) = \bigvee_{a' \in A} \sigma^A(a') \otimes E(a', a) = (\sigma^A \circ E)(a) = \sigma^A \circ E_a, \tag{18}$$

$$\tau^{A/E}(E_a) = \bigvee_{a' \in A} \tau^A(a') \otimes E(a', a) = (\tau^A \circ E)(a) = \tau^A \circ E_a, \tag{19}$$

for any $a \in A$. Evidently, $\delta^{A/E}$, $\sigma^{A/E}$ and $\tau^{A/E}$ are well-defined, and $\mathscr{A}/E = (A/E, X, \delta^{A/E}, \sigma^{A/E}, \tau^{A/E})$ is a fuzzy automaton, called the *factor fuzzy automaton* of \mathscr{A} with respect to *E*.

2.4. Position automata

In this section we recall the construction of the position automaton from a regular expression [30, 61].

Let α be a regular expression over an alphabet X. Denote by $\overline{\alpha}$ the expression obtained from α by marking each letter in α with its position. The same notation will be used for removing indices, that is, for a regular expression α we put $\alpha = \overline{\alpha}$. We define the following sets:

(i) $pos_0(\alpha) = \{0, 1, \dots, |\alpha|_X\},\$

- (ii) $first(\alpha) = \{i \mid x_i u \in ||\overline{\alpha}||\},\$
- (iii) $last(\alpha) = \{i \mid ux_i \in ||\overline{\alpha}||\},\$
- (iv) $follow(\alpha, i) = \{j \mid ux_i x_j v \in ||\overline{\alpha}||\},\$
- (v) $follow(\alpha, 0) = first(\alpha)$,

(vi)
$$last_0(\alpha) = \begin{cases} last(\alpha), & \varepsilon \notin ||\alpha|| \\ last(\alpha) \cup \{0\}, & \varepsilon \in ||\alpha|| \end{cases}$$

Define $\delta_{pos} \subseteq pos_0(\alpha) \times X \times pos_0(\alpha)$ by

$$(i, x, j) \in \delta_{pos} \iff \overline{x_j} = x \text{ and } j \in follow(\alpha, i).$$

Then $\mathscr{A}_{pos}(\alpha) = (pos_0(\alpha), X, \delta_{pos}, 0, last_0(\alpha))$ is a nondeterministic automaton called the *position automaton* of α . It was shown by Glushkov [30] and McNaughton and Yamada [61] that $L(\mathscr{A}_{pos}(\alpha)) = ||\alpha||$.

For the sake of simplicity, instead of $\mathcal{A}_{pos}(\alpha) = (pos_0(\alpha), X, \delta_{pos}, 0, last_0(\alpha))$, in the further text we will write $\mathcal{A}_p(\alpha) = (A_p, X, \delta^{A_p}, 0, \tau^{A_p})$.

3. Fuzzy automata from fuzzy regular expressions: Basic construction

For an ℓ -monoid $\mathscr{L} = (L, \land, \lor, \otimes, 0, 1, e), A, B \subseteq L$ and $\lambda \in L$ we will use the following notation

$$A \otimes B = \{a \otimes b \mid a \in A, b \in B\}, \qquad A \vee B = \{a \vee b \mid a \in A, b \in B\}, \qquad \lambda \otimes A = \{\lambda \otimes a \mid a \in A\}.$$

The following lemma will be useful in our further work.

Lemma 3.1. Let $\mathscr{L} = (L, \land, \lor, \otimes, 0, 1, e)$ be an ℓ -monoid, let $A, B \subseteq L$ and $\lambda \in L$. If there exist finite sets $C \subseteq A$ and $D \subseteq B$ such that

$$(\forall a \in A)(\exists c \in C) a \leq c \text{ and } (\forall b \in B)(\exists d \in D) b \leq d,$$

$$\bigvee A \otimes B = (\bigvee A) \otimes (\bigvee B), \qquad \bigvee A \vee B = (\bigvee A) \vee (\bigvee B), \qquad \bigvee \lambda \otimes A = \lambda \otimes (\bigvee A).$$

Proof. The proof of this lemma is elementary and will be omitted.

Let \mathscr{L} be an ℓ -monoid and let α be a fuzzy regular expression over a finite alphabet *X*. Let *K* be the set of all $\lambda \in L$ appearing in α (if α is a fuzzy regular expression without scalar multiplication then $K = \emptyset$) and let *Y* be an alphabet such that $Y \cap X = \emptyset$ and |K| = |Y|, and let $\lambda \mapsto \lambda'$ be an arbitrary bijective mapping from *K* to *Y*. We will call *Y* the *alphabet associated with* α . It is clear that *Y* is finite.

Let us denote by α_R the expression obtained from α by replacing each $\lambda \in K$ by the corresponding letter $\lambda' \in Y$. Obviously, α_R is a regular expression over the alphabet $X \cup Y$. Further, $||\alpha_R||$ is considered as a fuzzy language over an alphabet $X \cup Y$, taking values in the set $\{0, e\} \subseteq L$.

Let φ_{α} : $X \cup Y \rightarrow L$ be a mapping defined by

$$\varphi_{\alpha}(x) = \begin{cases} e, & \text{if } x \in X \\ \lambda, & \text{if } x = \lambda' \in Y \end{cases}$$
(20)

for any $x \in X \cup Y$. Denote by φ_{α}^* a homomorphism from the monoid $(X \cup Y)^*$ into the monoid (L, \otimes, e) defined by: $\varphi_{\alpha}^*(\varepsilon) = e$ and $\varphi_{\alpha}^*(u) = \varphi_{\alpha}(x_1) \otimes \varphi_{\alpha}(x_2) \otimes \cdots \otimes \varphi_{\alpha}(x_n)$, for any $u = x_1 x_2 \cdots x_n$ with $x_1, \ldots, x_n \in (X \cup Y)^*$.

Example 3.2. Let \mathscr{L} be an arbitrary ℓ -monoid and let α be a fuzzy regular expression over an alphabet X. If α is without scalars then $\alpha_R = \alpha$.

Example 3.3. Let \mathscr{L} be the Gödel structure. Consider $\alpha = 0.2((0.1(xy)^*)^* + y)$, a fuzzy regular expression over the alphabet $\{x, y\}$. An expression $\alpha_R = \lambda((\mu(xy)^*)^* + y)$ is a regular expression over the alphabet $\{x, y, \lambda, \mu\}$, obtained from α by replacing 0.2 with λ and 0.1 with μ . The mapping φ_{α} is given by

$$\varphi_{\alpha} = \begin{pmatrix} x & y & \lambda & \mu \\ 1 & 1 & 0.2 & 0.1 \end{pmatrix}.$$

Example 3.4. Consider a fuzzy regular expression $\alpha = (0.1x^*)(yx + 0.8y)^*$, where \mathscr{L} is the product structure. Then $\alpha_R = (\lambda x^*)(yx + \mu y)^*$ is a regular expression over the alphabet $\{x, y, \lambda, \mu\}$, where λ replaces 0.1 and μ replaces 0.8. The mapping φ_{α} is given by

$$\varphi_{\alpha} = \begin{pmatrix} x & y & \lambda & \mu \\ 1 & 1 & 0.1 & 0.8 \end{pmatrix}$$

Now we prove the following.

Lemma 3.5. Let \mathscr{L} be an integral ℓ -monoid, and let X be an arbitrary alphabet. Then every homomorphism φ from the monoid X^* into the monoid $(L, \otimes, 1)$ is antitone, i.e.,

$$u \leqslant_{em} v \implies \varphi(v) \leqslant \varphi(u), \tag{21}$$

for all $u, v \in X^*$. Furthermore, for any $U \subseteq X^*$ and any $\gamma : X^* \to \{0, 1\}$ there exists $\bigvee \{\varphi(u) \otimes \gamma(u) \mid u \in U\}$ and

$$\bigvee_{u \in U} \varphi(u) \otimes \gamma(u) = \bigvee_{u \in M(U')} \varphi(u) \otimes \gamma(u),$$
(22)

where $U' = \{u \in U \mid \gamma(u) = 1\}.$

Proof. If $u \leq_{em} v$, then by (9) we have

 $u = u_1 u_2 \cdots u_n$ and $v = v_0 u_1 v_1 u_2 \cdots v_{n-1} u_n v_n$,

where $n \in \mathbb{N}$ and $u, v, u_1, u_2, \ldots, u_n, v_0v_1, \ldots, v_n \in X^*$. Consequently, according to (2) ,we have

$$\varphi(v) = \varphi(v_0) \otimes \varphi(u_1) \otimes \varphi(v_1) \otimes \varphi(u_2) \otimes \dots \otimes \varphi(v_{n-1}) \otimes \varphi(u_n) \otimes \varphi(v_n)$$

$$\leq 1 \otimes \varphi(u_1) \otimes 1 \otimes \varphi(u_2) \otimes \dots \otimes 1 \otimes \varphi(u_n) \otimes 1 = \varphi(u_1) \otimes \varphi(u_2) \otimes \dots \otimes \varphi(u_n) = \varphi(u).$$

Therefore, $\varphi(v) \leq \varphi(u)$.

Further, for any $U \subseteq X^*$, $\gamma : X^* \to \{0, 1\}$, and $u \in U' = \{v \in U \mid \gamma(v) = 1\}$ there exists $w \in M(U')$ such that $w \leq_{em} u$, and by (21) it follows that $\varphi(u) \leq_{em} \varphi(w)$. According to Proposition 2.2, we have that M(U') is finite, and by Lemma 3.1 we obtain that $\bigvee \{\varphi(u) \otimes \gamma(u) \mid u \in U\} = \bigvee \{\varphi(u) \mid u \in U'\}$ exists and (22) holds. \Box

In particular, for a given regular expression α , the homomorphism φ_{α}^* satisfies (21) and (22). Let *Z* be an alphabet. The *shuffle* operation, denoted by \sqcup is defined in the following way

$$u \sqcup v = \{ u_1 v_1 u_2 v_2 \cdots u_n v_n \mid u = u_1 u_2 \dots u_n, v = v_1 v_2 \dots v_n, u_i, v_i \in \mathbb{Z}^*, 1 \le i \le n, n \in \mathbb{N} \},$$
(23)

where $u, v \in Z^*$.

The above operation is naturally extended to languages by the *shuffle* of languages, defined as

$$L_1 \sqcup L_2 = \bigcup_{u \in L_1, v \in L_2} u \sqcup v.$$
(24)

where $L_1, L_2 \subseteq Z^*$.

Let us return now to the fuzzy regular expression α over the alphabet X and the alphabet Y associated with α . Supposing $\emptyset^* = \{\varepsilon\}$, for any $u \in X^*$ we define a language $U_Y(u) \subseteq (X \cup Y)^*$ by

 $U_Y(u) = u \sqcup Y^*.$

It is easy to check that the following holds

$$U_{Y}(\varepsilon) = Y^{*},$$
If $Y = \emptyset$ then $U_{Y}(u) = \{u\}$, for every $u \in X^{*},$

$$U_{Y}(u)U_{Y}(v) = U_{Y}(uv),$$
 for all $u, v \in X^{*},$
(27)

$$U_Y(x) = Y^* x Y^*, \quad \text{for every } x \in X, \tag{28}$$

where the set $U_Y(u)U_Y(v)$ is the concatenation of sets $U_Y(u)$ and $U_Y(v)$, and Y^*xY^* is the concatenation of Y^* , $\{x\}$ and Y^* .

One of the main results of this paper is the following theorem.

Theorem 3.6. Let \mathscr{L} be an integral ℓ -monoid. Let α be a fuzzy regular expression over a finite alphabet X, and let Y be an alphabet associated with α . Then

$$\|\alpha\|(u) = \bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes \|\alpha_R\|(v),$$
⁽²⁹⁾

for every $u \in X^*$.

Proof. Consider an arbitrary $u \in X^*$.

For $U = U_Y(u)$, if the set $U' = \{v \in U \mid ||\alpha_R||(v) = 1\}$ is non-empty, then by Lemma 3.5 it follows that the supremum on the right side of (29) exists, and

$$\bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes ||\alpha_R||(v) = \bigvee_{v \in M(U')} \varphi_{\alpha}^*(v) \otimes ||\alpha_R||(v).$$
(30)

Otherwise, if $U' = \emptyset$, then

$$\bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes ||\alpha_R||(v) = 0.$$

Thus, we have proved that the supremum on the right side of (29) always exists.

Further, if α is a fuzzy regular expression without scalar multiplication, i.e., if $Y = \emptyset$, then $U_Y(u) = \{u\}$, $\alpha = \alpha_R$, $||\alpha|| = ||\alpha_R||$, and $\varphi_{\alpha}^*(v) = 1$ for every $v \in X^* = (X \cup Y)^*$. As a result, we have

$$\bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes ||\alpha_R||(v) = ||\alpha_R||(u) = ||\alpha||(u).$$

The rest of the proof will be done by induction of the length of the fuzzy regular expression α . Suppose that (29) holds for arbitrary fuzzy regular expressions whose length is less than the length of α .

Let $\alpha = \lambda \beta$, for $\lambda \in L$ and $\beta \in \mathcal{LR}$, and let $Y_1 \subseteq Y$ be the alphabet associated with β . For each $v \in (X \cup Y)^*$ we have

$$\|\alpha_R\|(v) = \begin{cases} \|\beta_R\|(w) & \text{if } v = \lambda'w, \text{ for some } w \in (X \cup Y)^* \\ 0 & \text{otherwise} \end{cases}.$$

For every $w \in (X \cup Y_1)^*$ we have that $\varphi_{\alpha}^*(w) = \varphi_{\beta}^*(w)$ and $\lambda' w \in U_Y(u)$ if and only if $w \in U_{Y_1}(u)$, and also, for every $w \in (X \cup Y)^*$ which contains a letter from $Y \setminus Y_1$ we have that $\|\beta_R\|(w) = 0$. Consequently,

$$\bigvee_{w \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes \|\alpha_R\|(v) = \bigvee_{w \in U_{Y_1}(u)} \varphi_{\alpha}^*(\lambda'w) \otimes \|\beta_R\|(w) = \lambda \otimes \bigvee_{w \in U_{Y_1}(u)} \varphi_{\beta}^*(w) \otimes \|\beta_R\|(w) = \lambda \otimes \|\beta\|(u) = \|\alpha\|(u).$$

The equality marked with * follows by Lemmas 3.1 and 3.5.

Let $\alpha = \beta + \gamma$, for $\beta, \gamma \in \mathcal{LR}$, let $Y_1 \subseteq Y$ be the alphabet associated with β , and let $Y_2 \subseteq Y$ be the alphabet associated with γ . For every $v \in (X \cup Y)^*$ we have that the following is true

$$\|\alpha_{R}\|(v) = \|\beta_{R}\|(v) \vee \|\gamma_{R}\|(v),$$

 $\varphi_{\alpha}^{*}(v) = \varphi_{\beta}^{*}(v), \text{ for } v \in (X \cup Y_{1})^{*}, \text{ and } \varphi_{\alpha}^{*}(v) = \varphi_{\gamma}^{*}(v), \text{ for } v \in (X \cup Y_{2})^{*},$
 $\|\beta_{R}\|(v) = 0, \text{ for } v \notin (X \cup Y_{1})^{*}, \text{ and } \|\gamma_{R}\|(v) = 0, \text{ for } v \notin (X \cup Y_{2})^{*}.$
(31)

Therefore,

$$\bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes ||\alpha_{R}||(v) = \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes (||\beta_{R}||(v) \vee ||\gamma_{R}||(v))$$

$$=^{*} \left(\bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes ||\beta_{R}||(v) \right) \vee \left(\bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes ||\gamma_{R}||(v) \right)$$

$$= \left(\bigvee_{v \in U_{Y_{1}}(u)} \varphi_{\beta}^{*}(v) \otimes ||\beta_{R}||(v) \right) \vee \left(\bigvee_{v \in U_{Y_{2}}(u)} \varphi_{\gamma}^{*}(v) \otimes ||\gamma_{R}||(v) \right) = ||\beta||(u) \vee ||\gamma||(u) = ||\alpha||(u),$$

The equality marked with * follows by Lemmas 3.1 and 3.5.

Next, let $\alpha = \beta \gamma$, for $\beta, \gamma \in \mathcal{LR}$, let $Y_1 \subseteq Y$ be the alphabet associated with β , and let $Y_2 \subseteq Y$ be the alphabet associated with γ . Then (31) and (32) hold, and $\|\alpha_R\|(v) = \bigvee_{v=wp} \|\beta_R\|(w) \otimes \|\gamma_R\|(p)$, for every

 $v \in (X \setminus Y)^*$. Thus

$$\begin{split} \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes \|\alpha_{R}\|(v) &= \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes \bigvee_{v = wp} (\|\beta_{R}\|(w) \otimes \|\gamma_{R}\|(p)) \\ &=^{*} \bigvee_{v \in U_{Y}(u)} \bigvee_{v = wp} \left(\varphi_{\alpha}^{*}(w) \otimes \|\beta_{R}\|(w) \right) \otimes \left(\varphi_{\alpha}^{*}(p) \otimes \|\gamma_{R}\|(p) \right) \\ &= \bigvee_{u = qr} \bigvee_{w \in U_{Y}(q), p \in U_{Y}(r)} \left(\varphi_{\alpha}^{*}(w) \otimes \|\beta_{R}\|(w) \right) \otimes \left(\varphi_{\alpha}^{*}(p) \otimes \|\gamma_{R}\|(p) \right) \\ &= \bigvee_{u = qr} \bigvee_{w \in U_{Y_{1}}(q), p \in U_{Y_{2}}(r)} \left(\varphi_{\beta}^{*}(w) \otimes \|\beta_{R}\|(w) \right) \otimes \left(\varphi_{\gamma}^{*}(p) \otimes \|\gamma_{R}\|(p) \right) \\ &= \sum_{u = qr} \left(\left(\bigvee_{w \in U_{Y_{1}}(q)} \varphi_{\beta}^{*}(w) \otimes \|\beta_{R}\|(w) \right) \otimes \left(\bigvee_{p \in U_{Y_{2}}(r)} \varphi_{\gamma}^{*}(p) \otimes \|\gamma_{R}\|(p) \right) \right) \\ &= \bigvee_{u = qr} \left\| \beta(q) \| \otimes \|\gamma(r)\| = \|\alpha\|(u), \end{split}$$

The equality marked with * follows by $\varphi_{\alpha}^{*}(p) \otimes ||\beta_{R}||(w) = ||\beta_{R}||(w) \otimes \varphi_{\alpha}^{*}(p)$, which is true since $||\beta_{R}||(w) \in \{0, 1\}$, and the equality marked with ** follows by Lemmas 3.1 and 3.5.

Finally, let $\alpha = \beta^*$, for $\beta \in \mathcal{LR}$, and for any $n \in \mathbb{N}$ let $\beta_n = \varepsilon + \beta + \cdots + \beta^n$. Clearly, β and β_n have the same associated alphabet as α , the alphabet Y. Also, $\|\beta_n\|(u) \leq \|\alpha\|(u)$, for all $u \in X^*$ and $n \in \mathbb{N}$. In addition, by the proof of Proposition 2.1 (cf. [55, p. 80]), we have that for every $u \in X^*$ there exists $n \in \mathbb{N}$ such that $\|\alpha\|(u) \leq \|\beta_n\|(u)$, and then

$$\|\alpha\|(u) \leq \|\beta_n\|(u) = \bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes \|(\beta_n)_R\|(v) \leq \bigvee_{v \in U_Y(u)} \varphi_{\alpha}^*(v) \otimes \|\alpha_R\|(v),$$

for every $u \in X^*$. Conversely, for every $u \in X^*$ and $v \in U_Y(u)$ there exists $m \in \mathbb{N}$ such that

$$\varphi_{\alpha}^{*}(v) \otimes \|\alpha_{R}\|(v) \leq \varphi_{\alpha}^{*}(v) \otimes \|(\beta_{m})_{R}\|(v) \leq \|\beta_{m}\|(u) \leq \|\alpha\|(u).$$

In conclusion,

$$\|\alpha\|(u) = \bigvee_{v \in U_Y(u)} \varphi^*_{\alpha}(v) \otimes \|\alpha_R\|(v),$$

which completes the proof of the theorem. \Box

For a fuzzy regular expression α over an alphabet *X*, let α_R be a regular expression over an alphabet $X \cup Y$, where *Y* is an alphabet associated with α . Now, let $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ be an arbitrary nondeterministic automaton recognizing the language $||\alpha_R||$. Evidently, the automaton \mathscr{A} , considered as a fuzzy automaton, recognizes the fuzzy language $||\alpha_R||$. Further, let $\mathscr{A}_{\alpha} = (A_{\alpha}, X, \delta^{A_{\alpha}}, a_0^{\alpha}, \tau^{A_{\alpha}})$ be a fuzzy automaton with $A_{\alpha} = A$, $a_0^{\alpha} = a_0$, and a fuzzy transition relation $\delta^{A_{\alpha}}$ defined by

$$\delta^{A_{\alpha}}(a,x,b) = \bigvee_{v \in U_{Y}(x)} \varphi^{*}_{\alpha}(v) \otimes \delta^{A}(a,v,b),$$
(33)

for all $a, b \in A_{\alpha}$ and $x \in X$,

$$\tau^{A_{\alpha}}(a) = \bigvee_{v \in Y^{*}} \bigvee_{b \in A} \varphi^{*}_{\alpha}(v) \otimes \delta^{A}(a, v, b) \otimes \tau^{A}(b),$$
(34)

or equivalently,

$$au^{A_{\alpha}}(a) = \bigvee_{v \in Y^*} \varphi^*_{\alpha}(v) \otimes (\delta^A_v \circ \tau^A)(a).$$

for each $a \in A_{\alpha}$. Note that the existence of the above suprema by $v \in U_Y(x)$ and $v \in Y^*$ follows immediately by equation (22) in Lemma 3.5.

We prove the following fundamental result.

Theorem 3.7. Let \mathscr{L} be an integral ℓ -monoid, let α be a fuzzy regular expression, and let $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ be an arbitrary nondeterministic automaton which recognizes $||\alpha_R||$.

Then $\mathscr{A}_{\alpha} = (A_{\alpha}, X, \delta^{A_{\alpha}}, a_0, \tau^{A_{\alpha}})$ is a well-defined fuzzy automaton and it recognizes the fuzzy language $||\alpha||$.

Proof. According to (15), we have

$$L(\mathscr{A}_{\alpha})(u) = \bigvee_{a \in A_{\alpha}} \delta^{A_{\alpha}}(a_0, u, a) \otimes \tau^{A_{\alpha}}(a).$$

Thus, for the empty word $\varepsilon \in X^*$, by Theorem 3.6, we have

$$\begin{split} L(\mathscr{A}_{\alpha})(\varepsilon) &= \tau^{A_{\alpha}}(a_{0}) = \bigvee_{v \in Y^{*}} \bigvee_{b \in A} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a_{0}, v, b) \otimes \tau^{A}(b) = \bigvee_{v \in Y^{*}} \varphi_{\alpha}^{*}(v) \otimes \left(\bigvee_{b \in A} \delta^{A}(a_{0}, v, b) \otimes \tau^{A}(b)\right) \\ &= \bigvee_{v \in U_{Y}(\varepsilon)} \varphi_{\alpha}^{*}(v) \otimes \|\alpha_{R}\|(v) = \|\alpha\|(\varepsilon). \end{split}$$

Suppose that $\delta^{A_{\alpha}}(a, u, b) = \bigvee \{ \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a, v, b) \mid v \in U_{Y}(u) \}$, for some $u \in X^{*}$ and all $a, b \in A_{\alpha}$. Then for any $x \in X$ we have

$$\begin{split} \delta^{A_{\alpha}}(a,ux,b) &= \bigvee_{c \in A_{\alpha}} \delta^{A_{\alpha}}(a,u,c) \otimes \delta^{A_{\alpha}}(c,x,b) = \bigvee_{c \in A} \left(\bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a,v,c) \right) \otimes \left(\bigvee_{w \in U_{Y}(x)} \varphi_{\alpha}^{*}(w) \otimes \delta^{A}(c,w,b) \right) \\ &= \bigvee_{c \in A} \bigvee_{v \in U_{Y}(w), \atop w \in U_{Y}(x)} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a,v,c) \otimes \varphi_{\alpha}^{*}(w) \otimes \delta^{A}(c,w,b) = \bigvee_{v \in U_{Y}(w), \atop w \in U_{Y}(x)} \varphi_{\alpha}^{*}(vw) \otimes \left(\bigvee_{c \in A} \delta^{A}(a,v,c) \otimes \delta^{A}(c,w,b) \right) \\ &= \bigvee_{v \in U_{Y}(w), \atop w \in U_{Y}(x)} \varphi_{\alpha}^{*}(vw) \otimes \delta^{A}(a,vw,b) = \bigvee_{v \in U_{Y}(ux)} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a,v,b). \end{split}$$

Observe that the above equalities follow by Lemmas 3.1 and 3.5, and equation (27). We have also used the equality $\delta^A(a, v, c) \otimes \varphi^*_{\alpha}(w) = \varphi^*_{\alpha}(w) \otimes \delta^A(a, v, c)$, which follows by the fact that $\delta^A(a, v, c) \in \{0, 1\}$.

Consequently, for any $u \in X^+$, due to Theorem 3.6, (33) and (34), we have

$$\begin{split} L(\mathscr{A}_{\alpha})(u) &= \bigvee_{a \in A_{\alpha}} \delta^{A_{\alpha}}(a_{0}, u, a) \otimes \tau^{A_{\alpha}}(a) = \bigvee_{a \in A} \left(\bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(a_{0}, v, a) \right) \otimes \left(\bigvee_{w \in Y^{*}} \bigvee_{b \in A} \varphi_{\alpha}^{*}(w) \otimes \delta^{A}(a, w, b) \otimes \tau^{A}(b) \right) \\ &= \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(vw) \otimes \left(\bigvee_{a,b \in A} \delta^{A}(a_{0}, v, a) \otimes \delta^{A}(a, w, b) \otimes \tau^{A}(b) \right) = \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes \left(\bigvee_{a \in A} \delta^{A}(a_{0}, v, b) \otimes \tau^{A}(b) \right) \\ &= \bigvee_{v \in U_{Y}(u)} \varphi_{\alpha}^{*}(v) \otimes ||\alpha_{R}||(v) = ||\alpha||(u). \end{split}$$

This completes the proof of the theorem. \Box

The fuzzy automaton $\mathscr{A}_{\alpha} = (A_{\alpha}, X, \delta^{A_{\alpha}}, a_0, \tau^{A_{\alpha}})$ will be called the *fuzzy automaton associated with* \mathscr{A} and α .

4. Fuzzy automata from fuzzy regular expressions: Effective construction

Let \mathscr{L} be an integral ℓ -monoid, and let α be an arbitrary fuzzy regular expression. Theorem 3.7 allows us to construct different types of fuzzyautomata from α , i.e., different fuzzy automata recognizing the fuzzy language $||\alpha||$. Namely, in the general case, by choosing different nondeterministic automata \mathscr{A} constructed from α_R , we obtain different fuzzy finite automata \mathscr{A}_{α} of α . Let us recall that there are many well-known constructions of small nondeterministic automata from a given regular expression. The most famous are those of Thompson [79], Glushkov [30] and McNaughton-Yamada [61]. The last one is known as the *position automaton*. In addition, Antimirov in [2] constructed the *partial derivative automaton*, which generalizes Brzozowski's *derivative automaton* [7]. However, in spite of improvements made by Brzozowski and Antimirov, the position automaton is the most often used, probably because of its simplicity and the fact that other constructions did not make any practical improvements. The latest nondeterministic automaton constructed from a regular exporession is the *follow automaton*, introduced by Ilie et al. [40–42]. It has been proved that the follow automaton is the quotient of the position automaton, and therefore it is smaller than the position automaton.

Let \mathscr{L} be an integral ℓ -monoid, and let α be an arbitrary fuzzy regular expression over an alphabet *X*. Consider a regular expression α_R over $X \cup Y$, where *Y* is an alphabet associated with α . Starting from the position automaton $\mathscr{A}_p(\alpha_R) = (A_p, X \cup Y, \delta^{A_p}, 0, \tau^{A_p})$ of α_R , by means of (33) and (34) we construct the fuzzy automaton associated with $\mathscr{A}_p(\alpha_R)$ and α , which will be denoted by $\mathscr{A}_{pf}(\alpha) = (A_{pf}, X, \delta^{A_{pf}}, 0, \tau^{A_{pf}})$. The computing of $\mathscr{A}_{pf}(\alpha)$ for a given fuzzy regular expression is described in Theorem 3.7, and Examples 4.1 and 4.5 clarify this construction.

Example 4.1. Let \mathscr{L} be the Gödel structure. Consider $\alpha = 0.2((0.1(xy)^*)^* + y)$, a fuzzy regular expression over the alphabet $\{x, y\}$ from Example 3.3. Here $\alpha_R = \lambda((\mu(xy)^*)^* + y)$ is a regular expression over the alphabet $\{x, y, \lambda, \mu\}$, obtained from α . The marked version of the expression α_R is $\overline{\alpha_R} = \lambda_1((\mu_2(x_3y_4)^*)^* + y_5)$, and φ_α is given by

 $\varphi_{\alpha} = \begin{pmatrix} x & y & \lambda & \mu \\ 1 & 1 & 0.2 & 0.1 \end{pmatrix}.$

The picture bellow represents the graph of the position automaton $\mathscr{A}_{p}(\alpha_{R})$:



Figure 1. The automaton $\mathscr{A}_{p}(\alpha_{R})$

Let us observe that

$$\delta^{A_{\mathrm{pf}}}(i,x,j) = \begin{cases} \bigvee_{u \in \mathcal{M}(i,x,j)} \varphi_{\alpha}^{*}(u) & \text{if } \mathcal{P}(i,x,j) \neq \emptyset\\ u \in \mathcal{M}(i,x,j) & 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$, $i, j \in A_p$ (in notation from the proof of Theorem 3.7).

Let us, for example, describe how to determine $\delta^{A_{pf}}(0, x, 3)$. For each word $u \in \mathcal{M}(0, x, 3)$ there is a path in the graph of $\mathscr{A}_p(\alpha_R)$, which starts in 0 and ends in 3, with a single edge marked with *x* and with all other edges marked with symbol λ or μ (see Figure 1.). Obviously, $\mathscr{M}(0, x, 3) = \{\lambda \mu x\}$. Now,

 $\delta^{A_{\rm pf}}(0,x,3) = \varphi^*_{\alpha}(\lambda \mu x) = 0.2 \otimes 0.1 \otimes 1 = 0.1$

Further, $\mathcal{M}(1, x, 3) = \{\mu x\}$, $\mathcal{M}(2, x, 3) = \{x\}$, $\mathcal{M}(4, x, 3) = \{x\}$ and $\mathcal{M}(i, x, j) = \emptyset$ in all other cases, and we have

 $\delta^{A_{\text{pf}}}(1, x, 3) = 0.1, \ \delta^{A_{\text{pf}}}(2, x, 3) = 1,$ $\delta^{A_{\text{pf}}}(4, x, 3) = 1, \text{ and } \delta^{A_{\text{pf}}}(i, x, j) = 0,$

for $(i, j) \notin \{(0, 3), (1, 3), (2, 3), (4, 3)\}.$

From Figure 1 one can see that $\mathcal{M}(0, y, 5) = \{\lambda y\}$, $\mathcal{M}(1, y, 5) = \{y\}$, $\mathcal{M}(3, y, 2) = \{y\mu\}$ and $\mathcal{M}(3, y, 4) = \{y\}$, whereas $\mathcal{M}(i, y, j) = \emptyset$ in all other cases. Therefore, we have

 $\delta^{A_{\rm pf}}(0,y,5)=0.2, \ \delta^{A_{\rm pf}}(1,y,5)=1, \ \delta^{A_{\rm pf}}(3,y,2)=0.1, \ \delta^{A_{\rm pf}}(3,y,4)=1, \ \delta^{A_{\rm pf}}(i,y,j)=0, \ \delta^{A_{\rm pf}}(i,y,4)=0.1, \ \delta^{A_{\rm pf}}(i,y,4)=0.1,$

for $(i, j) \notin \{(0, 5), (1, 5)(3, 2), (3, 4)\}.$

To summarize, fuzzy transition relations $\delta_x^{A_{pf}}$, $\delta_y^{A_{pf}}$, and the fuzzy set $\tau^{A_{pf}}$ of final states of the fuzzy automaton $\mathcal{A}_{pf}(\alpha)$ are:

	[0]	0	0	0.1	0	0		0	0	0	0	0	0.2	.]	[0.2]	1	
	0	0	0	0.1	0	0	$ \begin{array}{c c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} $	0	0	0	0	1			1		
$\delta_x^{A_{\mathrm{pf}}} =$	0	0	0	1	0	0		0	0	0	0	0	0		$ au^{A_{ m pf}} =$	1	
	0	0	0	0	0	0	, $O_y =$	0	0	0.1	0	1	0	'		0	ľ
	0	0	0	1	0	0	0	0	0	0	0	0	0			1	
	0	0	0	0	0	0		0	0	0	0	0	0			[1]	

and the graph of $\mathscr{A}_{pf}(\alpha)$ is presented by Figure 2.



Figure 2. $\mathscr{A}_{pf}(\alpha)$

It is important to note that the computing of the transition relation of the fuzzy automaton $\mathscr{A}_{pf}(\alpha)$, for a given regular expression, might be a problem. Namely, in the general case, for given $i, j \in A_{pf}$ and $x \in X$ the set $\mathscr{P}(i, x, j)$ of all words $u \in U_Y(x)$ such that $\delta^{A_p}(i, u, j) = 1$ is infinite, and hence, the computing of the set $\mathscr{M}(i, x, j)$ of all minimal words of $\mathscr{P}(i, x, j)$ with respect to \leq_{em} might be a difficult task. In the sequel we consider this problem.

The next lemma is the well-known result which, for instance, was proved in [25] for fuzzy relations with membership values in the real unit interval and the composition defined by means of a *t*-norm. In the same way it can be proved for fuzzy relations over an integral ℓ -monoid.

Lemma 4.2. Let \mathcal{L} be an integral ℓ -monoid and let R be a fuzzy relation on a finite set A with |A| = n. Then

$$\bigvee_{k=1}^{n} R^{k} \tag{35}$$

is the least transitive fuzzy relation on A which contains R.

In particular, if R is reflexive, then the least transitive fuzzy relation on A containing R is equal to R^n .

Let \mathscr{L} be an integral ℓ -monoid, and let α be an arbitrary fuzzy regular expression over an alphabet *X*. For a regular expression α_R over $X \cup Y$, where *Y* is an alphabet associated with α , and let $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ be an arbitrary nondeterministic automaton which recognizes the language $||\alpha_R||$. Let us define a reflexive fuzzy relation *R* on *A* as follows

$$R(a,b) = \begin{cases} 1 & \text{if } a = b \\ \bigvee_{\lambda' \in Y} \lambda \otimes \delta^A(a,\lambda',b) & \text{otherwise '} \end{cases}$$
(36)

and let us denote by $R_{\mathscr{A}}$ the least transitive relation containing *R*. By Lemma 4.2 we obtain that $R_{\mathscr{A}} = R^n$, where *n* is the number of states of \mathscr{A} . Now, we can prove the following:

Theorem 4.3. Let \mathscr{L} be an integral ℓ -monoid and let α be an arbitrary fuzzy regular expression. For an arbitrary nondeterministic automaton $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ recognizing the language $||\alpha_R||$ we have

$$R_{\mathscr{A}}(a,b) = \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes \delta^A(a,u,b), \tag{37}$$

for all $a, b \in A$.

Proof. First we note that the existence of the supremum on the right side of (37) follows by Lemma 3.5. Let *n* be the number of states of \mathscr{A} . If a = b then

$$R_{\mathscr{A}}(a,a) = 1 = \varphi_{\alpha}^{*}(\varepsilon) = \varphi_{\alpha}^{*}(\varepsilon) \otimes \delta^{A}(a,\varepsilon,a) \leq \bigvee_{u \in Y^{*}} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,u,a),$$

and hence, (37) holds. Otherwise, we have the following

$$\begin{split} R_{\mathscr{A}}(a,b) &= R^{n}(a,b) = \bigvee_{a_{1},\dots,a_{n-1} \in A} R(a,a_{1}) \otimes R(a_{1},a_{2}) \otimes \dots \otimes R(a_{n-1},b) \\ &= \bigvee_{a_{1},\dots,a_{k-1} \in A, \atop k \leq n, a_{1} \neq a_{i+1}} R(a,a_{1}) \otimes R(a_{1},a_{2}) \otimes \dots \otimes R(a_{k-1},b) \\ &= \bigvee_{a_{1},\dots,a_{k-1} \in A, \atop k \leq n, a_{1} \neq a_{i+1}} \bigvee_{\lambda_{1},\dots,\lambda_{k}' \in Y} \lambda_{1} \otimes \dots \otimes \lambda_{k} \otimes \delta^{A}(a,\lambda_{1}',a_{1}) \otimes \dots \otimes \delta^{A}(a_{k-1},\lambda_{k}',b) \\ &= \bigvee_{\lambda_{1}',\dots,\lambda_{k}' \in Y} \lambda_{1} \otimes \dots \otimes \lambda_{k} \otimes \left(\bigvee_{a_{1},\dots,a_{k-1} \in A, \atop k \leq n, a_{1} \neq a_{i+1}} \delta^{A}(a,\lambda_{1}',a_{1}) \otimes \dots \otimes \delta^{A}(a_{k-1},\lambda_{k}',b) \right) \\ &\leqslant \bigvee_{\lambda_{1}',\dots,\lambda_{k}' \in Y} \lambda_{1} \otimes \dots \otimes \lambda_{k} \otimes \delta^{A}(a,\lambda_{1}'\dots\lambda_{k}',b) = \bigvee_{\lambda_{1}',\dots,\lambda_{k}' \in Y} \varphi_{\alpha}^{*}(\lambda_{1}'\dots\lambda_{k}') \otimes \delta^{A}(a,\lambda_{1}'\dots\lambda_{k}',b) \\ &\leqslant \bigvee_{u \in Y^{*}} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,u,b). \end{split}$$

On the other hand, consider an arbitrary $u = \lambda'_1 \cdots \lambda'_k \in Y^*$, with $\lambda'_1, \dots, \lambda'_k \in Y$. Then

$$\begin{split} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a, u, b) &= \lambda_{1} \otimes \cdots \lambda_{k} \otimes \delta^{A}(a, \lambda_{1}' \cdots \lambda_{k}', b) \\ &= \lambda_{1} \otimes \cdots \lambda_{k} \otimes \bigvee_{a_{1}, \dots, a_{k-1} \in A} \delta^{A}(a, \lambda_{1}', a_{1}) \otimes \cdots \otimes \delta^{A}(a_{k-1}, \lambda_{k}', b) \\ &\leq \bigvee_{a_{1}, \dots, a_{k-1} \in A} \lambda_{1} \otimes \cdots \wedge_{k} \otimes \delta^{A}(a, \lambda_{1}', a_{1}) \otimes \cdots \otimes \delta^{A}(a_{k-1}, \lambda_{k}', b) \\ &\leq \bigvee_{a_{1}, \dots, a_{k-1} \in A} R(a, a_{1}) \otimes \cdots \otimes R(a_{k-1}, b) = R^{k}(a, b) \leq R_{\mathscr{A}}(a, b), \end{split}$$

whence it follows that

$$\bigvee_{u\in Y^*} \varphi^*_\alpha(u) \otimes \delta^A(a, u, b) \leq R_{\mathscr{A}}(a, b).$$

Therefore, (37) holds.

By the previous theorem we can conclude that $R_{\mathscr{A}}$ is the transitive closure of the adjacency matrix of the weighted graph obtained from the graph of the automaton \mathscr{A} by removing all the edges marked by the symbols from the alphabet *X*, in which the weight of the edge marked by $\lambda' \in Y$ equals $\varphi_{\alpha}^{*}(\lambda')$.

Next we prove the following.

Theorem 4.4. Let \mathscr{L} be an integral ℓ -monoid and let α be an arbitrary fuzzy regular expression. For an arbitrary nondeterministic automaton $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ recognizing the language $||\alpha_R||$ and the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α we have

$$\delta_x^{A_\alpha} = R_\mathscr{A} \circ \delta_x^A \circ R_\mathscr{A}, \text{ and } \tau^{A_\alpha} = R_\mathscr{A} \circ \tau^A, \tag{38}$$

for every $x \in X$.

Proof. By (33), (28) and (37), we have

$$\begin{split} \delta^{A_{\alpha}}(a,x,b) &= \bigvee_{u \in U_{Y}(x)} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,u,b) = \bigvee_{u,v \in Y^{*}} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,uxv,b) \otimes \varphi_{\alpha}^{*}(v) \\ &= \bigvee_{u,v \in Y^{*}} \bigvee_{c,d \in A} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,u,c) \otimes \delta^{A}(c,x,d) \otimes \delta^{A}(d,v,b) \otimes \varphi_{\alpha}^{*}(v) \\ &= \bigvee_{c,d \in A} \left(\bigvee_{u \in Y^{*}} \varphi_{\alpha}^{*}(u) \otimes \delta^{A}(a,u,c) \right) \otimes \delta^{A}(c,x,d) \otimes \left(\bigvee_{v \in Y^{*}} \varphi_{\alpha}^{*}(v) \otimes \delta^{A}(d,v,b) \right) \\ &= \bigvee_{c,d \in A} R_{\mathscr{A}}(a,c) \otimes \delta^{A}(c,x,d) \otimes R_{\mathscr{A}}(d,b), \end{split}$$

for all $a, b \in A$. Let us note that the existence of the above suprema follows by Lemmas 3.1 and 3.5. The rest of the proof follows immediately from (34) and Theorem 4.3. \Box

The previous theorem gives an efficient method for computing the fuzzy automaton corresponding to a given fuzzy regular expression α . Namely, for α and a nondeterministic automaton \mathscr{A} recognizing the language $||\alpha_R||$, the fuzzy transition relations of \mathscr{A}_{α} are just matrix products of $R_{\mathscr{A}}$ and the related fuzzy transition relation of \mathscr{A} (cf. Example 4.5).

Example 4.5. Consider $\alpha = 0.2((0.1(xy)^*)^* + y)$, the fuzzy regular expression from Example 4.1. It is easy to verify, using Figure 2, that fuzzy relations *R* and $R_{\mathcal{A}_p}$ are those given by matrices

	[1	0.2	0	0	0	0]		[1	0.2	0.1	0	0	[0
R =	0	1	0.1	0	0	0		0	1	0.1	0	0	0
	0	0	1	0	0	0	л	0	0	1	0	0	0
	0	0	0	1	0	0	, $\kappa_{\mathscr{A}_p} =$	0	0	0	1	0	0.
	0	0	0.1	0	1	0		0	0	0.1	0	1	0
	0	0	0	0	0	1		0	0	0	0	0	1

Now, by Theorem 4.4, we compute $\delta_x^{A_{pf}}$, $\delta_y^{A_{pf}}$ and $\tau^{A_{pf}}$ as follows:

5. Fuzzy automata from fuzzy regular expressions: Reduced construction

Let \mathscr{L} be an integral ℓ -monoid, and let α be an arbitrary fuzzy regular expression over an alphabet X. For a regular expression α_R over $X \cup Y$, where Y is an alphabet associated with α , let $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ be a nondeterministic automaton recognizing the language $||\alpha_R||$. Besides, let $\mathscr{A}_{\alpha} = (A_{\alpha}, X, \delta^{A_{\alpha}}, a_0, \tau^{A_{\alpha}})$ be the fuzzy automaton associated with \mathscr{A} and α . Set

$$A_{\alpha}^{\mathbf{r}} = \{a_0\} \cup \{a \in A_{\alpha} \mid (\exists b \in A_{\alpha}) (\exists x \in X) \ \delta^A(b, x, a) = 1\}$$

Let us denote by $\mathscr{A}_{\alpha}^{r} = (A_{\alpha}^{r}, X, \delta^{A_{\alpha}^{r}}, a_{0}, \tau^{A_{\alpha}^{r}})$ a fuzzy automaton defined by

$$\delta_x^{A_\alpha^i}(a,b) = (R_\mathscr{A} \circ \delta_x^A)(a,b), \quad \tau^{A_\alpha^i}(a) = (R_\mathscr{A} \circ \tau^A)(a). \tag{39}$$

for all $a, b \in A_{\alpha}^{r}$, and $x \in X$. The fuzzy automaton \mathscr{A}_{α}^{r} is called the *reduced fuzzy automaton* associated with \mathscr{A} and α .

Theorem 5.1. Let \mathscr{L} be an integral ℓ -monoid, let α be an arbitrary fuzzy regular expression, let \mathscr{A}_{α} be an arbitrary nondeterministic automaton recognizing the language $||\alpha_R||$, and let \mathscr{A}_{α}^r be the reduced fuzzy automaton defined as in (39). Then

$$L(\mathscr{A}^{\mathbf{r}}_{\alpha}) = \|\alpha\|. \tag{40}$$

Proof. First, we have that

$$L(\mathscr{A}_{\alpha}^{\mathrm{r}})(\varepsilon) = \tau^{A_{\alpha}^{\mathrm{r}}}(a_{0}) = \tau^{A_{\alpha}}(a_{0}) = ||\alpha||(\varepsilon)$$

Next, for every $u \in X^+$, where $u = x_1 x_2 \cdots x_n$, with $x_1, x_2, \dots, x_n \in X$, by (16), Theorems 3.7 and 4.4, and idempotency of $R_{\mathscr{A}}$ we obtain that

$$\begin{aligned} \|\alpha\|(u) &= L(\mathscr{A}_{\alpha})(u) = (\delta_{x_{1}}^{A_{\alpha}} \circ \cdots \circ \delta_{x_{n}}^{A_{\alpha}} \circ \tau^{A_{\alpha}})(a_{0}) = (R_{\mathscr{A}} \circ \delta_{x_{1}}^{A} \circ R_{\mathscr{A}}^{2} \circ \cdots \circ R_{\mathscr{A}}^{2} \circ \delta_{x_{n}}^{A} \circ R_{\mathscr{A}}^{2} \circ \tau^{A})(a_{0}) \\ &= (R_{\mathscr{A}} \circ \delta_{x_{1}}^{A} \circ R_{\mathscr{A}} \circ \cdots \circ R_{\mathscr{A}} \circ \delta_{x_{n}}^{A} \circ R_{\mathscr{A}} \circ \tau^{A})(a_{0}) \\ &= \bigvee_{a_{1},\dots,a_{n} \in A_{\alpha}} (R_{\mathscr{A}} \circ \delta_{x_{1}}^{A})(a_{0},a_{1}) \otimes \cdots \otimes (R_{\mathscr{A}} \circ \delta_{x_{n}}^{A})(a_{n-1},a_{n}) \otimes (R_{\mathscr{A}} \circ \tau^{A})(a_{n}) \\ &=^{*} (\delta_{x_{1}}^{A_{\alpha}^{*}} \circ \cdots \circ \delta_{x_{n}}^{A_{\alpha}^{*}} \circ \tau^{A_{\alpha}^{*}})(a_{0}) \\ &= L(\mathscr{A}_{\alpha}^{\mathsf{r}}). \end{aligned}$$

The equality marked by * follows from the fact that $(R_{\mathscr{A}} \circ \delta_x^A)(a, b) = 0$, for all $b \in A_\alpha \setminus A_\alpha^r$, and $x \in X$.

Obviously, Theorem 5.1 describes a method of construction a fuzzy automaton from a given fuzzy regular expression, which can be significantly smaller than the one made by the basic construction. Furthermore, if the starting nondeterministic automaton recognizing $||\alpha_R||$ is the position automaton $\mathscr{A}_p(\alpha_R)$, then $\mathscr{A}_{pf}^r(\alpha)$ has exactly $|\alpha|_X + 1$ states (Example 5.2 illustrates this fact). Accordingly, since the position automaton of a given regular expression has the number of states equal to the length of the considered regular expression, the fuzzy automaton $\mathscr{A}_{pf}^r(\alpha)$ is called the *position fuzzy automaton* of the given fuzzy regular expression α .

Example 5.2. Consider a fuzzy regular expression $\alpha = (0.1x^*)(yx+0.8y)^*$ from Example 3.4. Fuzzy transition relations $\delta_x^{A_{\text{pf}}}$, $\delta_y^{A_{\text{pf}}}$ and the fuzzy set $\tau^{A_{\text{pf}}}$ of terminal states of the fuzzy automaton $\mathscr{A}_{\text{pf}}(\alpha)$ are:

	[0	0	0.1	0	0	0.08	ן0	0]	0	0	0.1	0	0.064	0.08]		[0.1]	1
	0	0	1	0	0	0.8	0	0	0	0	1	0	0.64	0.8		1	
	0	0	1	0	0	0.8	0	0	0	0	1	0	0.64	0.8		1	
$\delta_x^{A_{\rm pf}} =$	0	0	0	0	1	0.8	0,	$\delta_{v}^{A_{\rm pf}} = 0$	0	0	0	0	0	0	, $\tau^{A_{\rm pf}} =$	0	
	0	0	0	0	0	0	0	0	0	0	1	0	0.64	0.8		1	
	0	0	0	0	0	0	0	0	0	0	0	0	0.8	1		0	
	0	0	0	0	0	0	0]	[0	0	0	1	0	0.64	0.8		1	

Evidently, $A_{pf}^{r} = \{0, 2, 3, 4, 6\}$, and hence, the fuzzy finite automaton \mathscr{A}_{pf}^{r} has two states less than the position fuzzy automaton $\mathscr{A}_{pf}(\alpha)$.

Fuzzy transition relations $\delta_x^{A_{pf}^r}$, $\delta_y^{A_{pf}^r}$, and the fuzzy set $\tau^{A_{pf}^r}$ of terminal states of $\mathscr{A}_{pf}^r(\alpha)$ are:

	٢0	0.1	0	0	ן0	٦J	0	0.1	0	0.08			[0.1]	
٨r	0	1	0	0	0	AT 0	0	1	0	0.8			1	
$\delta_x^{A_{\rm pf}} =$	0	0	0	1	0,	$\delta_v^{A_{\rm pf}} = 0$	0	0	0	0	,	$\tau^{A_{\rm pf}^{\rm r}} =$	0	
	0	0	0	0	0	9 0	0	1	0	0.8			1	
	[0	0	0	0	0	lo	0	1	0	0.8			1	

6. Reducing the size of position fuzzy automata by right invariant crisp equivalences

The reduction of the number of states of fuzzy automata with membership values in complete residuated lattices has been recently investigated in [23, 24, 78], where the state reduction problem has been related to the problem of solving particular systems of fuzzy relation equations and inequalities. Central place in the state reduction is held by right and left invariant fuzzy equivalences, as well as by right and left invariant fuzzy quasi-orders.

Complete residuated lattices have a rich algebraic structure that provides powerful tools for solving fuzzy relational equations and inequalities, including those that define right and left invariant fuzzy equivalences. Unfortunately, when we deal with fuzzy automata over lattice ordered monoids we do not have such tools, and we are forced to work with crisp equivalences. Therefore, here we reduce the number of states of fuzzy automata over ℓ -monoids using right invariant crisp equivalences.

The reduction of fuzzy automata over complete residuated lattices by means of right invariant fuzzy equivalences and fuzzy quasi-orders has been recently studied in [23, 24, 78]. It has been proved that better state reductions can be achieved employing right invariant fuzzy equivalences. Residuated lattices are rich algebraic structures supplied with operations called residuum and biresiduum, and satisfying many other important algebraic properties. In some sources residuated lattices are called integral, commutative, residuated ℓ -monoids. There, the operations of residuum and biresiduum play a very important role, and are used for modelling right invariant fuzzy equivalences and fuzzy quasi-orders. In this paper, however, we deal with ℓ -monoids, in which, due to the lack of algebraic properties and operations, construction of fuzzy equivalence relations is a problem. Therefore, here we investigate the problem of the reduction of fuzzy automata by right invariant crisp equivalences only. Note that the state reduction of fuzzy automata

by means of crisp equivalences has been already studied in [3, 17, 59, 62, 67], but in very special cases, and the algorithms provided there are based on computing and merging indistinguishable states.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over an ℓ -monoid \mathscr{L} , and let *E* be a fuzzy equivalence on its set of states *A*. If *E* is a solution to the system

$$E \circ \delta_x^A \leqslant \delta_x^A \circ E, \qquad x \in X,$$

$$E \circ \tau^A = \tau^A,$$
(41)

then it is called a *right invariant* fuzzy equivalence on \mathscr{A} . Dually we define *left invariant* fuzzy equivalences. A crisp equivalence on A which is a solution to (41) is called a *right invariant crisp equivalence* on \mathscr{A} . Note that ordinary crisp equivalences on A are considered here as fuzzy equivalences on A taking membership values in the set $\{0, e\} \subseteq L$.

It has been shown in [24] that right invariant fuzzy equivalences are immediate generalizations of right invariant equivalences on nondeterministic automata, studied in a series of papers by Ilie, Yu [40–45], as well as in [9–11], or well-behaved equivalences, studied by Calude et al. [8]. It has been also proved in [24] that congruences on fuzzy automata, studied by Petković in [67], are just right invariant crisp equivalences on fuzzy automata, in the terminology from this paper. Note that right invariant fuzzy equivalences have been called in [21, 22] *forward bisimulation* fuzzy equivalences, whereas left invariant ones were called *backward bisimulation* fuzzy equivalences.

In the same way as in [24] we can show that the inequality $E \circ \delta_x^A \leq \delta_x^A \circ E$ is equivalent to the equation $E \circ \delta_x^A \circ E = \delta_x^A \circ E$, for each $x \in X$. We can also prove that if $E \circ \delta_x^A \leq \delta_x^A \circ E$ or $E \circ \delta_x^A \circ E = \delta_x^A \circ E$ holds for every letter $x \in X$, then it also holds if we replace the letter x by an arbitrary word $u \in X^*$.

In the sequel we provide an algorithm for computing the greatest right invariant crisp equivalence on a fuzzy automaton with membership values in an integral ℓ -monoid. Such algorithm has been first given in [67], for fuzzy automata over the Gödel structure, and later in [23, 24], for fuzzy automata over a complete residuated lattice. The proof of the next theorem is the same as the proof of the corresponding theorem for fuzzy automata over a complete residuated lattice, so it will be omitted.

Theorem 6.1. [24, 67] Let \mathscr{L} be an integral ℓ -monoid, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy finite automaton over \mathscr{L} . Define inductively a sequence $\{E_k\}_{k \in \mathbb{N}}$ of crisp equivalences on A as follows:

$$E_1(a,b) = \begin{cases} 1 & \text{if } \tau^A(a) = \tau^A(b) \\ 0 & \text{otherwise} \end{cases}, \quad \text{for all } a, b \in A, \tag{42}$$

$$E_{k+1} = E_k \wedge E_k^r, \quad \text{for each } k \in \mathbb{N}, \tag{43}$$

where E_k^r is a crisp equivalence on A defined by

$$E_k^r(a,b) = \begin{cases} 1 & \text{if } (\delta_x \circ E_k)(a,c) = (\delta_x \circ E_k)(b,c), \text{ for all } x \in X \text{ and } c \in A \\ 0 & \text{otherwise} \end{cases}, \text{ for all } a, b \in A, \tag{44}$$

Then the sequence $\{E_k\}_{k \in \mathbb{N}}$ is finite and descending, there is the least $k \in \mathbb{N}$ such that $E_k = E_{k+m}$, for each $m \in \mathbb{N}$, and E_k is the greatest right invariant crisp equivalence on the fuzzy automaton \mathscr{A} .

Now we prove the following.

Proposition 6.2. Let \mathscr{L} be an integral ℓ -monoid and let α be an arbitrary fuzzy regular expression. For an arbitrary nondeterministic automaton $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ recognizing the language $||\alpha_R||$, the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α , and an arbitrary right invariant crisp equivalence E on \mathscr{A} we have that

- (a) $E \circ R_{\mathscr{A}} \leq R_{\mathscr{A}} \circ E$;
- (b) $R_{\mathscr{A}/E}(E_a, E_b) = (R_{\mathscr{A}} \circ E)(a, b)$ for all $a, b \in A$.

Proof. (a) First, by Theorem 4.3 and Lemmas 3.1 and 3.5 we have the following

$$(E \circ R_{\mathscr{A}})(a, b) = \bigvee_{c \in A} \bigvee_{u \in Y^*} E(a, c) \otimes \varphi_{\alpha}^*(u) \otimes \delta^A(c, u, b) = \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes (E \circ \delta_u^A)(a, b)$$
$$\leq \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes (\delta_u^A \circ E)(a, b) = \bigvee_{c \in A} \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes \delta^A(a, u, c) \otimes E(c, b)$$
$$= (R_{\mathscr{A}} \circ E)(a, b),$$

for every $a, b \in A$. Consequently, $E \circ R_{\mathscr{A}} \leq R_{\mathscr{A}} \circ E$.

(b) By Theorem 4.3, Lemmas 3.1 and 3.5, and (17), we obtain

$$\begin{aligned} R_{\mathscr{A}/E}(E_a, E_b) &= \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes \delta^{A/E}(E_a, u, E_b) = \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes (E \circ \delta_u^A \circ E)(a, b) \\ &= \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes (\delta_u^A \circ E)(a, b) = \bigvee_{c \in A} \bigvee_{u \in Y^*} \varphi_{\alpha}^*(u) \otimes \delta_u^A(a, c) \otimes E(c, b) \\ &= (R_{\mathscr{A}} \circ E)(a, b), \end{aligned}$$

for arbitrary $a, b \in A$. \square

Theorem 6.3. Let \mathscr{L} be an integral ℓ -monoid and let α be an arbitrary fuzzy regular expression. Moreover, consider an arbitrary nondeterministic automaton $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ recognizing the language $||\alpha_R||$, and the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α .

Then every right invariant equivalence E *on* \mathscr{A} *is a right invariant crisp equivalence on* \mathscr{A}_{α} *, and the fuzzy automaton* $(\mathscr{A}/E)_{\alpha}$ *is isomorphic to the factor fuzzy automaton* \mathscr{A}_{α}/E .

Proof. Let *E* be an arbitrary right invariant equivalence on \mathscr{A} . First, by Theorem 4.4 and statement (a) of Proposition 6.2 we obtain

$$E \circ \delta_x^{A_\alpha} = E \circ R_{\mathscr{A}} \circ \delta_x^A \circ R_{\mathscr{A}} \leqslant R_{\mathscr{A}} \circ E \circ \delta_x^A \circ R_{\mathscr{A}} \leqslant R_{\mathscr{A}} \circ \delta_x^A \circ E \circ R_{\mathscr{A}} \leqslant R_{\mathscr{A}} \circ \delta_x^A \circ R_{\mathscr{A}} \circ E = \delta_x^{A_\alpha} \circ E,$$

for every $x \in X$. In a similar way, we show that $E \circ \tau^{A_{\alpha}} = \tau^{A_{\alpha}}$. Consequently, *E* is a right invariant crisp equivalence on \mathscr{A}_{α} .

Next, by (17), Theorem 4.4, and statement (b) of Proposition 6.2 we have

$$\begin{split} \delta_x^{(A/E)_\alpha}(E_a, E_b) &= (R_{\mathscr{A}/E} \circ \delta_x^{A/E} \circ R_{\mathscr{A}/E})(E_a, E_b) = (R_{\mathscr{A}} \circ E \circ \delta_x^A \circ E \circ R_{\mathscr{A}} \circ E)(a, b) \\ &= (R_{\mathscr{A}} \circ \delta_x^A \circ R_{\mathscr{A}} \circ E)(a, b) = (\delta_x^{A_\alpha} \circ E)(a, b) = (E \circ \delta_x^{A_\alpha} \circ E)(a, b) = \delta_x^{A_\alpha/E}(E_a, E_b), \end{split}$$

for all $x \in X$ and $a, b \in A$. In addition, it is easy to check that

$$\tau^{(A/E)_{\alpha}}(E_a) = \tau^{A_{\alpha}/E}(E_a)$$

for every $a \in A$. Thus the identity function on A/E is an isomorphism from $(\mathscr{A}/E)_{\alpha}$ to \mathscr{A}_{α}/E .

According to the previous theorem, for an arbitrary fuzzy regular expression α , and any nondeterministic automaton \mathscr{A} recognizing $||\alpha||$, the greatest right invariant equivalence on the nondeterministic automaton \mathscr{A} is less or equal to the greatest right invariant crisp equivalence on the fuzzy automaton \mathscr{A}_{α} . The following example shows that even if the starting automaton \mathscr{A} is a minimal deterministic automaton of the language $||\alpha_R||$, the fuzzy automaton \mathscr{A}_{α} may be further reduced by right invariant crisp equivalences, i.e., the greatest right invariant crisp equivalence on \mathscr{A}_{α} .

Example 6.4. Let \mathscr{L} be Gödel structure, and $\alpha = x + 0.5x$, a fuzzy regular expression over the alphabet {*x*}. We have that $\alpha_R = x + \lambda x$, and the graph of the minimal deterministic automaton \mathscr{A} recognizing the language $||\alpha_R||$ is presented by Figure 3a.





Figure 3a. The automaton A

Figure 3b. The fuzzy automaton \mathscr{A}_{α}

Figure 3b presents the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α . The fuzzy set $\tau^{A_{\alpha}}$ of terminal states of \mathscr{A}_{α} , and the greatest right invariant crisp equivalence E^{cri} on \mathscr{A}_{α} are represented by:

$$\tau^{A_{\alpha}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad E^{\rm cri} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 6.5. Let \mathscr{L} be an integral ℓ -monoid and let α be an arbitrary fuzzy regular expression. Moreover, consider an arbitrary nondeterministic automaton $\mathscr{A} = (A, X \cup Y, \delta^A, a_0, \tau^A)$ recognizing the language $||\alpha_R||$, and the fuzzy automaton \mathscr{A}_{α} associated with \mathscr{A} and α .

Then for an arbitrary right invariant equivalence E on \mathscr{A} there exists a right invariant crisp equivalence E^r on \mathscr{A}^r_{α} , such that the factor fuzzy automata $(\mathscr{A}/E)^r_{\alpha}$ and $(\mathscr{A}^r_{\alpha})/E^r$ are isomorphic.

Proof. Let *E* be any right invariant equivalence on \mathscr{A} . Define a crisp relation E^r on A^r_{α} by $E^r(a, b) = E(a, b)$, for all $a, b \in A^r_{\alpha}$. Obviously, E^r is a crisp equivalence on the set A^r_{α} . Moreover, by (39) and Proposition 6.2, and we have that

$$(E^{\mathrm{r}} \circ \delta_{x}^{A_{\alpha}^{\mathrm{r}}})(a,b) = \bigvee_{c \in A_{\alpha}^{\mathrm{r}}} E(a,c) \otimes (R_{\mathscr{A}} \circ \delta_{x}^{A})(c,b) \leq \bigvee_{c \in A_{\alpha}} E(a,c) \otimes (R_{\mathscr{A}} \circ \delta_{x}^{A})(c,b)$$
$$= (E \circ R_{\mathscr{A}} \circ \delta_{x}^{A})(a,b) \leq (R_{\mathscr{A}} \circ \delta_{x}^{A} \circ E)(a,b) =^{*} (\delta_{x}^{A_{\alpha}^{\mathrm{r}}} \circ E^{\mathrm{r}})(a,b),$$

for all $a, b \in A_{\alpha}^{r}$. Note that the equality marked by * follows from the fact that $(R_{\mathcal{A}_{\alpha}} \circ \delta_{x}^{A_{\alpha}})(a, b) = 0$, for every $b \in A_{\alpha} \setminus A_{\alpha}^{r}$, and every $x \in X$. Thus E^{r} is a right invariant crisp equivalence on \mathscr{A}_{α}^{r} .

Define a mapping $\Phi : A_{\alpha}^{r}/E^{r} \to (A/E)_{\alpha}^{r}$ by $\Phi(E_{a}^{r}) = E_{a}$, for every $E_{a}^{r} \in A_{\alpha}^{r}/E^{r}$. For an arbitrary $E_{a}^{r} \in A_{\alpha}^{r}/E^{r}$, there are $b \in A$ and $x \in X$ such that $\delta^{A}(b, x, a) = 1$. Since we have that $\delta^{A/E}(E_{b}, x, E_{a}) \ge \delta^{A}(b, x, a)$, we obtain $E_{a} \in (A/E)_{\alpha}^{r}$. Moreover, from

$$E_a^{\mathbf{r}} = E_b^{\mathbf{r}} \Leftrightarrow E^{\mathbf{r}}(a,b) = 1 \Leftrightarrow E(a,b) = 1 \Leftrightarrow E_a = E_b,$$

for all $a, b \in A_{\alpha}^{r}$, we conclude that Φ is both a well-defined and an injective mapping.

Further, for any $E^a \in (A/E)^r_{\alpha}$ there are $E_b \in A/E$ and $x \in X$ such that $\delta^{A/E}(E_b, x, E_a) = 1$, which implies

$$\delta^A(c, x, d) = 1, E_b = E_c, E_d = E_a, \text{ for some } c, d \in A.$$

Thus $d \in A_{\alpha}^{r}$, and $\Phi(E_{d}^{r}) = E_{d} = E_{a}$. In conclusion, Φ is a bijective mapping.

Finally, by (17), (39), and Proposition 6.2, we have

$$\begin{split} \delta_x^{A_\alpha^r/E^r}(E_a^r, E_b^r) &= (\delta_x^{A_\alpha^r} \circ E^r)(a, b) = \bigvee_{c \in A_\alpha^r} (R_\mathscr{A} \circ \delta_x^A)(a, c) \otimes E^r(c, b) \\ &= \bigvee_{c \in A} (R_\mathscr{A} \circ \delta_x^A)(a, c) \otimes E^r(c, b) = (R_\mathscr{A} \circ \delta_x^A \circ E)(a, b) \\ &= (R_\mathscr{A} \circ E \circ \delta_x^A \circ E)(a, b) = (R_\mathscr{A}/E \circ \delta_x^{A/E})(E_a, E_b) = \delta_x^{(A/E)_\alpha^r}(\Phi(E_a^r), \Phi(E_b^r)) \end{split}$$

for all $E_a^r, E_b^r \in \mathscr{A}_{\alpha}^r / E$. Therefore, Φ is an isomorphism.

Let us recall that Theorem 5.1 gives us a simple method to construct various types of fuzzy automata from the fuzzy regular expression α . This method is based on choice of different nondeterministic automata \mathscr{A} recognizing $||\alpha_R||$, from which we obtain different fuzzy automata \mathscr{A}_{α} recognizing $||\alpha||$.

Let \mathscr{L} be an integral ℓ -monoid, and let α be an arbitrary fuzzy regular expression over an alphabet *X*. For a regular expression α_R over $X \cup Y$, where *Y* is an alphabet associated with α , let $\mathscr{A}_f(\alpha_R) = (A_f, X \cup Y, \delta^{A_f}, 0, \tau^{A_f})$ be the follow automaton of α_R . In this paper we will assume that the the follow automaton of α is exactly the factor automaton of the position automaton of α with respect to a particular right invariant equivalence *E*, called the *follow equivalence*. For the definition of the follow equivalence we refer to [40–43]. Starting from $\mathscr{A}_f(\alpha_R)$, by (33), (34) and (39) we obtain the reduced fuzzy automaton $\mathscr{A}_{ff}^r(\alpha)$ is called the *follow fuzzy automaton* of α .

Theorem 6.6. Let \mathscr{L} be an integral ℓ -monoid, let α be an arbitrary fuzzy regular expression, and let $\mathscr{A}_{pf}^{r}(\alpha)$ and $\mathscr{A}_{ff}^{r}(\alpha)$ be respectively the position fuzzy automaton and the follow fuzzy automaton of α .

Then the follow fuzzy automaton $\mathscr{A}_{\mathrm{ff}}^{\mathrm{r}}(\alpha)$ of α is isomorphic to the factor fuzzy automaton of the position fuzzy automaton $\mathscr{A}_{\mathrm{pf}}^{\mathrm{r}}(\alpha)$ of α with respect to some right invariant crisp equivalence on $\mathscr{A}_{\mathrm{pf}}^{\mathrm{r}}$.

Proof. The proof is an immediate consequence of Theorem 6.5.

By Theorem 6.6 we obtain that the follow fuzzy automaton is the reduced position fuzzy automaton with respect to some right invariant crisp equivalence, and therefore, it may be significantly smaller. However, Example 6.4 shows that, in the general case, follow equivalences are not necessarily the greatest right invariant crisp equivalences on the position fuzzy automata. Consequently, smaller fuzzy automata from a given α can be obtained by reducing the size of the position fuzzy automaton of α by means of the greatest right invariant crisp equivalence.

Example 6.7. Let \mathscr{L} be Gödel structure. Consider $\alpha = xx^* + 0.1x^*$, the fuzzy regular expression over the alphabet $X = \{x\}$. An expression $\alpha_R = xx^* + \lambda x^*$, over the alphabet $\{x, \lambda\}$, is the regular expression obtained from α .

The position automaton $\mathscr{A}_{p}(\alpha_{R})$ is given by the following fuzzy transition relations

	ſ٥	1	0	0	0			[0]	0	0	1	ן0			[0]	
	0	0	1	0	0			0	0	0	0	0			1	
$\delta_x^{A_p} =$	0	0	1	0	0	,	$\delta_{\lambda}^{A_{p}} =$	0	0	0	0	0	,	$\tau^{A_p} =$	1	,
	0	0	0	0	1		Л	0	0	0	0	0			1	
	0	0	0	0	1			0	0	0	0	0			$\begin{bmatrix} 1 \end{bmatrix}$	

and the position fuzzy automaton $\mathscr{A}_{pf}^{r}(\alpha)$ is given by the following fuzzy transition relations

	[0]	1	0	0.1]		[1]	
A^{r}_{pf}	0	0	1	0	$A_{\rm pf}^{\rm r}$	1	
$O_x^{\perp} \equiv$	0	0	1	0	, and $\tau^{p_1} =$	1	•
	0	0	0	1		[1]	

The follow relation E_f on $\mathscr{A}_p(\alpha_R)$, and the related right invariant crisp equivalence E_f^r on $A_{pf}^r(\alpha)$ are

$$E_{\rm f} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, E_{\rm f}^{\rm r} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and therefore follow fuzzy automaton $\mathscr{A}_{\text{ff}}^{\text{r}}(\alpha)$ has 3 states. However, since the greatest right invariant crisp equivalence E_1^{cri} on $\mathscr{A}_{\text{pf}}^{\text{r}}(\alpha)$ is given by

we conclude that the fuzzy finite automaton $\mathscr{A}_{pf}^{r}(\alpha)/E_{1}^{cri}$ has only 1 state, and is significantly smaller than $\mathscr{A}_{pf}^{r}(\alpha)$.

Let α be an regular expression. Observe that, starting from the partial derivative automaton of the regular expression α_R obtained from α , it is possible to construct the fuzzy partial derivative automaton of α . Since the partial derivative automaton is isomorphic to the factor automaton of the position automaton with respect to certain right invariant equivalence (cf. [14–16, 42]), the result which correspond to Theorem 6.6, concerning fuzzy partial derivative automata, can be easily derived.

7. Concluding remarks

In this paper we have discussed the problem of the effective construction of a fuzzy finite automaton from a given fuzzy regular expression. We have approached this problem by converting a given fuzzy regular expression α over an alphabet X in an ordinary regular expression α_R over a larger alphabet $X \cup Y$ obtained by adding new letters assigned to different scalars that appear in the fuzzy regular expression α . Starting from an arbitrary nondeterministic finite automaton \mathscr{A} that recognizes the language $||\alpha_R||$ represented by the regular expression α_R , we have constructed a fuzzy finite automaton \mathscr{A}_{α} associated with \mathscr{A} and α , which recognizes the fuzzy language $||\alpha||$ represented by α . The starting nondeterministic finite automaton \mathscr{A} can be obtained from α_R using any of the well-known constructions for converting regular expressions to nondeterministic finite automata, such as Glushkov-McNaughton-Yamada's position automaton, Brzozowski's derivative automaton, Antimirov's partial derivative automaton, or Ilie-Yu's follow automaton.

The fuzzy finite automaton \mathscr{A}_{α} that we have constructed has the same number of states as the starting nondeterministic finite automaton \mathscr{A} , but we have also given the reduced version of the fuzzy automaton \mathscr{A}_{α} which can have strictly less number of states than \mathscr{A} . Moreover, we have discussed the reduction of the number of states of the fuzzy automaton \mathscr{A}_{α} by means of right invariant crisp equivalences.

All the main results of the paper have been proved for fuzzy automata taking membership values in an integral lattice-ordered monoiod.

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