# STATE BCK-ALGEBRAS AND STATE-MORPHISM BCK-ALGEBRAS

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Dedicated to Prof. Ján Jakubík on the occasion of his 90<sup>th</sup> birthday

ABSTRACT. In the paper, we define the notion of a state BCK-algebra and a state-morphism BCKalgebra extending the language of BCK-algebras by adding a unary operator which models probabilistic reasoning. We present a relation between state operators and state-morphism operators and measures and states on BCK-algebras, respectively. We study subdirectly irreducible state (morphism) BCKalgebras. We introduce the concept of an adjoint pair in BCK-algebras and show that there is a oneto-one correspondence between adjoint pairs and state-morphism operators. In addition, we show the generators of quasivarieties of state-morphism BCK-algebras.

### 1. INTRODUCTION

In 1966, Imai and Iseki [18, 19] introduced two classes of abstract algebras: BCK-algebras and BCIalgebras. These algebras have been intensively studied by many authors. For a comprehensive overview on BCK-algebras, we recommend the book [22]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. MV-algebras were introduced by Chang in [6], in order to show that Lukasiewicz logic is complete with respect to evaluations of propositional variables in the real unit interval [0, 1]. It is well known that the class of MV-algebras is a proper subclass of the class of BCK-algebras. Therefore, both BCK-algebras and MV-algebras are important for the study of fuzzy logic.

In [24], Mundici introduced a *state* on MV-algebras as averaging the truth value in Lukasiewicz logic. States constitute measures on their associated MV-algebras which generalize the usual probability measures on Boolean algebras. Kroupa [20] and Panti [26] have recently shown that every state on an MV-algebra can be presented as a usual Lebesgue integral over an appropriate space. Kühr and Mundici [21] studied states via de Finetti's notion of a coherent state with motivation in Dutch book making. Their method is applicable to other structures besides MV-algebras. Measures on pseudo BCK-algebras were studied in [7].

Since MV-algebras with state are not universal algebras, they do not automatically induce an assertional logic. Recently, Flaminio and Montagna in [15, 16] presented an algebraizable logic using a probabilistic approach, and its equivalent algebraic semantics is precisely the variety of state MV-algebras. We recall that a *state MV-algebra* is an MV-algebra whose language is extended by adding an operator,  $\mu$  (also called an *internal state*), whose properties are inspired by ones of states. Analogues of extremal states are *state-morphism operators*, introduced in [8, 9], where by definition, a state-morphism is an idempotent endomorphism on an MV-algebra.

 $<sup>^1\!\</sup>mathrm{Keywords}:$  state-morphism operator, left state operator, right state operator, BCK-algebra, state BCK-algebra, state-morphism BCK-algebra, quasivariety, generator

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State MV-algebras generalize, for example, Hájek's approach, [17], to fuzzy logic with modality Pr (interpreted as *probably*) which has the following semantic interpretation: The probability of an event a is presented as the truth value of Pr(a). On the other hand, if s is a state, then s(a) is interpreted as the average appearance of the many valued event a.

In [15, 16], the authors found a relation between states on MV-algebras and state MV-algebras. In [8, 9], some results about characterizations of subdirectly irreducible state-morphism MV-algebras, simple, semisimple, and local state MV-algebras were shown. In [10], the authors study the variety of state-morphism MV-algebras together with a characterization of subdirectly irreducible state MV-algebras, and some interesting characterizations of some varieties of state-morphism MV-algebras were given. These results were generalized in [14, 13, 3].

In the present paper, we concentrate to the study of state BCK-algebras and state-morphism BCKalgebras. We show their basic properties and we characterize quasivarieties of state-morphism BCKalgebras and their generators. We present that the generator of a quasivariety of state-morphism BCKalgebras consists of diagonal state-morphism BCK-algebras. The goal of the present paper is to extend the study of state MV-algebras to state BCK-algebras. We note that in contrast to MV-algebras, in this case we have to deal with quasivarieties because the class of BCK-algebras forms a quasivariety and not a variety.

We note that a state-morphism BCK-algebra is a special case of algebras with a distinguished idempotent endomorphism and such algebras are not new: experts working in various areas (ranging from computer science, Baxter algebras, set theory, category theory and homotopy theory, see e.g. [28, 1, 27]) have considered such structures with a fixed endomorphism.

The paper is organized as follows. Section 2 gathers the elements of BCK-algebras. In Section 3, we introduce the concept of a state BCK-algebra and we study its properties. Then we verify a subdirectly irreducible state BCK-algebra and we characterize this structure. We show that if X is a bounded commutative BCK-algebra, then  $(X, \mu)$  is a state (morphism) MV-algebra if and only if  $(X, \mu)$  is a state (morphism) BCK-algebra such that  $\mu(1) = 1$ . In Section 4, we study state-morphism BCK-algebras and state ideals. Some relations between congruence relations on state-morphism BCK-algebra and describe a relation between state-morphism operators and adjoint pair in a BCK-algebra. Finally, Section 5 gives results on generators of quasivarieties of state-morphism BCK-algebras, and we present two open problems.

#### 2. Preliminaries

In the section, we gather some basic notions relevant to BCK-algebras and MV-algebras which will need in the next sections.

We say that an MV-algebra is an algebra  $(M, \oplus, ', 0)$  of type (2, 1, 0), where  $(M, \oplus, 0)$  is a commutative monoid with neutral element 0 and for all  $x, y \in M$ :

(i) 
$$x'' = x;$$

(ii)  $x \oplus 1 = 1$ , where 1 = 0';

(iii)  $x \oplus (x \oplus y')' = y \oplus (y \oplus x')'.$ 

In any MV-algebra  $(M, \oplus, ', 0)$ , we can define the following further operations:

$$x \odot y = (x' \oplus y')', \quad x \ominus y = (x' \oplus y)'.$$

A state *MV*-algebra is a pair  $(M, \sigma)$  such that  $(M, \oplus, ', 0)$  is an MV-algebra and  $\sigma$  is a unary operation on M satisfying:

(1)  $\sigma(1) = 1;$ (2)  $\sigma(x') = \sigma(x)';$ (3)  $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \oplus (x \odot y));$ (4)  $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y).$  In [8], Di Nola and Dvurečenskij have introduced a *state-morphism operator* on an MV-algebra  $(M, \oplus, ', 0)$  as an MV-homomorphism  $\sigma : M \to M$  such that  $\sigma^2 = \sigma$  and the pair  $(M, \sigma)$  is said to be a *state-morphism* MV-algebra. They have proved that the class of state-morphism MV-algebras is a proper subclass of state MV-algebras.

**Definition 2.1.** [18, 19] A *BCK-algebra* is an algebra (X, \*, 0) of type (2, 0) satisfying the following conditions:

(BCK1) ((x \* y) \* (x \* z)) \* (z \* y) = 0;

 $(BCK2) \ x * 0 = x;$ 

(BCK3) x \* y = 0 and y \* x = 0 imply y = x;

(BCK4) 0 \* x = 0.

A BCK-algebra X is called *non-trivial* if  $X \neq \{0\}$ . If X is a BCK-algebra, then the relation  $\leq$  defined by  $x \leq y \Leftrightarrow x * y = 0, x, y \in X$ , is a partial order on X. In addition, for all  $x, y, z \in X$ , the following hold:

(BCK5) x \* x = 0; (BCK6) (x \* y) \* z = (x \* z) \* y; (BCK7)  $x \le y$  implies  $x * z \le y * z$  and  $z * y \le z * x$ ; (BCK8) x \* (x \* (x \* y)) = x \* y; (BCK9)  $(x * y) * (x * z) \le z * y$  and  $(y * x) * (z * x) \le y * z$ .

In a BCK-algebra X, we define  $x * y^0 = x$  and  $x * y^n = (x * y^{n-1}) * y$  for any integer  $n \ge 1$  and all  $x, y \in X$ . A BCK-algebra (X, \*, 0) is called *bounded* if  $(X, \le)$  has the greatest element, where  $\le$  is the above defined partially order relation. Let use denote by 1 the greatest element of X (if it exits). In bounded BCK-algebras, we usually write Nx instead of 1 \* x. A BCK-algebra (X, \*, 0) is called a *commutative* BCK-algebra if x \* (x \* y) = y \* (y \* x) for all  $x, y \in X$ . Each commutative BCK-algebra is a lover semilattice and  $x \land y = x * (x * y)$  for all  $x, y \in X$  (see [22]). Let (X, \*, 0) and (Y, \*, 0) be two BCK-algebras. A map  $f : X \to Y$  is called a *homomorphism* if f(a \* b) = f(a) \* f(b) for all  $a, b \in X$ . Then f(0) = 0 (since f(0) = f(0 \* 0) = f(0) \* f(0) = 0).

A non-empty subset I of a BCK-algebra X is called an *ideal* if  $(1) \ 0 \in I$ ,  $(2) \ y * x \in I$  and  $x \in I$  imply that  $y \in I$  for all  $x, y \in X$ . We denote by I(X), the set of all ideals of X. An ideal I of a BCK-algebra X is called *proper* if  $I \neq X$ . Suppose that (X, \*, 0) and (Y, \*, 0) are two BCK-algebras and  $f : X \to Y$ is a homomorphism, then  $\operatorname{Ker}(f) = f^{-1}(\{0\})$  is an ideal of X. Let use denote by  $\langle S \rangle$  the least ideal of X containing S, where S is a subset of a BCK-algebra X. It is called the ideal generated by S. If S is a subset of more BCK-algebras, we will use  $\langle S \rangle_X$  to specify a concrete BCK-algebra X. Instead of  $\langle \{a\} \rangle$ we will write rather  $\langle a \rangle$ , where  $a \in X$ .

**Theorem 2.2.** [29] Let S be a subset of a BCK-algebra (X, \*, 0). Then

$$\langle S \rangle = \{ x \in X \mid (\cdots ((x * a_1) * a_2) * \cdots) * a_n = 0 \text{ for some } n \in \mathbb{N} \text{ and some } a_1, \ldots, a_n \in S \cup \{0\} \}.$$

Moreover, if I is an ideal of X, then

 $\langle I \cup S \rangle = \{ x \in X \mid (\cdots ((x * a_1) * a_2) * \cdots) * a_n \in I \text{ for some } n \in \mathbb{N} \text{ and some } a_1, \dots, a_n \in S \}.$ 

Let *I* be an ideal of a BCK-algebra (X, \*, 0). Then the relation  $\theta_I$ , defined by  $(x, y) \in \theta_I$  if and only if  $x * y, y * x \in I$ , is a congruence relation on *X*. Let us denote by x/I or [x] the set  $\{y \in X \mid (x, y) \in \theta_I\}$  for all  $x \in X$ . Then (X/I, \*, 0/I) is a BCK-algebra, when  $X/I := \{x/I \mid x \in X\}$  and x/I \* y/I := (x \* y)/I for all  $x, y \in X$  (see [22]).

An ideal I of a BCK-algebra (X, \*, 0) is called *commutative* if  $x * y \in I$  implies that  $x * (y * (y * x)) \in I$  for all  $x, y \in X$ . If I is a commutative ideal, the BCK-algebra X/I is a commutative BCK-algebra [29, Thm 2.5.6].

**Theorem 2.3.** Let (X, \*, 0) be a BCK-algebra and  $\theta$  be a congruence relation on X. Then  $[0]_{\theta}$  is an ideal of X. Moreover, if  $I = [0]_{\theta}$ , then  $\theta_I = \theta$ .

Proof. See [29, Prop 1.5.9, Prop. 1.5.11, Cor. 1.5.12].

**Definition 2.4.** [2, 29] Let I be a proper ideal of a BCK-algebra (X, \*, 0). Then I is called a

- prime ideal if  $\langle x \rangle \cap \langle y \rangle \subseteq I$  implies  $x \in I$  or  $y \in I$  for all  $x, y \in X$ ;
- maximal ideal if  $\langle I \cup \{x\} \rangle = X$  for all  $x \in X I$ .

We use Max(X) and Spec(X) to denote the set of all maximal and prime ideals of X, respectively. In each BCK-algebra X,  $Max(X) \subseteq Spec(X)$  (see [2, Thm 3.7]). A BCK-algebra (X, \*, 0) is called *simple* if it has only two ideals and it is called *semisimple* if  $Rad(X) := \bigcap Max(X) = \{0\}$ .

**Definition 2.5.** [29] A BCK-algebra (X, \*, 0) is *positive implicative* if (x \* y) \* z = (x \* z) \* (y \* z) for all  $x, y, z \in X$ .

If X = [0, a) or X = [0, a], where  $a \in \mathbb{R}$ , or  $X = [0, \infty)$ , we define the binary operation  $*_{\mathbb{R}}$  on X by  $x *_{\mathbb{R}} y = \max\{0, x - y\}$ . Then  $(X, *_{\mathbb{R}}, 0)$  is a commutative BCK-algebra (see [22]).

**Definition 2.6.** [12] Let (X, \*, 0) be a BCK-algebra and  $m : X \to [0, \infty]$  be a map such that, for all  $x, y \in [0, 1]$ ,

- (i) if m(x \* y) = m(x) m(y), whenever  $y \le x$ , then m is said to be a measure;
- (ii) if  $1 \in X$  and m is a measure with m(1) = 1, then m is said to be a state;
- (iii) if  $m(x * y) = \max\{0, m(x) m(y)\}$ , then m is said to be a measure-morphism;
- (iv) if  $1 \in X$  and m is a measure-morphism with m(1) = 1, then m is said to be a state-morphism.

# 3. State BCK-algebras

In the section, the concept of left and right state BCK-algebras is defined as a generalization of state MV-algebras, and its properties are studied. We introduce state ideals and congruence relations of right or left state BCK-algebras, and relations between them are obtained. Finally, we characterize subdirectly irreducible state BCK-algebras.

From now on, in this paper, (X, \*, 0) or simply X is a BCK-algebra, unless otherwise specified.

**Definition 3.1.** A map  $\mu : X \to X$  is called a *left (right) state operator* on X if it satisfies the following conditions:

- (S0) x \* y = 0 implies  $\mu(x) * \mu(y) = 0$ ;
- (S1)  $\mu(x * y) = \mu(x) * \mu(x * (x * y)) \quad (\mu(x * y) = \mu(x) * \mu(y * (y * x)));$
- (S2)  $\mu(\mu(x) * \mu(y)) = \mu(x) * \mu(y).$

A left (right) state BCK-algebra is a pair  $(X, \mu)$ , where X is a BCK-algebra and  $\mu$  is a left (right) state operator on X.

Clearly, if X is a commutative BCK-algebra, then  $\mu$  is a right state operator on X if and only if it is a left state operator. In the next proposition, we describe the basic properties of left (right) state operators.

**Proposition 3.2.** Let  $(X, \mu)$  be a left (right) state BCK-algebra. Then, for any  $x, y, x_1, \ldots, x_n \in X$ ,

- (i)  $\mu(0) = 0$  and  $\mu(\mu(x)) = \mu(x)$ .
- (ii)  $\mu(x) * \mu(y) \le \mu(x * y)$ . More generally,

 $(\cdots ((\mu(x) * \mu(x_1)) * \mu(x_2)) * \cdots) * \mu(x_n) \le \mu((\cdots ((x * x_1) * x_2) * \cdots) * x_n).$ 

- (iii)  $\text{Ker}(\mu) := \mu^{-1}(\{0\})$  is an ideal of X.
- (iv)  $\mu(X) := \{\mu(x) \mid x \in X\}$  is a subalgebra of X.
- (v)  $\text{Ker}(\mu) \cap \text{Im}(\mu) = \{0\}.$

*Proof.* We prove this theorem only for a left state BCK-algebra. The proof for a right state BCK-algebra is similar.

(i) By (BCK4) and (BCK8), we have  $\mu(0) = \mu(0*0) = \mu(0)*\mu(0*(0*0)) = \mu(0)*\mu(0) = 0$ . Moreover, by (S2) and (BCK2), we have  $\mu(\mu(x)) = \mu(\mu(x)*0) = \mu(\mu(x)*\mu(0)) = \mu(x)*\mu(0) = \mu(x)$ .

(ii) Let  $x, y \in X$ . Since  $x * (x * y) \le y$ , then  $\mu(x * (x * y)) \le \mu(y)$ , and so by (BCK7), we get that  $\mu(x) * \mu(y) \le \mu(x) * \mu(x * (x * y)) = \mu(x * y)$ . The proof of the second part follows from (BCK7).

(iii) By (i),  $0 \in \text{Ker}(\mu)$ . Let  $y * x, x \in \text{Ker}(\mu)$ , where  $x, y \in X$ . Then  $\mu(x) = \mu(y * x) = 0$ . It follows from (ii) that  $\mu(y) = \mu(y) * 0 = \mu(y) * \mu(x) \le \mu(y * x) = 0$ , hence  $y \in \text{Ker}(\mu)$ . Thus,  $\text{Ker}(\mu)$  is an ideal of X.

(iv) By (i),  $0 \in \mu(X)$ . Let  $a, b \in X$ . Then by (S2),  $\mu(\mu(a)*\mu(a)) = \mu(a)*\mu(b)$  and so  $\mu(a)*\mu(b) \in \mu(X)$ . Therefore,  $\mu(X)$  is a subalgebra of X.

(v) It is evident.

In Theorem 3.3, we attempt to find a relation between measures and states on BCK-algebras and state BCK-algebras.

**Theorem 3.3.** Let  $a \in [0,1]$ ,  $X = ([0,a), *_{\mathbb{R}}, 0)$  and  $(X, \mu)$  be a left state BCK-algebra. Then  $\mu : X \to [0,1]$  is a measure.

In addition, if  $X = ([0,1], *_{\mathbb{R}}, 0)$  and  $(X, \mu)$  is a left state BCK-algebra such that  $\mu(1) = 1$ , then  $\mu : X \to [0,1]$  is a state-morphism.

*Proof.* Let  $x, y \in X$  such that  $y \leq x$ . For simplicity, we will write  $* = *_{\mathbb{R}}$ . Then  $\mu(x * y) = \mu(x) * \mu(x * (x * y))$ . Since  $X = ([0, a), *_{\mathbb{R}}, 0)$  is a commutative BCK-algebra, then x \* (x \* y) = y \* (y \* x) = y \* 0 = y and so  $\mu(x * y) = \mu(x) * \mu(x * (x * y)) = \mu(x) * \mu(y)$ . Therefore,  $\mu : X \to [0, 1]$  is a measure.

Now, assume that  $X = ([0,1], *_{\mathbb{R}}, 0)$  and  $(X, \mu)$  is a left state BCK-algebra. Let  $x, y \in X$ . Then  $\mu(x * y) = \mu(x) * \mu(x * (x * y))$ . Since X is linearly ordered, we have two cases. If  $x \leq y$ , then  $\mu(x * y) = \mu(0) = 0$  and by Proposition 3.2(iv),  $\mu(x) * \mu(y) = 0$  and so  $\mu(x * y) = \mu(x) * \mu(y)$ . If  $y \leq x$ , then x \* (x \* y) = y (since  $([0,1], *_{\mathbb{R}}, 0)$  is a commutative BCK-algebra) and so  $\mu(x * y) = \mu(x) * \mu(y)$ . Therefore,  $\mu : X \to [0,1]$  is a state-morphism.

**Proposition 3.4.** Let  $(X, \mu)$  be a right state BCK-algebra. Then

- (i)  $y \le x$  implies  $\mu(x * y) = \mu(x) * \mu(y)$  for all  $x, y \in X$ .
- (ii)  $\mu^{-1}(\{0\})$  is a commutative ideal of X. Moreover, the map  $\overline{\mu} : X/\operatorname{Ker}(\mu) \to X/\operatorname{Ker}(\mu)$  defined by  $\overline{\mu}(x/\operatorname{Ker}(\mu)) = \mu(x)/\operatorname{Ker}(\mu)$  is both a right and left state operator on  $X/\operatorname{Ker}(\mu)$ .
- (iii)  $(X, \mu)$  is a left state BCK-algebra.

Proof. (i) Let x, y ∈ X such that y ≤ x. Then µ(x\*y) = µ(x)\*µ(y\*(y\*x)) = µ(x)\*µ(y\*0) = µ(x)\*µ(y).
(ii) By Proposition 3.3(i), 0 ∈ µ<sup>-1</sup>({0}). Let x, y \* x ∈ µ<sup>-1</sup>({0}). Then µ(x) = µ(y \* x) = 0 and so µ(y) \* µ(x \* (x \* y)) = 0. Since µ(x \* (x \* y)) ≤ µ(x) = 0, then µ(y) = 0. Hence, µ<sup>-1</sup>({0}) is an ideal of X. Now, let x \* y ∈ µ<sup>-1</sup>({0}). Since y \* (y \* x) ≤ x, by (i), we have

$$0 = \mu(x * y) = \mu(x) * \mu(y * (y * x)) = \mu(x * (y * (y * x))),$$

which concludes that  $x * (y * (y * x)) \in \mu^{-1}(\{0\})$ . Thus,  $\mu^{-1}(\{0\})$  is a commutative ideal of X.

It is easy to show that  $\overline{\mu}$ , defined by  $\overline{\mu}(x/\operatorname{Ker}(\mu)) := \mu(x)/\operatorname{Ker}(\mu)$ ,  $(x \in X)$ , is a right state operator on  $X/\operatorname{Ker}(\mu)$ . In fact, if  $x/\operatorname{Ker}(\mu) = y/\operatorname{Ker}(\mu)$ , then  $x * y, y * x \in \operatorname{Ker}(\mu)$  and so  $\mu(x * y) = \mu(y * x) = 0$ . Hence by Proposition 3.2(ii),  $\mu(x) * \mu(y) = \mu(y) * \mu(x) = 0$  and so  $\mu(x) = \mu(y)$ . Thus,  $\overline{\mu}$  is well defined. Since  $\operatorname{Ker}(\mu)$  is a commutative ideal of X, then  $X/\operatorname{Ker}(\mu)$  is a commutative BCK-algebra, hence  $\overline{\mu}$  is also a left state operator on  $X/\operatorname{Ker}(\mu)$ .

(iii) Let  $x, y \in X$ . By (ii), Ker( $\mu$ ) is a commutative ideal of X and so by [29, Thm 2.5.6],  $X/\text{Ker}(\mu)$  is a commutative BCK-algebra. Hence,  $(x * (x * y))/\text{Ker}(\mu) = (y * (y * x))/\text{Ker}(\mu)$ . Similarly to the proof of (ii), we obtain that  $\mu(x * (x * y)) = \mu(y * (y * x))$  and so  $\mu(x) * \mu(x * (x * y)) = \mu(x) * \mu(y * (y * x)) = \mu(x * y)$ . Therefore,  $(X, \mu)$  is a left state BCK-algebra.

**Corollary 3.5.** Let  $\mu : X \to X$  be a map. Then  $(X, \mu)$  is a right state BCK-algebra if and only if  $(X, \mu)$  is a left state BCK-algebra and Ker $(\mu)$  is a commutative ideal of X.

*Proof.* Suppose that  $(X, \mu)$  is a right state BCK-algebra. Then by Proposition 3.4,  $(X, \mu)$  is a left state and Ker $(\mu)$  is a commutative ideal. Conversely, let  $(X, \mu)$  be a left state BCK-algebra and let Ker $(\mu)$ 

be a commutative ideal of X. Then for all  $x, y \in X$ ,  $(x * (x * y))/\text{Ker}(\mu) = (y * (y * x))/\text{Ker}(\mu)$ , and similar to the proof of Proposition 3.4(ii), we have  $\mu(x * (x * y)) = \mu(y * (y * x))$ , hence  $\mu$  is a right state operator.

By Proposition 3.4(iii), every right state BCK-algebra is a left state BCK-algebra. In the following example, we show that the converse statement is not true, in general. We present a left state operator  $\mu$  on a BCK-algebra X which is not a right state operator because Ker( $\mu$ ) is not a commutative ideal of X.

**Example 3.6.** Let  $X = \{0, 1, 2, 3\}$ . Define a binary operation \* by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

Then (X, \*, 0) is a positive implicative BCK-algebra  $(P)B_{4-1-4}$  from [22] which is a chain  $(0 \le 1 \le 2 \le 3)$ . Let  $\mu : X \to X$  be defined by  $\mu(0) = \mu(1) = 0$  and  $\mu(2) = \mu(3) = 2$ . We claim that  $\mu$  is a left state operator on X. Clearly, it is a well defined and order preserving map. Let  $x, y \in X$ .

(1) If  $x \le y$ , then we have  $\mu(x * y) = \mu(0) = 0$  and  $\mu(x) * \mu(x * (x * y)) = \mu(x) * \mu(x) = 0$ .

(2) If y < x, then by definition of \*,  $\mu(x * y) = \mu(x)$ . Also,  $\mu(x) * \mu(x * (x * y)) = \mu(x) * \mu(x * x) = \mu(x) * \mu(0) = \mu(x)$ .

(3) It can be easily shown that  $\mu(x) * \mu(y) = \mu(\mu(x) * \mu(y))$ .

From (1)–(3), we conclude that  $\mu$  is a left state operator on X. But  $\operatorname{Ker}(\mu)$  is not a commutative ideal of X because  $2 * 3 \in \operatorname{Ker}(\mu)$ , but  $2 * (3 * (3 * 2)) = 2 * (3 * 3) = 2 * 0 = 2 \notin \operatorname{Ker}(\mu)$ . Hence,  $\mu$  is not a right state operator on X.

Let X be a set, we denote by  $Id_X : X \to X$  the identity on X. It also provides an example of a left state operator which is not necessarily a right state operator.

In each BCK-algebra X,  $Id_X$  is a left state operator. In fact,  $Id_X(x)*Id_X(x*(x*y)) = x*(x*(x*y)) = x*y$ . On the other hand,  $Id_X$  is a right state operator iff X is a commutative BCK-algebra. So it can be easily obtained that, X is a commutative BCK-algebra if and only if each left state operator on X is a right state operator.

By Proposition 3.4, each right state BCK-algebra is a left state BCK-algebra. So in the remainder of this paper, we will consider only left BCK-algebras. Moreover, we write simply a state BCK-algebra instead of a left state BCK-algebra.

**Definition 3.7.** Let  $(X, \mu)$  be a state BCK-algebra. An ideal I of a BCK-algebra X is called a *state ideal* if  $\mu(I) \subseteq I$ . If T is a subset of X, then  $\langle T \rangle_s$  is the least state ideal of X containing T. A state ideal I is said to be a *maximal state ideal* if  $\langle I \cup \{x\} \rangle_s = X$  for each  $x \in X - I$ . We denote by  $MaxS(X, \mu)$  the set of all maximal state ideals of  $(X, \mu)$ .

**Proposition 3.8.** Let I be a state ideal of a state BCK-algebra  $(X, \mu)$  and  $a \in X$ . Then

$$\langle I \cup \{a\} \rangle_s = \{x \in X \mid (x * a^n) * \mu(a)^m \in I \text{ for some } m, n \in \mathbb{N}\}.$$

*Proof.* Set  $A = \{x \in X \mid (x * a^n) * \mu(a)^m \in I \text{ for some } m, n \in \mathbb{N}\}$ . Clearly,  $I \cup \{a\} \subseteq A$ . Moreover, if J is a state ideal of  $(X, \mu)$  containing I and a, then by Theorem 2.2,  $A \subseteq J$ . It suffices to show that A is a state ideal. Let  $x, y * x \in A$ . Then there are  $m, n, s, t \in \mathbb{N}$  such that  $(x * a^n) * \mu(a)^m \in I$  and

$$\begin{aligned} ((y*x)*a^{s})*\mu(a)^{t} \in I. \\ (((y*a^{n+s})*\mu(a)^{m+t}) &* ((x*a^{n})*\mu(a)^{m}))*(((y*x)*a^{s})*\mu(a)^{t}) \\ &\leq (((y*a^{n+s})*\mu(a)^{t})*(x*a^{n}))*(((y*x)*a^{s})*\mu(a)^{t}), \text{ by (BCK9)} \\ &= (((y*a^{n+s})*(x*a^{n}))*\mu(a)^{t})*(((y*x)*a^{s})*\mu(a)^{t}), \text{ by (BCK6)} \\ &\leq (((y*a^{s})*x)*\mu(a)^{t})*(((y*x)*a^{s})*\mu(a)^{t}), \text{ by (BCK9)} \\ &= (((y*x)*a^{s})*\mu(a)^{t})*(((y*x)*a^{s})*\mu(a)^{t}), \text{ by (BCK6)} \\ &= 0 \in I. \end{aligned}$$

Since  $(x*a^n)*\mu(a)^m$ ,  $((y*x)*a^s)*\mu(a)^t \in I$  and I is an ideal of X, then we get  $(y*a^{n+s})*\mu(a)^{m+t} \in I$ and so  $y \in A$ . Hence, A is an ideal. Now, let x be an arbitrary element of A. Then there exist  $m, n \in \mathbb{N}$ such that  $(x*a^n)*\mu(a)^m \in I$ . Since I is a state ideal, then  $\mu((x*a^n)*\mu(a)^m) \in I$  and so by Proposition 3.2(ii),  $\mu(x)*\mu(a)^{n+m} = (\mu(x)*\mu(a)^n)*\mu(a)^m = (\mu(x)*\mu(a)^n)*\mu(\mu(a))^m \in I$ . Thus,  $\mu(x) \in A$ . Therefore, A is a state ideal of  $(X, \mu)$ .

Note that, if  $(X, \mu)$  is a state BCK-algebra, then  $\{0\}$  and X are state ideals of  $(X, \mu)$  and so by Proposition 3.8,  $J = \{x \in X \mid (x * a^n) * \mu(a)^m = 0 \text{ for some } m, n \in \mathbb{N}\}$  is a state ideal of X for any  $a \in X$ . Similarly, we can construct other state ideals of  $(X, \mu)$ .

**Corollary 3.9.** A state ideal I of a state BCK-algebra  $(X, \mu)$  is a maximal state ideal if and only if  $\{x \in X \mid (x * a^n) * \mu(a)^m \in I \text{ for some } m, n \in \mathbb{N}\} = X \text{ for all } a \in X - I.$ 

*Proof.* The proof is a straightforward corollary of Proposition 3.8.

By [2, Thm 3.7], we know that if M is a maximal ideal of X, then  $I \cap J \subseteq M$  implies that  $I \subseteq M$  or  $J \subseteq M$  for all  $I, J \in I(X)$ . In the next theorem, we show that if M is a maximal state ideal of a state BCK-algebra  $(X, \mu)$ , then  $I \cap J \subseteq M$  implies that  $I \subseteq M$  or  $J \subseteq M$  for all state ideals I and J of  $(X, \mu)$ .

**Theorem 3.10.** Let M be a maximal state ideal of a state BCK-algebra  $(X, \mu)$ . For for all state ideals I and J of  $(X, \mu)$ , we have  $I \cap J \subseteq M$  implies that  $I \subseteq M$  or  $J \subseteq M$ .

Proof. Let I and J be two state ideals of  $(X, \mu)$  such that  $I \cap J \subseteq M$ . Suppose that there are  $x \in I - M$  and  $y \in J - M$ . Then by Corollary 3.9,  $X = \langle M \cup \{x\} \rangle_s = \langle M \cup \{y\} \rangle_s$ . On the other hand, if  $a \in \langle M \cup \{x\} \rangle_s \cap \langle M \cup \{y\} \rangle_s$ , then by Proposition 3.8, there exist  $m, n, s, t \in \mathbb{N}$  such that  $(a * x^n) * \mu(x)^m = m_1 \in M$  and  $(a * y^s) * \mu(y)^t = m_2 \in M$  and so by (BCK2), (BCK4), (BCK6) and Proposition 3.8, we get  $(a * m_1) * m_2 \in I \cap J \subseteq M$  (since I and J are state ideals,  $x \in I$ ,  $y \in J$  and  $(((a * m_1) * m_2) * x^n) * \mu(x)^m = 0 * m_2 = 0$ ,  $(((a * m_1) * m_2) * y^s) * \mu(y)^t = 0 * m_1 = 0$ . Since  $m_1, m_2 \in M$  and M is an ideal of X, then we have  $a \in M$  and so  $X = \langle M \cup \{x\} \rangle_s \cap \langle M \cup \{y\} \rangle_s \subseteq M$ , which is a contradiction. Therefore,  $I \subseteq M$  or  $J \subseteq M$ .

In Theorem 3.11, we show a one-to-one relationship between congruence relations of a state BCKalgebra  $(X, \mu)$  and state ideals of  $(X, \mu)$ . We denote by SI(X) and Con $(X, \mu)$  the set of state ideals and the set of congruences, respectively, on a state BCK-algebra  $(X, \mu)$ .

**Theorem 3.11.** Let  $(X, \mu)$  be a state BCK-algebra.

- (i) If  $\theta$  is a congruence relation of  $(X, \mu)$ , then  $[0]_{\theta} = \{x \in X \mid (x, 0) \in \theta\}$  is a state ideal of  $(X, \mu)$ .
- (ii) If I is a state ideal of  $(X, \mu)$ , then  $\theta_I = \{(x, y) \in X \times X \mid x * y, y * x \in I\}$  is a congruence relation on  $(X, \mu)$ .
- (iii) There is a bijection between the set of all congruence relations of  $(X, \mu)$ ,  $Con(X, \mu)$ , and the set  $SI(X, \mu)$  of all state ideals of  $(X, \mu)$ .

*Proof.* (i) Let  $\theta$  be a congruence relation of  $(X, \mu)$ . Then by Theorem 2.3,  $[0]_{\theta}$  is an ideal of X. It suffices to show that  $[0]_{\theta}$  is a state ideal. Let  $x \in [0]_{\theta}$ . Then  $(x, 0) \in \theta$ . Since  $\theta$  is a congruence relation of  $(X, \mu)$ , then  $(\mu(x), \mu(0)) \in \theta$  and so by Proposition 3.2(i),  $(\mu(x), 0) \in \theta$ . Hence,  $\mu(x) \in [0]_{\theta}$ . That is,  $[0]_{\theta}$  is a state ideal.

(ii) Let I be a state ideal of X. Then  $\theta_I$  is a congruence relation on a BCK-algebra X. Let  $(x, y) \in \theta_I$ . Then  $x * y, y * x \in I$  and so by Proposition 3.2(ii),  $\mu(x) * \mu(y) \leq \mu(x * y) \in I$ . Thus,  $\mu(x) * \mu(y) \in I$ . In a similar way,  $\mu(y) * \mu(x) \in I$ , hence  $(\mu(x), \mu(y)) \in \theta_I$ , so  $\theta_I$  is a congruence relation of  $(X, \mu)$ .

(iii) We define a map  $f : SI(X, \mu) \to Con(X, \mu)$  by  $f(I) = \theta_I$ . Then it can be easily shown that f is a bijection map and its inverse is the map  $g : Con(X, \mu) \to SI(X, \mu)$ , which is defined by  $g(\theta) = [0]_{\theta}$ .  $\Box$ 

**Definition 3.12.** [4] An algebra A of type F is a subdirect product of an indexed family  $\{A_i\}_{i \in I}$  of algebras of type F if

- A is a subalgebra of  $\prod_{i \in I} A_i$ ,
- $\pi_i(A) = A_i$  for any  $i \in I$ , where  $\pi_i : \prod_{i \in I} A_i \to A_i$  is a natural projection map.

A one-to-one homomorphism  $\alpha : A \to \prod_{i \in I} A_i$  is called a *subdirect embedding* if  $\alpha(A)$  is a subdirect product of the family  $\{A_i\}_{i \in I}$ . An algebra A of type F is called *subdirectly irreducible* if, for every subdirect embedding  $\alpha : A \to \prod_{i \in I} A_i$ , there exists  $i \in I$  such that  $\pi_i \circ \alpha : A \to A_i$  is an isomorphism.

**Remark 3.13.** If *I* and *J* are two ideals of *X* such that  $I \subseteq J$ , then clearly,  $\theta_I \subseteq \theta_J$ . Let  $(X, \mu)$  be subdirectly irreducible. Then by [4, Thm II.8.4], the set  $\operatorname{Con}(X, \mu) - \Delta$  has a least element, where  $\Delta = \{(x, x) \mid x \in X\}$  and  $\nabla = X \times X$ . Suppose that  $\theta$  is the least element of  $\operatorname{Con}(X, \mu) - \Delta$ . Then by Theorem 3.11, there exists a state ideal of  $(X, \mu)$  such that  $\theta = \theta_I$  (so *I* is a non-zero ideal of *X*). It follows that *I* is the least non-zero state ideal of  $(X, \mu)$ . By Theorem 3.11 and [4, Thm II.8.4], we conclude that  $(X, \mu)$  is subdirectly irreducible if and only if  $\operatorname{SI}(X, \mu) - \{0\}$  has the least element.

In Theorem 3.14 and Theorem 3.15, we present characterizations of subdirectly irreducible state BCKalgebras. First, we show that if  $(X, \mu)$  is subdirectly irreducible, then the conditions (i) or (ii) of Theorem 3.14 hold. Then we prove that if  $(X, \mu)$  satisfies the condition (i) or (ii) in Theorem 3.14, then  $(X, \mu)$  must be subdirectly irreducible. We note that in the next theorem, we take an element a in the subalgebra  $\mu(X)$  of a BCK-algebra X, therefore,  $\langle a \rangle_X$  will denote the ideal of X generated by the element a.

**Theorem 3.14.** Let  $(X, \mu)$  be a subdirectly irreducible state BCK-algebra.

- (i) If  $\operatorname{Ker}(\mu) = \{0\}$ , then  $\mu(X)$  is a subdirectly irreducible subalgebra of X.
- (ii) If  $\operatorname{Ker}(\mu) \neq \{0\}$ , then  $\operatorname{Ker}(\mu)$  is a subdirectly irreducible subalgebra of X and  $\operatorname{Ker}(\mu) \cap \langle a \rangle_X \neq \{0\}$ for each non-zero element a of  $\mu(X)$ .

*Proof.* (i) Let  $(X, \mu)$  be subdirectly irreducible and  $\operatorname{Ker}(\mu) = \{0\}$ . By Remark 3.13, the set of all non-zero state ideals of  $(X, \mu)$  has the least element, I say. If  $I \cap \mu(X) = \{0\}$ , then by  $\mu(I) \subseteq I \cap \mu(X)$  (since I is a state ideal), we conclude that  $\mu(x) = 0$  for all  $x \in I$ . Thus,  $I \subseteq \operatorname{Ker}(\mu) = \{0\}$ , which is a contradiction. So,  $I \cap \mu(X) \neq \{0\}$ . Now, we show that  $I \cap \mu(X)$  is the least non-zero ideal of  $\mu(X)$ . Suppose that J is an ideal of  $\mu(X)$ .

(1) Let  $\langle J \rangle_X$  be the ideal of X generated by J, and choose an arbitrary element  $x \in \langle J \rangle_X$ . Then by Theorem 2.2, there exist  $b_1, \ldots, b_n \in J$  such that  $(\cdots ((x * b_1) * b_2) * \cdots) * b_n = 0$  and so by Proposition 3.2(i) and (ii), we get

$$(\cdots ((\mu(x) * \mu(b_1)) * \mu(b_2)) * \cdots) * \mu(b_n) \le \mu((\cdots ((x * b_1) * b_2) * \cdots) * b_n) = 0.$$

Since  $\mu^2 = \mu$  and  $b_1, \ldots, b_n \in J \subseteq \mu(X)$ , we get  $(\cdots ((\mu(x) * b_1) * b_2) * \cdots) * b_n = 0$ , hence  $\mu(x) \in J$ . Thus,  $\langle J \rangle_X$  is a state ideal of  $(X, \mu)$ .

(2) Clearly,  $J = \langle J \rangle_X \cap \mu(X)$ .

By (1), we get that  $I \subseteq \langle J \rangle_X$  and so by (2),  $I \cap \mu(X) \subseteq \langle J \rangle_X \cap \mu(X) = J$ . Hence,  $I \cap \mu(X)$  is the least non-zero ideal of  $\mu(X)$ . Therefore, by [4, Thm II.8.4], we conclude that  $\mu(X)$  is a subdirectly irreducible subalgebra of X.

(ii) Let  $\mu(X) \neq \{0\}$ . Again, let *I* be the least non-zero state ideal of the subdirectly irreducible state BCK-algebra  $(X, \mu)$ . Since *X* is a BCK-algebra, then every ideal of *X*, in particular Ker $(\mu)$ , is a subalgebra of *X*. Clearly, Ker $(\mu)$  is a state ideal of  $(X, \mu)$  and so  $I \subseteq \text{Ker}(\mu)$ . We show that *I* is the least non-zero ideal of Ker $(\mu)$ . Let *J* be a non-zero ideal of Ker $(\mu)$ . Then  $\mu(J) \subseteq \mu(\text{Ker}(\mu)) = \{0\} \subseteq J$ . For any  $x, y \in X$ , if  $y * x, x \in J$ , then by Proposition 3.2(ii),  $0 = \mu(y * x) \ge \mu(y) * \mu(x) = \mu(y) * 0 = \mu(y)$ . Thus,  $y \in \text{Ker}(\mu)$ , so  $y \in J$  (since J is an ideal of  $\text{Ker}(\mu)$ ). It follows that J is a state ideal of X and so  $I \subseteq J$ . Hence by [4, Thm II.8.4],  $\text{Ker}(\mu)$  is subdirectly irreducible.

Now, let a be a non-zero element of  $\mu(X)$  and let  $\langle a \rangle_X$  be the ideal generated by a in X. Then  $a = \mu(a)$ . Take an arbitrary element  $u \in \langle a \rangle_X$ . By Theorem 2.2, there exists  $n \in \mathbb{N}$  such that  $0 = u * a^n$ , and by by Proposition 3.2(ii),  $0 = \mu(0) = \mu(u * a^n) = \mu(x) * (\mu(a))^n = \mu(u) * a^n$ . Thus,  $\mu(u) \in \langle a \rangle_X$  and  $\mu(\langle a \rangle_X) \subseteq \langle a \rangle_X$ . This implies,  $\langle a \rangle_X$  is a non-zero state interval of  $(X, \mu)$  and, consequently,  $I \subseteq \langle a \rangle_X$ . Since also  $I \subseteq \operatorname{Ker}(\mu)$ , we have  $\{0\} \neq I \subseteq \operatorname{Ker}(\mu) \cap \langle a \rangle_X$ .

**Theorem 3.15.** Let  $(X, \mu)$  be a state BCK-algebra. If it satisfies the condition (i) or (ii) in Theorem 3.14, then  $(X, \mu)$  is subdirectly irreducible.

*Proof.* First, we assume that  $\operatorname{Ker}(\mu) = \{0\}$  and  $\mu(X)$  is a subdirectly irreducible subalgebra of X. Since  $\mu(X)$  is subdirectly irreducible, then by [4, Thm II.8.4],  $\bigcap(I(\mu(X)) - \{0\})$  is a non-zero ideal of  $\mu(X)$ . From  $I(\mu(X)) = \{I \cap \mu(X) \mid I \in I(X)\}$ , it follows that  $\bigcap(\{I \cap \mu(X) \mid I \in I(X)\} - \{0\})$  is non-zero, so  $\bigcap(I(X) - \{0\}) \neq \{0\}$ . Hence, the intersection of all non-zero state ideals of  $(X, \mu)$  is a non-zero ideal of X (clearly it is a state ideal), whence by Remark 3.13,  $(X, \mu)$  is subdirectly irreducible.

Now, let  $\operatorname{Ker}(\mu) \neq \{0\}$  and let  $\operatorname{Ker}(\mu)$  be a subdirectly irreducible subalgebra of X. Let I be the least non-zero ideal of  $\operatorname{Ker}(\mu)$ . Clearly, I is a state ideal (since  $\mu(I) = \{0\}$ ). We claim, for any non-zero state ideal H of  $(X, \mu)$ , we have  $I \subseteq H$ . Suppose that H is a non-zero state ideal of  $(X, \mu)$ . Then  $\mu(H) \subseteq H$ . If  $\mu(H) = \{0\}$ , then  $H \subseteq \operatorname{Ker}(\mu)$  and so  $I \subseteq H$ . Otherwise, there exists  $a \in \mu(H) - \{0\}$ . It follows that  $\{0\} \neq \operatorname{Ker}(\mu) \cap \langle a \rangle_X \subseteq \operatorname{Ker}(\mu) \cap H$  and so  $I \subseteq H \cap \operatorname{Ker}(\mu) \subseteq H$ . Thus, I is the least non-zero state ideal of  $(X, \mu)$ . Therefore,  $(X, \mu)$  is subdirectly irreducible.  $\Box$ 

In the final theorem of this section, we find a relation between state operators in BCK-algebras and MV-algebras. It is well known, if (X, \*, 0) is a bounded commutative BCK-algebra, then  $(X, \oplus, ', 0)$  is an MV-algebra, where  $x \oplus y = N(Nx * y)$  and x' = Nx for all  $x, y \in X$  (see [23]). Note that in each bounded BCK-algebra X, we have N(Nx) = x.

**Theorem 3.16.** Let (X, \*, 0) be a bounded commutative BCK-algebra and  $\mu$  be a left state BCK operator on X such that  $\mu(1) = 1$ . Then  $(X, \mu)$  is a state MV-algebra. The converse is also true.

*Proof.* Let  $x, y \in X$ . Then  $\mu(x') = \mu(1 * x) = \mu(1) * \mu(x * (x * 1)) = 1 * \mu(x) = \mu(x)'$ . Then

$$\begin{split} \mu(x) \oplus \mu(y \ominus (x \odot y)) &= \mu(x) \oplus \mu((y' \oplus (x \odot y))') \\ &= \mu(x) \oplus \mu(y * (x \odot y)) \\ &= \mu(x) \oplus \mu(y * (y \odot x)) \\ &= \mu(x) \oplus \mu(y * (y' \oplus x')') \\ &= \mu(x) \oplus \mu(y * (y * x')) \\ &= (\mu(x)' * \mu(y * (y * Nx)))' \\ &= (\mu(Nx) * \mu(y * (y * Nx)))' \\ &= \mu(Nx * y)', \text{ since } X \text{ is commutative and } \mu \text{ is a left state operator} \\ &= \mu(N(Nx * y)) \\ &= \mu(x \oplus y), \end{split}$$

so that,  $(X, \mu)$  is a state MV-algebra. Conversely, consider the MV-algebra  $(X, \oplus, ', 0)$ . If  $(X, \sigma)$  is a state MV-algebra, then we can easily show that  $\sigma : X \to X$  is a left state operator on a BCK-algebra (X, \*, 0), where  $x * y := x \odot y', x, y \in X$ . In fact, it follows from the following identity on X:

$$(y' \oplus (x' \odot y))' = y * (x' \odot y) = y * (y'' \odot x') = y * (y' \oplus x)' = y * (y * x).$$

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### 4. STATE-MORPHISM BCK-ALGEBRAS

In the section, we introduce and study state-morphism BCK-algebras which is an important subfamily of the family of state BCK-algebras.

**Definition 4.1.** Let (X, \*, 0) be a BCK-algebra. A homomorphism  $\mu : X \to X$  is called a *state-morphism* operator if  $\mu^2 = \mu$ , where  $\mu^2 = \mu \circ \mu$ , and the pair  $(X, \mu)$  is called a *state-morphism* BCK-algebra.

By (BCK8), every state-morphism BCK-algebra is a (left) state BCK-algebra. We note that not every state-morphism operator is also a right state operator. For example,  $Id_X$  is both a state-morphism operator and a left state operator, but it is a right state operator iff X is a commutative BCK-algebra.

**Example 4.2.** (i) For each BCK-algebra X, the identity map  $Id_X : X \to X$  and the zero operator  $O_X(x) = 0, x \in X$ , are state-morphism operators.

(ii) Let x be an element of X such that  $a * x = a * x^2$  for all  $a \in X$ . Define  $\alpha_x : X \to X$  by  $\alpha_x(a) = a * x$  for all  $a \in X$ . First, we show that  $\alpha_x$  is a homomorphism. By (BCK4), 0 \* x = 0 for all  $x \in X$ . Let  $a, b \in X$ . Then by (BCK6), we have (b \* x) \* b = (b \* b) \* x = 0 \* x = 0, so  $b * x \le b$ . Using (BCK6) and (BCK7), we obtain that  $(a * b) * x = (a * x) * b \le (a * x) * (b * x)$ . On the other hand, by (BCK1) and (BCK6),  $(a * x) * (b * x) = (a * x^2) * (b * x) \le (a * x) * b = (a * b) * x$ . Hence,  $\alpha_x$  is a homomorphism. Therefore,

$$\alpha_x(\alpha_x(a)) = (a * x) * x = a * x = \alpha_x(a),$$

so that,  $\alpha_x$  is a state-morphism operator on X. For example, if x = 0, then  $\alpha_0 = \text{Id}_X$ . In particular, if X is a positive implicative BCK-algebra, then by [29, Thm 3.1.1], for all  $a, x \in X$ , we have  $a * x^2 = a * x$  and so,  $\alpha_x$  is a state-morphism operator on X for all  $x \in X$ .

(iii) Every state operator  $\mu$  on a linearly ordered commutative BCK-algebra X is a state-morphism operator. Indeed, if  $x \leq y$ , then x \* y = 0 and by Proposition 3.2, we have  $0 \leq \mu(x) * \mu(y) \leq \mu(x * y) = \mu(0)$ . If  $y \leq x$ , by the definition of a state operator, we have  $\mu(x * y) = \mu(x) * \mu(x * (x * y)) = \mu(x) * \mu(y)$ . The both cases entail  $\mu$  is an endomorphism of the BCK-algebra X.

(iv) Every right state operator  $\mu$  on a linearly ordered commutative BCK-algebra X is a state-morphism operator. Indeed, by Proposition 3.4(iii),  $\mu$  is a left state operator, too. Take  $x, y \in X$ . Since X is a chain, then x \* y = 0 or y \* x = 0.

If x \* y = 0, then by Proposition 3.2(ii),  $0 \le \mu(x) * \mu(y) \le \mu(x * y) = \mu(0) = 0$ . So,  $\mu(x) * \mu(y) = \mu(x * y)$ . If y \* x = 0, then from Proposition 3.4 we have  $\mu(x * y) = \mu(x) * \mu(y)$ . Consequently,  $\mu$  is a homomorphism, and  $\mu$  is a state-morphism operator on X.

Example 3.6 shows that if  $\mu$  is a left state operator on a linearly ordered BCK-algebra, then  $\mu$  is not necessarily a state-morphism operator on X. Indeed, we have  $\mu(3*2) = \mu(3) = 2 \neq 0 = 2*2 = \mu(3)*\mu(2)$ . Thus  $\mu$  is not a state morphism operator.

As a consequence of Corollary 3.16, we have by [8], that there are also bounded commutative BCKalgebras X having state operators which are not state-morphism operators.

**Definition 4.3.** Let  $(X, \mu)$  be a state-morphism BCK-algebra. An ideal I of a BCK-algebra X is called a *state ideal* if  $\mu(I) \subseteq I$ . If T is a subset of X, then  $\langle T \rangle_s$  is the least state ideal of X containing T.

It can be easily shown that, if  $(X, \mu)$  is a state BCK-algebra, then  $\text{Ker}(\mu)$  is a state ideal of X. Clearly, the intersection of every arbitrary family of state ideals of X is a state ideal. So,

$$\langle T \rangle_s = \bigcap \{ I \mid T \subseteq I, \ I \text{ is a state ideal of } (X, \mu) \}$$

**Proposition 4.4.** Let I be an ideal of a state-morphism BCK-algebra  $(X, \mu)$ . Then

$$\langle I \rangle_s = \{ a \in X \mid (\cdots ((a * \mu(x_1)) * \mu(x_2)) * \cdots) * \mu(x_n) \in I, \exists n \in \mathbb{N}, \exists x_1, x_2, \dots, x_n \in I \}.$$

*Proof.* Let  $J = \langle I \rangle_s = \{a \in X \mid (\cdots ((a * \mu(x_1)) * \mu(x_2)) * \cdots) * \mu(x_n) \in I, \exists n \in \mathbb{N}, \exists x_1, x_2, \dots, x_n \in I\}$ . Then clearly,  $I \subseteq J$  (since  $0 \in I$  and  $\mu(0) = 0$ ). First, we show that J is a state ideal of X. Let  $a, b * a \in J$  for some  $a, b \in X$ . Then there exist  $m, n \in \mathbb{N}$  and  $x_1, \dots, x_n, y_1, \dots, y_m \in X$  such that

 $(\cdots ((a * \mu(x_1)) * \mu(x_2)) * \cdots) * \mu(x_n) \in I$  and  $(\cdots (((b * a) * \mu(y_1)) * \mu(y_2)) * \cdots) * \mu(y_m) = y \in I$ . By (BCK5) and (BCK6), we have

$$(\cdots(((b*y)*\mu(y_1))*\mu(y_2))*\cdots)*\mu(y_m) \le a$$

and so by (BCK7),

 $(\cdots (((\cdots (((b*y)*\mu(y_1))*\mu(y_2))*\cdots)*\mu(y_m))*\mu(x_1))*\cdots)*\mu(x_n) \le (\cdots ((a*\mu(x_1))*\mu(x_2))*\cdots)*\mu(x_n) \in I.$ Since  $y \in I$  and I is an ideal of X, then by (BCK6),

$$(\cdots (((\cdots ((b * \mu(y_1)) * \mu(y_2)) * \cdots) * \mu(y_m)) * \mu(x_1)) * \cdots) * \mu(x_n) \in I$$

and so  $b \in J$ . It follows that J is an ideal of X. Moreover, if  $c \in J$ , then there exist  $n \in \mathbb{N}$  and  $z_1, \ldots, z_n \in X$  such that  $(\cdots((c * \mu(z_1)) * \mu(z_2)) * \cdots) * \mu(z_n) = z \in I$ . Hence, by (BCK5) and (BCK6), we get that  $((\cdots((\mu(c) * \mu(z_1)) * \mu(z_2)) * \cdots) * \mu(z_n)) * \mu(z) = \mu(0) = 0 \in I$ . Also,  $z_1, \ldots, z_n, z \in I$ , so by definition of J,  $\mu(c) \in J$ . Thus,  $\mu(J) \subseteq J$  and so J is a state ideal of X containing I. Clearly, if K is a state ideal of X containing I, then  $J \subseteq K$ . Therefore, J is the least state ideal of X containing I. That is  $J = \langle I \rangle_s$ .

**Proposition 4.5.** Let  $(X, \mu)$  be a state-morphism BCK-algebra. Then the following hold:

- (i)  $\operatorname{Ker}(\mu) = \{x * \mu(x) \mid x \in X\} = \{\mu(x) * x \mid x \in X\}.$
- (ii)  $X = \langle \operatorname{Ker}(\mu) \cup \operatorname{Im}(\mu) \rangle.$

*Proof.* (i) Since  $\mu^2 = \mu$  and  $\mu$  is a homomorphism, we have  $\{x * \mu(x) \mid x \in X\} \subseteq \text{Ker}(\mu)$ . Also, for each  $x \in \text{Ker}(\mu), x = x * 0 = x * \mu(x) \in \{x * \mu(x) \mid x \in X\}$ , so  $\text{Ker}(\mu) = \{x * \mu(x) \mid x \in X\}$ . In a similar way, we can show that  $\text{Ker}(\mu) = \{\mu(x) * x \mid x \in X\}$ .

(ii) Let  $x \in X$ . By (i),  $x * \mu(x) \in \text{Ker}(\mu)$ . Since  $\mu(x) \in \text{Im}(\mu)$ , then by Theorem 2.2, we get that  $x \in \langle \text{Ker}(\mu) \cup \text{Im}(\mu) \rangle$ . Therefore,  $X = \langle \text{Ker}(\mu) \cup \text{Im}(\mu) \rangle$ .

Let X be a bounded BCK-algebra and  $m: X \to [0,1]$  be a state-morphism. Since m(1) = 1 and m is an order preserving map, then  $m(X) \subseteq [0,1]$ . Therefore, m is a homomorphism from the BCKalgebra X to the BCK-algebra ([0, 1],  $*_{\mathbb{R}}$ , 0). Hence, X/Ker(m) and m(X) are isomorphic. By [12, Thm 2.9],  $\operatorname{Ker}(m)$  is a commutative ideal of X and so  $X/\operatorname{Ker}(m)$  is a bounded commutative BCK-algebra. Since  $([0,1], *_{\mathbb{R}}, 0)$  is a simple BCK-algebra and m(X) is a subalgebra of it, then m(X) is simple, so X/Ker(m) is simple, too. It follows that Ker(m) is a maximal commutative ideal of X. Therefore,  $(X/\operatorname{Ker}(m), \oplus, ', 0/\operatorname{Ker}(m))$  is an MV-algebra, where  $x/\operatorname{Ker}(m) \oplus y/\operatorname{Ker}(m) = N(Nx * y)/\operatorname{Ker}(m)$  and  $(x/\operatorname{Ker}(m))' = Nx/\operatorname{Ker}(m)$  for all  $x, y \in X$ . It can be easily shown that the map  $f: X/\operatorname{Ker}(m) \to [0,1]$ defined by f(x/Ker(m)) = m(x) is an MV-homomorphism and X/Ker(m) is a simple MV-algebra (since I is a BCK-ideal of X/Ker(m) if and only if I is an MV-ideal of X/Ker(m)). By [25, Thm 1.1], there exists a unique one-to-one MV-homomorphism  $\tau : X/\text{Ker}(m) \to [0,1]$ . Thus,  $f = \tau$ . By summing up the above results, we get that  $m = \tau \circ \pi_{\text{Ker}(m)}$ , where  $\pi_{\text{Ker}(m)} : X \to X/\text{Ker}(m)$  is the canonical epimorphism. Conversely, let X be a bounded BCK-algebra such that X has at least one commutative ideal, I say. Then there exists a maximal ideal M of X such that  $I \subseteq M$ . In fact, M is a maximal element of the set  $\{H \mid H \text{ is an ideal of } X \text{ containing } I, 1 \notin H\}$ . Since I is a commutative ideal and  $I \subseteq M$ , then by [29, Thm 2.5.2], M is a commutative ideal and so X/M is a bounded commutative simple BCK-algebra. It follows that  $(X/M, \oplus, ', 0)$  is a simple MV-algebra. By [25, Thm 1.1], there exists a unique MV-homomorphism,  $\tau_M : (X/M, \oplus, ', 0) \to ([0, 1], \oplus, ', 0)$ . Clearly,  $\tau_M : X/M \to [0, 1]$  is a BCK-homomorphism and so  $\tau_M \circ \pi_M : X \to [0,1]$  is a state-morphism, where  $\pi_M : X \to X/M$  is the canonical epimorphism.

Now, let X be a bounded BCK-algebra and  $\mu : X \to X$  be a state-morphism operator on X such that Ker( $\mu$ ) is a commutative ideal of X. Then  $X/\text{Ker}(\mu)$  is a bounded commutative BCK-algebra. Thus,  $\mu(X)$  is an MV-algebra (since  $\mu(X) \cong X/\text{Ker}(\mu)$ ). Suppose that H is a maximal ideal of the MV-algebra  $\mu(X)$  and  $\pi_H : \mu(X) \to \mu(X)/H$  is the canonical epimorphism. Then  $\mu(X)/H$  is a simple MV-algebra and so by [25, Thm 1.1], there is a unique MV-homomorphism  $\tau_H : \mu(X)/H \to [0, 1]$ . Clearly,  $\tau_H \circ \pi_H \circ \mu : X \to [0, 1]$  is a measure-morphism. Moreover, if  $\mu(1) = 1$ , then  $\tau_H \circ \pi_H \circ \mu$  is a state-morphism.

**Remark 4.6.** Let  $\mu$  be a state-morphism operator on X such that  $\text{Ker}(\mu) = \{0\}$ . Then for all  $x \in X$ ,  $x * \mu(x), \mu(x) * x \in \text{Ker}(\mu) = \{0\}$  and so by (BCK3),  $\mu(x) = x$ . Therefore,  $\mu = \text{Id}_X$ .

**Corollary 4.7.** If X is a simple BCK-algebra, then  $Id_X$  and  $O_X$  are all state-morphism operators of X.

*Proof.* Let X be a simple BCK-algebra and  $\mu : X \to X$  be a state-morphism operator on X. Then  $\operatorname{Ker}(\mu) = \{0\}$  or  $\operatorname{Ker}(\mu) = X$ . Hence by Remark 4.6,  $\mu = \operatorname{Id}_X$  or  $\mu(x) = 0$  for all  $x \in X$ .

**Definition 4.8.** A state ideal I of a state-morphism BCK-algebra  $(X, \mu)$  is called a *prime state ideal* of  $(X, \mu)$  if, given state ideals A, B of  $(X, \mu), A \cap B \subseteq I$  implies that  $A \subseteq I$  or  $B \subseteq I$ .

**Theorem 4.9.** Let  $(X, \mu)$  be a subdirectly irreducible state-morphism BCK-algebra. Then  $\text{Ker}(\mu)$  is a prime state ideal.

Proof. Let I and J be two state ideals of  $(X, \mu)$  such that  $I \cap J \subseteq \operatorname{Ker}(\mu)$ . Define  $\phi : X/\operatorname{Ker}(\mu) \to \mu(X)/I \times \mu(X)/J$ , by  $\phi(x/\operatorname{Ker}(\mu)) = (x/I, x/J)$  for all  $x \in X$ . For each  $x, y \in X$ , if  $x/\operatorname{Ker}(\mu) = y/\operatorname{Ker}(\mu)$ , then  $x * y, y * x \in \operatorname{Ker}(\mu)$  and so  $\mu(x) * \mu(y) = \mu(x * y) = 0 = \mu(y * x) = \mu(y) * \mu(x)$ . Hence by (BCK3),  $\mu(x) = \mu(y)$ . Therefore,  $\phi$  is a well defined homomorphism. Thus, for each  $x, y \in X$ , if  $\phi(x/\operatorname{Ker}(\mu)) = \phi(y/\operatorname{Ker}(\mu))$ , then  $(\mu(x)/I, \mu(x)/J) = (\mu(y)/I, \mu(y)/J)$ , so that  $\mu(x) * \mu(y), \mu(y) * \mu(x) \in I \cap J$ . Hence,  $\mu(x) * \mu(y), \mu(y) * \mu(x) \in \operatorname{Ker}(\mu)$ . It follows that  $x/\operatorname{Ker}(\mu) = y/\operatorname{Ker}(\mu)$ , which implies that  $\phi$  is one-to-one. Clearly,  $\pi_1 \circ \phi(X/\operatorname{Ker}(\mu)) = \mu(X)/I$ , and  $\pi_2 \circ \phi(X/\operatorname{Ker}(\mu)) = \mu(X)/J$ , where  $\pi_1 : \mu(X)/I \times \mu(X)/J \to \mu(X)/I$  and  $\pi_2 : \mu(X)/I \times \mu(X)/J \to \mu(X)/J$  are natural projection maps. Since  $X/\operatorname{Ker}(\mu)$  and  $\mu(X)$  are isomorphic, then by Theorem 3.14(ii),  $X/\operatorname{Ker}(\mu)$  is a subdirectly irreducible BCK-algebra and so  $\pi_1 \circ \phi : X/\operatorname{Ker}(\mu) \to \mu(X)/I$  or  $\pi_2 \circ \phi : X/\operatorname{Ker}(\mu) \to \mu(X)/J$  is an isomorphism. Without lost of generality, we can assume that  $\pi_1 \circ \phi$  is an isomorphism. For any  $x \in I$ ,  $\pi_1(\phi(x/\operatorname{Ker}(\mu))) = \pi_1(\mu(x)/I, \mu(x)/J) = \mu(x)/I$ . Since I is a state ideal, then  $\mu(x) \in I$  and hence  $\mu(x)/I = 0/I$ . It follows that  $x/\operatorname{Ker}(\mu) = 0/\operatorname{Ker}(\mu)$  (since  $\pi_1 \circ \phi$  is an isomorphism) and  $x \in \operatorname{Ker}(\mu)$ . Therefore,  $I \subseteq \operatorname{Ker}(\mu)$  and so  $\operatorname{Ker}(\mu)$  is a prime ideal of X.

Now, let us to consider a commutative subdirectly irreducible state morphism BCK-algebra  $(X, \mu)$  satisfying the identity  $(x * y) \land (y * x) = 0$ .

**Proposition 4.10.** Let  $(X, \mu)$  be a subdirectly irreducible state-morphism BCK-algebra such that X is commutative and  $(x * y) \land (y * x) = 0$  for all  $x, y \in X$ . Then the following statements conditions hold:

- (i) For all  $x \in X$ , either  $x \leq \mu(x)$  or  $\mu(x) \leq x$ .
- (ii)  $\mu(X)$  is a chain.

*Proof.* (i) Since  $(X, \mu)$  is subdirectly irreducible, then by Theorem 3.14,  $\operatorname{Ker}(\mu) = \{0\}$  or  $\operatorname{Ker}(\mu) \neq \{0\}$ and it is a subdirectly irreducible subalgebra of X. If  $\operatorname{Ker}(\mu) = \{0\}$ , then by Remark 4.6,  $\mu(x) = x$  for all  $x \in X$ . Let  $\operatorname{Ker}(\mu) \neq \{0\}$ . Since  $(x * y) \land (y * x) = 0$  for all  $x, y \in X$ , then by Theorem 3.14 and [29, Thm 2.3.12],  $\operatorname{Ker}(X)$  must be a chain. Let  $x \in X$ . By Proposition 4.5,  $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)$  and so  $(x * \mu(x)) \land (\mu(x) * x) = 0$  implies that  $x * \mu(x) = 0$  or  $\mu(x) * x = 0$ . Therefore,  $x \leq \mu(x)$  or  $\mu(x) \leq x$ .

(ii) By the first isomorphism theorem,  $X/\text{Ker}(\mu) \cong \mu(X)$ . Since X is a commutative BCK-algebra and it satisfies the identity  $(x * y) \land (y * x) = 0$ , then by [22, Thm II.8.13] and Theorem 4.9,  $X/\text{Ker}(\mu)$  is a chain. Hence,  $\mu(X)$  is a chain.

Note that if (X, \*, 0) is a BCK-algebra such that  $(X, \leq)$  is a lattice, it is called a *BCK-lattice*. Then by [29, Thm 2.2.6], X satisfies the identity  $(x * y) \land (y * x) = 0$ .

**Definition 4.11.** A pair (A, I) is called an *adjoint pair* of a BCK-algebra X, if I is an ideal of X and A is a subalgebra of X satisfying the following conditions:

(Ap1)  $A \cap I = \{0\}$  and  $\langle A \cup I \rangle = X$ ;

(Ap2) for each  $x \in X$ , there exists an element  $a_x \in A$  such that  $(x, a_x) \in \theta_I$  (we say that  $a_x$  is a *component* of x in A with respect to I).

By Proposition 4.5(iii) and (iv), we conclude that if  $\mu$  is a state-morphism operator on X, then  $(\mu(X), \text{Ker}(\mu))$  satisfies (Ap1). In Theorem 4.14, a relation between state-morphism operators and adjoint pairs in any BCK-algebras will be found.

**Proposition 4.12.** Let (A, I) be an adjoint pair of X. Then, for all  $x \in X$ ,  $a_x$  is unique.

*Proof.* Let  $x \in X$  and  $a, b \in A$  be two components of x in A. Then  $(x, a), (x, b) \in \theta_I$  and so  $(a, b) \in \theta_I$ . Hence,  $a*b, b*a \in I$ . Also,  $a*b, b*a \in A$  (since A is a subalgebra of X), so by (Ap1),  $a*b, b*a \in I \cap A = \{0\}$ . Thus, by (BCK3), a = b. Therefore,  $a_x$  is the only component of x in A with respect to I.

Let  $\mu$  and  $\nu$  be two state-morphism operators on X such that  $\operatorname{Ker}(\mu) = \operatorname{Ker}(\nu)$  and  $\operatorname{Im}(\mu) = \operatorname{Im}(\nu)$ . For any  $x \in X$ , we have  $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu) = \operatorname{Ker}(\nu)$  and so  $\nu(x * \mu(x)) = 0 = \nu(\mu(x) * x)$ . Since  $\nu$  is a homomorphism and  $\mu(x) \in \operatorname{Im}(\mu) = \operatorname{Im}(\nu)$ , then  $\nu(\mu(x)) = \mu(x)$  and so  $\nu(x) * \mu(x) = 0 = \mu(x) * \nu(x)$ . From (BCK3), we obtain that  $\nu(x) = \mu(x)$  for all  $x \in X$ . Therefore,  $\mu = \nu$ . In Remark 4.13, we show that, there are state-morphism operators  $\mu$  and  $\nu$  on a BCK-algebra X such that  $\operatorname{Ker}(\mu) = \operatorname{Ker}(\nu)$ , but  $\mu \neq \nu$ .

**Remark 4.13.** Suppose that I is a maximal ideal of X such that |X/I| = 2 and  $2 \le |X - I|$ . Let a and b be two distinct elements of X - I. Define  $\mu_a : X \to X$  and  $\mu_b : X \to X$  by

$$\mu_a(x) = \begin{cases} 0 & \text{if } x \in I, \\ a & \text{if } x \in X - I. \end{cases} \qquad \mu_b(x) = \begin{cases} 0 & \text{if } x \in I, \\ b & \text{if } x \in X - I \end{cases}$$

(1) If  $x, y \in I$ , then  $x * y \in I$ , so  $\mu_a(x * y) = 0 = \mu_a(x) * \mu_b(y)$ .

(2) If  $x \in I$  and  $y \in X - I$ , then  $x * y \leq x$  and hence  $x * y \in I$ . It follows that  $\mu_a(x * y) = 0 = 0 * \mu_a(y) = \mu_a(x) * \mu_b(y)$ ,

(3) If  $x \in X - I$  and  $y \in I$ , then  $x * y \in X - I$  (since I is an ideal and  $x * y \in I$  implies  $x \in I$ ) and so  $\mu_a(x * y) = a = \mu_a(x) * 0 = \mu_a(x) * \mu_a(y)$ ,

(4) If  $x, y \in X - I$ , then by assumption, x/I = y/I (since |x/I| = 2), so  $x * y \in I$ . Thus,  $\mu_a(x * y) = 0 = a * a = \mu_a(x) * \mu_a(y)$ .

By (1)-(4), we obtain that  $\mu_a$  is a homomorphism. If  $x \in I$ , then  $\mu_a(\mu_a(x)) = \mu_a(x) = 0$ . Also, if  $x \in X - I$ , then  $\mu_a(\mu_a(x)) = \mu_a(a) = a = \mu_a(x)$  (since  $a \in X - I$ ), so  $\mu_a$  is a state-morphism operator. In a similar way, we can show that  $\mu_b$  is a state-morphism operator. Clearly,  $\operatorname{Ker}(\mu_a) = I = \operatorname{Ker}(\mu_b)$ , but  $\mu_a \neq \mu_b$ .

Note that if X is a non-trivial positive implicative BCK-algebra and I is a maximal ideal of X, then X/I is a simple positive implicative BCK-algebra and so by [29, Cor 3.1.7], |X/I| = 2. It follows that if  $2 \leq |X - I|$ , then X satisfies the conditions in Remark 4.13.

**Theorem 4.14.** There is a one-to-one correspondence between adjoint pairs of X and state-morphism operators on X.

*Proof.* Let  $\mu : X \to X$  be a state-morphism operator on X. We show that  $(\mu(X), \operatorname{Ker}(\mu))$  is an adjoint pair of X. By Proposition 4.5(iii) and (iv), (Ap1) holds. Let  $A = \mu(X)$  and x be an element of X. Then  $\mu(x) \in A$  and clearly,  $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)$  (since  $\mu^2 = \mu$ ). Hence,  $(x, \mu(x)) \in \theta_I$ . That is, for each  $x \in X, \mu(x)$  is a component of x in A and so (Ap2) holds. Therefore,  $(\mu(X), \operatorname{Ker}(\mu))$  is an adjoint pair of X.

Conversely, let (A, I) be an adjoint pair of X. Define  $\mu_{I,A} : X \to X$ , by  $\mu_{I,A}(x) = a_x$  for all  $x \in X$ . By Proposition 4.12,  $\mu_{I,A}$  is well defined. Let  $x, y \in X$ . Then  $(x, a_x) \in \theta_I$  and  $(y, a_y) \in \theta_I$  and so  $(x * y, a_x * a_y) \in \theta_I$ . By  $a_x * a_y \in A$ , we conclude that  $a_x * a_y$  is a component of x \* y in A, hence by Proposition 4.12,  $\mu_{I,A}(x * y) = a_{x*y} = a_x * a_y = \mu_{I,A}(x) * \mu_{I,A}(y)$ . Thus,  $\mu_{I,A}$  is a homomorphism. Moreover, for any  $a \in A$ ,  $a * a = 0 \in I$  and hence  $\mu_{I,A}(a) = a_a = a$ . It follows that  $\mu_{I,A}(\mu_{I,A}(x)) = \mu_{I,A}(x)$  for all  $x \in X$ . Therefore,  $\mu_{I,A}$  is a state-morphism operator on X. Let us

denote by  $\operatorname{Ad}(X)$  and  $\operatorname{SM}(X)$  the set of all adjoint pairs and the set of all state-morphism operators on X, respectively. Define  $f : \operatorname{Ad}(X) \to \operatorname{SM}(X)$ , by  $f(A, I) = \mu_{I,A}$  and  $g : \operatorname{SM}(X) \to \operatorname{Ad}(X)$  by  $g(\mu) = (\mu(X), \operatorname{Ker}(\mu))$ . Since  $\operatorname{Ker}(\mu_{I,A}) = I$  and  $\operatorname{Im}(\mu_{I,A}) = A$  for all  $(A, I) \in \operatorname{Ad}(X)$ , then by the paragraph just after Proposition 4.12, we conclude that  $f \circ g = \operatorname{Id}_{\operatorname{SM}(X)}$  and  $g \circ f = \operatorname{Id}_{\operatorname{Ad}(X)}$ .  $\Box$ 

In the sequel, we want to construct a state BCK-algebra from a state-morphism. Let  $m : X \to [0, 1]$  be a state-morphism. Then m is a homomorphism from X into the BCK-algebra  $([0, 1], *_{\mathbb{R}}, 0)$  and so  $X/\operatorname{Ker}(m) \cong m(X)$ . Let B = m(X) and  $C = \operatorname{Ker}(m)$ . Then B and C are BCK-algebras. Consider the BCK-algebra  $B \times C$ . Let  $A = \{(b, 0) | b \in B\}$  and  $I = \{(0, c) | c \in C\}$ . Then I is an ideal of  $B \times C$  and A is a subalgebra of  $B \times C$ . Also,

(1)  $A \cap I = \emptyset$ .

(2) For each  $(x, y) \in B \times C$ , we have ((x, y) \* (x, 0)) \* (0, y) = (0, 0), hence by Theorem 2.2,  $(x, y) \in \langle A \cup I \rangle$ . It follows that  $B \times C = \langle A \cup I \rangle$ .

(3) For each  $(x, y) \in B \times C$ , we have  $(x, y) * (x, 0) = (0, y) \in I$  and  $(x, 0) * (x, y) = (0, 0) \in I$ . Thus, (x, y)/I = (x, 0)/I.

So by Theorem 4.14, the map  $\mu : B \times C \to B \times C$  defined by  $\mu(x, y) = (x, 0)$  is a state-morphism operator on  $B \times C$ . Clearly,  $\operatorname{Ker}(\mu) = I$  and  $\operatorname{Im}(\mu) = A$ . Note that if  $m_{\mu} : B \times C \to [0, 1]$  is the statemorphism induced by  $\mu$  (see the paragraph before Remark 4.6), then  $(B \times C)/\operatorname{Ker}(m_{\mu}) \cong B \cong \operatorname{Im}(m)$ and  $\operatorname{Ker}(m_{\mu}) = C \cong \operatorname{Ker}(m)$ .

**Definition 4.15.** Let I be an ideal of X and  $\pi_I : X \to X/I$  be the canonical projection. Then I is called a *retract* ideal if there exists a homomorphism  $f : X/I \to X$  such that  $\pi_I \circ f = \operatorname{Id}_{X/I}$  (the identity map on X/I).

**Theorem 4.16.** An ideal I of X is a retract ideal if and only if there exists a subalgebra A of X such that (A, I) forms an adjoint pair.

Proof. Let I be a retract ideal of X. Then there exists a homomorphism  $f: X/I \to X$  such that  $\pi_I \circ f = \operatorname{Id}_{X/I}$ . Put A = f(X/I). Since f is a homomorphism, then A is a subalgebra of X. Let  $x \in I \cap A$ . Then there exists  $a \in X$  such that f(a/I) = x, so  $a/I = \pi_I \circ f(a/I) = \pi_I(x) = x/I$ . From  $x \in I$ , we get that  $a \in I$  and a/I = 0/I, whence x = f(0/I) = 0. Now, let  $x \in X$ . Then f(x/I) = a for some  $a \in A$ . It follows that  $x/I = \pi_I \circ f(x/I) = \pi_I(a) = a/I$ , which implies that  $x * a \in I$ . Hence,  $a \in \langle A \cup I \rangle$  and a is a component of x in A with respect to I. Therefore, (A, I) is an adjoint pair of X. Conversely, let (A, I) be an adjoint pair of X. Define  $f: X/I \to X$  by  $f(x/I) = a_x$  for all  $x \in X$  (see Definition 4.11). If x/I = y/I for some  $x, y \in X$ , then  $(x, y) \in \theta_I$  and  $(x, a_x) \in \theta_I$ , which yields  $a_x$  is a component of y in A. By Proposition 4.12, we get that  $a_y = a_x$ . Thus, f is well defined. In a similar way, we can show that f is a homomorphism. It follows from  $(x, a_x) \in \theta_I$  that  $\pi_I \circ f(x/I) = \pi_I(a_x) = a_x/I = x/I$ . Therefore, I is a retract ideal of X.

**Corollary 4.17.** There is a one-to-one correspondence between retract ideals and state-morphism operators of X.

*Proof.* The proof is a straightforward consequence of Theorem 4.14 and 4.16.

**Definition 4.18.** [4, Def II.8.8] A state BCK-algebra  $(X, \mu)$  is called

- simple if  $Con(X, \mu) = \{\Delta, \nabla\}.$
- semisimple if the intersection of all maximal congruence relations of  $(X, \mu)$  is  $\Delta$ .

By Theorem 3.11, we conclude that  $(X, \mu)$  is simple if and only if it has exactly, two state ideals ({0} and X) and it is semisimple if and only if the intersection of all maximal state ideals of  $(X, \mu)$  is the zero ideal.

**Theorem 4.19.** Let  $(X, \mu)$  be a state-morphism BCK-algebra. Then the following hold:

(i)  $\mu(X)$  is a simple (semisimple) subalgebra of X if and only if  $\text{Ker}(\mu) \in \text{Max}(X)$  ( $\text{Rad}(X) \subseteq \text{Ker}(\mu)$ ).

- (ii)  $(X,\mu)$  is a simple state-morphism BCK-algebra if and only if X is a simple BCK-algebra.
- (iii) If  $\mu(X)$  is a semisimple subalgebra of X, then the intersection of all maximal state ideals of  $(X, \mu)$  is a subset of Ker $(\mu)$ .
- (iv) If X is a non-trivial bounded BCK-algebra such that  $\mu(1) = 1$  and  $(X, \mu)$  is a semisimple state BCK-algebra, then  $\mu$  is the identity map.

*Proof.* (i) Let  $(X, \mu)$  be a state-morphism BCK-algebra. Then by the first isomorphism theorem,  $X/\text{Ker}(\mu)$  and  $\mu(X)$  are isomorphic (as BCK-algebras), whence the proof of (i) is straightforward.

(ii) Let  $(X, \mu)$  be a simple state-morphism BCK-algebra. By Proposition 3.2(iii), Ker $(\mu)$  is a state ideal of  $(X, \mu)$  and so Ker $(\mu) = \{0\}$  or Ker $(\mu) = X$ . By Corollary 4.7, we obtain that  $\mu = \operatorname{Id}_X$  or  $\mu(x) = 0$  for all  $x \in X$ . However, each ideal of X is a state ideal, so by assumption, X must have exactly two ideals. That is, X is a simple BCK-algebra. The proof of the converse is clear. In fact, any simple BCK-algebra X, has exactly two ideals, X and  $\{0\}$ , which are state ideals.

(iii) Let  $\mu(X)$  be a semisimple subalgebra of X. Since  $X/\operatorname{Ker}(\mu) \cong \mu(X)$ , we get that  $\operatorname{Rad}(X/\mu(X)) = \{0/\mu(X)\}$  and so  $\bigcap\{I/\operatorname{Ker}(\mu) \mid \operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\} = \{0/\operatorname{Ker}(\mu)\}$ , which implies that  $\bigcap\{I \mid \operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\} \subseteq \operatorname{Ker}(\mu)$ . Let H be a maximal ideal of X containing  $\operatorname{Ker}(\mu)$ . Since  $\mu(x) * x \in \operatorname{Ker}(\mu)$ , for each  $x \in H$ , then we have  $\mu(x) * x \in H$ , and so  $\mu(x) \in H$  for all  $x \in X$ . Thus, H is a state ideal of  $(X, \mu)$ . By summing up the above results, we have

$$\{I \mid I \text{ is a state ideal of } (X,\mu)\} \subseteq \{I \mid \operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\} \subseteq \operatorname{Ker}(\mu)$$

(iv) Let I be a maximal state ideal of X. Then we define  $\nu : X/I \to X/I$  by  $\nu(x/I) = \mu(x)/I$  for all  $x \in X$ . If x/I = y/I for some  $x, y \in X$ , then  $x * y, y * x \in I$ . By assumption,  $\mu(x) * \mu(y) \in I$ and  $\mu(y) * \mu(x) \in I$ , hence  $\mu(x)/I = \mu(y)/I$ , which implies that  $\nu(x/I) = \nu(y/I)$ . Clearly,  $\nu$  is a state operator on the BCK-algebra X/I. Since I is a maximal ideal, then X/I is a simple BCK-algebra, so by Corollary 4.7,  $\nu = \operatorname{Id}_{X/I}$  or  $\nu = 0$ . If  $\nu = 0$ , then  $\mu(x) \in I$  for all  $x \in X$ . It follows that  $1 \in I$ , which is a contradiction. So,  $\nu(x/I) = x/I$  for all  $x \in X$ . Hence,  $\mu(x) * x, x * \mu(x) \in I$  for all  $x \in X$ . Since I is an arbitrary maximal state ideal of  $(X, \mu)$ , then by Proposition 4.2, we conclude that  $\operatorname{Ker}(\mu) \subseteq \bigcap\{I \mid I \in \operatorname{MaxS}(X)\}$ . Now, let  $(X, \mu)$  be semisimple. Then  $\bigcap\{I \mid I \in \operatorname{MaxS}(X)\} = \{0\}$  and so,  $\operatorname{Ker}(\mu) = \{0\}$ . By Corollary 4.6,  $\mu = \operatorname{Id}_X$ .

Now we show a relation between state-morphism MV-algebras and state-morphism BCK-algebras.

**Theorem 4.20.** Let (X, \*, 0) be a bounded commutative BCK-algebra and  $\mu : X \to X$  be a statemorphism operator such that  $\mu(1) = 1$ . Then  $(X, \mu)$  is a state-morphism MV-algebra.

*Proof.* Let  $x, y \in X$ . Then  $\mu(x') = \mu(1 * x) = \mu(1) * \mu(x) = 1 * \mu(x) = \mu(x)'$ . Also,

$$\mu(x \oplus y) = \mu(N(Nx * y)) = 1 * \mu(Nx * y) = 1 * (\mu(Nx) * \mu(y)) = 1 * ((1 * \mu(x)) * \mu(y)) = \mu(x) \oplus \mu(y)$$

so,  $\mu(x)$  is a homomorphism of MV-algebras. Since  $\mu^2 = \mu$ , then  $\mu$  is a state-morphism operator on the MV-algebra  $(X, \oplus, ', 0)$ . That is,  $(X, \mu)$  is a state-morphism MV-algebra.

## 5. Generators of State-Morphism BCK-algebras

Let SMBCK be the quasivariety of state-morphism BCK-algebras. We note that the system of BCKalgebras is not a variety because it is not closed under homomorphic images, [22, Thm VI.4.1]. On the other side, the system of commutative BCK-algebras or of quasi-commutative BCK-algebra forms a variety, [22, Thm I.5.2, Thm I.9.2]. Since by [22, Thm I.9.4], every finite BCK-algebra is quasicommutative, we can define the variety generated by a system of finite BCK-algebras.

Let (X, \*, 0) be a BCK-algebra and on the direct product BCK-algebra  $X \times X$  we define a mapping  $\mu_X : X \times X \to X \times X$  by  $\mu_X(x, y) = (x, x), (x, y) \in X \times X$ . Then  $\mu_X$  is a state-morphism on the BCK-algebra  $X \times X$  and the state-morphism BCK-algebra  $D(X) := (X \times X, \mu_X)$  is a said to be a *diagonal state-morphism BCK-algebra*. In the same way we can define also  $\nu : X \times X \to X \times X$  by  $\nu(x, y) = (y, y), (x, y) \in X \times X$ , and  $(X \times X, \nu)$  is again a state-morphism BCK-algebra which is isomorphic to D(X) under the isomorphism  $h(x, y) = (y, x), (x, y) \in X \times X$ . For example, if X = [0, 1] is the MV-algebra of

the real interval, then it generates the variety of MV-algebras (as well as the quasivariety of MV-algebras), and by [13, Thm 5.4], D([0, 1]) generates the variety of state-morphism MV-algebras.

Given a quasivariety of BCK-algebras  $\mathcal{V}$ , let  $\mathcal{V}_{\mu}$  denote the class of state-morphism BCK-algebras  $(X, \mu)$  such that  $X \in \mathcal{V}$ . Then  $\mathcal{V}_{\mu}$  is a quasivariety, too.

As usual, given a class  $\mathcal{K}$  of algebras of the same type,  $I(\mathcal{K})$ ,  $H(\mathcal{K})$ ,  $S(\mathcal{K})$ ,  $P(\mathcal{K})$ , and  $P_R(\mathcal{K})$  will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products of algebras and of reduced products from  $\mathcal{K}$ , respectively. Moreover, let  $Q_V(\mathcal{K})$  and  $V(\mathcal{K})$  denote the quasivariety and the variety, respectively, generated by  $\mathcal{K}$ . We recall that a quasivariety is closed under isomorphic images, subalgebras, reduced products and containing the one-element algebras, see [4, Def V.2.24], and a variety is closed under homomorphic images, subalgebras and products.

Using methods from [13, Sec 5], which can be easily modified for state-morphism BCK-algebras instead of state-morphism MV-algebras, we can prove the following two lemmas and theorem on generators for a case when we have a variety of BCK-algebras as well as for a more general case - for quasivarieties of BCK-algebras; for reader's convenience, we present outlines of theirs proofs.

First we start with proofs concerning the case when a family of BCK-algebras belongs to some variety of BCK-algebras.

**Lemma 5.1.** (1) Let  $\mathcal{K}$  be a class of BCK-algebras belonging to some variety of BCK-algebras. Then  $V(D(\mathcal{K})) \subseteq V(\mathcal{K})_{\mu}$ .

(2) Let  $\mathcal{V}$  be any variety of BCK-algebras. Then  $\mathcal{V}_{\mu} = \mathsf{ISD}(\mathcal{V})$ .

Proof. (1) We have to prove that every BCK-reduct of a state-morphism BCK-algebra in  $V(D(\mathcal{K}))$  is in  $V(\mathcal{K})$ . Let  $\mathcal{K}_0$  be the class of all BCK-reducts of algebras in  $D(\mathcal{K})$ . Let  $X \in \mathcal{K}$ , then  $D(X) \in D(\mathcal{K})$ . Then the BCK-reduct of D(X) is  $X \times X$ , and since X is a homomorphic image (under the projection map) of  $X \times X$ ,  $\mathcal{K}_0 \subseteq P(\mathcal{K})$  and  $\mathcal{K} \subseteq H(\mathcal{K}_0)$ . Hence,  $\mathcal{K}_0$  and  $\mathcal{K}$  generate the same variety. Moreover, BCK-reducts of subalgebras (homomorphic images, direct products respectively) of algebras from  $D(\mathcal{K})$  are subalgebras (homomorphic images, direct products, respectively) of the corresponding BCK-reducts. Therefore, the BCK-reduct of any algebra in  $V(D(\mathcal{K}))$  is in  $HSP(\mathcal{K}_0) = HSP(\mathcal{K}) = V(\mathcal{K})$ .

(2) Let  $(X, \mu) \in \mathcal{V}_{\mu}$ . The map  $\Phi : a \mapsto (\mu(a), a)$  is an embedding of  $(X, \mu)$  into D(X). Moreover,  $\Phi(\mu(a)) = (\mu(\mu(a)), \mu(a)) = (\mu(a), \mu(a)) = \mu_X((\mu(a), a)) = \mu_X(\Phi(a))$ . Hence,  $\Phi$  is an injective homomorphism of state-morphism BCK-algebras, and  $(X, \mu) \in \mathsf{ISD}(\mathcal{V})$ . Conversely, the BCK-reduct of any algebra in  $\mathsf{D}(\mathcal{V})$  is in  $\mathcal{V}$ , and hence the BCK-reduct of any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathsf{IS}(\mathcal{V}) = \mathcal{V}$ . Hence, any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathcal{V}_{\mu}$ .

**Lemma 5.2.** Let  $\mathcal{K}$  be a class of BCK-algebras. Then: (1)  $\mathsf{DH}(\mathcal{K}) \subseteq \mathsf{HD}(\mathcal{K})$ . (2)  $\mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K})$ . (3)  $\mathsf{DP}(\mathcal{K}) \subseteq \mathsf{IPD}(\mathcal{K})$ . (4)  $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$ .

Proof. (1) Let  $D(C) \in \mathsf{DH}(\mathcal{K})$ . Then there are  $X \in \mathcal{K}$  and a BCK-homomorphism h from X onto C. Let, for all  $a, b \in X$ ,  $h^*(a, b) = (h(a), h(b))$ . We claim that  $h^*$  is a homomorphism from the diagonal statemorphism BCK-algebra D(X) onto D(C). That  $h^*$  is a BCK-homomorphism is clear. We verify that  $h^*$ is compatible with  $\mu_X$ . We have  $h^*(\mu_X(a, b)) = h^*(a, a) = (h(a), h(a)) = \mu_C(h(a), h(b)) = \mu_C(h^*(a, b))$ . Finally, since h is onto, given  $(c, d) \in C \times C$ , there are  $a, b \in X$  such that h(a) = c and h(b) = d. Hence,  $h^*(a, b) = (c, d), h^*$  is onto, and  $D(C) \in \mathsf{HD}(\mathcal{K})$ .

(2) It is trivial.

(3) Let  $X = \prod_{i \in I} X_i \in \mathsf{P}(\mathcal{K})$ , where each  $X_i$  is in  $\mathcal{K}$ . We assert the map

$$\Phi: \left( (a_i : i \in I), (b_i : i \in I) \right) \mapsto \left( (a_i, b_i) : i \in I \right)$$

is an isomorphism of state-morphism BCK-algebras from D(X) onto  $\prod_{i \in I} D(X_i)$ . Indeed, it is clear that  $\Phi$  is a BCK-isomorphism. Moreover, denoting the state-morphism of  $\prod_{i \in I} D(X_i)$  by  $\mu^*$ , we get:

$$\Phi\bigl(\mu_X\bigl((a_i:i\in I),(b_i:i\in I)\bigr)\bigr) = \Phi\bigl((a_i:i\in I),(a_i:i\in I)\bigr) =$$

$$= ((a_i, a_i) : i \in I) = (\mu_{X_i}(a_i, b_i) : i \in I) = \mu^* (\Phi((a_i : i \in I), (b_i : i \in I))),$$

and whence  $\Phi$  is an isomorphism of state-morphism BCK-algebras.

(4) By (1), (2) and (3),  $\mathsf{DV}(\mathcal{K}) = \mathsf{DHSP}(\mathcal{K}) \subseteq \mathsf{HSPD}(\mathcal{K}) = \mathsf{V}(\mathsf{D}(\mathcal{K}))$ , and hence  $\mathsf{ISDV}(\mathcal{K}) \subseteq \mathsf{ISV}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathsf{D}(\mathcal{K}))$ . Conversely, by Lemma 5.1(1),  $\mathsf{V}(\mathsf{D}(\mathcal{K})) \subseteq \mathsf{V}(\mathcal{K})_{\mu}$ , and by Lemma 5.1(2),  $\mathsf{V}(\mathcal{K})_{\mu} = \mathsf{ISDV}(\mathcal{K})$ . This proves the claim.

**Theorem 5.3.** If a system  $\mathcal{K}$  of BCK-algebras generates a variety  $\mathcal{V}$  of BCK-algebras, then  $D(\mathcal{K})$  generates the variety  $\mathcal{V}_{\mu}$  of state-morphism BCK-algebras.

*Proof.* By Lemma 5.2(4),  $V(D(\mathcal{K})) = ISD(V(\mathcal{K}))$ . Moreover, by Lemma 5.1(2),  $V(\mathcal{K})_{\mu} = ISDV(\mathcal{K})$ . Hence,  $V(D(\mathcal{K})) = V(\mathcal{K})_{\mu}$ .

Let [0,1] be the real interval. We endow it with the BCK-structure as before:  $s *_{\mathbb{R}} t = \max\{0, s - t\}$ ,  $s, t \in [0,1]$ . We denote by  $[0,1]_{BCK} := ([0,1], *_{\mathbb{R}}, 0)$  and it is a bounded commutative BCK-algebra. If, for bounded commutative BCK-algebras, we define a state-morphism operator  $\mu$  as a homomorphism of bounded commutative BCK-algebras  $\mu : X \to X$  such that  $\mu \circ \mu = \mu$  and  $\mu(1) = 1$ , we can obtain the following result.

**Corollary 5.4.** Let  $\mathcal{V}$  be the variety of bounded commutative BCK-algebras, and let  $\mathcal{V}_{BCK}$  be the variety of all bounded commutative state-morphism BCK-algebras. Then  $\mathcal{V}_{BCK} = \mathsf{V}(D([0,1]_{BCK}))$ .

*Proof.* We can repeat the proofs of Lemmas 5.1–5.2 and Theorem 5.3 also for state-morphism operators on bounded commutative BCK-algebras. We have  $\mathcal{V}_{BCK} = \mathcal{V}_{\mu}$ . By [23], the variety of bounded BCK-algebras is categorically equivalent to the variety of MV-algebras. Since the MV-algebra [0, 1] generates the variety of MV-algebras, we have that the BCK-algebra  $[0, 1]_{BCK}$  generates the variety of bounded commutative BCK-algebras. Then by Theorem 5.3, we have  $\mathcal{V}_{BCK} = \mathsf{V}(D([0, 1]_{BCK}))$ .

**Corollary 5.5.** There is uncountably many subvarieties of the variety  $\mathcal{V}_{BCK}$  of bounded commutative BCK-algebras with a state-morphism.

*Proof.* By [13, Thm 7.11], the variety of state-morphism MV-algebras is uncountable. Because the variety of bounded commutative BCK-algebras is categorically equivalent to the variety of MV-algebras, [23], we have the statement in question.  $\Box$ 

Now we present some analogous general results concerning quasivarieties. The proofs follow the similar ideas just used for varieties.

**Lemma 5.6.** (1) Let  $\mathcal{K}$  be a class of BCK-algebras. Then  $Q_V(D(\mathcal{K})) \subseteq Q_V(\mathcal{K})_{\mu}$ . (2) Let  $\mathcal{V}$  be any quasivariety of BCK-algebras. Then  $\mathcal{V}_{\mu} = \mathsf{ISD}(\mathcal{V})$ .

*Proof.* (1) We have to prove that every BCK-reduct of a state-morphism BCK-algebra in  $Q_V(\mathcal{K})$  is in  $Q_V(\mathcal{K})$ .

Let  $\mathcal{K}_0$  be the class of all BCK-reducts of algebras in  $\mathsf{D}(\mathcal{K})$ . Let  $X \in \mathcal{K}$ , and let  $\{0\}$  be the one-element BCK-algebra which is a subalgebra of X. Then  $D(X) \in \mathsf{D}(\mathcal{K})$ . The BCK-reduct of D(X) is  $X \times X$ , and since X is isomorphic to the BCK-algebra  $\{0\} \times X$ , which is a subalgebra of  $X \times X$ , we have  $X \in \mathsf{IS}(\mathcal{K}_0)$ . Thus  $\mathcal{K}_0 \subseteq \mathsf{P}(\mathcal{K})$  and  $\mathcal{K} \subseteq \mathsf{IS}(\mathcal{K}_0)$ . By [4, Thm 2.23, 2.25], we have  $\mathsf{Q}_V(\mathcal{K}_0) = \mathsf{ISP}_R(\mathcal{K}_0) \subseteq \mathsf{ISP}_R(\mathcal{K}) \subseteq$  $\mathsf{ISIP}_R(\mathcal{K}) \subseteq \mathsf{IISP}_R(\mathcal{K}) = \mathsf{ISP}_R(\mathcal{K}) = \mathsf{Q}_V(\mathcal{K})$ . Similarly,  $\mathsf{Q}_V(\mathcal{K}) = \mathsf{ISP}_R(\mathcal{K}) \subseteq \mathsf{ISP}_R\mathsf{IS}(\mathcal{K}_0) \subseteq \mathsf{ISIP}_R(\mathcal{K}_0) \subseteq \mathsf{ISP}_R(\mathcal{K}_0) = \mathsf{ISP}_R(\mathcal{K}_0) = \mathsf{Q}_V(\mathcal{K}_0)$ . Hence,  $\mathcal{K}$  and  $\mathcal{K}_0$  generates the same quasivariety.

Moreover, BCK-reducts of subalgebras (isomorphic images, reduced products, respectively) of algebras from  $D(\mathcal{K})$  are subalgebras (isomorphic images, reduced products, respectively) of the corresponding BCK-reducts. Therefore, the BCK-reduct of any algebra in  $Q_V(D(\mathcal{K}))$  is in  $Q_V(\mathcal{K}_0) = Q_V(\mathcal{K}) = Q_V(\mathcal{K})$ , which proves (1).

(2) Let  $(X, \mu) \in \mathcal{V}_{\mu}$ . The map  $\Phi : a \mapsto (\mu(a), a)$  is an embedding of  $(X, \mu)$  into D(X). Moreover,  $\Phi(\mu(a)) = (\mu(\mu(a)), \mu(a)) = (\mu(a), \mu(a)) = \mu_X((\mu(a), a)) = \mu_X(\Phi(a))$ . Hence,  $\Phi$  is an injective homomorphism of state-morphism BCK-algebras, and  $(X, \mu) \in \mathsf{ISD}(\mathcal{V})$ . Conversely, let  $X \in \mathcal{V}$ . Then the BCK-reduct of D(X) is  $X \times X$ , and  $X \times X$  is isomorphic with the reduced product  $(X \times X)/F$ , where Fis the one-element filter  $F = \{1, 2\}$  of the set  $I = \{1, 2\}$ . Hence,  $X \times X$  is in  $\mathcal{V}$ , and the BCK-reduct of any algebra in  $\mathsf{D}(\mathcal{V})$  is in  $\mathcal{V}$ , whence the BCK-reduct of any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathsf{IS}(\mathcal{V}) = \mathcal{V}$ . Therefore, any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathcal{V}_{\mu}$ .

**Lemma 5.7.** Let  $\mathcal{K}$  be a class of BCK-algebras. Then:

 $\begin{array}{l} (1) \ \mathsf{DI}(\mathcal{K}) \subseteq \mathsf{ID}(\mathcal{K}). \\ (2) \ \mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K}). \\ (3) \ \mathsf{DP}_{\mathsf{R}}(\mathcal{K}) \subseteq \mathsf{IP}_{\mathsf{R}}\mathsf{D}(\mathcal{K}). \\ (4) \ \mathsf{Q}_{\mathsf{V}}(\mathsf{D}(\mathcal{K})) = \mathsf{ISD}(\mathsf{Q}_{\mathsf{V}}(\mathcal{K})). \end{array}$ 

Proof. (1) Let  $D(C) \in \mathsf{DI}(\mathcal{K})$ . Then there are  $X \in \mathcal{K}$  and an isomorphism h from X onto C. Let, for all  $a, b \in X$ ,  $h^*(a, b) = (h(a), h(b))$ . We claim that  $h^*$  is an isomorphism from D(X) onto D(C). That  $h^*$  is an isomorphism of BCK-algebras is clear. We verify that  $h^*$  is compatible with  $\mu_X$ . We have  $h^*(\mu_X(a, b)) = h^*(a, a) = (h(a), h(a)) = \mu_C(h(a), h(b)) = \mu_C(h^*(a, b))$ . Finally, since h is onto, given  $(c, d) \in C \times C$ , there are  $a, b \in X$  such that h(a) = c and h(b) = d. Hence,  $h^*(a, b) = (c, d)$ ,  $h^*$  is onto, and  $D(C) \in \mathsf{ID}(\mathcal{K})$ .

(2) It is trivial.

(3) Let  $X = \prod_{i \in I} X_i / F \in \mathsf{P}_{\mathsf{R}}(\mathcal{K})$ , where each  $X_i$  is in  $\mathcal{K}$ , and F is a filter over I. We claim the map

 $\Phi: \left( (a_i : i \in I) / F, (b_i : i \in I) / F \right) \mapsto \left( (a_i, b_i) : i \in I \right) / F$ 

is an isomorphism from D(X) onto  $\prod_{i \in I} D(X_i)/F$ . Indeed, it is clear that  $\Phi$  is a BCK-isomorphism: let  $((a_i, b_i) : i \in I)/F = ((a'_i, b'_i) : i \in I)/F$ . Then  $[\![a_i = a'_i]\!] \cap [\![b_i = b'_i]\!] = [\![(a_i, b_i) = (a'_i, b'_i)]\!] \in F$ , so that  $[\![a_i = a'_i]\!], [\![b_i = b'_i]\!] \in F$  and hence  $((a_i, b_i) : i \in I)/F = ((a'_i, b_i) : i \in I)/F$ . Moreover, denoting the state-morphism of  $\prod_{i \in I} D(X_i)$  by  $\mu^*$ , we get:

$$\Phi(\mu_X((a_i:i\in I)/F,(b_i:i\in I)/F)) = \Phi((a_i:i\in I)/F,(a_i:i\in I))/F = \\ = ((a_i,a_i):i\in I)/F = (\mu_{X_i}(a_i,b_i):i\in I) = \mu^*(\Phi((a_i:i\in I)/F,(b_i:i\in I)/F)),$$

and hence,  $\Phi$  is a state-morphism isomorphism.

(4) By (1), (2) and (3),  $\mathsf{DQ}_{\mathsf{V}}(\mathcal{K}) = \mathsf{DISP}_{\mathsf{R}}(\mathcal{K}) \subseteq \mathsf{IISP}_{\mathsf{R}}\mathsf{D}(\mathcal{K}) \subseteq \mathsf{ISP}_{\mathsf{R}}\mathsf{D}(\mathcal{K}) = \mathsf{Q}_{\mathsf{V}}(\mathsf{D}(\mathcal{K})),$ and hence,  $\mathsf{ISDQ}_{\mathsf{V}}(\mathcal{K}) \subseteq \mathsf{ISQ}_{\mathsf{V}}(\mathsf{D}(\mathcal{K})) = \mathsf{Q}_{\mathsf{V}}(\mathsf{D}(\mathcal{K})).$  Conversely, by Lemma 5.6(1),  $\mathsf{Q}_{\mathsf{V}}(\mathsf{D}(\mathcal{K})) \subseteq \mathsf{Q}_{\mathsf{V}}(\mathcal{K})_{\mu},$ and by Lemma 5.6(2),  $\mathsf{Q}_{\mathsf{V}}(\mathcal{K})_{\mu} = \mathsf{ISDQ}_{\mathsf{V}}(\mathcal{K}).$  This proves the claim.

Finally, we present the main result of the section about generators of quasivarieties of state-morphism BCK-algebras which is an analogue of Theorem 5.3.

**Theorem 5.8.** If a system  $\mathcal{K}$  of BCK-algebras generates a quasivariety  $\mathcal{V}$  of BCK-algebras, then  $D(\mathcal{K})$  generates the quasivariety  $\mathcal{V}_{\mu}$  of state-morphism BCK-algebras.

*Proof.* By Lemma 5.7(4),  $Q_V(D(\mathcal{K})) = ISD(Q_V(\mathcal{K}))$ . Moreover, by Lemma 5.6(2),  $Q_V(\mathcal{K})_\mu = ISD(Q_V(\mathcal{K}))$ . Hence,  $Q_V(D(\mathcal{K})) = Q_V(\mathcal{K})_\mu$ .

Since the interval [0, 1] generates the class  $\mathcal{MV}$  of MV-algebras as both a variety and a quasivariety, due to the categorical equivalence of MV-algebras and bounded commutative BCK-algebras, [23], by Theorem 5.8 and Corollary 5.4, we have the following corollary.

**Corollary 5.9.** If  $[0,1]_{BCK} = ([0,1], *_{\mathbb{R}}, 0)$  is the bounded commutative BCK-algebra of the real interval [0,1], then  $D([0,1]_{BCK})$  generates both as the variety and as the quasivariety of state-morphism BCK-algebras whose BCK-reduct is a bounded commutative BCK-algebra. In other words,  $V(D([0,1]_{BCK})) = \mathcal{V}_{BCK} = Q_V(D([0,1]_{BCK})).$ 

Finally, we formulate two open problems.

**Problem 1.** Describe some interesting generators of the quasivariety of state BCK-algebras.

We note that we do not know yet any interesting generator for the variety of state MV-algebras.

(2) If X is a subdirectly irreducible BCK-algebra, then the diagonal state-morphism BCK-algebra D(X) is subdirectly irreducible. Similarly, if X is linearly ordered and subdirectly irreducible, then  $(X, \mathrm{Id}_X)$  is subdirectly irreducible. If X is an MV-algebra, the third category of subdirectly irreducible state-morphism MV-algebra  $(X, \mu)$  is the case when X has a unique maximal ideal. Inspired by that, we formulate the second open problem:

**Problem 2.** Characterize (bounded) subdirectly irreducible state-morphism BCK-algebras as it was done in [8, 11, 13].

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