# STATE BCK-ALGEBRAS AND STATE-MORPHISM BCK-ALGEBRAS 

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#### Abstract

In the paper, we define the notion of a state BCK-algebra and a state-morphism BCKalgebra extending the language of BCK-algebras by adding a unary operator which models probabilistic reasoning. We present a relation between state operators and state-morphism operators and measures and states on BCK-algebras, respectively. We study subdirectly irreducible state (morphism) BCKalgebras. We introduce the concept of an adjoint pair in BCK-algebras and show that there is a one-to-one correspondence between adjoint pairs and state-morphism operators. In addition, we show the generators of quasivarieties of state-morphism BCK-algebras.


## 1. Introduction

In 1966, Imai and Iseki [18, 19 introduced two classes of abstract algebras: BCK-algebras and BCIalgebras. These algebras have been intensively studied by many authors. For a comprehensive overview on BCK-algebras, we recommend the book [22. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. MV-algebras were introduced by Chang in [6], in order to show that Lukasiewicz logic is complete with respect to evaluations of propositional variables in the real unit interval $[0,1]$. It is well known that the class of MV-algebras is a proper subclass of the class of BCK-algebras. Therefore, both BCK-algebras and MV-algebras are important for the study of fuzzy logic.

In [24, Mundici introduced a state on MV-algebras as averaging the truth value in Eukasiewicz logic. States constitute measures on their associated MV-algebras which generalize the usual probability measures on Boolean algebras. Kroupa [20 and Panti [26] have recently shown that every state on an MV-algebra can be presented as a usual Lebesgue integral over an appropriate space. Kühr and Mundici [21] studied states via de Finetti's notion of a coherent state with motivation in Dutch book making. Their method is applicable to other structures besides MV-algebras. Measures on pseudo BCK-algebras were studied in [7].

Since MV-algebras with state are not universal algebras, they do not automatically induce an assertional logic. Recently, Flaminio and Montagna in [15, 16] presented an algebraizable logic using a probabilistic approach, and its equivalent algebraic semantics is precisely the variety of state MV-algebras. We recall that a state MV-algebra is an MV-algebra whose language is extended by adding an operator, $\mu$ (also called an internal state), whose properties are inspired by ones of states. Analogues of extremal states are state-morphism operators, introduced in [8, 9], where by definition, a state-morphism is an idempotent endomorphism on an MV-algebra.

[^0]State MV-algebras generalize, for example, Hájek's approach, [17, to fuzzy logic with modality Pr (interpreted as probably) which has the following semantic interpretation: The probability of an event $a$ is presented as the truth value of $\operatorname{Pr}(a)$. On the other hand, if $s$ is a state, then $s(a)$ is interpreted as the average appearance of the many valued event $a$.

In [15, 16], the authors found a relation between states on MV-algebras and state MV-algebras. In [8, 9, some results about characterizations of subdirectly irreducible state-morphism MV-algebras, simple, semisimple, and local state MV-algebras were shown. In [10, the authors study the variety of statemorphism MV-algebras together with a characterization of subdirectly irreducible state MV-algebras, and some interesting characterizations of some varieties of state-morphism MV-algebras were given. These results were generalized in [14, 13, 3].

In the present paper, we concentrate to the study of state BCK-algebras and state-morphism BCKalgebras. We show their basic properties and we characterize quasivarieties of state-morphism BCKalgebras and their generators. We present that the generator of a quasivariety of state-morphism BCKalgebras consists of diagonal state-morphism BCK-algebras. The goal of the present paper is to extend the study of state MV-algebras to state BCK-algebras. We note that in contrast to MV-algebras, in this case we have to deal with quasivarieties because the class of BCK-algebras forms a quasivariety and not a variety.

We note that a state-morphism BCK-algebra is a special case of algebras with a distinguished idempotent endomorphism and such algebras are not new: experts working in various areas (ranging from computer science, Baxter algebras, set theory, category theory and homotopy theory, see e.g. [28, 1, 27]) have considered such structures with a fixed endomorphism.

The paper is organized as follows. Section 2 gathers the elements of BCK-algebras. In Section 3, we introduce the concept of a state BCK-algebra and we study its properties. Then we verify a subdirectly irreducible state BCK-algebra and we characterize this structure. We show that if $X$ is a bounded commutative BCK-algebra, then $(X, \mu)$ is a state (morphism) MV-algebra if and only if $(X, \mu)$ is a state (morphism) BCK-algebra such that $\mu(1)=1$. In Section 4, we study state-morphism BCK-algebras and state ideals. Some relations between congruence relations on state-morphism BCK-algebras and state ideals are also obtained. Then we introduce the concept of an adjoint pair in a BCK-algebra and describe a relation between state-morphism operators and adjoint pairs in BCK-algebras. Finally, Section 5 gives results on generators of quasivarieties of state-morphism BCK-algebras, and we present two open problems.

## 2. Preliminaries

In the section, we gather some basic notions relevant to BCK-algebras and MV-algebras which will need in the next sections.

We say that an MV-algebra is an algebra $\left(M, \oplus,{ }^{\prime}, 0\right)$ of type $(2,1,0)$, where $(M, \oplus, 0)$ is a commutative monoid with neutral element 0 and for all $x, y \in M$ :
(i) $x^{\prime \prime}=x$;
(ii) $x \oplus 1=1$, where $1=0^{\prime}$;
(iii) $x \oplus\left(x \oplus y^{\prime}\right)^{\prime}=y \oplus\left(y \oplus x^{\prime}\right)^{\prime}$.

In any MV-algebra $\left(M, \oplus,^{\prime}, 0\right)$, we can define the following further operations:

$$
x \odot y=\left(x^{\prime} \oplus y^{\prime}\right)^{\prime}, \quad x \ominus y=\left(x^{\prime} \oplus y\right)^{\prime}
$$

A state $M V$-algebra is a pair $(M, \sigma)$ such that $\left(M, \oplus,^{\prime}, 0\right)$ is an MV-algebra and $\sigma$ is a unary operation on $M$ satisfying:
(1) $\sigma(1)=1$;
(2) $\sigma\left(x^{\prime}\right)=\sigma(x)^{\prime}$;
(3) $\sigma(x \oplus y)=\sigma(x) \oplus \sigma(y \ominus(x \odot y))$;
(4) $\sigma(\sigma(x) \oplus \sigma(y))=\sigma(x) \oplus \sigma(y)$.

In [8], Di Nola and Dvurečenskij have introduced a state-morphism operator on an MV-algebra $\left(M, \oplus,{ }^{\prime}, 0\right)$ as an MV-homomorphism $\sigma: M \rightarrow M$ such that $\sigma^{2}=\sigma$ and the pair $(M, \sigma)$ is said to be a state-morphism MV-algebra. They have proved that the class of state-morphism MV-algebras is a proper subclass of state MV-algebras.

Definition 2.1. [18, 19 A $B C K$-algebra is an algebra $(X, *, 0)$ of type $(2,0)$ satisfying the following conditions:
(BCK1) $((x * y) *(x * z)) *(z * y)=0$;
(BCK2) $x * 0=x$;
(BCK3) $x * y=0$ and $y * x=0$ imply $y=x$;
(BCK4) $0 * x=0$.
A BCK-algebra $X$ is called non-trivial if $X \neq\{0\}$. If $X$ is a BCK-algebra, then the relation $\leq$ defined by $x \leq y \Leftrightarrow x * y=0, x, y \in X$, is a partial order on $X$. In addition, for all $x, y, z \in X$, the following hold:
(BCK5) $x * x=0$;
(BCK6) $(x * y) * z=(x * z) * y$;
(BCK7) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$;
(BCK8) $x *(x *(x * y))=x * y$;
(BCK9) $(x * y) *(x * z) \leq z * y$ and $(y * x) *(z * x) \leq y * z$.
In a BCK-algebra $X$, we define $x * y^{0}=x$ and $x * y^{n}=\left(x * y^{n-1}\right) * y$ for any integer $n \geq 1$ and all $x, y \in X$. A BCK-algebra $(X, *, 0)$ is called bounded if $(X, \leq)$ has the greatest element, where $\leq$ is the above defined partially order relation. Let use denote by 1 the greatest element of $X$ (if it exits). In bounded BCK-algebras, we usually write $N x$ instead of $1 * x$. A BCK-algebra $(X, *, 0)$ is called a commutative BCK-algebra if $x *(x * y)=y *(y * x)$ for all $x, y \in X$. Each commutative BCK-algebra is a lover semilattice and $x \wedge y=x *(x * y)$ for all $x, y \in X$ (see [22]). Let $(X, *, 0)$ and ( $Y, *, 0)$ be two BCK-algebras. A map $f: X \rightarrow Y$ is called a homomorphism if $f(a * b)=f(a) * f(b)$ for all $a, b \in X$. Then $f(0)=0$ (since $f(0)=f(0 * 0)=f(0) * f(0)=0)$.

A non-empty subset $I$ of a BCK-algebra $X$ is called an ideal if (1) $0 \in I,(2) y * x \in I$ and $x \in I$ imply that $y \in I$ for all $x, y \in X$. We denote by $\mathrm{I}(X)$, the set of all ideals of $X$. An ideal $I$ of a BCK-algebra $X$ is called proper if $I \neq X$. Suppose that $(X, *, 0)$ and $(Y, *, 0)$ are two BCK-algebras and $f: X \rightarrow Y$ is a homomorphism, then $\operatorname{Ker}(f)=f^{-1}(\{0\})$ is an ideal of $X$. Let use denote by $\langle S\rangle$ the least ideal of $X$ containing $S$, where $S$ is a subset of a BCK-algebra $X$. It is called the ideal generated by $S$. If $S$ is a subset of more BCK-algebras, we will use $\langle S\rangle_{X}$ to specify a concrete BCK-algebra $X$. Instead of $\langle\{a\}\rangle$ we will write rather $\langle a\rangle$, where $a \in X$.
Theorem 2.2. 29] Let $S$ be a subset of a BCK-algebra $(X, *, 0)$. Then
$\langle S\rangle=\left\{x \in X \mid\left(\cdots\left(\left(x * a_{1}\right) * a_{2}\right) * \cdots\right) * a_{n}=0\right.$ for some $n \in \mathbb{N}$ and some $\left.a_{1}, \ldots, a_{n} \in S \cup\{0\}\right\}$.
Moreover, if $I$ is an ideal of $X$, then

$$
\langle I \cup S\rangle=\left\{x \in X \mid\left(\cdots\left(\left(x * a_{1}\right) * a_{2}\right) * \cdots\right) * a_{n} \in I \text { for some } n \in \mathbb{N} \text { and some } a_{1}, \ldots, a_{n} \in S\right\}
$$

Let $I$ be an ideal of a BCK-algebra $(X, *, 0)$. Then the relation $\theta_{I}$, defined by $(x, y) \in \theta_{I}$ if and only if $x * y, y * x \in I$, is a congruence relation on $X$. Let us denote by $x / I$ or $[x]$ the set $\left\{y \in X \mid(x, y) \in \theta_{I}\right\}$ for all $x \in X$. Then $(X / I, *, 0 / I)$ is a BCK-algebra, when $X / I:=\{x / I \mid x \in X\}$ and $x / I * y / I:=(x * y) / I$ for all $x, y \in X$ (see [22]).

An ideal $I$ of a BCK-algebra $(X, *, 0)$ is called commutative if $x * y \in I$ implies that $x *(y *(y * x)) \in I$ for all $x, y \in X$. If $I$ is a commutative ideal, the BCK-algebra $X / I$ is a commutative BCK-algebra [29, Thm 2.5.6].

Theorem 2.3. Let $(X, *, 0)$ be a BCK-algebra and $\theta$ be a congruence relation on $X$. Then $[0]_{\theta}$ is an ideal of $X$. Moreover, if $I=[0]_{\theta}$, then $\theta_{I}=\theta$.

Proof. See [29, Prop 1.5.9, Prop. 1.5.11, Cor. 1.5.12].
Definition 2.4. [2, 29] Let $I$ be a proper ideal of a BCK-algebra $(X, *, 0)$. Then $I$ is called a

- prime ideal if $\langle x\rangle \cap\langle y\rangle \subseteq I$ implies $x \in I$ or $y \in I$ for all $x, y \in X$;
- maximal ideal if $\langle I \cup\{x\}\rangle=X$ for all $x \in X-I$.

We use $\operatorname{Max}(X)$ and $\operatorname{Spec}(X)$ to denote the set of all maximal and prime ideals of $X$, respectively. In each BCK-algebra $X, \operatorname{Max}(X) \subseteq \operatorname{Spec}(X)$ (see [2, Thm 3.7]). A BCK-algebra $(X, *, 0)$ is called simple if it has only two ideals and it is called semisimple if $\operatorname{Rad}(X):=\bigcap \operatorname{Max}(X)=\{0\}$.
Definition 2.5. [29] A BCK-algebra $(X, *, 0)$ is positive implicative if $(x * y) * z=(x * z) *(y * z)$ for all $x, y, z \in X$.

If $X=[0, a)$ or $X=[0, a]$, where $a \in \mathbb{R}$, or $X=[0, \infty)$, we define the binary operation $*_{\mathbb{R}}$ on $X$ by $x *_{\mathbb{R}} y=\max \{0, x-y\}$. Then $\left(X, *_{\mathbb{R}}, 0\right)$ is a commutative BCK-algebra (see [22]).
Definition 2.6. [12] Let $(X, *, 0)$ be a BCK-algebra and $m: X \rightarrow[0, \infty]$ be a map such that, for all $x, y \in[0,1]$,
(i) if $m(x * y)=m(x)-m(y)$, whenever $y \leq x$, then $m$ is said to be a measure;
(ii) if $1 \in X$ and $m$ is a measure with $m(1)=1$, then $m$ is said to be a state;
(iii) if $m(x * y)=\max \{0, m(x)-m(y)\}$, then $m$ is said to be a measure-morphism;
(iv) if $1 \in X$ and $m$ is a measure-morphism with $m(1)=1$, then $m$ is said to be a state-morphism.

## 3. State BCK-algebras

In the section, the concept of left and right state BCK-algebras is defined as a generalization of state MV-algebras, and its properties are studied. We introduce state ideals and congruence relations of right or left state BCK-algebras, and relations between them are obtained. Finally, we characterize subdirectly irreducible state BCK-algebras.

From now on, in this paper, $(X, *, 0)$ or simply $X$ is a BCK-algebra, unless otherwise specified.
Definition 3.1. A map $\mu: X \rightarrow X$ is called a left (right) state operator on $X$ if it satisfies the following conditions:
(S0) $x * y=0$ implies $\mu(x) * \mu(y)=0$;
(S1) $\mu(x * y)=\mu(x) * \mu(x *(x * y)) \quad(\mu(x * y)=\mu(x) * \mu(y *(y * x)))$;
(S2) $\mu(\mu(x) * \mu(y))=\mu(x) * \mu(y)$.
A left (right) state BCK-algebra is a pair $(X, \mu)$, where $X$ is a BCK-algebra and $\mu$ is a left (right) state operator on $X$.

Clearly, if $X$ is a commutative BCK-algebra, then $\mu$ is a right state operator on $X$ if and only if it is a left state operator. In the next proposition, we describe the basic properties of left (right) state operators.

Proposition 3.2. Let $(X, \mu)$ be a left (right) state BCK-algebra. Then, for any $x, y, x_{1}, \ldots, x_{n} \in X$,
(i) $\mu(0)=0$ and $\mu(\mu(x))=\mu(x)$.
(ii) $\mu(x) * \mu(y) \leq \mu(x * y)$. More generally,

$$
\left(\cdots\left(\left(\mu(x) * \mu\left(x_{1}\right)\right) * \mu\left(x_{2}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \leq \mu\left(\left(\cdots\left(\left(x * x_{1}\right) * x_{2}\right) * \cdots\right) * x_{n}\right)
$$

(iii) $\operatorname{Ker}(\mu):=\mu^{-1}(\{0\})$ is an ideal of $X$.
(iv) $\mu(X):=\{\mu(x) \mid x \in X\}$ is a subalgebra of $X$.
(v) $\operatorname{Ker}(\mu) \cap \operatorname{Im}(\mu)=\{0\}$.

Proof. We prove this theorem only for a left state BCK-algebra. The proof for a right state BCK-algebra is similar.
(i) By (BCK4) and (BCK8), we have $\mu(0)=\mu(0 * 0)=\mu(0) * \mu(0 *(0 * 0))=\mu(0) * \mu(0)=0$. Moreover, by (S2) and (BCK2), we have $\mu(\mu(x))=\mu(\mu(x) * 0)=\mu(\mu(x) * \mu(0))=\mu(x) * \mu(0)=\mu(x)$.
(ii) Let $x, y \in X$. Since $x *(x * y) \leq y$, then $\mu(x *(x * y)) \leq \mu(y)$, and so by (BCK7), we get that $\mu(x) * \mu(y) \leq \mu(x) * \mu(x *(x * y))=\mu(x * y)$. The proof of the second part follows from (BCK7).
(iii) By (i), $0 \in \operatorname{Ker}(\mu)$. Let $y * x, x \in \operatorname{Ker}(\mu)$, where $x, y \in X$. Then $\mu(x)=\mu(y * x)=0$. It follows from (ii) that $\mu(y)=\mu(y) * 0=\mu(y) * \mu(x) \leq \mu(y * x)=0$, hence $y \in \operatorname{Ker}(\mu)$. Thus, $\operatorname{Ker}(\mu)$ is an ideal of $X$.
(iv) By (i), $0 \in \mu(X)$. Let $a, b \in X$. Then by (S2), $\mu(\mu(a) * \mu(a))=\mu(a) * \mu(b)$ and so $\mu(a) * \mu(b) \in \mu(X)$. Therefore, $\mu(X)$ is a subalgebra of $X$.
(v) It is evident.

In Theorem 3.3, we attempt to find a relation between measures and states on BCK-algebras and state BCK-algebras.

Theorem 3.3. Let $a \in[0,1], X=\left([0, a), *_{\mathbb{R}}, 0\right)$ and $(X, \mu)$ be a left state BCK-algebra. Then $\mu: X \rightarrow$ $[0,1]$ is a measure.

In addition, if $X=\left([0,1], *_{\mathbb{R}}, 0\right)$ and $(X, \mu)$ is a left state BCK-algebra such that $\mu(1)=1$, then $\mu: X \rightarrow[0,1]$ is a state-morphism.
Proof. Let $x, y \in X$ such that $y \leq x$. For simplicity, we will write $*=*_{\mathbb{R}}$. Then $\mu(x * y)=\mu(x) * \mu(x *$ $(x * y))$. Since $X=\left([0, a), *_{\mathbb{R}}, 0\right)$ is a commutative BCK-algebra, then $x *(x * y)=y *(y * x)=y * 0=y$ and so $\mu(x * y)=\mu(x) * \mu(x *(x * y))=\mu(x) * \mu(y)$. Therefore, $\mu: X \rightarrow[0,1]$ is a measure.

Now, assume that $X=\left([0,1], *_{\mathbb{R}}, 0\right)$ and $(X, \mu)$ is a left state BCK-algebra. Let $x, y \in X$. Then $\mu(x * y)=\mu(x) * \mu(x *(x * y))$. Since $X$ is linearly ordered, we have two cases. If $x \leq y$, then $\mu(x * y)=\mu(0)=0$ and by Proposition 3.2(iv), $\mu(x) * \mu(y)=0$ and so $\mu(x * y)=\mu(x) * \mu(y)$. If $y \leq x$, then $x *(x * y)=y$ (since $\left([0,1], *_{\mathbb{R}}, 0\right)$ is a commutative BCK-algebra) and so $\mu(x * y)=\mu(x) * \mu(y)$. Therefore, $\mu: X \rightarrow[0,1]$ is a state-morphism.

Proposition 3.4. Let $(X, \mu)$ be a right state BCK-algebra. Then
(i) $y \leq x$ implies $\mu(x * y)=\mu(x) * \mu(y)$ for all $x, y \in X$.
(ii) $\mu^{-1}(\{0\})$ is a commutative ideal of $X$. Moreover, the map $\bar{\mu}: X / \operatorname{Ker}(\mu) \rightarrow X / \operatorname{Ker}(\mu)$ defined by $\bar{\mu}(x / \operatorname{Ker}(\mu))=\mu(x) / \operatorname{Ker}(\mu)$ is both a right and left state operator on $X / \operatorname{Ker}(\mu)$.
(iii) $(X, \mu)$ is a left state BCK-algebra.

Proof. (i) Let $x, y \in X$ such that $y \leq x$. Then $\mu(x * y)=\mu(x) * \mu(y *(y * x))=\mu(x) * \mu(y * 0)=\mu(x) * \mu(y)$.
(ii) By Proposition 3.3(i), $0 \in \mu^{-1}(\{0\})$. Let $x, y * x \in \mu^{-1}(\{0\})$. Then $\mu(x)=\mu(y * x)=0$ and so $\mu(y) * \mu(x *(x * y))=0$. Since $\mu(x *(x * y)) \leq \mu(x)=0$, then $\mu(y)=0$. Hence, $\mu^{-1}(\{0\})$ is an ideal of $X$. Now, let $x * y \in \mu^{-1}(\{0\})$. Since $y *(y * x) \leq x$, by (i), we have

$$
0=\mu(x * y)=\mu(x) * \mu(y *(y * x))=\mu(x *(y *(y * x)))
$$

which concludes that $x *(y *(y * x)) \in \mu^{-1}(\{0\})$. Thus, $\mu^{-1}(\{0\})$ is a commutative ideal of $X$.
It is easy to show that $\bar{\mu}$, defined by $\bar{\mu}(x / \operatorname{Ker}(\mu)):=\mu(x) / \operatorname{Ker}(\mu),(x \in X)$, is a right state operator on $X / \operatorname{Ker}(\mu)$. In fact, if $x / \operatorname{Ker}(\mu)=y / \operatorname{Ker}(\mu)$, then $x * y, y * x \in \operatorname{Ker}(\mu)$ and so $\mu(x * y)=\mu(y * x)=0$. Hence by Proposition 3.2 (ii), $\mu(x) * \mu(y)=\mu(y) * \mu(x)=0$ and so $\mu(x)=\mu(y)$. Thus, $\bar{\mu}$ is well defined. Since $\operatorname{Ker}(\mu)$ is a commutative ideal of $X$, then $X / \operatorname{Ker}(\mu)$ is a commutative BCK-algebra, hence $\bar{\mu}$ is also a left state operator on $X / \operatorname{Ker}(\mu)$.
(iii) Let $x, y \in X$. By (ii), $\operatorname{Ker}(\mu)$ is a commutative ideal of $X$ and so by [29, Thm 2.5.6], $X / \operatorname{Ker}(\mu)$ is a commutative BCK-algebra. Hence, $(x *(x * y)) / \operatorname{Ker}(\mu)=(y *(y * x)) / \operatorname{Ker}(\mu)$. Similarly to the proof of (ii), we obtain that $\mu(x *(x * y))=\mu(y *(y * x))$ and so $\mu(x) * \mu(x *(x * y))=\mu(x) * \mu(y *(y * x))=\mu(x * y)$. Therefore, $(X, \mu)$ is a left state BCK-algebra.

Corollary 3.5. Let $\mu: X \rightarrow X$ be a map. Then $(X, \mu)$ is a right state BCK-algebra if and only if $(X, \mu)$ is a left state $B C K$-algebra and $\operatorname{Ker}(\mu)$ is a commutative ideal of $X$.
Proof. Suppose that $(X, \mu)$ is a right state BCK-algebra. Then by Proposition 3.4 $(X, \mu)$ is a left state and $\operatorname{Ker}(\mu)$ is a commutative ideal. Conversely, let $(X, \mu)$ be a left state BCK-algebra and let $\operatorname{Ker}(\mu)$
be a commutative ideal of $X$. Then for all $x, y \in X,(x *(x * y)) / \operatorname{Ker}(\mu)=(y *(y * x)) / \operatorname{Ker}(\mu)$, and similar to the proof of Proposition 3.4(ii), we have $\mu(x *(x * y))=\mu(y *(y * x))$, hence $\mu$ is a right state operator.

By Proposition 3.4(iii), every right state BCK-algebra is a left state BCK-algebra. In the following example, we show that the converse statement is not true, in general. We present a left state operator $\mu$ on a BCK-algebra $X$ which is not a right state operator because $\operatorname{Ker}(\mu)$ is not a commutative ideal of $X$.

Example 3.6. Let $X=\{0,1,2,3\}$. Define a binary operation $*$ by the following table:

| $*$ | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Then $(X, *, 0)$ is a positive implicative BCK-algebra $(P) B_{4-1-4}$ from [22] which is a chain $(0 \leq 1 \leq 2 \leq 3)$. Let $\mu: X \rightarrow X$ be defined by $\mu(0)=\mu(1)=0$ and $\mu(2)=\mu(3)=2$. We claim that $\mu$ is a left state operator on $X$. Clearly, it is a well defined and order preserving map. Let $x, y \in X$.
(1) If $x \leq y$, then we have $\mu(x * y)=\mu(0)=0$ and $\mu(x) * \mu(x *(x * y))=\mu(x) * \mu(x)=0$.
(2) If $y<x$, then by definition of $*, \mu(x * y)=\mu(x)$. Also, $\mu(x) * \mu(x *(x * y))=\mu(x) * \mu(x * x)=$ $\mu(x) * \mu(0)=\mu(x)$.
(3) It can be easily shown that $\mu(x) * \mu(y)=\mu(\mu(x) * \mu(y))$.

From (1)-(3), we conclude that $\mu$ is a left state operator on $X$. But $\operatorname{Ker}(\mu)$ is not a commutative ideal of $X$ because $2 * 3 \in \operatorname{Ker}(\mu)$, but $2 *(3 *(3 * 2))=2 *(3 * 3)=2 * 0=2 \notin \operatorname{Ker}(\mu)$. Hence, $\mu$ is not a right state operator on $X$.

Let $X$ be a set, we denote by $\operatorname{Id}_{X}: X \rightarrow X$ the identity on $X$. It also provides an example of a left state operator which is not necessarily a right state operator.

In each BCK-algebra $X, \operatorname{Id}_{X}$ is a left state operator. In fact, $\operatorname{Id}_{X}(x) * \operatorname{Id}_{X}(x *(x * y))=x *(x *(x * y))=$ $x * y$. On the other hand, $\operatorname{Id}_{X}$ is a right state operator iff $X$ is a commutative BCK-algebra. So it can be easily obtained that, $X$ is a commutative BCK-algebra if and only if each left state operator on $X$ is a right state operator.

By Proposition 3.4 each right state BCK-algebra is a left state BCK-algebra. So in the remainder of this paper, we will consider only left BCK-algebras. Moreover, we write simply a state BCK-algebra instead of a left state BCK-algebra.

Definition 3.7. Let $(X, \mu)$ be a state BCK-algebra. An ideal $I$ of a BCK-algebra $X$ is called a state ideal if $\mu(I) \subseteq I$. If $T$ is a subset of $X$, then $\langle T\rangle_{s}$ is the least state ideal of $X$ containing $T$. A state ideal $I$ is said to be a maximal state ideal if $\langle I \cup\{x\}\rangle_{s}=X$ for each $x \in X-I$. We denote by $\operatorname{MaxS}(X, \mu)$ the set of all maximal state ideals of $(X, \mu)$.

Proposition 3.8. Let $I$ be a state ideal of a state BCK-algebra $(X, \mu)$ and $a \in X$. Then

$$
\langle I \cup\{a\}\rangle_{s}=\left\{x \in X \mid\left(x * a^{n}\right) * \mu(a)^{m} \in I \text { for some } m, n \in \mathbb{N}\right\} .
$$

Proof. Set $A=\left\{x \in X \mid\left(x * a^{n}\right) * \mu(a)^{m} \in I\right.$ for some $\left.m, n \in \mathbb{N}\right\}$. Clearly, $I \cup\{a\} \subseteq A$. Moreover, if $J$ is a state ideal of $(X, \mu)$ containing $I$ and $a$, then by Theorem [2.2, $A \subseteq J$. It suffices to show that $A$ is a state ideal. Let $x, y * x \in A$. Then there are $m, n, s, t \in \mathbb{N}$ such that $\left(x * a^{n}\right) * \mu(a)^{m} \in I$ and

$$
\begin{aligned}
\left((y * x) * a^{s}\right) * \mu(a)^{t} \in I . & \\
\left(\left(\left(y * a^{n+s}\right) * \mu(a)^{m+t}\right)\right. & \left.*\left(\left(x * a^{n}\right) * \mu(a)^{m}\right)\right) *\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right) \\
& \leq\left(\left(\left(y * a^{n+s}\right) * \mu(a)^{t}\right) *\left(x * a^{n}\right)\right) *\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right), \text { by (BCK9) } \\
& =\left(\left(\left(y * a^{n+s}\right) *\left(x * a^{n}\right)\right) * \mu(a)^{t}\right) *\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right), \text { by (BCK6) } \\
& \leq\left(\left(\left(y * a^{s}\right) * x\right) * \mu(a)^{t}\right) *\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right), \text { by (BCK9) } \\
& =\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right) *\left(\left((y * x) * a^{s}\right) * \mu(a)^{t}\right), \text { by (BCK6) } \\
& =0 \in I .
\end{aligned}
$$

Since $\left(x * a^{n}\right) * \mu(a)^{m},\left((y * x) * a^{s}\right) * \mu(a)^{t} \in I$ and $I$ is an ideal of $X$, then we get $\left(y * a^{n+s}\right) * \mu(a)^{m+t} \in I$ and so $y \in A$. Hence, $A$ is an ideal. Now, let $x$ be an arbitrary element of $A$. Then there exist $m, n \in \mathbb{N}$ such that $\left(x * a^{n}\right) * \mu(a)^{m} \in I$. Since $I$ is a state ideal, then $\mu\left(\left(x * a^{n}\right) * \mu(a)^{m}\right) \in I$ and so by Proposition 3.2(ii), $\mu(x) * \mu(a)^{n+m}=\left(\mu(x) * \mu(a)^{n}\right) * \mu(a)^{m}=\left(\mu(x) * \mu(a)^{n}\right) * \mu(\mu(a))^{m} \in I$. Thus, $\mu(x) \in A$. Therefore, $A$ is a state ideal of $(X, \mu)$.

Note that, if $(X, \mu)$ is a state BCK-algebra, then $\{0\}$ and $X$ are state ideals of $(X, \mu)$ and so by Proposition 3.8, $J=\left\{x \in X \mid\left(x * a^{n}\right) * \mu(a)^{m}=0\right.$ for some $\left.m, n \in \mathbb{N}\right\}$ is a state ideal of $X$ for any $a \in X$. Similarly, we can construct other state ideals of $(X, \mu)$.
Corollary 3.9. A state ideal I of a state BCK-algebra $(X, \mu)$ is a maximal state ideal if and only if $\left\{x \in X \mid\left(x * a^{n}\right) * \mu(a)^{m} \in I\right.$ for some $\left.m, n \in \mathbb{N}\right\}=X$ for all $a \in X-I$.

Proof. The proof is a straightforward corollary of Proposition 3.8
By [2, Thm 3.7], we know that if $M$ is a maximal ideal of $X$, then $I \cap J \subseteq M$ implies that $I \subseteq M$ or $J \subseteq M$ for all $I, J \in \mathrm{I}(X)$. In the next theorem, we show that if $M$ is a maximal state ideal of a state BCK-algebra $(X, \mu)$, then $I \cap J \subseteq M$ implies that $I \subseteq M$ or $J \subseteq M$ for all state ideals $I$ and $J$ of $(X, \mu)$.

Theorem 3.10. Let $M$ be a maximal state ideal of a state $B C K$-algebra $(X, \mu)$. For for all state ideals $I$ and $J$ of $(X, \mu)$, we have $I \cap J \subseteq M$ implies that $I \subseteq M$ or $J \subseteq M$.

Proof. Let $I$ and $J$ be two state ideals of $(X, \mu)$ such that $I \cap J \subseteq M$. Suppose that there are $x \in$ $I-M$ and $y \in J-M$. Then by Corollary 3.9, $X=\langle M \cup\{x\}\rangle_{s}=\langle M \cup\{y\}\rangle_{s}$. On the other hand, if $a \in\langle M \cup\{x\}\rangle_{s} \cap\langle M \cup\{y\}\rangle_{s}$, then by Proposition 3.8, there exist $m, n, s, t \in \mathbb{N}$ such that $\left(a * x^{n}\right) * \mu(x)^{m}=m_{1} \in M$ and $\left(a * y^{s}\right) * \mu(y)^{t}=m_{2} \in M$ and so by (BCK2), (BCK4), (BCK6) and Proposition 3.8, we get $\left(a * m_{1}\right) * m_{2} \in I \cap J \subseteq M$ (since $I$ and $J$ are state ideals, $x \in I, y \in J$ and $\left.\left(\left(\left(a * m_{1}\right) * m_{2}\right) * x^{n}\right) * \mu(x)^{m}=0 * m_{2}=0,\left(\left(\left(a * m_{1}\right) * m_{2}\right) * y^{s}\right) * \mu(y)^{t}=0 * m_{1}=0\right)$. Since $m_{1}, m_{2} \in M$ and $M$ is an ideal of $X$, then we have $a \in M$ and so $X=\langle M \cup\{x\}\rangle_{s} \cap\langle M \cup\{y\}\rangle_{s} \subseteq M$, which is a contradiction. Therefore, $I \subseteq M$ or $J \subseteq M$.

In Theorem 3.11, we show a one-to-one relationship between congruence relations of a state BCKalgebra $(X, \mu)$ and state ideals of $(X, \mu)$. We denote by $\operatorname{SI}(X)$ and $\operatorname{Con}(X, \mu)$ the set of state ideals and the set of congruences, respectively, on a state BCK-algebra $(X, \mu)$.

Theorem 3.11. Let $(X, \mu)$ be a state BCK-algebra.
(i) If $\theta$ is a congruence relation of $(X, \mu)$, then $[0]_{\theta}=\{x \in X \mid(x, 0) \in \theta\}$ is a state ideal of $(X, \mu)$.
(ii) If $I$ is a state ideal of $(X, \mu)$, then $\theta_{I}=\{(x, y) \in X \times X \mid x * y, y * x \in I\}$ is a congruence relation on $(X, \mu)$.
(iii) There is a bijection between the set of all congruence relations of $(X, \mu), \operatorname{Con}(X, \mu)$, and the set $\mathrm{SI}(X, \mu)$ of all state ideals of $(X, \mu)$.

Proof. (i) Let $\theta$ be a congruence relation of $(X, \mu)$. Then by Theorem[2.3, $[0]_{\theta}$ is an ideal of $X$. It suffices to show that $[0]_{\theta}$ is a state ideal. Let $x \in[0]_{\theta}$. Then $(x, 0) \in \theta$. Since $\theta$ is a congruence relation of $(X, \mu)$, then $(\mu(x), \mu(0)) \in \theta$ and so by Proposition [3.2(i), $(\mu(x), 0) \in \theta$. Hence, $\mu(x) \in[0]_{\theta}$. That is, $[0]_{\theta}$ is a state ideal.
(ii) Let $I$ be a state ideal of $X$. Then $\theta_{I}$ is a congruence relation on a BCK-algebra $X$. Let $(x, y) \in \theta_{I}$. Then $x * y, y * x \in I$ and so by Proposition 3.2(ii), $\mu(x) * \mu(y) \leq \mu(x * y) \in I$. Thus, $\mu(x) * \mu(y) \in I$. In a similar way, $\mu(y) * \mu(x) \in I$, hence $(\mu(x), \mu(y)) \in \theta_{I}$, so $\theta_{I}$ is a congruence relation of $(X, \mu)$.
(iii) We define a map $f: \operatorname{SI}(X, \mu) \rightarrow \operatorname{Con}(X, \mu)$ by $f(I)=\theta_{I}$. Then it can be easily shown that $f$ is a bijection map and its inverse is the map $g: \operatorname{Con}(X, \mu) \rightarrow \operatorname{SI}(X, \mu)$, which is defined by $g(\theta)=[0]_{\theta}$.

Definition 3.12. [4] An algebra $A$ of type $F$ is a subdirect product of an indexed family $\left\{A_{i}\right\}_{i \in I}$ of algebras of type $F$ if

- $A$ is a subalgebra of $\Pi_{i \in I} A_{i}$,
- $\pi_{i}(A)=A_{i}$ for any $i \in I$, where $\pi_{i}: \Pi_{i \in I} A_{i} \rightarrow A_{i}$ is a natural projection map.

A one-to-one homomorphism $\alpha: A \rightarrow \Pi_{i \in I} A_{i}$ is called a subdirect embedding if $\alpha(A)$ is a subdirect product of the family $\left\{A_{i}\right\}_{i \in I}$. An algebra $A$ of type $F$ is called subdirectly irreducible if, for every subdirect embedding $\alpha: A \rightarrow \Pi_{i \in I} A_{i}$, there exists $i \in I$ such that $\pi_{i} \circ \alpha: A \rightarrow A_{i}$ is an isomorphism.

Remark 3.13. If $I$ and $J$ are two ideals of $X$ such that $I \subseteq J$, then clearly, $\theta_{I} \subseteq \theta_{J}$. Let $(X, \mu)$ be subdirectly irreducible. Then by [4, Thm II.8.4], the set $\operatorname{Con}(X, \mu)-\Delta$ has a least element, where $\Delta=\{(x, x) \mid x \in X\}$ and $\nabla=X \times X$. Suppose that $\theta$ is the least element of $\operatorname{Con}(X, \mu)-\Delta$. Then by Theorem 3.11 there exists a state ideal of $(X, \mu)$ such that $\theta=\theta_{I}$ (so $I$ is a non-zero ideal of $X$ ). It follows that $I$ is the least non-zero state ideal of $(X, \mu)$. By Theorem 3.11 and [4. Thm II.8.4], we conclude that $(X, \mu)$ is subdirectly irreducible if and only if $\operatorname{SI}(X, \mu)-\{0\}$ has the least element.

In Theorem 3.14 and Theorem 3.15 we present characterizations of subdirectly irreducible state BCKalgebras. First, we show that if ( $X, \mu$ ) is subdirectly irreducible, then the conditions (i) or (ii) of Theorem 3.14 hold. Then we prove that if ( $X, \mu$ ) satisfies the condition (i) or (ii) in Theorem 3.14 then ( $X, \mu$ ) must be subdirectly irreducible. We note that in the next theorem, we take an element $a$ in the subalgebra $\mu(X)$ of a BCK-algebra $X$, therefore, $\langle a\rangle_{X}$ will denote the ideal of $X$ generated by the element $a$.
Theorem 3.14. Let $(X, \mu)$ be a subdirectly irreducible state BCK-algebra.
(i) If $\operatorname{Ker}(\mu)=\{0\}$, then $\mu(X)$ is a subdirectly irreducible subalgebra of $X$.
(ii) If $\operatorname{Ker}(\mu) \neq\{0\}$, then $\operatorname{Ker}(\mu)$ is a subdirectly irreducible subalgebra of $X$ and $\operatorname{Ker}(\mu) \cap\langle a\rangle_{X} \neq\{0\}$ for each non-zero element a of $\mu(X)$.
Proof. (i) Let $(X, \mu)$ be subdirectly irreducible and $\operatorname{Ker}(\mu)=\{0\}$. By Remark 3.13] the set of all non-zero state ideals of ( $X, \mu$ ) has the least element, $I$ say. If $I \cap \mu(X)=\{0\}$, then by $\mu(I) \subseteq I \cap \mu(X)$ (since $I$ is a state ideal), we conclude that $\mu(x)=0$ for all $x \in I$. Thus, $I \subseteq \operatorname{Ker}(\mu)=\{0\}$, which is a contradiction. So, $I \cap \mu(X) \neq\{0\}$. Now, we show that $I \cap \mu(X)$ is the least non-zero ideal of $\mu(X)$. Suppose that $J$ is an ideal of $\mu(X)$.
(1) Let $\langle J\rangle_{X}$ be the ideal of $X$ generated by $J$, and choose an arbitrary element $x \in\langle J\rangle_{X}$. Then by Theorem 2.2] there exist $b_{1}, \ldots, b_{n} \in J$ such that $\left(\cdots\left(\left(x * b_{1}\right) * b_{2}\right) * \cdots\right) * b_{n}=0$ and so by Proposition 3.2(i) and (ii), we get

$$
\left(\cdots\left(\left(\mu(x) * \mu\left(b_{1}\right)\right) * \mu\left(b_{2}\right)\right) * \cdots\right) * \mu\left(b_{n}\right) \leq \mu\left(\left(\cdots\left(\left(x * b_{1}\right) * b_{2}\right) * \cdots\right) * b_{n}\right)=0 .
$$

Since $\mu^{2}=\mu$ and $b_{1}, \ldots, b_{n} \in J \subseteq \mu(X)$, we get $\left(\cdots\left(\left(\mu(x) * b_{1}\right) * b_{2}\right) * \cdots\right) * b_{n}=0$, hence $\mu(x) \in J$. Thus, $\langle J\rangle_{X}$ is a state ideal of $(X, \mu)$.
(2) Clearly, $J=\langle J\rangle_{X} \cap \mu(X)$.

By (1), we get that $I \subseteq\langle J\rangle_{X}$ and so by (2), $I \cap \mu(X) \subseteq\langle J\rangle_{X} \cap \mu(X)=J$. Hence, $I \cap \mu(X)$ is the least non-zero ideal of $\mu(X)$. Therefore, by [4. Thm II.8.4], we conclude that $\mu(X)$ is a subdirectly irreducible subalgebra of $X$.
(ii) Let $\mu(X) \neq\{0\}$. Again, let $I$ be the least non-zero state ideal of the subdirectly irreducible state BCK-algebra $(X, \mu)$. Since $X$ is a BCK-algebra, then every ideal of $X$, in particular $\operatorname{Ker}(\mu)$, is a subalgebra of $X$. Clearly, $\operatorname{Ker}(\mu)$ is a state ideal of $(X, \mu)$ and so $I \subseteq \operatorname{Ker}(\mu)$. We show that $I$ is the least non-zero ideal of $\operatorname{Ker}(\mu)$. Let $J$ be a non-zero ideal of $\operatorname{Ker}(\mu)$. Then $\mu(J) \subseteq \mu(\operatorname{Ker}(\mu))=\{0\} \subseteq J$. For any $x, y \in X$, if $y * x, x \in J$, then by Proposition 3.2(ii), $0=\mu(y * x) \geq \mu(y) * \mu(x)=\mu(y) * 0=\mu(y)$.

Thus, $y \in \operatorname{Ker}(\mu)$, so $y \in J$ (since $J$ is an ideal of $\operatorname{Ker}(\mu)$ ). It follows that $J$ is a state ideal of $X$ and so $I \subseteq J$. Hence by [4, Thm II.8.4], $\operatorname{Ker}(\mu)$ is subdirectly irreducible.

Now, let $a$ be a non-zero element of $\mu(X)$ and let $\langle a\rangle_{X}$ be the ideal generated by $a$ in $X$. Then $a=\mu(a)$. Take an arbitrary element $u \in\langle a\rangle_{X}$. By Theorem 2.2 there exists $n \in \mathbb{N}$ such that $0=u * a^{n}$, and by by Proposition 3.2(ii), $0=\mu(0)=\mu\left(u * a^{n}\right)=\mu(x) *(\mu(a))^{n}=\mu(u) * a^{n}$. Thus, $\mu(u) \in\langle a\rangle_{X}$ and $\mu\left(\langle a\rangle_{X}\right) \subseteq\langle a\rangle_{X}$. This implies, $\langle a\rangle_{X}$ is a non-zero state interval of $(X, \mu)$ and, consequently, $I \subseteq\langle a\rangle_{X}$. Since also $I \subseteq \operatorname{Ker}(\mu)$, we have $\{0\} \neq I \subseteq \operatorname{Ker}(\mu) \cap\langle a\rangle_{X}$.

Theorem 3.15. Let $(X, \mu)$ be a state BCK-algebra. If it satisfies the condition (i) or (ii) in Theorem 3.14, then $(X, \mu)$ is subdirectly irreducible.

Proof. First, we assume that $\operatorname{Ker}(\mu)=\{0\}$ and $\mu(X)$ is a subdirectly irreducible subalgebra of $X$. Since $\mu(X)$ is subdirectly irreducible, then by [4, Thm II.8.4], $\bigcap(\mathrm{I}(\mu(X))-\{0\})$ is a non-zero ideal of $\mu(X)$. From $\mathrm{I}(\mu(X))=\{I \cap \mu(X) \mid I \in \mathrm{I}(X)\}$, it follows that $\bigcap(\{I \cap \mu(X) \mid I \in \mathrm{I}(X)\}-\{0\})$ is non-zero, so $\bigcap(\mathrm{I}(X)-\{0\}) \neq\{0\}$. Hence, the intersection of all non-zero state ideals of $(X, \mu)$ is a non-zero ideal of $X$ (clearly it is a state ideal), whence by Remark 3.13, $(X, \mu)$ is subdirectly irreducible.

Now, let $\operatorname{Ker}(\mu) \neq\{0\}$ and let $\operatorname{Ker}(\mu)$ be a subdirectly irreducible subalgebra of $X$. Let $I$ be the least non-zero ideal of $\operatorname{Ker}(\mu)$. Clearly, $I$ is a state ideal (since $\mu(I)=\{0\}$ ). We claim, for any non-zero state ideal $H$ of $(X, \mu)$, we have $I \subseteq H$. Suppose that $H$ is a non-zero state ideal of $(X, \mu)$. Then $\mu(H) \subseteq H$. If $\mu(H)=\{0\}$, then $H \subseteq \operatorname{Ker}(\mu)$ and so $I \subseteq H$. Otherwise, there exists $a \in \mu(H)-\{0\}$. It follows that $\{0\} \neq \operatorname{Ker}(\mu) \cap\langle a\rangle_{X} \subseteq \operatorname{Ker}(\mu) \cap H$ and so $I \subseteq H \cap \operatorname{Ker}(\mu) \subseteq H$. Thus, $I$ is the least non-zero state ideal of $(X, \mu)$. Therefore, $(X, \mu)$ is subdirectly irreducible.

In the final theorem of this section, we find a relation between state operators in BCK-algebras and MV-algebras. It is well known, if $(X, *, 0)$ is a bounded commutative BCK-algebra, then $\left(X, \oplus,{ }^{\prime}, 0\right)$ is an MV-algebra, where $x \oplus y=N(N x * y)$ and $x^{\prime}=N x$ for all $x, y \in X$ (see [23]). Note that in each bounded BCK-algebra $X$, we have $N(N x)=x$.

Theorem 3.16. Let $(X, *, 0)$ be a bounded commutative $B C K$-algebra and $\mu$ be a left state $B C K$ operator on $X$ such that $\mu(1)=1$. Then $(X, \mu)$ is a state $M V$-algebra. The converse is also true.

Proof. Let $x, y \in X$. Then $\mu\left(x^{\prime}\right)=\mu(1 * x)=\mu(1) * \mu(x *(x * 1))=1 * \mu(x)=\mu(x)^{\prime}$. Then

$$
\begin{aligned}
\mu(x) \oplus \mu(y \ominus(x \odot y)) & =\mu(x) \oplus \mu\left(\left(y^{\prime} \oplus(x \odot y)\right)^{\prime}\right) \\
& =\mu(x) \oplus \mu(y *(x \odot y)) \\
& =\mu(x) \oplus \mu(y *(y \odot x)) \\
& =\mu(x) \oplus \mu\left(y *\left(y^{\prime} \oplus x^{\prime}\right)^{\prime}\right) \\
& =\mu(x) \oplus \mu\left(y *\left(y * x^{\prime}\right)\right) \\
& =\left(\mu(x)^{\prime} * \mu(y *(y * N x))\right)^{\prime} \\
& =(\mu(N x) * \mu(y *(y * N x)))^{\prime} \\
& =\mu(N x * y)^{\prime}, \text { since } X \text { is commutative and } \mu \text { is a left state operator } \\
& =\mu(N(N x * y)) \\
& =\mu(x \oplus y)
\end{aligned}
$$

so that, $(X, \mu)$ is a state MV-algebra. Conversely, consider the MV-algebra $\left(X, \oplus,{ }^{\prime}, 0\right)$. If $(X, \sigma)$ is a state MV-algebra, then we can easily show that $\sigma: X \rightarrow X$ is a left state operator on a BCK-algebra $(X, *, 0)$, where $x * y:=x \odot y^{\prime}, x, y \in X$. In fact, it follows from the following identity on $X$ :

$$
\left(y^{\prime} \oplus\left(x^{\prime} \odot y\right)\right)^{\prime}=y *\left(x^{\prime} \odot y\right)=y *\left(y^{\prime \prime} \odot x^{\prime}\right)=y *\left(y^{\prime} \oplus x\right)^{\prime}=y *(y * x)
$$

## 4. State-morphism BCK-algebras

In the section, we introduce and study state-morphism BCK-algebras which is an important subfamily of the family of state BCK-algebras.

Definition 4.1. Let $(X, *, 0)$ be a BCK-algebra. A homomorphism $\mu: X \rightarrow X$ is called a state-morphism operator if $\mu^{2}=\mu$, where $\mu^{2}=\mu \circ \mu$, and the pair $(X, \mu)$ is called a state-morphism BCK-algebra.

By (BCK8), every state-morphism BCK-algebra is a (left) state BCK-algebra. We note that not every state-morphism operator is also a right state operator. For example, $\mathrm{Id}_{X}$ is both a state-morphism operator and a left state operator, but it is a right state operator iff $X$ is a commutative BCK-algebra.

Example 4.2. (i) For each BCK-algebra $X$, the identity map $\operatorname{Id}_{X}: X \rightarrow X$ and the zero operator $O_{X}(x)=0, x \in X$, are state-morphism operators.
(ii) Let $x$ be an element of $X$ such that $a * x=a * x^{2}$ for all $a \in X$. Define $\alpha_{x}: X \rightarrow X$ by $\alpha_{x}(a)=a * x$ for all $a \in X$. First, we show that $\alpha_{x}$ is a homomorphism. By (BCK4), $0 * x=0$ for all $x \in X$. Let $a, b \in X$. Then by (BCK6), we have $(b * x) * b=(b * b) * x=0 * x=0$, so $b * x \leq b$. Using (BCK6) and (BCK7), we obtain that $(a * b) * x=(a * x) * b \leq(a * x) *(b * x)$. On the other hand, by (BCK1) and (BCK6), $(a * x) *(b * x)=\left(a * x^{2}\right) *(b * x) \leq(a * x) * b=(a * b) * x$. Hence, $\alpha_{x}$ is a homomorphism. Therefore,

$$
\alpha_{x}\left(\alpha_{x}(a)\right)=(a * x) * x=a * x=\alpha_{x}(a),
$$

so that, $\alpha_{x}$ is a state-morphism operator on $X$. For example, if $x=0$, then $\alpha_{0}=\operatorname{Id}_{X}$. In particular, if $X$ is a positive implicative BCK-algebra, then by [29, Thm 3.1.1], for all $a, x \in X$, we have $a * x^{2}=a * x$ and so, $\alpha_{x}$ is a state-morphism operator on $X$ for all $x \in X$.
(iii) Every state operator $\mu$ on a linearly ordered commutative BCK-algebra $X$ is a state-morphism operator. Indeed, if $x \leq y$, then $x * y=0$ and by Proposition3.2, we have $0 \leq \mu(x) * \mu(y) \leq \mu(x * y)=\mu(0)$. If $y \leq x$, by the definition of a state operator, we have $\mu(x * y)=\mu(x) * \mu(x *(x * y))=\mu(x) * \mu(y)$. The both cases entail $\mu$ is an endomorphism of the BCK-algebra $X$.
(iv) Every right state operator $\mu$ on a linearly ordered commutative BCK-algebra $X$ is a state-morphism operator. Indeed, by Proposition 3.4(iii), $\mu$ is a left state operator, too. Take $x, y \in X$. Since $X$ is a chain, then $x * y=0$ or $y * x=0$.

If $x * y=0$, then by Proposition 3.2 (ii), $0 \leq \mu(x) * \mu(y) \leq \mu(x * y)=\mu(0)=0$. So, $\mu(x) * \mu(y)=\mu(x * y)$.
If $y * x=0$, then from Proposition 3.4 we have $\mu(x * y)=\mu(x) * \mu(y)$. Consequently, $\mu$ is a homomorphism, and $\mu$ is a state-morphism operator on $X$.

Example 3.6 shows that if $\mu$ is a left state operator on a linearly ordered BCK-algebra, then $\mu$ is not necessarily a state-morphism operator on $X$. Indeed, we have $\mu(3 * 2)=\mu(3)=2 \neq 0=2 * 2=\mu(3) * \mu(2)$. Thus $\mu$ is not a state morphism operator.

As a consequence of Corollary 3.16, we have by [8], that there are also bounded commutative BCKalgebras $X$ having state operators which are not state-morphism operators.

Definition 4.3. Let $(X, \mu)$ be a state-morphism BCK-algebra. An ideal $I$ of a BCK-algebra $X$ is called a state ideal if $\mu(I) \subseteq I$. If $T$ is a subset of $X$, then $\langle T\rangle_{s}$ is the least state ideal of $X$ containing $T$.

It can be easily shown that, if $(X, \mu)$ is a state BCK-algebra, then $\operatorname{Ker}(\mu)$ is a state ideal of $X$. Clearly, the intersection of every arbitrary family of state ideals of $X$ is a state ideal. So,

$$
\langle T\rangle_{s}=\bigcap\{I \mid T \subseteq I, I \text { is a state ideal of }(X, \mu)\}
$$

Proposition 4.4. Let $I$ be an ideal of a state-morphism BCK-algebra $(X, \mu)$. Then

$$
\langle I\rangle_{s}=\left\{a \in X \mid\left(\cdots\left(\left(a * \mu\left(x_{1}\right)\right) * \mu\left(x_{2}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \in I, \quad \exists n \in \mathbb{N}, \quad \exists x_{1}, x_{2}, \ldots, x_{n} \in I\right\}
$$

Proof. Let $J=\langle I\rangle_{s}=\left\{a \in X \mid\left(\cdots\left(\left(a * \mu\left(x_{1}\right)\right) * \mu\left(x_{2}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \in I, \quad \exists n \in \mathbb{N}, \quad \exists x_{1}, x_{2}, \ldots, x_{n} \in\right.$ $I\}$. Then clearly, $I \subseteq J$ (since $0 \in I$ and $\mu(0)=0$ ). First, we show that $J$ is a state ideal of $X$. Let $a, b * a \in J$ for some $a, b \in X$. Then there exist $m, n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m} \in X$ such that
$\left(\cdots\left(\left(a * \mu\left(x_{1}\right)\right) * \mu\left(x_{2}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \in I$ and $\left(\cdots\left(\left((b * a) * \mu\left(y_{1}\right)\right) * \mu\left(y_{2}\right)\right) * \cdots\right) * \mu\left(y_{m}\right)=y \in I$. By (BCK5) and (BCK6), we have

$$
\left(\cdots\left(\left((b * y) * \mu\left(y_{1}\right)\right) * \mu\left(y_{2}\right)\right) * \cdots\right) * \mu\left(y_{m}\right) \leq a
$$

and so by (BCK7),
$\left.\left.\left(\cdots\left(\left((\cdots)\left((b * y) * \mu\left(y_{1}\right)\right) * \mu\left(y_{2}\right)\right) * \cdots\right) * \mu\left(y_{m}\right)\right) * \mu\left(x_{1}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \leq\left(\cdots\left(\left(a * \mu\left(x_{1}\right)\right) * \mu\left(x_{2}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \in I$.
Since $y \in I$ and $I$ is an ideal of $X$, then by (BCK6),

$$
\left.\left(\cdots\left(\left((\cdots)\left(\left(b * \mu\left(y_{1}\right)\right) * \mu\left(y_{2}\right)\right) * \cdots\right) * \mu\left(y_{m}\right)\right) * \mu\left(x_{1}\right)\right) * \cdots\right) * \mu\left(x_{n}\right) \in I
$$

and so $b \in J$. It follows that $J$ is an ideal of $X$. Moreover, if $c \in J$, then there exist $n \in \mathbb{N}$ and $z_{1}, \ldots, z_{n} \in X$ such that $\left(\cdots\left(\left(c * \mu\left(z_{1}\right)\right) * \mu\left(z_{2}\right)\right) * \cdots\right) * \mu\left(z_{n}\right)=z \in I$. Hence, by (BCK5) and (BCK6), we get that $\left(\left(\cdots\left(\left(\mu(c) * \mu\left(z_{1}\right)\right) * \mu\left(z_{2}\right)\right) * \cdots\right) * \mu\left(z_{n}\right)\right) * \mu(z)=\mu(0)=0 \in I$. Also, $z_{1}, \ldots, z_{n}, z \in I$, so by definition of $J, \mu(c) \in J$. Thus, $\mu(J) \subseteq J$ and so $J$ is a state ideal of $X$ containing $I$. Clearly, if $K$ is a state ideal of $X$ containing $I$, then $J \subseteq K$. Therefore, $J$ is the least state ideal of $X$ containing $I$. That is $J=\langle I\rangle_{s}$.

Proposition 4.5. Let $(X, \mu)$ be a state-morphism BCK-algebra. Then the following hold:
(i) $\operatorname{Ker}(\mu)=\{x * \mu(x) \mid x \in X\}=\{\mu(x) * x \mid x \in X\}$.
(ii) $X=\langle\operatorname{Ker}(\mu) \cup \operatorname{Im}(\mu)\rangle$.

Proof. (i) Since $\mu^{2}=\mu$ and $\mu$ is a homomorphism, we have $\{x * \mu(x) \mid x \in X\} \subseteq \operatorname{Ker}(\mu)$. Also, for each $x \in \operatorname{Ker}(\mu), x=x * 0=x * \mu(x) \in\{x * \mu(x) \mid x \in X\}$, so $\operatorname{Ker}(\mu)=\{x * \mu(x) \mid x \in X\}$. In a similar way, we can show that $\operatorname{Ker}(\mu)=\{\mu(x) * x \mid x \in X\}$.
(ii) Let $x \in X$. By (i), $x * \mu(x) \in \operatorname{Ker}(\mu)$. Since $\mu(x) \in \operatorname{Im}(\mu)$, then by Theorem 2.2, we get that $x \in\langle\operatorname{Ker}(\mu) \cup \operatorname{Im}(\mu)\rangle$. Therefore, $X=\langle\operatorname{Ker}(\mu) \cup \operatorname{Im}(\mu)\rangle$.

Let $X$ be a bounded BCK-algebra and $m: X \rightarrow[0,1]$ be a state-morphism. Since $m(1)=1$ and $m$ is an order preserving map, then $m(X) \subseteq[0,1]$. Therefore, $m$ is a homomorphism from the BCKalgebra $X$ to the BCK-algebra $\left([0,1], *_{\mathbb{R}}, 0\right)$. Hence, $X / \operatorname{Ker}(m)$ and $m(X)$ are isomorphic. By [12, Thm 2.9], $\operatorname{Ker}(m)$ is a commutative ideal of $X$ and so $X / \operatorname{Ker}(m)$ is a bounded commutative BCK-algebra. Since $\left([0,1], *_{\mathbb{R}}, 0\right)$ is a simple BCK-algebra and $m(X)$ is a subalgebra of it, then $m(X)$ is simple, so $X / \operatorname{Ker}(m)$ is simple, too. It follows that $\operatorname{Ker}(m)$ is a maximal commutative ideal of $X$. Therefore, $\left(X / \operatorname{Ker}(m), \oplus,^{\prime}, 0 / \operatorname{Ker}(m)\right)$ is an MV-algebra, where $x / \operatorname{Ker}(m) \oplus y / \operatorname{Ker}(m)=N(N x * y) / \operatorname{Ker}(m)$ and $(x / \operatorname{Ker}(m))^{\prime}=N x / \operatorname{Ker}(m)$ for all $x, y \in X$. It can be easily shown that the map $f: X / \operatorname{Ker}(m) \rightarrow[0,1]$ defined by $f(x / \operatorname{Ker}(m))=m(x)$ is an MV-homomorphism and $X / \operatorname{Ker}(m)$ is a simple MV-algebra (since $I$ is a BCK-ideal of $X / \operatorname{Ker}(m)$ if and only if $I$ is an MV-ideal of $X / \operatorname{Ker}(m))$. By [25, Thm 1.1], there exists a unique one-to-one MV-homomorphism $\tau: X / \operatorname{Ker}(m) \rightarrow[0,1]$. Thus, $f=\tau$. By summing up the above results, we get that $m=\tau \circ \pi_{\operatorname{Ker}(m)}$, where $\pi_{\operatorname{Ker}(m)}: X \rightarrow X / \operatorname{Ker}(m)$ is the canonical epimorphism. Conversely, let $X$ be a bounded BCK-algebra such that $X$ has at least one commutative ideal, $I$ say. Then there exists a maximal ideal $M$ of $X$ such that $I \subseteq M$. In fact, $M$ is a maximal element of the set $\{H \mid H$ is an ideal of $X$ containing $I, 1 \notin H\}$. Since $I$ is a commutative ideal and $I \subseteq M$, then by [29, Thm 2.5.2], $M$ is a commutative ideal and so $X / M$ is a bounded commutative simple BCK-algebra. It follows that $\left(X / M, \oplus,^{\prime}, 0\right)$ is a simple MV-algebra. By [25, Thm 1.1], there exists a unique MV-homomorphism, $\tau_{M}:\left(X / M, \oplus,^{\prime}, 0\right) \rightarrow\left([0,1], \oplus,{ }^{\prime}, 0\right)$. Clearly, $\tau_{M}: X / M \rightarrow[0,1]$ is a BCK-homomorphism and so $\tau_{M} \circ \pi_{M}: X \rightarrow[0,1]$ is a state-morphism, where $\pi_{M}: X \rightarrow X / M$ is the canonical epimorphism.

Now, let $X$ be a bounded BCK-algebra and $\mu: X \rightarrow X$ be a state-morphism operator on $X$ such that $\operatorname{Ker}(\mu)$ is a commutative ideal of $X$. Then $X / \operatorname{Ker}(\mu)$ is a bounded commutative BCK-algebra. Thus, $\mu(X)$ is an MV-algebra (since $\mu(X) \cong X / \operatorname{Ker}(\mu)$ ). Suppose that $H$ is a maximal ideal of the MV-algebra $\mu(X)$ and $\pi_{H}: \mu(X) \rightarrow \mu(X) / H$ is the canonical epimorphism. Then $\mu(X) / H$ is a simple MV-algebra and so by [25, Thm 1.1], there is a unique MV-homomorphism $\tau_{H}: \mu(X) / H \rightarrow[0,1]$.

Clearly, $\tau_{H} \circ \pi_{H} \circ \mu: X \rightarrow[0,1]$ is a measure-morphism. Moreover, if $\mu(1)=1$, then $\tau_{H} \circ \pi_{H} \circ \mu$ is a state-morphism.

Remark 4.6. Let $\mu$ be a state-morphism operator on $X$ such that $\operatorname{Ker}(\mu)=\{0\}$. Then for all $x \in X$, $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)=\{0\}$ and so by (BCK3), $\mu(x)=x$. Therefore, $\mu=\operatorname{Id}_{X}$.
Corollary 4.7. If $X$ is a simple BCK-algebra, then $\operatorname{Id}_{X}$ and $O_{X}$ are all state-morphism operators of $X$.
Proof. Let $X$ be a simple BCK-algebra and $\mu: X \rightarrow X$ be a state-morphism operator on $X$. Then $\operatorname{Ker}(\mu)=\{0\}$ or $\operatorname{Ker}(\mu)=X$. Hence by Remark 4.6, $\mu=\operatorname{Id}_{X}$ or $\mu(x)=0$ for all $x \in X$.
Definition 4.8. A state ideal $I$ of a state-morphism BCK-algebra $(X, \mu)$ is called a prime state ideal of $(X, \mu)$ if, given state ideals $A, B$ of $(X, \mu), A \cap B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.
Theorem 4.9. Let $(X, \mu)$ be a subdirectly irreducible state-morphism BCK-algebra. Then $\operatorname{Ker}(\mu)$ is a prime state ideal.
Proof. Let $I$ and $J$ be two state ideals of $(X, \mu)$ such that $I \cap J \subseteq \operatorname{Ker}(\mu)$. Define $\phi: X / \operatorname{Ker}(\mu) \rightarrow$ $\mu(X) / I \times \mu(X) / J$, by $\phi(x / \operatorname{Ker}(\mu))=(x / I, x / J)$ for all $x \in X$. For each $x, y \in X$, if $x / \operatorname{Ker}(\mu)=y / \operatorname{Ker}(\mu)$, then $x * y, y * x \in \operatorname{Ker}(\mu)$ and so $\mu(x) * \mu(y)=\mu(x * y)=0=\mu(y * x)=\mu(y) * \mu(x)$. Hence by (BCK3), $\mu(x)=\mu(y)$. Therefore, $\phi$ is a well defined homomorphism. Thus, for each $x, y \in X$, if $\phi(x / \operatorname{Ker}(\mu))=\phi(y / \operatorname{Ker}(\mu))$, then $(\mu(x) / I, \mu(x) / J)=(\mu(y) / I, \mu(y) / J)$, so that $\mu(x) * \mu(y), \mu(y) * \mu(x) \in$ $I \cap J$. Hence, $\mu(x) * \mu(y), \mu(y) * \mu(x) \in \operatorname{Ker}(\mu)$. It follows that $x / \operatorname{Ker}(\mu)=y / \operatorname{Ker}(\mu)$, which implies that $\phi$ is one-to-one. Clearly, $\pi_{1} \circ \phi(X / \operatorname{Ker}(\mu))=\mu(X) / I$, and $\pi_{2} \circ \phi(X / \operatorname{Ker}(\mu))=\mu(X) / J$, where $\pi_{1}: \mu(X) / I \times \mu(X) / J \rightarrow \mu(X) / I$ and $\pi_{2}: \mu(X) / I \times \mu(X) / J \rightarrow \mu(X) / J$ are natural projection maps. Since $X / \operatorname{Ker}(\mu)$ and $\mu(X)$ are isomorphic, then by Theorem3.14(ii), $X / \operatorname{Ker}(\mu)$ is a subdirectly irreducible BCK-algebra and so $\pi_{1} \circ \phi: X / \operatorname{Ker}(\mu) \rightarrow \mu(X) / I$ or $\pi_{2} \circ \phi: X / \operatorname{Ker}(\mu) \rightarrow \mu(X) / J$ is an isomorphism. Without lost of generality, we can assume that $\pi_{1} \circ \phi$ is an isomorphism. For any $x \in I, \pi_{1}(\phi(x / \operatorname{Ker}(\mu)))=$ $\pi_{1}(\mu(x) / I, \mu(x) / J)=\mu(x) / I$. Since $I$ is a state ideal, then $\mu(x) \in I$ and hence $\mu(x) / I=0 / I$. It follows that $x / \operatorname{Ker}(\mu)=0 / \operatorname{Ker}(\mu)$ (since $\pi_{1} \circ \phi$ is an isomorphism) and $x \in \operatorname{Ker}(\mu)$. Therefore, $I \subseteq \operatorname{Ker}(\mu)$ and so $\operatorname{Ker}(\mu)$ is a prime ideal of $X$.

Now, let us to consider a commutative subdirectly irreducible state morphism BCK-algebra $(X, \mu)$ satisfying the identity $(x * y) \wedge(y * x)=0$.
Proposition 4.10. Let $(X, \mu)$ be a subdirectly irreducible state-morphism BCK-algebra such that $X$ is commutative and $(x * y) \wedge(y * x)=0$ for all $x, y \in X$. Then the following statements conditions hold:
(i) For all $x \in X$, either $x \leq \mu(x)$ or $\mu(x) \leq x$.
(ii) $\mu(X)$ is a chain.

Proof. (i) Since $(X, \mu)$ is subdirectly irreducible, then by Theorem 3.14, $\operatorname{Ker}(\mu)=\{0\}$ or $\operatorname{Ker}(\mu) \neq\{0\}$ and it is a subdirectly irreducible subalgebra of $X$. If $\operatorname{Ker}(\mu)=\{0\}$, then by Remark 4.6, $\mu(x)=x$ for all $x \in X$. Let $\operatorname{Ker}(\mu) \neq\{0\}$. Since $(x * y) \wedge(y * x)=0$ for all $x, y \in X$, then by Theorem 3.14 and [29, Thm 2.3.12], $\operatorname{Ker}(X)$ must be a chain. Let $x \in X$. By Proposition 4.5, $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)$ and so $(x * \mu(x)) \wedge(\mu(x) * x)=0$ implies that $x * \mu(x)=0$ or $\mu(x) * x=0$. Therefore, $x \leq \mu(x)$ or $\mu(x) \leq x$.
(ii) By the first isomorphism theorem, $X / \operatorname{Ker}(\mu) \cong \mu(X)$. Since $X$ is a commutative BCK-algebra and it satisfies the identity $(x * y) \wedge(y * x)=0$, then by [22, Thm II.8.13] and Theorem 4.9, $X / \operatorname{Ker}(\mu)$ is a chain. Hence, $\mu(X)$ is a chain.

Note that if $(X, *, 0)$ is a BCK-algebra such that $(X, \leq)$ is a lattice, it is called a BCK-lattice. Then by [29, Thm 2.2.6], $X$ satisfies the identity $(x * y) \wedge(y * x)=0$.
Definition 4.11. A pair $(A, I)$ is called an adjoint pair of a BCK-algebra $X$, if $I$ is an ideal of $X$ and $A$ is a subalgebra of $X$ satisfying the following conditions:
(Ap1) $A \cap I=\{0\}$ and $\langle A \cup I\rangle=X ;$
(Ap2) for each $x \in X$, there exists an element $a_{x} \in A$ such that $\left(x, a_{x}\right) \in \theta_{I}$ (we say that $a_{x}$ is a component of $x$ in $A$ with respect to $I$ ).

By Proposition 4.5 (iii) and (iv), we conclude that if $\mu$ is a state-morphism operator on $X$, then $(\mu(X), \operatorname{Ker}(\mu))$ satisfies (Ap1). In Theorem4.14, a relation between state-morphism operators and adjoint pairs in any BCK-algebras will be found.

Proposition 4.12. Let $(A, I)$ be an adjoint pair of $X$. Then, for all $x \in X$, $a_{x}$ is unique.
Proof. Let $x \in X$ and $a, b \in A$ be two components of $x$ in $A$. Then $(x, a),(x, b) \in \theta_{I}$ and so $(a, b) \in \theta_{I}$. Hence, $a * b, b * a \in I$. Also, $a * b, b * a \in A$ (since $A$ is a subalgebra of $X$ ), so by (Ap1), $a * b, b * a \in I \cap A=\{0\}$. Thus, by (BCK3), $a=b$. Therefore, $a_{x}$ is the only component of $x$ in $A$ with respect to $I$.

Let $\mu$ and $\nu$ be two state-morphism operators on $X$ such that $\operatorname{Ker}(\mu)=\operatorname{Ker}(\nu)$ and $\operatorname{Im}(\mu)=\operatorname{Im}(\nu)$. For any $x \in X$, we have $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)=\operatorname{Ker}(\nu)$ and so $\nu(x * \mu(x))=0=\nu(\mu(x) * x)$. Since $\nu$ is a homomorphism and $\mu(x) \in \operatorname{Im}(\mu)=\operatorname{Im}(\nu)$, then $\nu(\mu(x))=\mu(x)$ and so $\nu(x) * \mu(x)=0=\mu(x) * \nu(x)$. From (BCK3), we obtain that $\nu(x)=\mu(x)$ for all $x \in X$. Therefore, $\mu=\nu$. In Remark 4.13, we show that, there are state-morphism operators $\mu$ and $\nu$ on a BCK-algebra $X$ such that $\operatorname{Ker}(\mu)=\operatorname{Ker}(\nu)$, but $\mu \neq \nu$.
Remark 4.13. Suppose that $I$ is a maximal ideal of $X$ such that $|X / I|=2$ and $2 \leq|X-I|$. Let $a$ and $b$ be two distinct elements of $X-I$. Define $\mu_{a}: X \rightarrow X$ and $\mu_{b}: X \rightarrow X$ by

$$
\mu_{a}(x)=\left\{\begin{array}{ll}
0 & \text { if } x \in I, \\
a & \text { if } x \in X-I .
\end{array} \quad \mu_{b}(x)= \begin{cases}0 & \text { if } x \in I \\
b & \text { if } x \in X-I\end{cases}\right.
$$

(1) If $x, y \in I$, then $x * y \in I$, so $\mu_{a}(x * y)=0=\mu_{a}(x) * \mu_{b}(y)$.
(2) If $x \in I$ and $y \in X-I$, then $x * y \leq x$ and hence $x * y \in I$. It follows that $\mu_{a}(x * y)=0=$ $0 * \mu_{a}(y)=\mu_{a}(x) * \mu_{b}(y)$,
(3) If $x \in X-I$ and $y \in I$, then $x * y \in X-I$ (since $I$ is an ideal and $x * y \in I$ implies $x \in I$ ) and so $\mu_{a}(x * y)=a=\mu_{a}(x) * 0=\mu_{a}(x) * \mu_{a}(y)$,
(4) If $x, y \in X-I$, then by assumption, $x / I=y / I$ (since $|x / I|=2$ ), so $x * y \in I$. Thus, $\mu_{a}(x * y)=$ $0=a * a=\mu_{a}(x) * \mu_{a}(y)$.

By (1)-(4), we obtain that $\mu_{a}$ is a homomorphism. If $x \in I$, then $\mu_{a}\left(\mu_{a}(x)\right)=\mu_{a}(x)=0$. Also, if $x \in X-I$, then $\mu_{a}\left(\mu_{a}(x)\right)=\mu_{a}(a)=a=\mu_{a}(x)$ (since $a \in X-I$ ), so $\mu_{a}$ is a state-morphism operator. In a similar way, we can show that $\mu_{b}$ is a state-morphism operator. Clearly, $\operatorname{Ker}\left(\mu_{a}\right)=I=\operatorname{Ker}\left(\mu_{b}\right)$, but $\mu_{a} \neq \mu_{b}$.

Note that if $X$ is a non-trivial positive implicative BCK-algebra and $I$ is a maximal ideal of $X$, then $X / I$ is a simple positive implicative BCK-algebra and so by [29, Cor 3.1.7], $|X / I|=2$. It follows that if $2 \leq|X-I|$, then $X$ satisfies the conditions in Remark 4.13,
Theorem 4.14. There is a one-to-one correspondence between adjoint pairs of $X$ and state-morphism operators on $X$.
Proof. Let $\mu: X \rightarrow X$ be a state-morphism operator on $X$. We show that $(\mu(X), \operatorname{Ker}(\mu))$ is an adjoint pair of $X$. By Proposition 4.5 (iii) and (iv), (Ap1) holds. Let $A=\mu(X)$ and $x$ be an element of $X$. Then $\mu(x) \in A$ and clearly, $x * \mu(x), \mu(x) * x \in \operatorname{Ker}(\mu)$ (since $\left.\mu^{2}=\mu\right)$. Hence, $(x, \mu(x)) \in \theta_{I}$. That is, for each $x \in X, \mu(x)$ is a component of $x$ in $A$ and so (Ap2) holds. Therefore, $(\mu(X), \operatorname{Ker}(\mu))$ is an adjoint pair of $X$.

Conversely, let $(A, I)$ be an adjoint pair of $X$. Define $\mu_{I, A}: X \rightarrow X$, by $\mu_{I, A}(x)=a_{x}$ for all $x \in X$. By Proposition 4.12, $\mu_{I, A}$ is well defined. Let $x, y \in X$. Then $\left(x, a_{x}\right) \in \theta_{I}$ and $\left(y, a_{y}\right) \in \theta_{I}$ and so $\left(x * y, a_{x} * a_{y}\right) \in \theta_{I}$. By $a_{x} * a_{y} \in A$, we conclude that $a_{x} * a_{y}$ is a component of $x * y$ in $A$, hence by Proposition 4.12, $\mu_{I, A}(x * y)=a_{x * y}=a_{x} * a_{y}=\mu_{I, A}(x) * \mu_{I, A}(y)$. Thus, $\mu_{I, A}$ is a homomorphism. Moreover, for any $a \in A, a * a=0 \in I$ and hence $\mu_{I, A}(a)=a_{a}=a$. It follows that $\mu_{I, A}\left(\mu_{I, A}(x)\right)=\mu_{I, A}(x)$ for all $x \in X$. Therefore, $\mu_{I, A}$ is a state-morphism operator on $X$. Let us
denote by $\operatorname{Ad}(X)$ and $\operatorname{SM}(X)$ the set of all adjoint pairs and the set of all state-morphism operators on $X$, respectively. Define $f: \operatorname{Ad}(X) \rightarrow \mathrm{SM}(X)$, by $f(A, I)=\mu_{I, A}$ and $g: \operatorname{SM}(X) \rightarrow \operatorname{Ad}(X)$ by $g(\mu)=(\mu(X), \operatorname{Ker}(\mu))$. Since $\operatorname{Ker}\left(\mu_{I, A}\right)=I$ and $\operatorname{Im}\left(\mu_{I, A}\right)=A$ for all $(A, I) \in A d(X)$, then by the paragraph just after Proposition 4.12, we conclude that $f \circ g=\operatorname{Id}_{\mathrm{SM}(\mathrm{X})}$ and $g \circ f=\operatorname{Id}_{\operatorname{Ad}(\mathrm{X})}$.

In the sequel, we want to construct a state BCK-algebra from a state-morphism. Let $m: X \rightarrow[0,1]$ be a state-morphism. Then $m$ is a homomorphism from $X$ into the BCK-algebra $\left([0,1], *_{\mathbb{R}}, 0\right)$ and so $X / \operatorname{Ker}(m) \cong m(X)$. Let $B=m(X)$ and $C=\operatorname{Ker}(m)$. Then $B$ and $C$ are BCK-algebras. Consider the BCK-algebra $B \times C$. Let $A=\{(b, 0) \mid b \in B\}$ and $I=\{(0, c) \mid c \in C\}$. Then $I$ is an ideal of $B \times C$ and $A$ is a subalgebra of $B \times C$. Also,
(1) $A \cap I=\emptyset$.
(2) For each $(x, y) \in B \times C$, we have $((x, y) *(x, 0)) *(0, y)=(0,0)$, hence by Theorem 2.2, $(x, y) \in$ $\langle A \cup I\rangle$. It follows that $B \times C=\langle A \cup I\rangle$.
(3) For each $(x, y) \in B \times C$, we have $(x, y) *(x, 0)=(0, y) \in I$ and $(x, 0) *(x, y)=(0,0) \in I$. Thus, $(x, y) / I=(x, 0) / I$.

So by Theorem 4.14, the map $\mu: B \times C \rightarrow B \times C$ defined by $\mu(x, y)=(x, 0)$ is a state-morphism operator on $B \times C$. Clearly, $\operatorname{Ker}(\mu)=I$ and $\operatorname{Im}(\mu)=A$. Note that if $m_{\mu}: B \times C \rightarrow[0,1]$ is the statemorphism induced by $\mu$ (see the paragraph before Remark 4.6), then $(B \times C) / \operatorname{Ker}\left(m_{\mu}\right) \cong B \cong \operatorname{Im}(m)$ and $\operatorname{Ker}\left(m_{\mu}\right)=C \cong \operatorname{Ker}(m)$.

Definition 4.15. Let $I$ be an ideal of $X$ and $\pi_{I}: X \rightarrow X / I$ be the canonical projection. Then $I$ is called a retract ideal if there exists a homomorphism $f: X / I \rightarrow X$ such that $\pi_{I} \circ f=\operatorname{Id}_{X / I}$ (the identity map on $X / I$ ).

Theorem 4.16. An ideal $I$ of $X$ is a retract ideal if and only if there exists a subalgebra $A$ of $X$ such that $(A, I)$ forms an adjoint pair.

Proof. Let $I$ be a retract ideal of $X$. Then there exists a homomorphism $f: X / I \rightarrow X$ such that $\pi_{I} \circ f=\operatorname{Id}_{X / I}$. Put $A=f(X / I)$. Since $f$ is a homomorphism, then $A$ is a subalgebra of $X$. Let $x \in I \cap A$. Then there exists $a \in X$ such that $f(a / I)=x$, so $a / I=\pi_{I} \circ f(a / I)=\pi_{I}(x)=x / I$. From $x \in I$, we get that $a \in I$ and $a / I=0 / I$, whence $x=f(0 / I)=0$. Now, let $x \in X$. Then $f(x / I)=a$ for some $a \in A$. It follows that $x / I=\pi_{I} \circ f(x / I)=\pi_{I}(a)=a / I$, which implies that $x * a \in I$. Hence, $a \in\langle A \cup I\rangle$ and $a$ is a component of $x$ in $A$ with respect to $I$. Therefore, $(A, I)$ is an adjoint pair of $X$. Conversely, let $(A, I)$ be an adjoint pair of $X$. Define $f: X / I \rightarrow X$ by $f(x / I)=a_{x}$ for all $x \in X$ (see Definition 4.11). If $x / I=y / I$ for some $x, y \in X$, then $(x, y) \in \theta_{I}$ and $\left(x, a_{x}\right) \in \theta_{I}$, which yields $a_{x}$ is a component of $y$ in $A$. By Proposition 4.12, we get that $a_{y}=a_{x}$. Thus, $f$ is well defined. In a similar way, we can show that $f$ is a homomorphism. It follows from $\left(x, a_{x}\right) \in \theta_{I}$ that $\pi_{I} \circ f(x / I)=\pi_{I}\left(a_{x}\right)=a_{x} / I=x / I$. Therefore, $I$ is a retract ideal of $X$.

Corollary 4.17. There is a one-to-one correspondence between retract ideals and state-morphism operators of $X$.
Proof. The proof is a straightforward consequence of Theorem 4.14 and 4.16
Definition 4.18. [4, Def II.8.8] A state BCK-algebra $(X, \mu)$ is called

- simple if $\operatorname{Con}(X, \mu)=\{\Delta, \nabla\}$.
- semisimple if the intersection of all maximal congruence relations of $(X, \mu)$ is $\Delta$.

By Theorem 3.11, we conclude that $(X, \mu)$ is simple if and only if it has exactly, two state ideals (\{0\} and $X)$ and it is semisimple if and only if the intersection of all maximal state ideals of $(X, \mu)$ is the zero ideal.
Theorem 4.19. Let $(X, \mu)$ be a state-morphism BCK-algebra. Then the following hold:
(i) $\mu(X)$ is a simple (semisimple) subalgebra of $X$ if and only if $\operatorname{Ker}(\mu) \in \operatorname{Max}(X)(\operatorname{Rad}(X) \subseteq$ $\operatorname{Ker}(\mu))$.
(ii) $(X, \mu)$ is a simple state-morphism BCK-algebra if and only if $X$ is a simple $B C K$-algebra.
(iii) If $\mu(X)$ is a semisimple subalgebra of $X$, then the intersection of all maximal state ideals of $(X, \mu)$ is a subset of $\operatorname{Ker}(\mu)$.
(iv) If $X$ is a non-trivial bounded BCK-algebra such that $\mu(1)=1$ and $(X, \mu)$ is a semisimple state $B C K$-algebra, then $\mu$ is the identity map.

Proof. (i) Let $(X, \mu)$ be a state-morphism BCK-algebra. Then by the first isomorphism theorem, $X / \operatorname{Ker}(\mu)$ and $\mu(X)$ are isomorphic (as BCK-algebras), whence the proof of (i) is straightforward.
(ii) Let $(X, \mu)$ be a simple state-morphism BCK-algebra. By Proposition [3.2(iii), $\operatorname{Ker}(\mu)$ is a state ideal of $(X, \mu)$ and so $\operatorname{Ker}(\mu)=\{0\}$ or $\operatorname{Ker}(\mu)=X$. By Corollary 4.7, we obtain that $\mu=\operatorname{Id}_{X}$ or $\mu(x)=0$ for all $x \in X$. However, each ideal of $X$ is a state ideal, so by assumption, $X$ must have exactly two ideals. That is, $X$ is a simple BCK-algebra. The proof of the converse is clear. In fact, any simple BCK-algebra $X$, has exactly two ideals, $X$ and $\{0\}$, which are state ideals.
(iii) Let $\mu(X)$ be a semisimple subalgebra of $X$. Since $X / \operatorname{Ker}(\mu) \cong \mu(X)$, we get that $\operatorname{Rad}(X / \mu(X))=$ $\{0 / \mu(X)\}$ and so $\bigcap\{I / \operatorname{Ker}(\mu) \mid \operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\}=\{0 / \operatorname{Ker}(\mu)\}$, which implies that $\bigcap\{I \mid$ $\operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\} \subseteq \operatorname{Ker}(\mu)$. Let $H$ be a maximal ideal of $X$ containing $\operatorname{Ker}(\mu)$. Since $\mu(x) * x \in$ $\operatorname{Ker}(\mu)$, for each $x \in H$, then we have $\mu(x) * x \in H$, and so $\mu(x) \in H$ for all $x \in X$. Thus, $H$ is a state ideal of $(X, \mu)$. By summing up the above results, we have

$$
\bigcap\{I \mid I \text { is a state ideal of }(X, \mu)\} \subseteq \bigcap\{I \mid \operatorname{Ker}(\mu) \subseteq I \in \operatorname{MaxS}(X)\} \subseteq \operatorname{Ker}(\mu)
$$

(iv) Let $I$ be a maximal state ideal of $X$. Then we define $\nu: X / I \rightarrow X / I$ by $\nu(x / I)=\mu(x) / I$ for all $x \in X$. If $x / I=y / I$ for some $x, y \in X$, then $x * y, y * x \in I$. By assumption, $\mu(x) * \mu(y) \in I$ and $\mu(y) * \mu(x) \in I$, hence $\mu(x) / I=\mu(y) / I$, which implies that $\nu(x / I)=\nu(y / I)$. Clearly, $\nu$ is a state operator on the BCK-algebra $X / I$. Since $I$ is a maximal ideal, then $X / I$ is a simple BCK-algebra, so by Corollary 4.7, $\nu=\operatorname{Id}_{X / I}$ or $\nu=0$. If $\nu=0$, then $\mu(x) \in I$ for all $x \in X$. It follows that $1 \in I$, which is a contradiction. So, $\nu(x / I)=x / I$ for all $x \in X$. Hence, $\mu(x) * x, x * \mu(x) \in I$ for all $x \in X$. Since $I$ is an arbitrary maximal state ideal of $(X, \mu)$, then by Proposition 4.2, we conclude that $\operatorname{Ker}(\mu) \subseteq \bigcap\{I \mid I \in \operatorname{MaxS}(X)\}$. Now, let $(X, \mu)$ be semisimple. Then $\bigcap\{I \mid I \in \operatorname{MaxS}(X)\}=\{0\}$ and so, $\operatorname{Ker}(\mu)=\{0\}$. By Corollary 4.6, $\mu=\operatorname{Id}_{X}$.

Now we show a relation between state-morphism MV-algebras and state-morphism BCK-algebras.
Theorem 4.20. Let $(X, *, 0)$ be a bounded commutative BCK-algebra and $\mu: X \rightarrow X$ be a statemorphism operator such that $\mu(1)=1$. Then $(X, \mu)$ is a state-morphism MV-algebra.

Proof. Let $x, y \in X$. Then $\mu\left(x^{\prime}\right)=\mu(1 * x)=\mu(1) * \mu(x)=1 * \mu(x)=\mu(x)^{\prime}$. Also,

$$
\mu(x \oplus y)=\mu(N(N x * y))=1 * \mu(N x * y)=1 *(\mu(N x) * \mu(y))=1 *((1 * \mu(x)) * \mu(y))=\mu(x) \oplus \mu(y)
$$

so, $\mu(x)$ is a homomorphism of MV-algebras. Since $\mu^{2}=\mu$, then $\mu$ is a state-morphism operator on the MV-algebra $\left(X, \oplus,^{\prime}, 0\right)$. That is, $(X, \mu)$ is a state-morphism MV-algebra.

## 5. Generators of State-Morphism BCK-algebras

Let $\mathcal{S M B C K}$ be the quasivariety of state-morphism BCK-algebras. We note that the system of BCKalgebras is not a variety because it is not closed under homomorphic images, [22, Thm VI.4.1]. On the other side, the system of commutative BCK-algebras or of quasi-commutative BCK-algebra forms a variety, [22, Thm I.5.2, Thm I.9.2]. Since by [22, Thm I.9.4], every finite BCK-algebra is quasicommutative, we can define the variety generated by a system of finite BCK-algebras.

Let $(X, *, 0)$ be a BCK-algebra and on the direct product BCK-algebra $X \times X$ we define a mapping $\mu_{X}: X \times X \rightarrow X \times X$ by $\mu_{X}(x, y)=(x, x),(x, y) \in X \times X$. Then $\mu_{X}$ is a state-morphism on the BCKalgebra $X \times X$ and the state-morphism BCK-algebra $D(X):=\left(X \times X, \mu_{X}\right)$ is a said to be a diagonal state-morphism BCK-algebra. In the same way we can define also $\nu: X \times X \rightarrow X \times X$ by $\nu(x, y)=(y, y)$, $(x, y) \in X \times X$, and $(X \times X, \nu)$ is again a state-morphism BCK-algebra which is isomorphic to $D(X)$ under the isomorphism $h(x, y)=(y, x),(x, y) \in X \times X$. For example, if $X=[0,1]$ is the MV-algebra of
the real interval, then it generates the variety of MV-algebras (as well as the quasivariety of MV-algebras), and by [13, Thm 5.4], $D([0,1])$ generates the variety of state-morphism MV-algebras.

Given a quasivariety of BCK-algebras $\mathcal{V}$, let $\mathcal{V}_{\mu}$ denote the class of state-morphism BCK-algebras $(X, \mu)$ such that $X \in \mathcal{V}$. Then $\mathcal{V}_{\mu}$ is a quasivariety, too.

As usual, given a class $\mathcal{K}$ of algebras of the same type, $\mathrm{I}(\mathcal{K}), \mathrm{H}(\mathcal{K}), \mathrm{S}(\mathcal{K}), \mathrm{P}(\mathcal{K})$, and $\mathrm{P}_{\mathrm{R}}(\mathcal{K})$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products of algebras and of reduced products from $\mathcal{K}$, respectively. Moreover, let $\mathrm{Q}_{\mathrm{V}}(\mathcal{K})$ and $\mathrm{V}(\mathcal{K})$ denote the quasivariety and the variety, respectively, generated by $\mathcal{K}$. We recall that a quasivariety is closed under isomorphic images, subalgebras, reduced products and containing the one-element algebras, see [4, Def V.2.24], and a variety is closed under homomorphic images, subalgebras and products.

Using methods from [13, Sec 5], which can be easily modified for state-morphism BCK-algebras instead of state-morphism MV-algebras, we can prove the following two lemmas and theorem on generators for a case when we have a variety of BCK-algebras as well as for a more general case - for quasivarieties of BCK-algebras; for reader's convenience, we present outlines of theirs proofs.

First we start with proofs concerning the case when a family of BCK-algebras belongs to some variety of BCK-algebras.

Lemma 5.1. (1) Let $\mathcal{K}$ be a class of BCK-algebras belonging to some variety of BCK-algebras. Then $\mathrm{V}(\mathrm{D}(\mathcal{K})) \subseteq \mathrm{V}(\mathcal{K})_{\mu}$.
(2) Let $\mathcal{V}$ be any variety of BCK-algebras. Then $\mathcal{V}_{\mu}=\operatorname{ISD}(\mathcal{V})$.

Proof. (1) We have to prove that every BCK-reduct of a state-morphism BCK-algebra in $\mathrm{V}(\mathrm{D}(\mathcal{K}))$ is in $\mathrm{V}(\mathcal{K})$. Let $\mathcal{K}_{0}$ be the class of all BCK-reducts of algebras in $\mathrm{D}(\mathcal{K})$. Let $X \in \mathcal{K}$, then $D(X) \in \mathrm{D}(\mathcal{K})$. Then the BCK-reduct of $D(X)$ is $X \times X$, and since $X$ is a homomorphic image (under the projection map) of $X \times X, \mathcal{K}_{0} \subseteq \mathrm{P}(\mathcal{K})$ and $\mathcal{K} \subseteq \mathrm{H}\left(\mathcal{K}_{0}\right)$. Hence, $\mathcal{K}_{0}$ and $\mathcal{K}$ generate the same variety. Moreover, BCK-reducts of subalgebras (homomorphic images, direct products respectively) of algebras from $\mathrm{D}(\mathcal{K})$ are subalgebras (homomorphic images, direct products, respectively) of the corresponding BCK-reducts. Therefore, the BCK-reduct of any algebra in $\mathrm{V}(\mathrm{D}(\mathcal{K}))$ is in $\operatorname{HSP}\left(\mathcal{K}_{0}\right)=\operatorname{HSP}(\mathcal{K})=\mathrm{V}(\mathcal{K})$.
(2) Let $(X, \mu) \in \mathcal{V}_{\mu}$. The map $\Phi: a \mapsto(\mu(a), a)$ is an embedding of $(X, \mu)$ into $D(X)$. Moreover, $\Phi(\mu(a))=(\mu(\mu(a)), \mu(a))=(\mu(a), \mu(a))=\mu_{X}((\mu(a), a))=\mu_{X}(\Phi(a))$. Hence, $\Phi$ is an injective homomorphism of state-morphism BCK-algebras, and $(X, \mu) \in \operatorname{ISD}(\mathcal{V})$. Conversely, the BCK-reduct of any algebra in $\mathrm{D}(\mathcal{V})$ is in $\mathcal{V}$, and hence the BCK-reduct of any member of $\operatorname{ISD}(\mathcal{V})$ is in $\operatorname{IS}(\mathcal{V})=\mathcal{V}$. Hence, any member of $\operatorname{ISD}(\mathcal{V})$ is in $\mathcal{V}_{\mu}$.

Lemma 5.2. Let $\mathcal{K}$ be a class of BCK-algebras. Then:
(1) $\mathrm{DH}(\mathcal{K}) \subseteq \mathrm{HD}(\mathcal{K})$.
(2) $\operatorname{DS}(\mathcal{K}) \subseteq \operatorname{ISD}(\mathcal{K})$.
(3) $\mathrm{DP}(\mathcal{K}) \subseteq \operatorname{IPD}(\mathcal{K})$.
(4) $\mathrm{V}(\mathrm{D}(\mathcal{K}))=\operatorname{ISD}(\mathrm{V}(\mathcal{K}))$.

Proof. (1) Let $D(C) \in \mathrm{DH}(\mathcal{K})$. Then there are $X \in \mathcal{K}$ and a BCK-homomorphism $h$ from $X$ onto $C$. Let, for all $a, b \in X, h^{*}(a, b)=(h(a), h(b))$. We claim that $h^{*}$ is a homomorphism from the diagonal statemorphism BCK-algebra $D(X)$ onto $D(C)$. That $h^{*}$ is a BCK-homomorphism is clear. We verify that $h^{*}$ is compatible with $\mu_{X}$. We have $h^{*}\left(\mu_{X}(a, b)\right)=h^{*}(a, a)=(h(a), h(a))=\mu_{C}(h(a), h(b))=\mu_{C}\left(h^{*}(a, b)\right)$. Finally, since $h$ is onto, given $(c, d) \in C \times C$, there are $a, b \in X$ such that $h(a)=c$ and $h(b)=d$. Hence, $h^{*}(a, b)=(c, d), h^{*}$ is onto, and $D(C) \in \operatorname{HD}(\mathcal{K})$.
(2) It is trivial.
(3) Let $X=\prod_{i \in I} X_{i} \in \mathrm{P}(\mathcal{K})$, where each $X_{i}$ is in $\mathcal{K}$. We assert the map

$$
\Phi:\left(\left(a_{i}: i \in I\right),\left(b_{i}: i \in I\right)\right) \mapsto\left(\left(a_{i}, b_{i}\right): i \in I\right)
$$

is an isomorphism of state-morphism BCK-algebras from $D(X)$ onto $\prod_{i \in I} D\left(X_{i}\right)$. Indeed, it is clear that $\Phi$ is a BCK-isomorphism. Moreover, denoting the state-morphism of $\prod_{i \in I} D\left(X_{i}\right)$ by $\mu^{*}$, we get:

$$
\begin{gathered}
\Phi\left(\mu_{X}\left(\left(a_{i}: i \in I\right),\left(b_{i}: i \in I\right)\right)\right)=\Phi\left(\left(a_{i}: i \in I\right),\left(a_{i}: i \in I\right)\right)= \\
=\left(\left(a_{i}, a_{i}\right): i \in I\right)=\left(\mu_{X_{i}}\left(a_{i}, b_{i}\right): i \in I\right)=\mu^{*}\left(\Phi\left(\left(a_{i}: i \in I\right),\left(b_{i}: i \in I\right)\right)\right),
\end{gathered}
$$

and whence $\Phi$ is an isomorphism of state-morphism BCK-algebras.
(4) $\mathrm{By}(1),(2)$ and $(3), \operatorname{DV}(\mathcal{K})=\operatorname{DHSP}(\mathcal{K}) \subseteq \operatorname{HSPD}(\mathcal{K})=\mathrm{V}(\mathrm{D}(\mathcal{K}))$, and hence $\operatorname{ISDV}(\mathcal{K}) \subseteq \operatorname{ISV}(\mathrm{D}(\mathcal{K}))=$ $\mathrm{V}(\mathrm{D}(\mathcal{K}))$. Conversely, by Lemma 5.1(1), $\mathrm{V}(\mathrm{D}(\mathcal{K})) \subseteq \mathrm{V}(\mathcal{K})_{\mu}$, and by Lemma 5.1(2), $\mathrm{V}(\mathcal{K})_{\mu}=\operatorname{ISDV}(\mathcal{K})$. This proves the claim.

Theorem 5.3. If a system $\mathcal{K}$ of $B C K$-algebras generates a variety $\mathcal{V}$ of $B C K$-algebras, then $\mathrm{D}(\mathcal{K})$ generates the variety $\mathcal{V}_{\mu}$ of state-morphism BCK-algebras.

Proof. By Lemma5.2(4), $\mathrm{V}(\mathrm{D}(\mathcal{K}))=\operatorname{ISD}(\mathrm{V}(\mathcal{K}))$. Moreover, by Lemma5.1(2), $\mathrm{V}(\mathcal{K})_{\mu}=\operatorname{ISDV}(\mathcal{K})$. Hence, $\mathrm{V}(\mathrm{D}(\mathcal{K}))=\mathrm{V}(\mathcal{K})_{\mu}$.

Let $[0,1]$ be the real interval. We endow it with the BCK-structure as before: $s *_{\mathbb{R}} t=\max \{0, s-t\}$, $s, t \in[0,1]$. We denote by $[0,1]_{B C K}:=\left([0,1], *_{\mathbb{R}}, 0\right)$ and it is a bounded commutative BCK-algebra. If, for bounded commutative BCK-algebras, we define a state-morphism operator $\mu$ as a homomorphism of bounded commutative BCK-algebras $\mu: X \rightarrow X$ such that $\mu \circ \mu=\mu$ and $\mu(1)=1$, we can obtain the following result.

Corollary 5.4. Let $\mathcal{V}$ be the variety of bounded commutative BCK-algebras, and let $\mathcal{V}_{B C K}$ be the variety of all bounded commutative state-morphism BCK-algebras. Then $\mathcal{V}_{B C K}=\mathrm{V}\left(D\left([0,1]_{B C K}\right)\right)$.

Proof. We can repeat the proofs of Lemmas 5.15 .2 and Theorem 5.3 also for state-morphism operators on bounded commutative BCK-algebras. We have $\mathcal{V}_{B C K}=\mathcal{V}_{\mu}$. By [23], the variety of bounded BCKalgebras is categorically equivalent to the variety of MV-algebras. Since the MV-algebra $[0,1]$ generates the variety of MV-algebras, we have that the BCK-algebra $[0,1]_{B C K}$ generates the variety of bounded commutative BCK-algebras. Then by Theorem 5.3, we have $\mathcal{V}_{B C K}=\mathrm{V}\left(D\left([0,1]_{B C K}\right)\right)$.
Corollary 5.5. There is uncountably many subvarieties of the variety $\mathcal{V}_{B C K}$ of bounded commutative BCK-algebras with a state-morphism.

Proof. By [13, Thm 7.11], the variety of state-morphism MV-algebras is uncountable. Because the variety of bounded commutative BCK-algebras is categorically equivalent to the variety of MV-algebras, [23], we have the statement in question.

Now we present some analogous general results concerning quasivarieties. The proofs follow the similar ideas just used for varieties.

Lemma 5.6. (1) Let $\mathcal{K}$ be a class of BCK-algebras. Then $\operatorname{Q}_{\vee}(\mathrm{D}(\mathcal{K})) \subseteq \mathrm{Q}_{\mathrm{V}}(\mathcal{K})_{\mu}$.
(2) Let $\mathcal{V}$ be any quasivariety of BCK-algebras. Then $\mathcal{V}_{\mu}=\operatorname{ISD}(\mathcal{V})$.

Proof. (1) We have to prove that every BCK-reduct of a state-morphism BCK-algebra in $\mathrm{Q}_{\mathrm{V}}(\mathcal{K})$ is in $\mathrm{Qv}_{\mathrm{V}}(\mathcal{K})$.

Let $\mathcal{K}_{0}$ be the class of all BCK-reducts of algebras in $\mathrm{D}(\mathcal{K})$. Let $X \in \mathcal{K}$, and let $\{0\}$ be the one-element BCK-algebra which is a subalgebra of $X$. Then $D(X) \in \mathrm{D}(\mathcal{K})$. The BCK-reduct of $D(X)$ is $X \times X$, and since $X$ is isomorphic to the BCK-algebra $\{0\} \times X$, which is a subalgebra of $X \times X$, we have $X \in \operatorname{IS}\left(\mathcal{K}_{0}\right)$. Thus $\mathcal{K}_{0} \subseteq \mathrm{P}(\mathcal{K})$ and $\mathcal{K} \subseteq \operatorname{IS}\left(\mathcal{K}_{0}\right)$. By [4, Thm 2.23, 2.25], we have $\mathrm{Qv}_{\mathrm{V}}\left(\mathcal{K}_{0}\right)=\operatorname{ISP}_{\mathrm{R}}\left(\mathcal{K}_{0}\right) \subseteq \operatorname{ISP} \mathrm{R}_{\mathrm{R}} \mathrm{P}(\mathcal{K}) \subseteq$ $\operatorname{ISIP}_{\mathrm{R}}(\mathcal{K}) \subseteq \operatorname{IISP}_{\mathrm{R}}(\mathcal{K})=\operatorname{ISP}_{\mathrm{R}}(\mathcal{K})=\mathrm{Qv}_{\mathrm{V}}(\mathcal{K})$. Similarly, $\mathrm{Qv}(\mathcal{K})=\operatorname{ISP}(\mathcal{K}) \subseteq \operatorname{ISP} \operatorname{R} \operatorname{IS}\left(\mathcal{K}_{0}\right) \subseteq \operatorname{ISIP}_{\mathrm{R}} \mathrm{S}\left(\mathcal{K}_{0}\right) \subseteq$ $\operatorname{ISISP} \mathrm{R}_{\mathrm{R}}\left(\mathcal{K}_{0}\right) \subseteq \operatorname{IISSP} \mathrm{R}_{\mathrm{R}}\left(\mathcal{K}_{0}\right)=\operatorname{ISP}_{\mathrm{R}}\left(\mathcal{K}_{0}\right)=\mathrm{Qv}\left(\mathcal{K}_{0}\right)$. Hence, $\mathcal{K}$ and $\mathcal{K}_{0}$ generates the same quasivariety.

Moreover, BCK-reducts of subalgebras (isomorphic images, reduced products, respectively) of algebras from $D(\mathcal{K})$ are subalgebras (isomorphic images, reduced products, respectively) of the corresponding BCK-reducts. Therefore, the BCK-reduct of any algebra in $\mathrm{Qv}_{\mathrm{v}}(\mathrm{D}(\mathcal{K}))$ is in $\mathrm{Qv}_{\mathrm{v}}\left(\mathcal{K}_{0}\right)=\mathrm{Q}_{\mathrm{v}}(\mathcal{K})=\mathrm{Q}_{\mathrm{v}}(\mathcal{K})$, which proves (1).
(2) Let $(X, \mu) \in \mathcal{V}_{\mu}$. The map $\Phi: a \mapsto(\mu(a), a)$ is an embedding of $(X, \mu)$ into $D(X)$. Moreover, $\Phi(\mu(a))=(\mu(\mu(a)), \mu(a))=(\mu(a), \mu(a))=\mu_{X}((\mu(a), a))=\mu_{X}(\Phi(a))$. Hence, $\Phi$ is an injective homomorphism of state-morphism BCK-algebras, and $(X, \mu) \in \operatorname{ISD}(\mathcal{V})$. Conversely, let $X \in \mathcal{V}$. Then the BCK-reduct of $D(X)$ is $X \times X$, and $X \times X$ is isomorphic with the reduced product ( $X \times X$ )/F, where $F$ is the one-element filter $F=\{1,2\}$ of the set $I=\{1,2\}$. Hence, $X \times X$ is in $\mathcal{V}$, and the BCK-reduct of any algebra in $\mathrm{D}(\mathcal{V})$ is in $\mathcal{V}$, whence the BCK-reduct of any member of $\operatorname{ISD}(\mathcal{V})$ is in $\operatorname{IS}(\mathcal{V})=\mathcal{V}$. Therefore, any member of $\operatorname{ISD}(\mathcal{V})$ is in $\mathcal{V}_{\mu}$.

Lemma 5.7. Let $\mathcal{K}$ be a class of BCK-algebras. Then:
(1) $\mathrm{DI}(\mathcal{K}) \subseteq \operatorname{ID}(\mathcal{K})$.
(2) $\operatorname{DS}(\mathcal{K}) \subseteq \operatorname{ISD}(\mathcal{K})$.
(3) $\mathrm{DP}_{\mathrm{R}}(\mathcal{K}) \subseteq \operatorname{IP}_{\mathrm{R}} \mathrm{D}(\mathcal{K})$.
(4) $\operatorname{Qv}_{\mathrm{v}}(\mathrm{D}(\mathcal{K}))=\operatorname{ISD}(\operatorname{Qv}(\mathcal{K}))$.

Proof. (1) Let $D(C) \in \operatorname{DI}(\mathcal{K})$. Then there are $X \in \mathcal{K}$ and an isomorphism $h$ from $X$ onto $C$. Let, for all $a, b \in X, h^{*}(a, b)=(h(a), h(b))$. We claim that $h^{*}$ is an isomorphism from $D(X)$ onto $D(C)$. That $h^{*}$ is an isomorphism of BCK-algebras is clear. We verify that $h^{*}$ is compatible with $\mu_{X}$. We have $h^{*}\left(\mu_{X}(a, b)\right)=h^{*}(a, a)=(h(a), h(a))=\mu_{C}(h(a), h(b))=\mu_{C}\left(h^{*}(a, b)\right)$. Finally, since $h$ is onto, given $(c, d) \in C \times C$, there are $a, b \in X$ such that $h(a)=c$ and $h(b)=d$. Hence, $h^{*}(a, b)=(c, d), h^{*}$ is onto, and $D(C) \in \operatorname{ID}(\mathcal{K})$.
(2) It is trivial.
(3) Let $X=\prod_{i \in I} X_{i} / F \in \mathrm{P}_{\mathrm{R}}(\mathcal{K})$, where each $X_{i}$ is in $\mathcal{K}$, and $F$ is a filter over $I$. We claim the map

$$
\Phi:\left(\left(a_{i}: i \in I\right) / F,\left(b_{i}: i \in I\right) / F\right) \mapsto\left(\left(a_{i}, b_{i}\right): i \in I\right) / F
$$

is an isomorphism from $D(X)$ onto $\prod_{i \in I} D\left(X_{i}\right) / F$. Indeed, it is clear that $\Phi$ is a BCK-isomorphism: let $\left(\left(a_{i}, b_{i}\right): i \in I\right) / F=\left(\left(a_{i}^{\prime}, b_{i}^{\prime}\right): i \in I\right) / F$. Then $\llbracket a_{i}=a_{i}^{\prime} \rrbracket \cap \llbracket b_{i}=b_{i}^{\prime} \rrbracket=\llbracket\left(a_{i}, b_{i}\right)=\left(a_{i}^{\prime}, b_{i}^{\prime}\right) \rrbracket \in F$, so that $\llbracket a_{i}=a_{i}^{\prime} \rrbracket, \llbracket b_{i}=b_{i}^{\prime} \rrbracket \in F$ and hence $\left(\left(a_{i}, b_{i}\right): i \in I\right) / F=\left(\left(a_{i}^{\prime}, b_{i}\right): i \in I\right) / F$. Moreover, denoting the state-morphism of $\prod_{i \in I} D\left(X_{i}\right)$ by $\mu^{*}$, we get:

$$
\begin{gathered}
\Phi\left(\mu_{X}\left(\left(a_{i}: i \in I\right) / F,\left(b_{i}: i \in I\right) / F\right)\right)=\Phi\left(\left(a_{i}: i \in I\right) / F,\left(a_{i}: i \in I\right)\right) / F= \\
=\left(\left(a_{i}, a_{i}\right): i \in I\right) / F=\left(\mu_{X_{i}}\left(a_{i}, b_{i}\right): i \in I\right)=\mu^{*}\left(\Phi\left(\left(a_{i}: i \in I\right) / F,\left(b_{i}: i \in I\right) / F\right)\right),
\end{gathered}
$$

and hence, $\Phi$ is a state-morphism isomorphism.
(4) $\mathrm{By}(1),(2)$ and $(3), \mathrm{DQv}_{\mathrm{V}}(\mathcal{K})=\operatorname{DISP}_{\mathrm{R}}(\mathcal{K}) \subseteq \operatorname{IISIP}_{\mathrm{R}} \mathrm{D}(\mathcal{K}) \subseteq \operatorname{IISP}_{\mathrm{R}} \mathrm{D}(\mathcal{K})=\operatorname{ISP}_{\mathrm{R}} \mathrm{D}(\mathcal{K})=\mathrm{Q}_{\mathrm{V}}(\mathrm{D}(\mathcal{K}))$, and hence, $\operatorname{ISDQv}(\mathcal{K}) \subseteq \operatorname{ISQv}(\mathrm{D}(\mathcal{K}))=\operatorname{Qv}_{\mathrm{V}}(\mathrm{D}(\mathcal{K}))$. Conversely, by Lemma 5.6(1), $\operatorname{Qv}_{\mathrm{V}}(\mathrm{D}(\mathcal{K})) \subseteq \operatorname{Q}_{\mathrm{V}}(\mathcal{K})_{\mu}$, and by Lemma 5.6(2), $\operatorname{Qv}_{\mathrm{V}}(\mathcal{K})_{\mu}=\operatorname{ISDQv}(\mathcal{K})$. This proves the claim.

Finally, we present the main result of the section about generators of quasivarieties of state-morphism BCK-algebras which is an analogue of Theorem 5.3

Theorem 5.8. If a system $\mathcal{K}$ of BCK-algebras generates a quasivariety $\mathcal{V}$ of BCK-algebras, then $\mathrm{D}(\mathcal{K})$ generates the quasivariety $\mathcal{V}_{\mu}$ of state-morphism BCK-algebras.
Proof. By Lemma [5.7(4), $\mathrm{Q}_{\mathrm{V}}(\mathrm{D}(\mathcal{K}))=\operatorname{ISD}\left(\mathrm{Q}_{\mathrm{V}}(\mathcal{K})\right)$. Moreover, by Lemma $5.6(2), \mathrm{Q}_{\mathrm{V}}(\mathcal{K})_{\mu}=\operatorname{ISD}\left(\mathrm{Q}_{\mathrm{V}}(\mathcal{K})\right)$. Hence, $\mathrm{Qv}_{\mathrm{V}}(\mathrm{D}(\mathcal{K}))=\mathrm{Qv}_{\mathrm{V}}(\mathcal{K})_{\mu}$.

Since the interval $[0,1]$ generates the class $\mathcal{M V}$ of MV-algebras as both a variety and a quasivariety, due to the categorical equivalence of MV-algebras and bounded commutative BCK-algebras, [23], by Theorem 5.8 and Corollary 5.4 we have the following corollary.

Corollary 5.9. If $[0,1]_{B C K}=\left([0,1], *_{\mathbb{R}}, 0\right)$ is the bounded commutative BCK-algebra of the real interval $[0,1]$, then $D\left([0,1]_{B C K}\right)$ generates both as the variety and as the quasivariety of state-morphism BCKalgebras whose BCK-reduct is a bounded commutative BCK-algebra. In other words, $\mathrm{V}\left(D\left([0,1]_{B C K}\right)\right)=$ $\mathcal{V}_{B C K}=\operatorname{Qv}\left(D\left([0,1]_{B C K}\right)\right)$.

Finally, we formulate two open problems.
Problem 1. Describe some interesting generators of the quasivariety of state BCK-algebras.
We note that we do not know yet any interesting generator for the variety of state MV-algebras.
(2) If $X$ is a subdirectly irreducible BCK-algebra, then the diagonal state-morphism BCK-algebra $D(X)$ is subdirectly irreducible. Similarly, if $X$ is linearly ordered and subdirectly irreducible, then ( $X, \operatorname{Id}_{X}$ ) is subdirectly irreducible. If $X$ is an MV-algebra, the third category of subdirectly irreducible state-morphism MV-alegbra $(X, \mu)$ is the case when $X$ has a unique maximal ideal. Inspired by that, we formulate the second open problem:
Problem 2. Characterize (bounded) subdirectly irreducible state-morphism BCK-algebras as it was done in [8, 11, 13].

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