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Abstract

Yager (Fuzzy Sets and Systems, 117(1): 1-12, 2001) proposed an algorithm to combine multi-agent preference orderings of several alternatives into a single consensus fused ordering, when the agents' importance is expressed through a rank-ordering and not a set of weights. This algorithm is simple and automatable but has some limitations which reduce its range of application, e.g., (i) preference orderings should not include incomparabilities between alternative and/or omissions of some of them, and (ii) the fused ordering may sometimes not reflect the majority of the multi-agent preference orderings.

The aim of this article is to present an enhanced version of the Yager's algorithm, which overcomes the above limitations. Some practical examples support the description of the new algorithm is supported by practical examples.

Keywords: Decision making, Multi-agent, Preference ordering, Fusion, Ordinal semi-democratic, Partial ordering.

1. Introduction

A general problem, which may concern practical contexts of different nature, is to aggregate multi-agent orderings of different alternatives into a single consensus fused ordering. Assume that there are M decision-making agents D_1, D_2, …, D_M, each of which defines an ordering of n alternatives a, b, c, etc.. For any two alternatives a and b, this ordering allows statements like $a > b$, $a \sim b$, $b > a$, where symbols ">" and "~" respectively mean "strictly preferred to" and "indifferent to".

This decision-making problem is very diffused (Zhu, 2003; Xu, 2004) in a variety of real-life contexts, ranging from *multi-criteria decision aiding* (Figueira et al., 2005; Etzioni, Weld, 1995) to *social choice* (Kelly, 1991) and *voting theory* (Bouyssou et al., 2000; Colomer, 2004). Two of the reasons for this diffusion are that (i) preference orderings are probably the most intuitive and effective way to represent preference judgments of alternatives, and (ii) they do not require a common scale – neither numeric, linguistic or ordinal – to be shared by the interacting agents (Yager, 2001; Chen et al., 2012).

The literature embraces a variety of aggregation techniques, which are relatively interchangeable among the fields of application. Despite this variety, they can generally be divided in two categories

(Arrows and Rayanaud, 1986):

1. Methods in which all agents have the same importance (Zhu, 2003); e.g., let us consider the classical approaches in the *voting theory* field (Borda, 1781; Condorcet, 1785; Lepelley and Martin, 2001);
2. Methods in which agents have recognised abilities and attributes and/or privileged positions of power, represented by weights (Xu, 2004; Dubois et al., 2012); e.g., let us consider the ELECTRE or the PROMETHEE methods, in the *multicriteria decision aiding* field (Brans and Mareschal, 2005; Figueira et al., 2005).

Considering the second category methods, the definition of the agents' weights is sometimes controversial, because there are no indisputable criteria for this operation. Although the literature includes several techniques about the quantification of weights – for example, the AHP procedure (Saaty, 1980) or the method proposed by Martel and Ben Khelifa (2000) – these techniques are often neglected because of their complexity.

In some situations weights are not available or cannot be defined. In this contexts agent's importance may be modelled by a simple rank-ordering, such as $D_1 > (D_2 \sim D_3) > \ldots > D_M$ (Roberts, 1979; Franceschini et al., 2007). The formulation of such a rank-ordering is certainly simpler and more intuitive than that of weights, especially when the agent importance prioritization is doubtful (Chen et al., 2012). This more specific decision-making framework can be denominated as "ordinal semi-democratic"; the adjective "semi-democratic" indicates that agents do not necessarily have the same importance, while "ordinal" indicates that their hierarchy is defined by a crude ordering. The set of the possible solutions to the problem may range between the two extremes of (i) *full dictatorship* – in which the fused ordering coincides with the preference ordering by the most important agent (dictator) – and (ii) *full democracy* – where all agents' orderings are considered as equi-important.

Despite the adaptability to a large number of practical contexts, this specific decision-making problem has received relatively little attention in the literature. Some years ago, Yager (2001) proposed an algorithm to address this problem in a relatively simple, fast and automatable way; this algorithm will be hereafter abbreviated as YA, which stands for "Yager's Algorithm". Unfortunately, this algorithm has two major limitations: (i) the resulting fused ordering may sometimes not reflect the preference ordering for the majority of agents (Jianqiang, 2007) and (ii) it is only applicable to (non-strict) *linear* orderings (such as that one exemplified in Fig. 1(a)) without incomparabilities and omissions of the alternatives of interest. These and other limitations will be clarified in the next section.

The objective of this paper is to enhance the YA so as to overcome its limitations and adapt to less stringent preference orderings (such as that one exemplified in Fig.1(b)). The new algorithm will be denominated as “Enhanced (Yager’s) Algorithm” – hereafter abbreviated as EYA – and can be interpreted as a generalization of that by Yager.

The remaining of the paper is organized into three sections. Sect. 2 recalls the YA in detail, with special attention to its limitations. Sect. 3 illustrates the EYA, highlighting its advantages with respect to the YA. The description of both algorithms is supported by practical examples. The concluding section summarizes the original contributions of the paper and its practical implications, limitations and suggestions for future research.

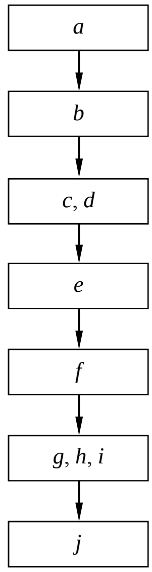
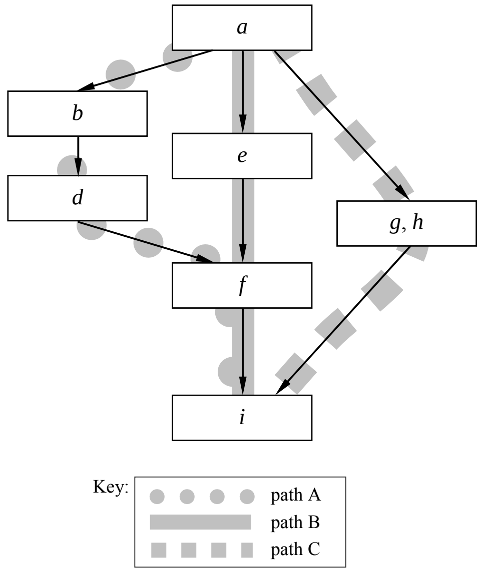
Type of ordering	(a) linear	(b) partial
Ordering (analytic form)	$a > b > (c \sim d) > e > f > (g \sim h \sim i) > j$	$a > \{\{[(b > d) \| e] > f\} \| (g \sim h)\} > i$
Graph		
Alternatives of interest	$\{a, b, c, d, e, f, g, h, i, j\}$	$\{a, b, d, e, f, g, h, i\}$
Omitted alternatives	Null	$\{c, j\}$
Incomparable alternatives	Null	(*b* and *d*) with (*e*, *g* and *h*); *e* with (*g* and *h*)

Fig. 1. (a) example of *linear* and (b) *partial* preference ordering. In the latter, two alternatives are omitted (i.e., *c* and *j*) and some are incomparable with each other. Symbols “>”, “~” and “||”depict respectively the strict preference, indifference and incomparability relationship.

2. Yager’s Algorithm (YA)

In Sect. 2.1 we take the liberty to illustrate the YA from a “pedagogical” point of view. For a more rigorous description, we refer the reader to the original contribution by Yager (2001). Sect. 2.2 discusses the (dis)advantages of this algorithm.

2.1 YA description

The algorithm can be schematized in the following three basic phases, which are described individually in the Sects. 2.1.1 to 2.1.3:

- construction and reorganization of preference vectors;
- definition of the reading sequence;

- construction of the fused ordering.

Sect. 2.1.4 illustrates a variant of the YA, in which the construction of the fused ordering is based on a *top-down* – instead of *bottom-up* – analysis of the preference orderings.

2.1.1 Construction and reorganization of preference vectors

The goal of this phase is building preference vectors based on the (linear) preference orderings by the agents. For each agent's vector, we place the alternatives as they appear in the ordering, with the most preferred one(s) in the top positions. If at any point $t > 1$ alternatives are tied (i.e., indifferent), we place them in the same element and then place the null set ("Null") in the next $t - 1$ lower positions. For example, when considering three alternatives (a, b and c) with the ordering $(a \sim b) > c$, the resulting vector will conventionally be $[\{a \sim b\}, \text{Null}, \{c\}]^T$. By adopting this convention, the number (n) of elements of a vector will coincide with the number of alternatives of interest.

Considering four fictitious agents (D_1 to D_4), with four relevant orderings of six alternatives (a, b, c, d, e and f), and assuming a certain hierarchical ordering between agents (i.e., $D_4 > (D_2 \sim D_3) > D_1$), the resulting preference vectors can be constructed as shown in Tab. 1. For simplicity, vectors will be denominated as the relevant agents (i.e., D_i).

Tab. 1. Construction of preference vectors related to the orderings by four fictitious agents (D_1 to D_4).

Agents			D_1	D_2	D_3	D_4
Orderings			$b > a > (d \sim e) > f > c$	$c > b > (a \sim d \sim e) > f$	$b > (a \sim c) > f > (d \sim e)$	$a > c > b > d > e > f$
Preference vectors	j	$f_j = j/n$	Elem.	Elem.	Elem.	Elem.
	6	$f_6 = 6/6$	$\{b\}$	$\{c\}$	$\{b\}$	$\{a\}$
	5	$f_5 = 5/6$	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{c\}$
	4	$f_4 = 4/6$	$\{d, e\}$	$\{a, d, e\}$	Null	$\{b\}$
	3	$f_3 = 3/6$	Null	Null	$\{f\}$	$\{d\}$
	2	$f_2 = 2/6$	$\{f\}$	Null	$\{d, e\}$	$\{e\}$
	1	$f_1 = 1/6$	$\{c\}$	$\{f\}$	Null	$\{f\}$

n=6 total alternatives are considered: a, b, c, d, e and f.
The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.
$f_{i,j}$ is the cumulative relative frequency of a certain vector element.

Each vector element can be associated with an indicator (f_j), given by the ratio between the position/level (j) of the element (from the bottom) and the total number of elements ($n = 6$ in the case exemplified). f_j can be also interpreted as a relative-position indicator for the alternatives contained in a certain element, since it corresponds to their cumulative relative frequency in the preference vector.

Next, preference vectors are transformed into "reorganized" vectors, conventionally denominated as D_i^*. This transformation consists in (i) sorting the D_i vectors decreasingly with respect to the agents' importance and (ii) aggregating those with indifferent importance (e.g., D_2 and D_3 in the example) into a single vector. This aggregation is performed through a level-by-level union of the vector elements, where alternatives in elements with the same (j-th) position are considered as indifferent. The resulting D_i^* vectors will therefore have a strictly decreasing importance ordering.

Going back to the example in Tab. 1, the four vectors (D_1 and D_4) are turned into three reorganised vectors (D_1^* to D_3^*, see Tab. 2). It can be noted that D_2^* – given by the aggregation of two vectors with equal importance (i.e., D_2 and D_3) – contains two occurrences for each alternative.
Of course, the total number (m) of "reorganized" vectors will be smaller than or equal to the number (M) of initial preference orderings (3 against 4 in the example presented).

Tab. 2. Reorganized vectors (D_i^*) related to the four preference vectors in Tab. 1 and relevant sequence numbers (S).

Agents		D_1^* (D_4)		D_2^* ($D_2 \sim D_3$)		D_3^* (D_1)	
	$f_j = j/n$	S	Elem.	S	Elem.	S	Elem.
Reorganized vectors	$f_6 = 6/6$	16	$\{a\}$	17	$\{b, c\}$	18	$\{b\}$
	$f_5 = 5/6$	13	$\{c\}$	14	$\{a, b, c\}$	15	$\{a\}$
	$f_4 = 4/6$	10	$\{b\}$	11	$\{a, d, e\}$	12	$\{d, e\}$
	$f_3 = 3/6$	7	$\{d\}$	8	$\{f\}$	9	Null
	$f_2 = 2/6$	4	$\{e\}$	5	$\{d, e\}$	6	$\{f\}$
	$f_1 = 1/6$	1	$\{f\}$	2	$\{f\}$	3	$\{c\}$

2.1.2 Definition of the reading sequence

This phase defines a sequence for reading the elements of the D_i^* vectors, according to the following pseudo-code:

1. Initialise the sequence number to $S = 0$.
2. Consider the elements with lowest position, by setting $j = 1$.
3. Consider the most important D_i^* vector, by setting $i = 1$.
4. Set $S = S + 1$.
5. Associate the element of interest with the sequence number S.
6. If $i < m$ (i.e., total number of D_i^* vectors), then:
7. Set $i = i + 1$.
8. Consider the element with position j, related to the i-th D_i^* vector.
9. Go To Step 4.
10. End If.
11. If $j < n$ (i.e., total number of alternatives), then:
12. Set $j = j + 1$.
13. Go To Step 3.
14. End If.
15. End.

The sequence defines a *bottom-up* level-by-level reading of vector elements. The first elements read are those with lowest position (j = 1). When considering elements with the same (j-th) position, priority is given to the vectors of greater importance. After having read all the elements with (j-th) position, we move up to the (j+1)-th position, repeating the reading sequence. Tab. 2 reports the sequence numbers (S) associated with each element of the reorganized vectors.

2.1.3 Construction of the fused ordering

This phase is aimed at determining a consensus fused ordering through a gradual selection of the alternatives. The following pseudo-code illustrates the algorithm for constructing the fused ordering:

1. Initialise the gradual ordering to "Null".
2. Initialise $S = 1$.
3. Consider the element with sequence number S.
4. If the element is not "Null", then:
5. Identify the alternative(s) in the element of interest.
6. If all these alternatives are not yet present in the gradual ordering, then:
7. Include the alternative(s) not yet present at the top of the gradual ordering. Tied alternatives should be considered as indifferent (~).
8. If the gradual ordering includes all the (n) alternatives, then:
9. Go to Step 15.
10. End If.
11. End If.
12. End If.
13. Increment $S = S + 1$.
14. Go to Step 3.
15. The final fused ordering is given by the gradual ordering.
16. End.

The YA can be classified as an AND-ing type as for an alternative to be in a higher positions of the fused ordering, it should be in a higher position for any of the individual orderings (i.e., AND relationship). Reversing the perspective, a generic alternative is excluded from the higher positions of the fused ordering when it is in a lower position in (at least) one of the individual preference orderings.

Applying the algorithm to the vectors in Tab. 2, the resulting fused ordering is $a > b > d > e > c > f$. Tab. 3 shows the gradual construction of the fused ordering; the first two columns report the S value of the element of interest and the alternative(s) that it contains, while the last two report the alternatives not yet included in the gradual ordering and the gradual ordering itself.

Tab. 3. Step-by-step construction of the fused ordering when applying the YA to the example illustrated in Tab. 1.

Step (S)	Element	Residual alternatives	Gradual ordering
0	-	$\{a, b, c, d, e, f\}$	Null
1	$\{f\}$	$\{a, b, c, d, e\}$	f
2	$\{f\}$	$\{a, b, c, d, e\}$	f
3	$\{c\}$	$\{a, b, d, e\}$	$c > f$
4	$\{e\}$	$\{a, b, d\}$	$e > c > f$
5	$\{d, e\}$	$\{a, b\}$	$d > e > c > f$
6	$\{f\}$	$\{a, b\}$	$d > e > c > f$
4	$\{d\}$	$\{a, b\}$	$d > e > c > f$
8	$\{f\}$	$\{a, b\}$	$d > e > c > f$
9	Null	$\{a, b\}$	$d > e > c > f$
10	$\{b\}$	$\{a\}$	$b > d > e > c > f$
11	$\{a, d, e\}$	Null	$a > b > d > e > c > f$
End	-	-	-

2.1.4 Top-down variant of the YA

Yager (2001) points out that the construction of the fused ordering could in theory be based on a top-down reading sequence, instead of bottom-up. In this other case, the level-by-level reading of the vector elements would be analogous to that one illustrated in Sect. 2.1.2, except that it would begin from the elements with highest position ($j = n$), gradually moving down to the lower levels (i.e., $j = n - 1$, $j = n - 2$, and so on). The pseudo-code for determining the reading sequence would be identical to that shown in Sect. 2.1.2, except that steps 2, 11 and 12 should be replaced with the following ones:

2.’ Consider the elements with highest position, by setting $j = n$.

11.’ If $j >1$, then:

12.’ Set $j = j - 1$.

E.g., Tab. 4 reports the top-down reading sequence related to the vectors in Tab. 2.

Tab. 4. Reorganized vectors (D_i^*) related to the four preference vectors in Tab. 1 and relevant sequence numbers (S), in the case of top-down (instead of bottom-up) reading sequence.

Agents		D_1^* (D_4)		D_2^* ($D_2 \sim D_3$)		D_3^* (D_1)	
Reorganized vectors	$f_j = j/n$	S	Elem.	S	Elem.	S	Elem.
	$f_6 = 6/6$	1	$\{a\}$	2	$\{b, c\}$	3	$\{b\}$
	$f_5 = 5/6$	4	$\{c\}$	5	$\{a, b, c\}$	6	$\{a\}$
	$f_4 = 4/6$	7	$\{b\}$	8	$\{a, d, e\}$	9	$\{d, e\}$
	$f_3 = 3/6$	10	$\{d\}$	11	$\{f\}$	12	Null
	$f_2 = 2/6$	13	$\{e\}$	14	$\{d, e\}$	15	$\{f\}$
	$f_1 = 1/6$	16	$\{f\}$	17	$\{f\}$	18	$\{c\}$

The gradual insertion of the alternatives into the fused ordering should be performed according to an increasing preference order (i.e., from the most preferred to the least preferred), instead of decreasing. The relevant pseudo-code would be identical to that shown in Sect. 2.1.3, except that step 7 should be replaced with:

7.’ Include the alternative(s) not yet present at the bottom of the gradual ordering. Tied alternatives should be considered as indifferent (~).

For the purpose of example, the fused ordering resulting from this top-down approach would be: $a > (b \sim c) > (d \sim e) > f$; Tab. 5 shows the step-by-step construction. This approach would be classified as OR-ing type, as for an alternative to be in a higher positions of the fused ordering, it should be in a higher position for at least one of the preference orderings (i.e., OR relationship). Yager states that the implementation of an OR-ing would produce a fused ordering “that is compatible with *at least one* of the individual (preference) orderings”, while that of an AND-ing a fused ordering “that is compatible with *all* the individual orderings” (Yager, 2001, page 4), suggesting that the latter approach is probably more appropriate than the former.

Tab. 5. Step-by-step construction of the fused ordering when applying the top-down variant of the YA to the example illustrated in Tab. 1.

Step (S)	Element	Residual alternatives	Gradual ordering
0	-	$\{a, b, c, d, e, f\}$	Null
1	$\{a\}$	$\{b, c, d, e, f\}$	a
2	$\{b, c\}$	$\{d, e, f\}$	$a > (b \sim c)$
3	$\{b\}$	$\{d, e, f\}$	$a > (b \sim c)$
4	$\{c\}$	$\{d, e, f\}$	$a > (b \sim c)$
5	$\{a, b, c\}$	$\{d, e, f\}$	$a > (b \sim c)$
6	$\{a\}$	$\{d, e, f\}$	$a > (b \sim c)$
4	$\{b\}$	$\{d, e, f\}$	$a > (b \sim c)$
8	$\{a, d, e\}$	$\{f\}$	$a > (b \sim c) > (d \sim e)$
9	$\{d, e\}$	$\{f\}$	$a > (b \sim c) > (d \sim e)$
10	$\{d\}$	$\{f\}$	$a > (b \sim c) > (d \sim e)$
11	$\{f\}$	Null	$a > (b \sim c) > (d \sim e) > f$
End	-	-	-

2.2 Discussion

The fusion technique proposed by Yager is simple, automatable and potentially interesting for a number of applications. Unfortunately, it is grounded on several assumptions that, to some extent, may limit its applicability. We describe them in the following paragraphs.

- One of the strongest assumptions of Yager’s model is that alternatives from different preference vectors are compared on level-by-level basis. In other words, the alternatives are prioritized according to a potentially questionable lexicographical order based on two dimensions: (i) relative position in the preference vectors and (ii) importance of the agent.
- The selection of one alternative is performed at the first occurrence in the reading sequence. This logic is rather drastic as the occurrence of one alternative in a lower position – even for a single preference ordering – can determine a lower positioning in the fused ordering. E.g., in Tab. 1, c is in the penultimate position of the fused ordering as it was “relegated” by D_1 at the bottom of the preference ordering. A practical consequence is that the fused ordering by Yager does not necessarily reflect the preference ordering for the majority of agents (Jianqiang, 2007): e.g., for

agents D_2, D_3 and D_4 (which, by the way, are all more important than D_1), $c > d$ and $c > e$, while in the fused ordering these relationships are reversed. Similar considerations may be applied to the top-down variant of the YA. In this case, the occurrence of one alternative in a higher position – even for a single preference ordering – could determine a higher position in the fused ordering.

- Another consequence of the previous point is that the (bottom-up) YA tends to overlook the upper positions of the preference orderings; e.g., considering the example in Tab. 3, the fused ordering was determined after having read just ten out of seventeen total non-null elements; in particular, the two upper levels of the preference vectors have been totally ignored. Similarly, the top-down variant of the YA tends to overlook the lower levels of the preference orderings.
- In the case the totality of the agents have the same importance, the YA may lose its effectiveness. For example, let us assume that – in the example in Tab. 1 – agents (D_1 to D_4) are equi-important. In the case of bottom-up reading of the preference vectors, the resulting fused ordering would be $(a \sim b) > (d \sim e) > (c \sim f)$, which lacks in discrimination power since it contains nothing less than three relationships of indifference (for six total alternatives).
- As anticipated in Sect. 1, the YA is applicable to linear orderings only, where no alternatives are omitted and any two alternatives are comparable.

The last point deserves special attention. In formal terms, borrowed from Mathematics' Order Theory, a (non-strict) linear ordering satisfies three properties (Nederpelt and Kamareddine, 2004):

Totality: $a \geq b$ or $b \geq a$, (1)

Antisymmetry: if $a \geq b$ and $b \geq a$ then $a = b$, (2)

Transitivity: if $a \geq b$ and $b \geq c$ then $a \geq c$, (3)

where a, b and c are three generic alternatives and the symbol "≥" denotes the "strict preference or indifference" relationship.

A generic linear ordering can be diagrammed as an acyclic line or chain of elements containing the alternatives of interest, linked by arrows depicting the strict preference relationship (see the example in Fig. 1(a)). In this conventional representation, the most preferred alternatives are positioned at the top. Two generic alternatives are always comparable, since there exist a path from the first to the second one (or *vice versa*) that is directed downwards. In other words, each element has an immediate predecessor and successor element, except the first and the last one, with no predecessor and successor respectively.

Having said that, the authors believe that – to fit a relatively large amount of practical contexts – the decision-making problem of interest should admit orderings with omitted and/or incomparable alternatives. In formal terms, any two alternatives in one agent's ordering should not necessarily satisfy the property of *totality* (in Eq. 1) – which states that each alternative should be comparable with the other ones – but only the property of *reflexivity* – which states that each alternative should be comparable with itself:

Reflexivity: $a \geq a$. (4)

According to the Mathematics' Order theory, an ordering that satisfies the three properties of *reflexivity*, *antisymmetry* and *transitivity* (in Eqs. 4, 2 and 3 respectively) is classified as *(non-strict) partial* ordering (Nederpelt and Kamareddine, 2004). This type of ordering can be diagrammed as a graph with branches, which determine different possible paths from the top to the bottom element(s) (see the example in Fig. 1(b)). If two alternatives are not comparable, there exists no direct path from the first to the second one (or viceversa); e.g., in Fig. 1(b), *b* and *h* are incomparable since they lie along paths A and C respectively.

Based on this consideration, authors think that the algorithm by Yager should be enhanced significantly, in order to overcome its limitations and be adaptable to a wider range of practical contexts.

3. Enhanced Yager's Algorithm (EYA)

In this section we introduce the EYA, which is supposed to meet the requirements described in Tab. 6. Likewise the YA, the construction of the fused ordering can be based on a bottom-up or top-town reading sequence of the preference orderings. The choice of a specific approach depends on the specific decision-making problem. Being focused on the lower positions of the preference orderings, the bottom-up approach might be more appropriate for decision-making problems aimed at excluding a relatively small number of "low-rank alternatives" from those of interest. On the other hand, being focused on the upper positions of the preference orderings, the top-down approach might be more appropriate for decision-making problems aimed at isolating and prioritizing a relatively small number of "top-rank alternatives". The remaining description of the EYA will refer to the top-down approach.

Tab. 6. Requirements of the EYA. Symbols "✓" and "×" identify those satisfied and non-satisfied by the YA.

Requirements	Yager's
1. Each agent can have its own individual preference ordering over the alternatives.	✓
2. Agents (can) have a hierarchical importance ordering.	✓
3. The algorithm should be automatable.	✓
4. The preference orderings should include the possibility of ties between two or more alternatives.	✓
5. The preference orderings should include the possibility of omitting one or more alternatives.	×
6. The preference orderings should include the possibility of incomparability between two or more alternatives.	×
7. The logic for selecting alternatives should reflect the preference ordering for the majority of agents.	×
8. All agents (can) have the same importance (*full democracy*).	×

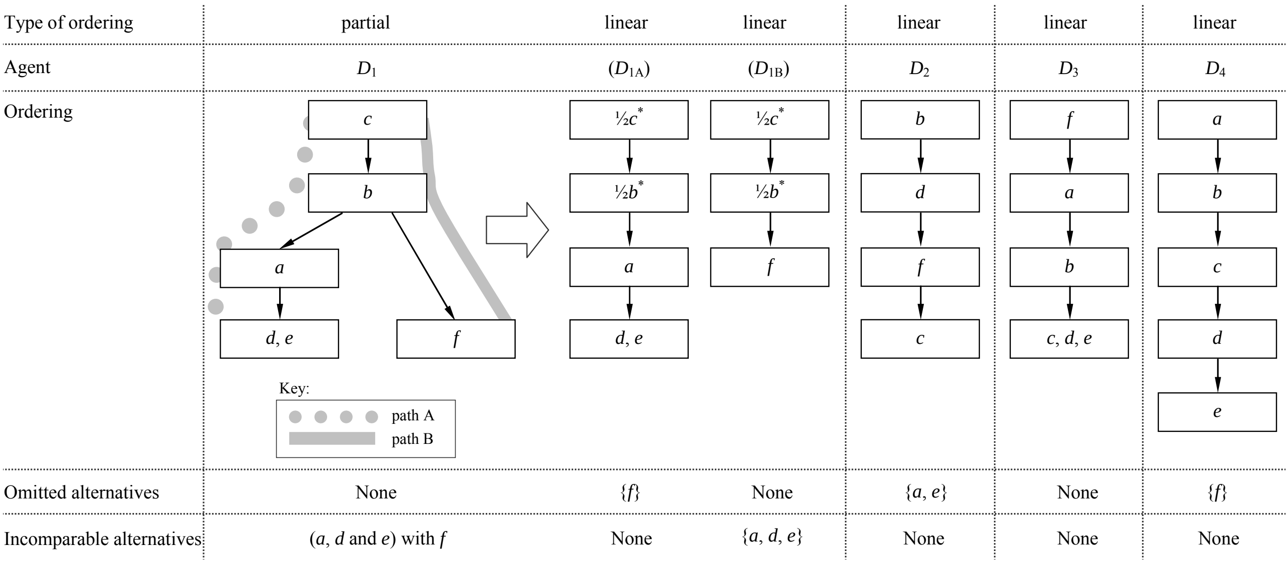
3.1 EYA description

Similarly to the YA, the EYA can be decomposed in three phases, which are individually described in the following sub-sections:

- construction, normalization and reorganization of preference vectors;
- definition of the reading sequence;
- construction of the fused ordering.

3.1.1 Construction, normalization and reorganization of preference vectors

When preference orderings contain incomparable alternatives, the construction of preference vectors is more complicated than for the YA. The first step is to transform each preference ordering with incomparabilities into a set of linear sub-orderings. Precisely, each of these orderings can be artificially split into p linear sub-orderings, corresponding to the possible paths from the top to the bottom element(s). Obviously, the number of paths depends on the configuration of the relevant graph (e.g., amount and position of the branches). For the purpose of example, let us consider the preference orderings illustrated in Fig. 2; the agents' importance ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$. It can be noticed that the (partial) ordering by agent D_1 includes $p = 2$ possible paths (A and B); therefore, the ordering is turned into two linear sub-orderings, D_{1A} and D_{1B}.



(*) Coefficient "½" means that the alternative of interest has a (fractional) number of occurrences in that vector element equals 1/2.

Fig. 2. Graphical representation of the preference orderings by four fictitious agents (D_1 to D_4). The alternatives of interest are *a*, *b*, *c*, *d*, *e* and *f*. The ordering by D_1 has two paths, therefore it is turned into two linear sub-orderings (D_{1A} and D_{1B}). The agents' importance ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$.

The authors are aware that the existing literature includes several techniques for turning partial orderings into linear ones (Marczewski, 1930). It was decided to adopt the above-described technique since it is simple and well-suited to the next steps.

Each alternative in the sub-orderings is associated with a conventional number of occurrences, fractionalised with respect to the number of sub-orderings where the alternative is present. E.g., for *c* and *b*, the fractional number of occurrences is 1/2 as these alternatives are contained in both the sub-orderings D_{1A} and D_{1B}. The importance associated with each *linear* sub-ordering is that of the relevant source (partial) ordering.

The decomposition illustrated must be applied to all the preference orderings with incomparabilities. As an additional example, the ordering in Fig. 1(b) would produce three sub-orderings, related to the three possible paths: i.e., $a > b > d > f > i$ for path A, $a > e > f > i$ for path B, and $a > (g \sim h) > i$ for path C. The fractional number of occurrences of *a* and *i* would be 1/3 while that of *f* would be 1/2 .

Next, the linear (sub-)orderings are turned into preference vectors, according to the convention seen in Sect. 2.1.1. Tab. 7 exemplifies the construction of the preference vectors from the orderings in Fig. 2. Although there are six total alternatives (*a*, *b*, *c*, *d*, *e* and *f*), some of them may be omitted in a certain vector; therefore the number of elements (n_i) can change from a vector to one other. Each vector element is associated with a relative-position indicator, which represents the cumulative relative frequency $f_{i,j}$ – i.e., the ratio between the position (j) of an element, starting from the bottom, and n_i. The set $F_i = \{f_{i,1}, f_{i,2}, ..., f_{i,n_i}\}$ relating to a certain *i*-th preference vector can vary from a vector to one other (in fact, it depends on n_i). For this reason, the subscript "*i*" has been introduced in the notation concerning relative-position indicators (i.e., $f_{i,j}$).

Tab. 7. Construction of preference vectors for the linear (sub-)orderings in Fig. 2.

Agent	D_{1A}			D_{1B}			D_2			D_3			D_4		
Orderings	$c > b > a > (d \sim e)$			$c > b > f$			$b > d > f > c$			$f > a > b > (c \sim d \sim e)$			$a > b > c > d > e$		
No. of alternatives (n_i)	5			3			4			6			5		
Omitted alternative(s)	{*f*}			{*a*, *d*, *e*}			{*a*, *e*}			Null			{*f*}		
Preference vectors	j	$f_{1A,j} = j/n_{1A}$	Elem.	j	$f_{1B,j} = j/n_{1B}$	Elem.	j	$f_{2,j} = j/n_2$	Elem.	j	$f_{3,j} = j/n_3$	Elem.	j	$f_{4,j} = j/n_4$	Elem.
	5	5/5=1.00	{½*c*}	3	3/3=1.00	{½*c*}	4	4/4=1.00	{*b*}	6	6/6=1.00	{*f*}	5	5/5=1.00	{*a*}
	4	4/5=0.80	{½*b*}	2	2/3=0.67	{½*b*}	3	3/4=0.75	{*d*}	5	5/6=0.83	{*a*}	4	4/5=0.80	{*b*}
	3	3/5=0.60	{*a*}	1	1/3=0.33	{*f*}	2	2/4=0.50	{*f*}	4	4/6=0.67	{*b*}	3	3/5=0.60	{*c*}
	2	2/5=0.40	{*d*, *e*}				1	1/4=0.25	{*c*}	3	3/6=0.50	{*c*, *d*, *e*}	2	2/5=0.40	{*d*}
	1	1/5=0.20	Null							2	2/6=0.33	Null	1	1/5=0.20	{*e*}
										1	1/6=0.17	Null			

Six total alternatives are considered: *a*, *b*, *c*, *d*, *e* and *f*.
The agents' importance ordering is $D_4 > (D_2 \sim D_3) > (D_{1A} \sim D_{1B})$.
j denotes the position of an element, starting from the bottom.
$f_{i,j} = j/n_i$ is the cumulative relative frequency referring to the *j*-th element of an *i*-th vector.

Before being reorganized, vectors should be normalized in terms of length, i.e., turned into new vectors with the same number (n_T) of elements. This operation was not present in the YA. We define the set F^*, given by the *union* of the $f_{i,j}$ values relating to the vectors of interest, sorted in ascending order:

$$F^* = sort\left(\bigcup_{\forall i} F_i\right), \tag{5}$$

in which $F_i = \{f_{i,1}, f_{i,2}, ..., f_{i,n_i}\}$ is the set of relative-position indicators relating to a certain *i*-th preference vector and the "*sort*" operator represents the ascending order permutation. For example, considering the five preference vectors in Tab. 7, it is obtained:

$$F^* = \{0.17, 0.20, 0.25, 0.33, 0.40, 0.50, 0.60, 0.67, 0.75, 0.80, 0.83, 1.00\}. \tag{6}$$

Being independent on a particular *i*-th vector, the elements in F^* can be conventionally renamed as f_j^* (without subscript "*i*"):

$$F^* = \{f_1^*, f_2^*, ...\}. \tag{7}$$

Next, we define:

$$n_T = card(F^*), \tag{8}$$

i.e., the total number of elements of F^*; e.g., considering the ordered set in Eq. 6, $n_T = 12$. Obviously, $n_i \leq n_T \ \forall i$.

Each *i*-th preference vector can be now normalized by adding a "Null" element for each f_j^* value that is included in the set F^* but not included in F_i, following a decreasing sequence. Tab. 8 exemplifies this mechanism for the preference vectors in Tab. 7.

Tab. 8. Normalization of the preference vectors in Tab. 7.

Agents		D_{1A}	D_{1B}	D_2	D_3	D_4
	f_j^*	Elem.	Elem.	Elem.	Elem.	Elem.
	1.00	{½*c*}	{½*c*}	{*b*}	{*f*}	{*a*}
	0.83	Null	Null	Null	{*a*}	Null
	0.80	{½*b*}	Null	Null	Null	{*b*}
	0.75	Null	Null	{*d*}	Null	Null
Normalized vectors	0.67	Null	{½*b*}	Null	{*b*}	Null
	0.60	{*a*}	Null	Null	Null	{*c*}
	0.50	Null	Null	{*f*}	{*c*, *d*, *e*}	Null
	0.40	{*d*, *e*}	Null	Null	Null	{*d*}
	0.33	Null	{*f*}	Null	Null	Null
	0.25	Null	Null	{*c*}	Null	Null
	0.20	Null	Null	Null	Null	{*e*}
	0.17	Null	Null	Null	Null	Null

Six total alternatives are considered: *a*, *b*, *c*, *d*, *e* and *f*.
The agents' importance ordering is $D_4 > (D_2 \sim D_3) > (D_{1A} \sim D_{1B})$.
The elements highlighted in grey have been added to normalize the number of elements of each vector to $n_T = 12$.

The mechanism for normalizing the preference vectors deserves special attention. Since the elements of a vector have an ordinal relationship, the concept of "distance" is meaningless (Roberts, 1979). Vector normalization is performed by using the information on the relative position of the elements in the source vectors. It's interesting to note that, in the case preference vectors have the same number of elements (case of the YA), this normalization is not required.

Next, the normalized vectors are turned into reorganized vectors (D_i^*). The aggregation is identical to that of the YA, i.e., the normalized D_i vectors are sorted decreasingly with respect to the agents' importance and (ii) vectors with indifferent importance (e.g., D_{1A}, D_{1B} and D_2, D_3 in the example) are aggregated into a single one, through a level-by-level union of their elements. Going back to the example in Tab. 7, the resulting reorganized vectors are three (i.e., D_1^* to D_3^*, see Tab. 9).

Tab. 9. Construction of reorganized vectors related to the preference vectors in Tab. 7. *S* are the resulting sequence numbers, obtained by applying the logic illustrated in Sect. 3.1.2. The elements of D_2 and D_3 are merged into D_2^*, while those of D_{1A} and D_{1B} into D_3^*.

Agents		D_1^* (D_4)		D_2^* ($D_2 \sim D_3$)		D_3^* ($D_{1A} \sim D_{1B}$)	
	f_j^*	S	Elem.	S	Elem.	S	Elem.
Reorganized vectors	1.00	1	{a}	2	{b, f}	3	{c}
	0.83	4	Null	5	{a}	6	Null
	0.80	7	{b}	8	Null	9	{½b}
	0.75	10	Null	11	{d}	12	Null
	0.67	13	Null	14	{b}	15	{½b}
	0.60	16	{c}	17	Null	18	{a}
	0.50	19	Null	20	{c, d, e, f}	21	Null
	0.40	22	{d}	23	Null	24	{d, e}
	0.33	25	Null	26	Null	27	{f}
	0.25	28	Null	29	{c}	30	Null
	0.20	31	{e}	32	Null	33	Null
	0.17	34	Null	35	Null	36	Null

Again, the rationale behind this operation is that the degree of preference of the alternatives from different preference vectors depends mainly on their relative position. For a certain aggregated vector, the resulting f_j^* values reflect the position of the elements in the source vectors but they should not be interpreted as indicators of cumulative relative frequency of the aggregated vector's elements. In other words, they do not necessarily correspond to j/n_T, being j the position of an element in that aggregated vector.

The strong assumption is that, for a certain aggregated vector, the alternatives contained in elements with identical f_j^* values are considered as indifferent, while those in elements with non-identical f_j^* values (although very close to each other) not. This aggregation mechanism could be refined by introducing suitable preference/indifference thresholds.

3.1.2 Definition of the reading sequence

The object of this phase is determining a sequence for reading the elements of the reorganized vectors. Likewise Yager's algorithm, (i) vector elements can be read according to a bottom-up or top-down sequence and (ii) the importance of D_i^* vectors is taken into account when establishing the reading sequence. The following pseudo-code illustrates the algorithm, in the case of a top-down approach:

1. Initialise the sequence number to $S = 0$.
2. Consider the elements with highest position, by setting $j = n_T$.
3. Consider the most important D_i^* vector, by setting $i = 1$.

4. Set $S = S + 1$.

5. Associate the element of interest with the sequence number S.

6. If $i < m$ (i.e., total number of D_i^* vectors), then:

7. Set $i = i + 1$.

8. Consider the element with position j, related to the i-th D_i^* vector.

9. Go To Step 4.

10. End If.

11. If $j > 1$, then:

12. Set $j = j – 1$.

13. Go To Step 3.

14. End If.

15. End.

Tab. 9 reports the full sequence numbers (S) associated with each element of the reorganized vectors. The suggested sequencing strategy is midway between the two extremes of *full dictatorship* – in which the fused ordering coincides with the preference ordering by the most important agent (dictator), neglecting the others – and (ii) *full democracy* – where all agents' orderings are considered as equi-important. A practical consequence of this strategy is that it gives priority to the preference vectors related to the most important agents and with a relatively high number of alternatives.

The authors are aware that this is just one of the possible mechanisms for determining a reading sequence. However, we chose this because it is simple, easy to automate and, in the case preference vectors have the same number of elements, it "degenerates" into that suggested by Yager.

3.1.3 Construction of the fused ordering

The following pseudo-code illustrates the procedure for determining the fused ordering in the case of top-down reading sequence:

1. Initialise the gradual ordering to "Null".

2. For each (k-th) alternative, initialise the counter of the occurrences (O_k) to 0

3. Initialise $S = 1$.

4. Consider the element with sequence number S.

5. If the element is not "Null", then:

6. Identify the alternative(s) in the element of interest.

7. Initialize the set (E) of the alternatives to be included in the general ordering to "Null".

8. For each of the alternatives in the element of interest:

9. $O_k = O_k + O_{k,S}$, being $O_{k,S}$ the (fractional) number of occurrences of the k-th alternative in that element (associated with S).

10. If $O_k \geq T_{k,x}$ (i.e., the threshold value) and the alternative is not yet present in the gradual ordering, then:

11. Insert the alternative of interest in the set (E) of those to be included in the general ordering.

12. End If.

13. End For.

14. If E is not "Null", then:

15. Insert the alternatives contained in E at the bottom of the gradual ordering. In case of multiple alternatives, consider them as indifferent.

16. If the totality of the alternatives have been included in the gradual ordering, then:

17. Go To step 23.

18. End If.

19. End If.

20. End If.

21. Increment $S = S + 1$.

22. Go To step 4

23. The final fused ordering is given by the gradual ordering.

24. End.

We remark that a k-th alternative is included into the fused ordering when – during the reading sequence – the gradual number of occurrences (O_k) reaches a certain threshold, i.e.:

$$T_{k,x} = x \cdot O_k^T \, , \tag{9}$$

being x a conventional percentage of the total number of occurrences (O_k^T) in the D_i^* vectors' elements. x can be considered as an indirect measure of the level of democracy in the decision. Low values of x entail a low involvement of agents; on the contrary high values of x imply a high involvement of agents. Tab. 10 shows the $T_{k,x}$ values related to the alternatives; x was conventionally set to 50% (simple majority). We remark that the standard algorithm by Yager works as if $T_{k,x} = 1, \ \forall k$.

Tab. 10. Thresholds for the selection of the alternatives; *x* was conventionally set to 50%.

Alternatives	a	b	c	d	e	f
Total no. of occurrences O_k^{TOT}	3	4	4	4	3	3
$T_{k,50\%}$	1.5	2	2	2	1.5	1.5

Applying the algorithm to the orderings in Tab. 7 and using the thresholds in Tab. 10, the fused preference ordering is $a > b > c > (d \sim f) > e$. Tab. 11 shows the step-by-step results; the last columns contains the gradual ordering.

Tab. 11. Step-by-step construction of the fused ordering in the case of top-down sequence. S is the sequence number, while the subsequent columns refer to the construction of the gradual ordering. We remark that an alternative is added to the gradual ordering when the cumulative number of occurrences (O_k) reaches $T_{k,x}$ (see the numeric values in Tab. 10).

Step (S)	Element	Occurrences (O_k)						Residual alternatives	Gradual ordering
		a	*b*	*c*	*d*	*e*	*f*		
0	-	-	-	-	-	-	-	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
1	{*a*}	1	0	0	0	0	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
2	{*b*, *f*}	1	1	0	0	0	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
3	{*c*}	1	1	1	0	0	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
4	Null	1	1	1	0	0	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
5	{*a*}	2	1	1	0	0	1	{*b*, *c*, *d*, *e*, *f*}	*a*
6	Null	2	1	1	0	0	1	{*b*, *c*, *d*, *e*, *f*}	*a*
7	{*b*}	2	2	1	0	0	1	{*c*, *d*, *e*, *f*}	$a > b$
8	Null	2	2	1	0	0	1	{*c*, *d*, *e*, *f*}	$a > b$
9	{½*b*}	2	2.5	1	0	0	1	{*c*, *d*, *e*, *f*}	$a > b$
10	Null	2	2.5	1	0	0	1	{*c*, *d*, *e*, *f*}	$a > b$
11	{*d*}	2	2.5	1	1	0	1	{*c*, *d*, *e*, *f*}	$a > b$
12	Null	2	2.5	1	1	0	1	{*c*, *d*, *e*, *f*}	$a > b$
13	Null	2	2.5	1	1	0	1	{*c*, *d*, *e*, *f*}	$a > b$
14	{*b*}	2	3.5	1	1	0	1	{*c*, *d*, *e*, *f*}	$a > b$
15	{½*b*}	2	4	1	1	0	1	{*c*, *d*, *e*, *f*}	$a > b$
16	{*c*}	2	4	2	1	0	1	{*d*, *e*, *f*}	$a > b > c$
17	Null	2	4	2	1	0	1	{*d*, *e*, *f*}	$a > b > c$
18	{*a*}	3	4	2	1	0	1	{*d*, *e*, *f*}	$a > b > c$
19	Null	3	4	2	1	0	1	{*d*, *e*, *f*}	$a > b > c$
20	{*c*, *d*, *e*, *f*}	3	4	3	2	1	2	{*e*}	$a > b > c > (d \sim f)$
21	Null	3	4	3	2	1	2	{*e*}	$a > b > c > (d \sim f)$
22	{*d*}	3	4	3	3	1	2	{*e*}	$a > b > c > (d \sim f)$
23	Null	3	4	3	3	1	2	{*e*}	$a > b > c > (d \sim f)$
24	{*d*, *e*}	3	4	3	4	2	2	Null	$a > b > c > (d \sim f) > e$
End	-	-	-	-	-	-	-	-	-

3.2 Discussion

In the following paragraphs, we discuss the major features of the EYA, focussing on its advantages with respect to the YA.

- The OR-ing philosophy – which characterises the top-down variant of YA – is mitigated significantly: a *k*-th alternative is included in the higher positions of the fused ordering when a predetermined portion (x) of its occurrences (not just a single one!) are in a higher position of the individual preference orderings. Reversing the perspective, for an alternative to be in a lower position of the fused ordering, a portion of the occurrences larger than $(1 – x)$ should be in lower positions of the individual preference orderings. This strategy makes it possible to construct a fused ordering that does not overlook the lower positions of the preference orderings; e.g., in the example illustrated in Tab. 11, the fused ordering is determined after having read more than the 80% of the non-null vector elements (i.e., fourteen out of seventeen), not only those in the upper positions. As a further example, applying the EYA (with top-down approach) to the linear orderings in Tab. 1, the resulting fused ordering would be: $b > c > a > (d \sim e) > f$.

- It can be shown that the EYA provides a fused ordering which is consistent with the input preference orderings. Tab. 12 compares the resulting fused ordering with the input preference orderings, at the level of paired comparisons between alternatives. For each paired comparison (in the first column), we report the preference ordering(s) in which it is involved, specifying the corresponding relation(s) (">", "<" or "~"). With rare exceptions, the relation obtained from the fused ordering reflects the relations in the majority of the preference orderings. This result demonstrates the consistency of the EYA solution, even in situations – such as that exemplified – in which some alternatives are characterised by relatively large fluctuations in the preference orderings (e.g., consider the fluctuations of *c* and *f*).

Tab. 12. Comparison between the fused ordering and the preference orderings by individual respondents, at the level of paired comparisons. Symbols ">", "<", "~" and "||"denote respectively the strict preference, reverse strict preference, indifference and incomparability relationship. The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.

Paired comparison	Relations in the preference ordering(s) >	<	~	||	Relation in the fused ordering
a, b	D_3, D_4	D_1	Null	Null	>
a, c	D_3, D_4	D_1	Null	Null	>
a, d	D_1, D_3, D_4	Null	Null	Null	>
a, e	D_1, D_3, D_4	Null	Null	Null	>
a, f	Null	D_3	Null	D_1	>
b, c	D_2, D_3, D_4	D_1	Null	Null	>
b, d	D_1, D_2, D_3, D_4	Null	Null	Null	>
b, e	D_1, D_3, D_4	Null	Null	Null	>
b, f	D_1, D_2	D_3	Null	Null	>
c, d	D_1, D_4	D_2	D_3	Null	>
c, e	D_1, D_4	Null	D_3	Null	>
c, f	D_1	D_2, D_3	Null	Null	>
d, e	D_4	Null	D_1, D_3	Null	>
d, f	D_2	D_3	Null	D_1	~
e, f	Null	D_3	Null	D_1	<

- With a few minor changes, the top-down reading sequence could be turned into a bottom-up one. As regards the determination of the reading sequence (see the pseudo-code in Sect. 3.1.2), steps 2, 11 and 12 should be replaced with:

  2.' Consider the elements with lowest position, by setting $j = 1$.

  11.' If $j < n_T$, then:

  12.' Set $j = j + 1$.

  As regards the gradual construction of the fused ordering (see the pseudo-code in Sect. 3.1.3), step 15 should be replaced with:

  15.' Insert the alternatives contained in E at the top of the gradual ordering. In case of multiple alternatives, consider them as indifferent.

  For the purpose of example, the fused ordering obtained by applying the bottom-up approach to the preference vectors in Tab. 7, would be: $a > b > c > f > d > e$.

- The EYA can be applied effectively even in the case all agents are equi-important (full democracy). For the purpose of example, let us consider the preference orderings in Tab. 7 under this assumption. The individual orderings would be merged into a single reorganized vector (D_1^* in Tab. 13(a)) and the reading sequence of the vector elements would be trivial: i.e., from the top to the bottom. When using x = 50%, the resulting fused ordering would be $a > b > c > (d \sim f) > e$ (see the step-by-step construction in Tab. 13(b)). This solution seems to have an acceptable discrimination power (i.e., it only contains one indifference relationships) and, coincidentally, is identical to that obtained when considering the agents' importance ranking (see Tab. 11).

Tab. 13. Application of the EYA to the four preference orderings in Fig. 2, in the case of top-down sequence, assuming that agents are equi-important: (a) single reorganized vector; (b) step-by-step construction of the fused ordering.

(a)

D_1^*		
f_j^*	S	Elements
1.00	1	{a, b, c, f}
0.83	2	{a}
0.80	3	{3/2b}
0.75	4	{d}
0.67	5	{3/2b}
0.60	6	{a, c}
0.50	7	{c, d, e, f}
0.40	8	{2d, e}
0.33	9	{f}
0.25	10	{c}
0.20	11	{e}
0.17	12	Null

(b)

Step (S)	Occurrences (O_k)						Residual alternatives	Gradual ordering
	a	b	c	d	e	f		
0	-	-	-	-	-	-	{a, b, c, d, e, f}	Null
1	1	1	1	0	0	1	{a, b, c, d, e, f}	Null
2	2	1	1	0	0	1	{b, c, d, e, f}	a
3	2	2.5	1	0	0	1	{c, d, e, f}	$a > b$
4	2	2.5	1	1	0	1	{c, d, e, f}	$a > b$
5	2	4	1	1	0	1	{c, d, e, f}	$a > b$
6	3	4	2	1	0	1	{d, e, f}	$a > b > c$
7	3	4	3	2	1	2	{e}	$a > b > c > (d \sim f)$
8	3	4	3	4	2	2	Null	$a > b > c > (d \sim f) > e$
End	-	-	-	-	-	-	-	-

- A potentially controversial aspect of the EYA is choosing the $T_{k,x}$ threshold values. As $T_{k,x}$ values decrease, the algorithm tends to degenerate into the YA (where $T_{k,x} = 1 \quad \forall k$), incurring the disadvantages described in Sect. 2.2. On the other hand, as $T_{k,x}$ values increase, preference vectors are read in more depth, focussing the attention on the lower-level elements (when adopting a top-down approach).

  As regards the percentages of occurrences x, we think that all the alternatives should have the same x value, in order not to favour the selection of some alternatives (i.e., those with relatively lower x values) to the detriment of others. We re-emphasize that x can be considered as a proxy of the level of democracy in the decision. From an operating point of view, it does not seem unreasonable to set x to a value roughly in the middle of the range [0, 1]. In this case it conveys the consensus of the majority of agents (simple majority).

  The fact remains that the choice of the $T_{k,x}$ values, as that of every conventional threshold, is arbitrary. In general, it would be advisable to evaluate the robustness of the fused ordering through a *sensitivity analysis* with respect to small variations in the $T_{k,x}$ values, analyzing the stability of the EYA solution. Tab. 14 reports three different groups of $T_{k,x}$ values and the resulting fused orderings. It can be noticed that, despite the use of different $T_{k,x}$ values, the three fused orderings are not so dissimilar from each other. This result confirms the robustness of the EYA (Franceschini et al., 2014).

Tab. 14. Data concerning sensitivity analysis: $T_{k,x}$ values relating to x = 33%, 50% and 67%, and resulting fused orderings.

x	$T_{k,x}$						Fused ordering
	a	b	c	d	e	f	
33%	1	1.3	1.3	1.3	1	1	$a > f > b > c > (d \sim e)$
50%	1.5	2	2	2	1.5	1.5	$a > b > c > (d \sim f) > e$
67%	2	2.7	2.7	2.7	2	2	$a > b > (c \sim d \sim f) > e$

4. Conclusions

This paper proposed an enhanced version of the YA, which has two main advantages: (i) it better reflects the multi-agent preference orderings and (ii) it is more flexible, since it admits preference orderings with omitted or incomparable alternatives. Also, it is automatable and can be applied to a larger variety of practical contexts, providing more reasonable results.

A delicate feature of the new algorithm is related to the mechanism for aggregating and/or comparing elements from different preference vectors. The underlying assumption is that the degree of preference of the alternatives in different preference vectors mainly depends on their relative position, depicted by $f_{i,j}$ indicators. Another potentially controversial aspect of the EYA is the choice of the $T_{k,x}$ values. It was shown that a sensitivity analysis can help to tackle this problem.

Future research go in several directions: (i) application of the algorithm to various decision-making frameworks (Franceschini et al., 2014) and (ii) revision of the logic for aggregating the preference vectors by more sophisticated/refined mechanisms.

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