The solution of Hutník's open problem

Michał Boczek [1], Marek Kaluszka

Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland

Abstract

In this note, we give a solution to Problem 9.2, which was presented by Mesiar and Stupňanová [*Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications*, Fuzzy Sets and Systems (2014), http://dx.doi.org/10.1016/j.fss.2014.07.012]. We show that the class of semicopulas solving Problem 9.2 contains only the Łukasiewicz t-norm.

Keywords: Integral; Capacity; Semicopula; Łukasiewicz t-norm.

1 The main result

Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of non-empty set X, and let $\mathcal{S}$ be the family of all measurable spaces. The class of all $\mathcal{A}$-measurable functions $f\colon X \to [0,1]$ is denoted by $\mathcal{F}_{(X,\mathcal{A})}$. A *capacity* on $\mathcal{A}$ is a non-decreasing set function $\mu\colon \mathcal{A} \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$. We denote by $\mathcal{M}_{(X,\mathcal{A})}$ the class of all capacities on $\mathcal{A}$. Let $S\colon [0,1]^2 \to [0,1]$ be a semicopula (also called a *t-seminorm*), i.e., a non-decreasing function in both coordinates with the neutral element equal to 1, and satisfying the inequality $S(x,y) \leqslant x \wedge y$ for all $x, y \in [0,1]$, where $x \wedge y = \min(x,y)$ (see [1], [2] and [4]). By the above assumptions it follows that $S(x,0) = 0 = S(0,x)$ for all x. There are three important examples of semicopulas: M, Π and S_L, where $M(a,b) = a \wedge b$, $\Pi(a,b) = ab$ and $S_L(a,b) = (a+b-1) \vee 0$; S_L is called the *Łukasiewicz t-norm* [5]. Hereafter, $a \vee b = \max(a,b)$.

A class of the smallest semicopula-based integrals is given by

$$\mathbf{I}_S(\mu, f) = \sup_{t \in [0,1]} S\Big(t, \mu\big(\{f \geqslant t\}\big)\Big),$$

where $\{f \geqslant t\} = \{x \in X \colon f(x) \geqslant t\}$, $(X, \mathcal{A}) \in \mathcal{S}$ and $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$. Replacing semicopula S with M, we get the Sugeno integral [8]. Moreover, if $S = \Pi$, then S is called the Shilkret integral [7].

Below we present Problem 9.2 from [6], which was posed by Hutník during the conference FSTA 2014 *The Twelfth International Conference on Fuzzy Set Theory and Applications* held from January 26 to January 31, 2014 in Liptovský Ján, Slovakia.

[1] Corresponding author. E-mail adress: 800401@edu.p.lodz.pl;

Problem 1. (*O. Hutník*) *Characterize a class of semicopulas* S, *for which the equality*

$$\mathbf{I}_S(\mu, f+a) = \mathbf{I}_S(\mu, f) + a \tag{1}$$

holds for each $(X, \mathcal{A}) \in \mathcal{S}$, *each* $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$ *and* $a \in [0,1]$ *such that* $f + a \in [0,1]$.

Halčinová, Hutník and Molnárová [3] showed that the integral equality (1) holds if S is the Łukasiewicz t-norm. We show that the class of semicopulas solving Problem 1 contains only the Łukasiewicz t-norm.

Firstly, we can observe that $\mu\big(\{f+a \geqslant t\}\big) = 1$ for $t \in [0,a]$, so the left-hand side of (1) has the form

$$\begin{aligned}\mathbf{I}_S(\mu, f+a) &= \sup_{t\in[0,1]} S\Big(t, \mu\big(\{f+a \geqslant t\}\big)\Big)\\ &= \max\Big[\sup_{t\in[0,a]} S(t,1),\ \sup_{t\in(a,1]} S\Big(t, \mu\big(\{f+a \geqslant t\}\big)\Big)\Big]\\ &= a \vee \sup_{t\in(0,1-a]} S\Big(t+a, \mu\big(\{f \geqslant t\}\big)\Big),\end{aligned} \tag{2}$$

since $S(a,1) = a$ for all a. Let

$$\mu\big(\{f \geqslant t\}\big) = \begin{cases} 1 & \text{if } t = 0,\\ b & \text{if } t \in (0, 1-a],\\ 0 & \text{if } t \in (1-a, 1],\end{cases} \tag{3}$$

where $b \in [0,1]$. By (2) and (3), equality (1) can be rewritten as follows

$$a \vee \sup_{t\in(0,1-a]} S(t+a, b) = \sup_{t\in(0,1-a]} S(t,b) + a.$$

As S is a non-decreasing function, we have

$$a \vee S(1,b) = S(1-a, b) + a$$

for all $a, b \in [0,1]$. Now, let $c = 1-a \in [0,1]$. Thus $(1-c) \vee b = S(c,b) + (1-c)$. If $1-c \geqslant b$, then $S(c,b) = 0$. If $1-c < b$, then $S(c,b) = c+b-1$. Summing up, $S(c,b) = S_L(c,b)$ for all $c, b \in [0,1]$.

Acknowledgments

This paper has been partially supported by the grant for young researchers from Lodz University of Technology, grant number 501\17-2-2-7154\2014.

References

[1] B. Bassan, F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Anal. 93 (2005), 313–339.

[2] F. Durante, C. Sempi, Semicopulæ. Kybernetika 41 (2005), 315-328.

[3] J. Borzowá-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals I: properties and characterizations, Fuzzy Sets and Systems (2014) http://dx.doi.org/10.1016/j.fss.2014.09.0232014.

[4] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer Academic Publishers, Dordrecht, 2000.

[5] J. Łukasiewicz, O logice trójwartościowej (in Polish). Ruch filozoficzny (1920) 170–171. English translation: On three-valued logic, in L. Borkowski (ed.), Selected works by Jan Łukasiewicz, North–Holland, Amsterdam, (1970), 87–88.

[6] R. Mesiar, A. Stupňanová, Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications, Fuzzy Sets and Systems (2014), http://dx.doi.org/10.1016/j.fss.2014.07.012

[7] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109-116.

[8] M. Sugeno, Theory of Fuzzy Integrals and its Applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.