Fuzzy Sets and Formal Logics

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– Dedicated to the memory of Franco Montagna –

Abstract

The paper discusses the relationship between fuzzy sets and formal logics as well as the influences fuzzy set theory had on the development of particular formal logics. Our focus is on the historical side of these developments.

Key words: mathematical fuzzy logics, fuzzy sets, graded membership, graded entailment, monoidal logic, basic fuzzy logic, monoidal fuzzy logic, R-implications,

1 Introduction

The theory of standard, i.e. crisp sets is strongly tied with classical logic. This becomes particularly obvious if one looks at the usual set algebraic operations like intersection and union. These can for crisp sets A, B be characterized by the conditions

$$x \in A \cap B \quad \Leftrightarrow \quad x \in A \wedge x \in B\,, \tag{1}$$
$$x \in A \cup B \quad \Leftrightarrow \quad x \in A \vee x \in B\,. \tag{2}$$

The theory of fuzzy sets, as initiated in 1965 by Lotfi A. Zadeh [114], started with quite similar definitions for the membership degrees of the set algebraic operations:

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$$\begin{aligned} \mu_{A\cap B}(x) &= \min\{\mu_A(x), \mu_B(x)\}\,, && (3)\\ \mu_{A\cup B}(x) &= \max\{\mu_A(x), \mu_B(x)\}\,, && (4) \end{aligned}$$

but offered also other operations for fuzzy sets, called "algebraic" by Zadeh, as, e.g., an algebraic product AB and an algebraic sum $A + B$ defined via the equations [1]

$$\begin{aligned} \mu_{AB}(x) &= \mu_A(x) \cdot \mu_B(x)\,, && (5)\\ \mu_{A+B}(x) &= \min\{\mu_A(x) + \mu_B(x), 1\}\,. && (6) \end{aligned}$$

Zadeh [114] designed the fuzzy sets as a mathematical tool for the modeling of vague notions. Essentially he did not relate his fuzzy sets to non-classical logics. There was only a minor exception. In discussing the meaning of the membership degrees he explained (in a "comment" pp. 341–342; and with reference to the monograph [92] and Kleene's three valued logic) with respect to two thresholds $0 < \beta < \alpha < 1$ that one may

> "say that (1) "x belongs to A" if $\mu_A(x) \geq \alpha$; (2) "x does not belong to A" if $\mu_A(x) \leq \beta$; and (3) "x has an indeterminate status relative to A" if $\beta < \mu_A(x) < \alpha$." [2]

Also the overwhelming majority of fuzzy set papers that followed [114] treated fuzzy sets in the standard mathematical context, i.e. with an implicit reference to a naively understood classical logic as argumentation structure.

Here we sketch the way fuzzy sets and the idea of membership grading have been strongly related to non-classical, particularly many-valued logics.

This has not been an obvious development. Even philosophically oriented predecessors of Zadeh in the discussion of vague notions, like Max Black in 1937 [4] and Carl Hempel in 1939 [82], did refer only to classical logic, even in those parts of these papers in which they discuss the problem of some incompatibilities of the naively correct use of vague notions and principles of classical logic, e.g., concerning the treatment of negation-like statements. And also the most direct forerunner of fuzzy set theory, Karl Menger used in 1951 only classical logic [96].

This paper is structured as follows. After this introduction, we start from the early relationship between fuzzy sets and Łukasiewicz logic up to the introduc-

[1] The reader should be aware that equation (6) is not Zadeh's original formulation. He introduced the algebraic sum as a partial operation for fuzzy sets, defined only if $\mu_A(x) + \mu_B(x) \leq 1$ was always satisfied. We disregard this minor(?) difference here.
[2] The cautious reader should be aware that we use here the more common notation μ_A instead of f_A from [114].

tion of t-norms into fuzzy set theory. Then we discuss t-norm based logics for fuzzy set theory, their semantics and their axiomatizations. These ideas give rise to a whole zoo of related formal logics, as well as to the idea of logics with graded notions of inference. Finally we take a look at two important, more application oriented developments: fuzzy logics for reasoning about probabilities, and a formal-logical treatment of Zadeh's basic ideas for approximate reasoning.

2 Fuzzy sets and Łukasiewicz logic

In parallel, and independent of the approach by Zadeh, the German mathematician Dieter Klaua presented in 1965/66 two versions [88,89] for a cumulative hierarchy of so-called *many-valued* sets. [3] These many-valued sets had the fuzzy sets of Zadeh as a particular case.

Historically, Zadeh's approach proved to be much more influential than that of Klaua.

In Klaua's two versions [88,89] for a cumulative hierarchy of fuzzy sets he considered as membership degrees the real unit interval $\mathcal{W}_\infty = [0,1]$ or a finite, m-element set $\mathcal{W}_m = \{\frac{k}{m-1} \mid 0 \leq k < m\}$ of equidistant points of $[0,1]$. He also started his cumulative hierarchies from sets U of urelements. The infinite-valued case with membership degree set $\mathcal{W}_\infty = [0,1]$ gives, in both cases, on the first level of these hierarchies just the fuzzy sets over the universe of discourse U in the sense of Zadeh.

Furthermore Klaua understood the membership degrees as the truth degrees of the corresponding Łukasiewicz systems L_∞ or L_m, respectively.

And indeed, the majority of results in [89,90] were presented using the language of these Łukasiewicz systems. Some examples are:

$$
\begin{aligned}
&\models \quad A \subseteqq B \,\&\, B \subseteqq C \to_{\mathsf{L}} A \subseteqq C\,, \\
&\models \quad a \,\varepsilon\, B \,\&\, B \subseteqq C \to_{\mathsf{L}} a \,\varepsilon\, C\,, \\
&\models \quad A \equiv B \,\&\, B \subseteqq C \to_{\mathsf{L}} A \subseteqq C\,.
\end{aligned}
$$

Here $\to_{\mathsf{L}}$ is the Łukasiewicz implication with truth degree function $(u,v) \mapsto \min\{1, 1-u+v\}$, & the strong (or: arithmetical) conjunction with truth degree function $(u,v) \mapsto \max\{0, u+v-1\}$, ε the graded membership predicate, and

[3] The German language name for these objects was "mehrwertige Mengen". The stimulus for these investigations came from discussions following a colloquium talk which K. Menger had given in Berlin (East) in the first half of the 1960s, cf. [91].

$\models \varphi$ means that the formula φ of the language of Łukasiewicz logic is logically valid, i.e. assumes always truth degree 1.

Furthermore, a graded inclusion relation $\subseteqq$ is defined (for fuzzy sets of the same level in the hierarchy) as

$$A \subseteqq B \ =_{\mathrm{df}} \ \forall x (x \,\varepsilon\, A \rightarrow_{\mathsf{L}} x \,\varepsilon\, B)\,, \tag{7}$$

and a graded equality $\equiv$ for fuzzy sets is defined as

$$A \equiv B \ =_{\mathrm{df}} \ A \subseteqq B \ \wedge \ B \subseteqq A\,. \tag{8}$$

where $\wedge$ is the weak conjunction with truth degree function $(u, v) \mapsto \min\{u, v\}$.

These are prototypical examples for fuzzy, i.e. graded, relationships which appear quite naturally in a fuzzy sets context.

This line of approach was continued in the early 1970s, e.g., in the second author's papers [58,59]. The topic of [58] is the formulation of (crisp) properties of fuzzy relations. The natural continuation, to consider graded properties of fuzzy relations, was realized for the particular cases of the graded uniqueness of fuzzy relations and the graded equipollence of fuzzy sets in [59,60]. A more general approach toward graded properties of fuzzy relations was sketched in [63] and more explicitly presented in [64].

Another author who pointed out a strong relationship between fuzzy sets and many-valued logic is Robin Giles. Starting in 1975, he proposed in a series of papers [50–52], and again in [53], a general treatment of reasoning with vague predicates by means of a formal system based upon a convenient dialogue interpretation. He had already used this dialogue interpretation in other papers, like [49], with the aim of dealing with subjective belief and the foundations of physics. The main idea is to let "*a sentence represent a belief by expressing it tangibly in the form of a bet*". In this setting then

> "... a sentence ψ is considered to follow from sentences $\varphi_1, \cdots, \varphi_n$ just when who accepts the bets on $\varphi_1, \cdots, \varphi_n$ can at the same time bet on ψ without fear of loss."

The (formal) language obtained in this way is closely related to Łukasiewicz's infinite-valued logic L_∞: in fact the two systems coincide if one assigns to a sentence φ the truth value $1 - \langle\varphi\rangle$, with $\langle\varphi\rangle$ being the risk value of asserting φ. And he even adds this remark:

> " ... with this dialogue interpretation, Łukasiewicz logic is exactly appropriate for the formulation of the 'fuzzy set theory' first described by L.A. Zadeh [114]; indeed, it is not too much to claim that L_∞ is related to fuzzy set theory exactly as classical logic is related to ordinary set theory."

It is worth mentioning here that Christian Fermüller and colleagues have recently pushed forward Giles' as well as other dialogue games as foundational semantics for different fuzzy logics, see e.g. [39,38,37,36].

3 Toward more general settings

3.1 Discussing algebraic structures for the membership degrees

It was Joseph A. Goguen who, starting without being aware of Klaua's papers and only from Zadeh's approach, was the first among Zadeh's immediate followers who emphasized an intimate relationship between fuzzy sets and non-classical logics. In his 1969 paper [57], he considers membership degrees as generalized truth values, i.e. as truth degrees. Additionally he sketches a "solution" of the sorites paradox, i.e. the heap paradox, using – but only implicitly – the ordinary product $*$ in $[0,1]$ as a generalized conjunction operation. Based upon these ideas, and having in mind suitable analogies to the situation for intuitionistic logic, he proposes completely distributive lattice ordered monoids, called *closg*'s by him, enriched with an operation $\rightarrow$ which is the (right) residuum to the monoidal operation $*$ and hence characterized by the well known adjointness condition

$$a * b \leq c \quad \Leftrightarrow \quad b \leq a \rightarrow c\,, \tag{9}$$

and with the "implies falsum"-negation, as suitable structures for the membership degrees of fuzzy sets. Goguen introduces in this context the notion of tautology, with the neutral element of the monoid as the only designated truth degree. He defines a graded notion of inclusion in the same natural way as Klaua (7) did, of course with the residual implication $\rightarrow$ instead of the implication of the Łukasiewicz systems. But he does not mention any results for this graded implication. Furthermore, he does not see a possibility to develop a suitable formalized logic of closg's, as may be seen from his statement:

> "Tautologies have the advantage of independence of truth set, but no list of tautologies can encompass the entire system because we want to perform calculations with degrees of validity between 0 and 1. In this sense the logic of inexact concepts does not have a *purely* syntactic form. Semantics, in the form of specific truth values of certain assertions, is sometimes required."

3.2 Invoking t-norms

In the beginning 1980s it became common use in the mathematical fuzzy community to consider t-norms as suitable candidates for connectives upon which generalized intersection operations for fuzzy sets should be based, see [1,27,106] or a bit later [93,112]. These t-norms, a shorthand for "triangular norms", first became important in discussions of the triangle inequality within probabilistic metric spaces, see [110,94]. They are binary operations in the real unit interval which make this interval into an ordered commutative monoid with 1 as unit element of the monoid.

By $T_{\mathsf{L}}, T_{\mathsf{P}}, T_{\mathsf{G}}$ we denote the basic t-norms, i.e. the Łukasiewicz, the product, and the Gödel t-norm, respectively. For arbitrary $x, y \in [0,1]$ this means $T_{\mathsf{L}}(x,y) = \max\{x+y-1, 0\}$, $T_{\mathsf{P}}(x,y) = x \cdot y$, and $T_{\mathsf{G}}(x,y) = \min\{x,y\}$.

The general understanding in the context of fuzzy connectives is that t-norms form a suitable class of generalized conjunction operators.

For logical considerations the class of left-continuous t-norms is of particular interest. Here left-continuity for a t-norm $T : [0,1]^2 \to [0,1]$ means that for each $a \in [0,1]$ the unary function $T_a(x) = T(a,x)$ is left-continuous. The core result, which motivates the interest in left-continuous t-norms, is the fact that just for left-continuous t-norms $*$ a suitable implication function, usually called R-implication, is uniquely determined via the adjointness condition (9). Suitability of an implication function here means that it allows for a corresponding sound detachment, or *modus ponens* rule: to infer a formula ψ from formulas $\varphi \to \psi$ and φ *salva veritate*. In the present context this means the logical validity

$$\models \ \varphi \,\&\, (\varphi \to \psi) \to \psi \,. \tag{10}$$

It was almost immediately clear that a propositional language with connectives $\wedge, \vee$ for the truth degree functions min, max, and with connectives $\&, \to$ for a left-continuous t-norm T and its residuation operation offered a suitable framework to do fuzzy set theory within – at least as long as the complementation of fuzzy sets remains out of scope.

With this limitation, i.e. disregarding complementation, this framework offers a suitable extension of Zadeh's standard set-algebraic operations.

Additionally, this framework, with the "implies falsum" construction, yields a natural way to define a negation, i.e. to introduce a t-norm related complementation operation for fuzzy sets, via the definition $-_T\varphi =_{\mathrm{def}} \varphi \to \overline{0}$ using a truth degree constant $\overline{0}$ for the truth degree 0. However, this particular complementation operation does not always become the standard complementation of Zadeh's approach.

If one starts from the t-norm T_{L}, this t-norm based construction gives the infinite-valued Łukasiewicz system L_∞ together with the negation which describes Zadeh's complementation. If one starts with the t-norm T_{G}, this construction gives the infinite-valued Gödel system G_∞. And this approach gives the product logic [75] if one starts with the t-norm T_P. The "implies falsum" negations of the latter two systems coincide, but are different from the negation operation of the Łukasiewicz system L_∞. So these two cases do not offer Zadeh's complementation. But this can be reached if one adds the Łukasiewicz negation to these systems, as first discussed in [32] and later extended e.g. in [18,44]. Subsection 2.2.2 of [9] offers a good survey.

It was essentially a routine matter to develop this type of t-norm based logic to some suitable extent, as was done in 1984 in [61]. Also the development of fuzzy set theory on this basis did not offer problems, and it was done in [62], including essential parts of fuzzy set algebra, some fuzzy relation theory up to a fuzzified version of the Szpilrajn order extension theorem, and some solvability considerations for systems of fuzzy relation equations. All these considerations have later been included into the monograph [64].

4 The logics of continuous and of left-continuous t-norms

What was missing in all the previously mentioned approaches toward a suitable logic for fuzzy set theory, as long as this logic should be different from the infinite-valued Łukasiewicz system L_∞ or from the infinite-valued Gödel system G_∞,[4] was an adequate axiomatization of such a logic. All these approaches offered interesting semantics, but had not been, in general, presented in an axiomatic way, and hence did not provide suitable logical calculi – neither for the propositional nor for the first-order level.

The first proposal to fill in this gap was made by Ulrich Höhle [83–85] who offered in 1994 his *monoidal logic*. This common generalization of the Łukasiewicz logic L_∞, the intuitionistic logic, and Girard's integral, commutative linear logic [54] was determined by an algebraic semantics, viz. the class of all M-algebras, i.e. of all integral residuated commutative lattice-ordered monoids with the unit element of the monoid, i.e. the universal upper bound of the lattice, as the only designated element. So this monoidal logic was determined by a particular subclass of Goguen's closg's, indeed by a variety of residuated lattices. At this point, it is interesting to notice that Höhle's monoidal logic belongs to the family of substructural logics, namely M-algebras are nothing but the algebras of the logic FL_{ew}, i.e. Full Lambek calculus with exchange

[4] In 1996 the product logic [75] was added to this list.

and weakening. [5] And adequate axiomatizations for the propositional as well as for the first-order version of this logic were given in [83,85].

Of course, this monoidal logic intended to grasp the relationship between fuzzy set theory and the t-norm based setting of their set-algebraic operations. But it was not strongly enough tied with this background.

4.1 The logic of all continuous t-norms

The use of t-norm based logics in fuzzy set theory, particularly those ones based upon left-continuous t-norms, happened throughout the 1980s and beginning 1990s in a naive way: there was only the naive semantics available, but in general any logical calculus was missing.

To discuss the case of a single corresponding logic based upon an arbitrary left-continuous t-norm seemed to be a very hard problem.

Different from Höhle's quite general approach, and guided by the idea that it would be sufficiently general to restrict the considerations to the case of continuous t-norms, instead of allowing also non-continuous but left-continuous ones, it was the idea of Petr Hájek to ask in 1998 for the *common part* of all those t-norm based logics which refer to a continuous t-norm: in short, to ask for the logic of all continuous t-norms [66].

This logic was called *basic logic* by Hájek [66,67], later he used also *basic fuzzy logic* or *basic t-norm logic.* [6] This logic is usually denoted BL. It is based upon an algebraic semantics.

There are two crucial observations which pave the way to the original algebraic semantics for BL. The first one is that for any t-norm $*$ and their residuation operation $\rightarrow$ one has

$$(u \rightarrow v) \vee (v \rightarrow u) = 1\,, \tag{11}$$

with $\vee$ denoting the lattice join here, i.e. the max-operation for a linearly ordered carrier. This *prelinearity* condition (11) is a first restriction on the M-algebras which determine the monoidal logic, and it yields the MTL-algebras

[5] The reader is referred to [31] for a description on how different systems of fuzzy logics can be placed in Ono's hierarchy of extensions of the Full Lambek Calculus.

[6] In the fuzzy logic community also "Hájek's basic logic" is in use. The simple name "basic logic" has a certain disadvantage because it is also in use in a completely different sense: as some weakening of the standard system of intuitionistic logic, e.g. in [2,108]. For these reasons, very recently the authors in [23] even propose to replace the names of BL logic and BL algebras by *Hájek logic* (HL) and *Hájek algebras* (HL-algebras).

– now with $*$ denoting the semigroup operation.

Moreover, if this condition is imposed upon the Heyting algebras, which form an adequate algebraic semantics for intuitionistic logic, the resulting class of prelinear Heyting algebras is an adequate algebraic semantics for the infinite-valued Gödel logic.

The second observation is that the continuity condition can be given in algebraic terms: for any t-norm $*$ and its residuum $\rightarrow$ one has that the *divisibility* condition

$$u * (u \rightarrow v) = u \wedge v \tag{12}$$

is satisfied if and only if $*$ is a continuous t-norm, see [84]. Condition (12), again with $*$ denoting the semigroup operation and $\wedge$ the lattice meet, is the second restriction here. The BL-algebras are just those MTL-algebras which satisfy this divisibility condition (12).

Hájek characterized his basic fuzzy logic by the class $\mathsf{BL\text{-}alg}$ of BL-algebras as algebraic semantics – again with the universal upper bound of the lattice as the only designated element. And he gave adequate axiomatizations for the propositional version BL as well as for the first-order version $\mathsf{BL}\forall$ of this basic fuzzy logic in his highly influential monograph [67] via the axiom schemata:

($\mathsf{Ax_{BL}}1$) $\quad(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\,,$
($\mathsf{Ax_{BL}}2$) $\quad\varphi \,\&\, \psi \rightarrow \varphi\,,$
($\mathsf{Ax_{BL}}3$) $\quad\varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi\,,$
($\mathsf{Ax_{BL}}4$) $\quad\varphi \,\&\, (\varphi \rightarrow \psi) \rightarrow \psi \,\&\, (\psi \rightarrow \varphi)\,,$
($\mathsf{Ax_{BL}}5a$) $\quad(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \psi \rightarrow \chi)\,,$
($\mathsf{Ax_{BL}}5b$) $\quad(\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\,,$
($\mathsf{Ax_{BL}}6$) $\quad((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\,,$
($\mathsf{Ax_{BL}}7$) $\quad\overline{0} \rightarrow \varphi\,,$

and with the rule of detachment (modus ponens) as its only inference rule.

Routine calculations show that the axioms $\mathsf{Ax_{BL}}5a$ and $\mathsf{Ax_{BL}}5b$ express the adjointness condition (9). Also by elementary calculations one can show that $\mathsf{Ax_{BL}}6$ formulates the prelinearity condition (11). This was one of the interesting reformulations Hájek gave to the standard algebraic properties. Another one was that he recognized that the weak disjunction, i.e. the connective which corresponds to the lattice join operation in the truth degree structures, could be defined as

$$\varphi \vee \psi \;=_{\mathrm{def}}\; ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)\,. \tag{13}$$

Here $\wedge$ is the weak conjunction with the lattice meet as truth degree function which can, according to the divisibility condition, be defined as

$$\varphi \wedge \psi \ =_{\text{def}} \ \varphi \,\&\, (\varphi \to \psi)\,. \tag{14}$$

Hájek's eight axioms are well chosen and give a very concise system. In comparison, Höhle offers 14 axioms for his system ML and, additionally, has to include the conditions of prelinearity and of divisibility into his system. Nevertheless, also in this axiom system the axioms ($\mathsf{Ax_{BL}}2$) and ($\mathsf{Ax_{BL}}3$) are redundant, i.e. can be proved from the remaining ones. Even more, the remaining axioms then are mutually independent, as shown in [13].

But Hájek's presentation of the basic fuzzy logic BL was only a partial realization of the plan to give the logic of all continuous t-norms. Intuitively, the most natural algebraic semantics for such a logic of all continuous t-norms would be the subclass of all T-algebras, i.e. of all BL-algebras with carrier $[0,1]$. [7]

Hájek guessed that this *standard semantics*, determined by the class of all T-algebras, should be an adequate semantics for the fuzzy logic BL too. He was able to reduce the problem to the BL-provability of two particular formulas [66]. Finally, this guess proved to be correct: Roberto Cignoli et al. [14] proved in 2000 that BL completely axiomatizes logical validity w.r.t. this standard semantics.

And yet another fundamental property of BL logics could be proved [33]: all the t-norm based residuated many-valued logics with a continuous t-norm algebra as their standard semantics can be adequately axiomatized as finite extensions of BL. The proof comes by algebraic methods, viz. through a study of the variety of all BL-algebras and their subvarieties which are generated by continuous t-norm algebras: for each one of these subvarieties a finite system of defining equations is algorithmically determined. Actually, this result was later improved by Zuzana Haniková [81] where she shows that the equational theory of an arbitrary class of continuous t-norm algebras is finitely based as well.

4.2 The logic of all left-continuous t-norms

Only a short time after Hájek's axiomatization of the logic of continuous t-norms, also the logic of all left-continuous t-norms was adequately axiomatized. It was the guess of Francesc Esteva and Lluís Godo [30] that the class of

[7] If a BL-algebra has carrier $[0,1]$ with its natural ordering then its semigroup operation is automatically a continuous t-norm.

MTL-algebras should give an adequate semantics for this logic. First they offered an adequate axiomatization of the logic MTL, a shorthand for *monoidal t-norm logic*, which is determined by the class of MTL-algebras. And later on Sándor Jenei and Franco Montagna [86] proved that this is really the logic of all left-continuous t-norms: the logical calculus MTL has an adequate algebraic semantics formed by the class of all MTL-algebras with carrier $[0, 1]$. [8]

4.3 First-order logics

The extensions of these propositional logics to first-order ones follows the standard lines of approach: one has to start from a first-order language $\mathcal{L}$ with the two standard quantifiers $\forall, \exists$. and a suitable commutative, residuated, integral lattice ordered monoid $\mathbf{A}$ over a bounded lattice, [9] and has to define $\mathbf{A}$-interpretations $\mathbf{M}$ by fixing a nonempty domain $M = |\mathbf{M}|$ and by assigning to each predicate symbol of $\mathcal{L}$ an $\mathbf{A}$-valued relation in M (of suitable arity) and to each constant an element from (the carrier of) $\mathbf{A}$.

The satisfaction relation is also defined in the standard way. The quantifiers $\forall$ and $\exists$ are interpreted as taking the infimum or supremum, respectively, of all the values of the relevant instances.

Unfortunately, infima and suprema do not always exist in lattices. So one could suppose to consider only complete lattices. A less restrictive assumption is to assume that, for any formula, all the infima and suprema do exist which have to be considered for any evaluation this formula. Interpretations which satisfy this last condition are called *safe* by Hájek [67].

For the logic BL of continuous t-norms, Hájek [67] added the axioms

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ,
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ,
($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,
($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ,
($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ,

and the *rule of generalization* to the propositional system BL yielding the system $\mathsf{BL}\forall$.

Here, substitutability and the rule of generalization have the same meaning

[8] If a MTL-algebra has carrier $[0, 1]$ with its natural ordering then its semigroup operation is automatically a left-continuous t-norm.
[9] Integrality means that the monoidal unit coincides with the upper bound of the lattice.

as in classical first-order logic.

Then he was able to prove the following general *chain completeness theorem*: A first-order formula φ is $\mathsf{BL}\forall$-provable iff it is valid in all safe interpretations over BL-chains.

This result can be extended to elementary theories as well as to a lot of other first-order fuzzy logics, e.g. to $\mathsf{MTL}\forall$. And for this logic $\mathsf{MTL}\forall$ one has also a strong *standard completeness* result [100]: a formula φ is $\mathsf{MTL}\forall$-provable from a set T of formulas iff φ holds true in all safe $\mathbf{A}$-interpretations $\mathbf{M}$ which are models of T and are based upon a t-algebra $\mathbf{A}$ determined by a left-continuous t-norm.

We will not discuss further completeness results here but refer to the survey paper [19] or the more recent extended survey [9].

But it should be mentioned that, as suprema are not always maxima and infima not always minima, the truth degree of an existentially/universally quantified formula may not be the maximum/minimum of the truth degrees of the instances. It is, however, interesting to have conditions which characterize models in which the truth degrees of each existentially/universally quantified formula is witnessed as the truth degree of an instance. This problem was first considered by Petr Hájek in the framework of fuzzy description logics [71], later studied by Petr Cintula and Petr Hájek in [17], and surveyed in [9] as well.

5 Graded inferences

Having in mind that fuzzy logics, even understood them as formal logical systems, should be a (mathematical) tool for approximative reasoning, makes it desirable they should be able to deal with graded inferences.

The systems of t-norm based logics discussed up to now have been designed to formalize the logical background for fuzzy sets, and they allow themselves for degrees of truth of their formulas. But they all have crisp notions of consequence, i.e. of entailment and of provability.

It is natural to ask whether it is possible to generalize these considerations to the case that one starts from *fuzzy sets of formulas*, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by Jan Pavelka [105] and later further developed by Vilém Novák et al. [104,102].

However, it should be mentioned that there is also another, more algebraically oriented approach toward consequence operations for the classical case, originating from Alfred Tarski [111] and presented e.g. in [113]. This approach treats consequence operations as closure operations. And this type of approach has been generalized to closure operations in classes of fuzzy sets of formulas by Giangiacomo Gerla [48].

The Pavelka-style approach deals with fuzzy sets $\Sigma^\sim$ of formulas, i.e. besides formulas φ also their membership degrees $\Sigma^\sim(\varphi)$ in $\Sigma^\sim$. And these membership degrees are just the truth degrees.

Within the Pavelka-style approach a graded entailment notion arises in a quite natural way. An evaluation e is a *model* of a fuzzy set $\Sigma^\sim$ of formulas iff

$$\Sigma^\sim(\varphi) \leqslant e(\varphi) \tag{15}$$

holds for each formula φ. This immediately yields as definition of the entailment relation that the semantic consequence hull of $\Sigma^\sim$ should be characterized by the membership degrees

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^\sim)(\psi) = \bigwedge\{e(\psi) \mid e \text{ model of } \Sigma^\sim\} \tag{16}$$

for each formula ψ.

For a syntactic characterization of this entailment relation it is necessary to have some calculus $\mathbb{K}$ which treats formulas of the language together with truth degrees. So the language of this calculus has to extend the language of the basic logical system by having also symbols for the truth degrees. Depending upon the truth degree structure, this may mean that the language of this calculus becomes an uncountable one.

Further on we indicate these symbols by overlined letters like $\overline{a}, \overline{c}$. And we realize the common treatment of formulas and truth degrees by considering *evaluated formulas*, i.e. ordered pairs $(\overline{a}, \varphi)$ consisting of a truth degree symbol and a formula. This trick transforms in a natural way each fuzzy set $\Sigma^\sim$ of formulas into a (crisp) set of evaluated formulas, again denoted by $\Sigma^\sim$.

So $\mathbb{K}$ has to allow to derive evaluated formulas out of sets of evaluated formulas, of course using suitable axioms and rules of inference. These axioms are usually only formulas φ which, however, are used in the derivations as the corresponding evaluated formulas $(\overline{1}, \varphi)$. Derivations in $\mathbb{K}$ out of some set $\Sigma^\sim$ of evaluated formulas are finite sequences of evaluated formulas which either are axioms, or elements of (the support of) $\Sigma^\sim$, or result from former evaluated formulas by application of one of the inference rules.

Each $\mathbb{K}$-derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of φ *to the degree* $a \in L$. The *provability degree* of φ from $\Sigma^\sim$ in $\mathbb{K}$ is the supremum over all these degrees. This now yields that the syntactic consequence hull of

$\Sigma^\sim$ should be the fuzzy set $\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}$ of formulas characterized by the membership function

$$\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}(\Sigma^\sim)(\psi) = \bigvee\{a \in L \mid \mathbb{K} \text{ derives } (\overline{a}, \psi) \text{ out of } \Sigma^\sim\} \tag{17}$$

for each formula ψ.

Unfortunately, this is an *infinitary* notion of provability.

For the infinite-valued Łukasiewicz logic L_∞ this machinery works particularly well because it needs in an essential way the continuity of the residuation operation. In this case we can form a calculus $\mathbb{K}_{\mathsf{L}}$ which gives an adequate axiomatization for the graded notion of entailment in the sense that one has suitable soundness and completeness results.

This calculus $\mathbb{K}_{\mathsf{L}}$ has as axioms any axiom system of the infinite-valued Łukasiewicz logic L_∞ which provides together with the rule of detachment an adequate axiomatization of L_∞, but $\mathbb{K}_{\mathsf{L}}$ replaces this standard rule of detachment by the generalized form

$$\frac{(\overline{a}, \varphi) \qquad (\overline{c}, \varphi \to \psi)}{(\overline{a * c}, \psi)} \tag{18}$$

for evaluated formulas.

The soundness result for this calculus $\mathbb{K}_{\mathsf{L}}$ yields the fact that the $\mathbb{K}_{\mathsf{L}}$-provability of an evaluated formula $(\overline{a}, \varphi)$ says that $a \leq e(\varphi)$ holds for every valuation e, i.e. that the formula $\overline{a} \to \varphi$ is valid—however as a formula of an *extended* propositional language which has all the truth degree constants among its vocabulary. Of course, now the evaluations e have also to satisfy $e(\overline{a}) = a$ for each $a \in [0, 1]$.

And the soundness and completeness results for $\mathbb{K}_{\mathsf{L}}$ say that a *strong completeness theorem* holds true giving

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^\sim)(\psi) \;=\; \mathcal{C}^{\mathsf{syn}}_{\mathbb{K}_{\mathsf{L}}}(\Sigma^\sim)(\psi) \tag{19}$$

for each formula ψ and each fuzzy set $\Sigma^\sim$ of formulas.

If one takes the previously mentioned turn and extends the standard language of propositional L by truth degree constants for all degrees $a \in [0, 1]$, and if one reads each evaluated formula $(\overline{a}, \varphi)$ as the formula $\overline{a} \to \varphi$, then a slight modification of the former calculus again provides an adequate axiomatization: one has to add the *bookkeeping axioms*

$$\begin{aligned} (\overline{a} \mathbin{\&} \overline{c}) &\equiv \overline{a * c}\,, \\ (\overline{a} \to \overline{c}) &\equiv \overline{a \to_{\mathsf{L}} c}\,, \end{aligned}$$

as explained e.g. in [104]. And if one is interested to have evaluated formulas together with the extension of the language by truth degree constants, one has also to add the *degree introduction rule*

$$\frac{(\overline{a}, \varphi)}{\overline{a} \to \varphi}\ .$$

However, even a stronger result is available which refers only to a notion of derivability over a countable language. The completeness result (19), for $\mathbb{K}_{\mathsf{L}}^{+}$ instead of $\mathbb{K}_{\mathsf{L}}$, becomes already provable if one adds truth degree constants only for all the *rationals* in $[0, 1]$, as was shown in [67]. And this extension of L is even only a conservative one, cf. [78], i.e. proves only such *constant-free* formulas of the language with rational constants which are already provable in the standard infinite-valued Łukasiewicz logic.

Similar *rational* expansions for other t-norm based fuzzy logics can be analogously defined, but unfortunately Pavelka-style completeness cannot be obtained since Łukasiewicz logic is the only fuzzy logic whose truth-functions (conjunction and implication) are continuous functions [3].

To overcome the discontinuity problems one way out is to include infinitary rules for each such discontinuity points in the calculus, as it has been systematically studied by Cintula in [16] where he provides adequate axiomatizations for Pavelka-style extensions of fuzzy logics that are expansions of MTL given by a fixed standard algebra.

An alternative approach that has also been developed goes along the line of providing traditional algebraic semantics for these fuzzy logics expansions with (rational) truth-constants together with their corresponding book-keeping axioms, and studies completeness results with respect to the usual (finitary) notion of proof. In fact, only the case of Łukasiewicz logic is dealt with by Hájek in [67]. Using this algebraic approach the expansions of the other two distinguished fuzzy logics, Gödel and Product logics, with countable sets of truth-constants were later reported by Esteva et al. in [34] and in [109] respectively, and further generalized to the case of other fuzzy logics in [29] for the propositional case and in [35] for the first order case.

6 Further generalizations

6.1 Non-commutative and non-integral fuzzy logics

While MTL is obtained from BL by removing divisibility, one may wonder what happens if one removes commutativity of the conjunction. BL deprived of commutativity has been investigated e.g. in [47] and [45]. In the paper [69], Hájek finds adequate axiomatizations for these logics and proves a completeness theorem for them. Moreover in [68], Hájek proves that each BL-algebra given by a continuous t-norm is a subalgebra of a non-commutative pseudo-BL-algebra on a 'non-standard' interval $[0,1]^*$. The corresponding non-commutative version of MTL, called pseudo-MTL, has also been studied by S. Jenei and F. Montagna in [87]. These logics have two implications, corresponding to the left and right residuum of the conjunction. The algebraic counterpart are the so-called pseudo-BL and pseudo-MTL algebras. Interestingly enough, while there are pseudo-MTL algebras over the real unit interval $[0,1]$, defined by left continuous pseudo-t-norms (i.e. operations satisfying all properties of t-norms but the commutativity), there are no pseudo-BL algebras, since continuous pseudo-t-norms are necessarily commutative.

The logic BL was already an attempt to generalize the three main fuzzy logics, that is, Łukasiewicz, Gödel and Product logics. Probably Hájek didn't imagine such an amount of generalizations obtained by removing either connectives or the divisibility axiom, or the commutativity axiom. In his paper *Fleas and fuzzy logic* [70], Hájek finds a common generalization of the logic of basic hoops and the logic pseudo-MTL of non-commutative pseudo-t-norms. He presents a general completeness theorem and he discusses the relations to the logic of pseudo-BCK algebras. The reference to fleas in the title is due to the following story:

> Some scientists make experiments on a flea: they remove one of its legs and tell it: *Jump!* The flea can still jump. Then they repeat the experiment over and over again, and, although with some difficulty, the flea still jumps. But once all legs are removed, the flea is no longer able to jump. Then the doctors come to the conclusion that a flea without legs becomes deaf. Now the attitude of logicians who remove more and more axioms and symbols and still expect to be able to derive interesting properties, is compared to the attitude of the scientists of the story.

Another weakening of MTL in a different direction was proposed by George Metcalfe and Franco Montagna in his paper [99], where they remove the integrality of the conjunction, i.e. in algebraic terms, they do not require any longer that the neutral element of the monoidal operation to be the maximum of the order. They introduce the logic UL, which is shown to be the logic of all to left-continuous *uninorms*, its involutive version IUL, as well as the logics UML and IUML extensions of UL and IUL with an idempotency axiom. Algebraic semantics for these logics are provided by subvarieties of pointed bounded commutative residuated lattices.

6.2 Weakly implicative semilinear logics

In the last years there have been a increasing variety of fuzzy logics studied in the literature. The evolution, exemplified in the previous subsection, even has gone further on generalizing systems of fuzzy logic, for instance P. Cintula [15] has introduced the framework of weakly implicative fuzzy logics. The main idea behind this class of logics is to capture the notion of comparative truth common to all fuzzy logics. Roughly speaking, they are logics close to Rasiowa's implicative logics [107] but satisfying a proof-by-cases property. This tendency shows that almost no property of those systems was essential. Nevertheless, there is one that has remained untouched so far: completeness with respect to a semantics based on linearly ordered algebras. It actually corresponds to the main thesis of [8] that defends the claim that fuzzy logics are the logics of chains, pointing to a roughly defined class of logics, rather than a precise mathematical description of what fuzzy logics are, since there could be many different ways in which a logic might enjoy a complete semantics based on chains.

With the aim of dealing in a uniform way with the increasing variety of fuzzy logics studied in the literature, Petr Cintula and Carles Noguera provide in [25,24] a new framework (the hierarchy of the so-called *implicational logics*) where one can develop in a natural way a technical notion corresponding to the intuition of fuzzy logics as the logics of chains. Indeed, they introduce the notion of implicational *semilinear* logic as a property related to the implication, namely a logic L is an implicational semilinear logic iff it has an implication such that L is complete w.r.t. the class of logical matrices where the implication induces a linear order on the set of truth-values. The above mentioned hierarchy, when restricted to the semilinear case, provides a classification of implicational semilinear logics that encompasses almost all the known examples of fuzzy logics.

Even if Cintula-Noguera's framework is more general (encompassing e.g. even non-associative logics [22]), we restrict ourselves here to the class of the so-called *weakly implicative logics*, that is, logics with a weak implication given a single binary (either primitive or definable) connective $\rightarrow$ satisfiyng the following conditions (where $\mathcal{L}$ is the language of the logic):

(R) $\vdash_{\mathrm{L}} \varphi \to \varphi$,

(MP) $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$,

(T) $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$,

(sCNG) $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$ for each n-ary $c \in \mathcal{L}$ and each $i < n$,

If we add the *weakening* condition

(W) $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$.

then we get Rasiowa-implicative logics, which are algebraizable in the sense [12] and its equivalent algebraic semantics, the class of L-algebras, is a quasivariety. L is called a *semilinear logic* iff it is strongly complete with respect to the semantics given by L-chains or, equivalently, if every L-algebra is representable as subdirect product of L-chains. In [24], they prove that L is semilinear iff the following proof-by-cases like property is satisfied:

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

They also link the property of being semilinear to the property of having a well-behaved disjunction in the language. A (primitive or definable) binary connective $\vee$ is called a *disjunction* in L whenever it satisfies:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \vee \psi$,

(PCP) If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$.

Now let L be a finitary Rasiowa implicative logic with a binary connective $\vee$ satisfying (PD) and consider the following two properties:

(PRL) $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$,

(DMP) $\varphi \to \psi, \varphi \vee \psi \vdash_{\mathrm{L}} \psi$ and $\varphi \to \psi, \psi \vee \varphi \vdash_{\mathrm{L}} \psi$.

Then the following are equivalent:

(i) $\vee$ is a disjunction and satisfies (PRL)
(ii) L is semilinear and satisfies (DMP)

Notice that the class of core fuzzy logics (axiomatic expansions of MTL satisfying (Cong) for any possible new connective) fall into the class of Rasiowa-implicative semilinear logics, and hence they are strongly complete with respect the class of their chains.

For further details on the state-of-the-art and recent developments of mathematical fuzzy logic, the interested reader is referred to the Handbook volumes [20,21].

7 Some applications

In this last section we illustrate, by way of two examples, how formal systems of fuzzy logic can also be used for more application oriented purposes, in particular we show how they have been used to devise systems for probabilistic reasoning and how to they can cope with some patterns of Zadeh's approximate reasoning machinery.

7.1 Fuzzy probability logic

Already in a 1994, Petr Hájek and Dagmar Harmancová [77] noticed that one can safely interpret a probability degree on a Boolean proposition φ as a truth degree, not of φ itself but of another (modal) formula $P\varphi$, read as "φ is *probable*". The point is that "being probable" is actually a fuzzy predicate, which can be more or less true, depending on how much probable is φ. Hence, it is meaningful to take the truth-degree of $P\varphi$ as the probability-degree of φ. The second important observation is the fact that the standard Łukasiewicz logic connectives provide a proper modeling of the Kolmogorov axioms of finitely additive probabilities. For instance, the following axiom

$$P(\varphi \vee \psi) \leftrightarrow ((P\varphi \to P(\varphi \wedge \psi)) \to P\psi)$$

faithfully captures the finite-additive property when $\to$ is interpreted by the standard Łukasiewicz logic implication. Indeed, these were the key issues that are behind the first probability logic defined as a theory over Rational Pavelka logic in P. Hájek, L. Godo and F. Esteva's paper [74]. This was later described with an improved presentation in Hájek's monograph [67] where P is introduced as a (fuzzy) modality. Exactly the same approach works to capture uncertainty reasoning with necessity measures, replacing the above axiom by $N\varphi \wedge N\psi \to N(\varphi \wedge \psi)$. More interesting was the generalization of the approach to deal with Dempster-Shafer belief functions proposed in the paper [56] by L. Godo, P. Hájek and F. Esteva. There, to get a complete axiomatization, the authors use one of possible definitions of Dempster-Shafer belief functions in terms of probability of knowing (in the epistemic sense), and hence they combine the above approach to probabilistic reasoning with the modal logic S5 to introduce a modality B for belief such that $B\varphi$ is defined as $P\Box\varphi$, where $\Box$ is a S5 modality and φ is a propositional modality-free formula. The complexity of the fuzzy probability logics over Łukasiewicz and ŁΠ logics was studied by Petr Hájek and Sandro Tulipani in [79].

This line of research has been followed in a number of papers where analogs of these uncertainty logics have been extended over different fuzzy logics, mainly Łukasiewicz and Gödel logics, see e.g. [40,42,43,41]. A recent paper [26] treats most of all the above mentioned logics in a uniform way. Hájek himself wrote another very interesting paper [72], generalizing [79], about the complexity of general fuzzy probability logics defined over logics whose standard set of truth values is the real unit interval $[0, 1]$ and the truth functions of its (finitely many) connectives are definable by open formulas in the ordered field of reals.

7.2 Formalizing approximate reasoning

In the literature one can find several approaches to carry Zadeh's main approximate reasoning constructs in a formal logical framework. In particular, Novák and colleagues have done much in this direction, using the model of fuzzy logic with evaluated syntax, fully elaborated in the monograph [104] (see the references therein and also [28]), and more recently he has developed a very powerful and sophisticated model of fuzzy type theory [101,103]. In his monograph, Hájek [67] also has a part devoted to this task. Here below, we show how to capture at a syntactical level, namely in a many-sorted version of predicate fuzzy logic calculus, say MTL∀, some of Zadeh's basic approximate reasoning patterns. These ideas were mainly presented in [67,55].

Consider the simplest and most usual expressions in Zadeh's fuzzy logic

$$\text{“}\mathbf{x} \text{ is } A\text{”},$$

with the intended meaning the variable x takes the value in A, represented by a fuzzy set μ_A on a certain domain U. The representation of this statement in possibility theoretic terms is the constraint

$$(\forall u)(\pi_{\mathbf{x}}(u) \leq \mu_A(u)),$$

where $\pi_{\mathbf{x}}$ stands for the possibility distribution for the variable $\mathbf{x}$. But such a constraint is very easy to represent in MTL∀ as the formula [10]

$$(\forall x)(X(x) \to A(x))$$

where A and X are many-valued *predicates* of the same sort in each particular model $\mathbf{M}$. Their interpretations (as fuzzy relations on their common domain)

[10] Caution: do not confuse the logical variable x in this logical expression from the linguistic (extra-logical) variable $\mathbf{x}$ in "$\mathbf{x}$ is A".

can be understood as the membership function $\mu_A : U \longrightarrow [0,1]$ and the possibility distribution $\pi_{\mathbf{x}}$ respectively. Indeed, one can easily observe that the truth degree equation $\| (\forall x)(X(x) \rightarrow A(x)) \|_M = 1$ holds if and only if the truth degree inequality $\| X(x) \|_{M,e} \leq \| A(x) \|_{M,e}$ holds for all x and any evaluation e. From now on, variables ranging over universes will be $\mathbf{x}, \mathbf{y}, \mathbf{z}$; "$\mathbf{x}$ is A" becomes $(\forall x)(X(x) \rightarrow A(x))$ or just $X \subseteq A$; if $\mathbf{z}$ is 2-dimensional variable $(\mathbf{x}, \mathbf{y})$, then an expression "$\mathbf{z}$ is R" becomes $(\forall x, y)(Z(x,y) \rightarrow R(x,y))$ or just $Z \subseteq R$.

In what follows, only two (linguistic) variables will be involved $\mathbf{x}, \mathbf{y}$ and $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. Therefore we assume that X, Y (corresponding to the possibility distributions $\pi_{\mathbf{x}}$ and $\pi_{\mathbf{y}}$) are projections of a binary fuzzy predicate Z (corresponding to the joint possibility distribution $\pi_{\mathbf{x},\mathbf{y}}$). The axiom we need to state in order to formalize this assumption is:

Proj: $(\forall x)(X(x) \equiv (\exists y)Z(x,y))\ \&\ (\forall y)(Y(y) \equiv (\exists x)Z(x,y))$

As a mater of example, we consider next several approximate reasoning patterns, and for each pattern we shall present a provable tautology (in MTL$\forall$) and its corresponding derived deduction rule, which will automatically be sound.

(1) *Entailment Principle:* From "$\mathbf{x}$ is A" infer "$\mathbf{x}$ is A^*", whenever $\mu_A(u) \leq \mu_{A^*}(u)$ for all u.

Provable tautology: $(A \subseteq A^*) \rightarrow (X \subseteq A \rightarrow X \subseteq A^*)$

Sound rule: $\dfrac{A \subseteq A^*, X \subseteq A}{X \subseteq A^*}$

(2) *Truth-qualification:* From "$\mathbf{x}$ is A" infer that "($\mathbf{x}$ is A^*) is α-true", where $\alpha = \inf_u \mu_A(u) \Rightarrow \mu_{A^*}(u)$.

Provable tautology: $(X \subseteq A) \rightarrow (A \subseteq A^* \rightarrow X \subseteq A^*)$

Sound rule: $\dfrac{X \subseteq A}{A \subseteq A^* \rightarrow X \subseteq A^*}$

(3) *Truth-modification:* From "($\mathbf{x}$ is A) is α-true" infer that "$\mathbf{x}$ is A^*", where $\mu_{A^*}(u) = \alpha \Rightarrow \mu_A(u)$.

Provable tautology: [11] $(\overline{\alpha} \rightarrow (X \subseteq A)) \rightarrow (X \subseteq (\overline{\alpha} \rightarrow A)$

Sound rule: $\dfrac{(\overline{\alpha} \rightarrow (X \subseteq A)}{X \subseteq (\overline{\alpha} \rightarrow A)}$

(4) *min–Combination*: From "$\mathbf{x}$ is A_1" and "$\mathbf{x}$ is A_2" infer "$\mathbf{x}$ is $A_1 \cap A_2$", where $\mu_{A_1 \cap A_2}(u) = \min(\mu_{A_1}(u), \mu_{A_2}(u))$.

[11] Where $\overline{\alpha}$ denotes a truth-constant

Provable tautology: $(X \subseteq A_1) \to ((X \subseteq A_2) \to (X \subseteq (A_1 \wedge A_2)))$

Sound rule: $\dfrac{X \subseteq A_1, X \subseteq A_2}{X \subseteq (A_1 \wedge A_2)}$,

where $(A_1 \wedge A_2)(x)$ is an abbreviation for $A_1(x) \wedge A_2(x)$.

(5) *Compositional rule of inference*: From "$(\mathbf{x}, \mathbf{y})$ is R_1" and "$(\mathbf{y}, \mathbf{z})$ is R_2" infer "$(\mathbf{x}, \mathbf{z})$ is $R_1 \circ R_2$", where $\mu_{R_1 \circ R_2}(u, w) = \sup_v \min(\mu_{R_1}(u, v), \mu_{R_2}(v, w))$.

Provable tautology: $(Z_1 \subseteq R_1) \to ((Z_2 \subseteq R_2) \to (Z_3 \subseteq (R_1 \circ R_2)))$

Sound rule: $\dfrac{Z_1 \subseteq R_1, Z_2 \subseteq R_2}{Z_3 \subseteq (R_1 \circ R_2)}$

where $(R_1 \circ R_2)(x, z)$ is an abbreviation for $(\exists y)(R_1(x, y) \wedge R_2(y, z))$.

Note that the following rule

$$\frac{Cond, Proj,\ X \subseteq A,\ Z \subseteq R}{Y \subseteq B},$$

where $Cond$ is the formula $(\forall y)(B(y) \equiv (\exists x)(A(x) \wedge R(x, y)))$, formalizing the particular instance of *max–min composition rule*, from "$\mathbf{x}$ is A" and "$(\mathbf{x}, \mathbf{y})$ is R" infer "$\mathbf{y}$ is B", where $\mu_B(y) = \sup_u \min(\mu_A(u), \mu_R(u, v))$, is indeed a derived rule from the above ones.

8 Concluding Remarks

This paper sketches core developments, which have created the actual understanding of the intimate relationship of fuzzy set theory with non-classical, particularly many-valued, logics.

Within the chosen topics we intended to present and discuss the core approaches and results. But this is not the place to strive for completeness. Hence there are quite a lot of interesting further aspects of the relationship between fuzzy set theory and formal logics we did not include here. Nevertheless, we will give a few hints to at least some of the missing topics here.

A first topic is proof theory: the formalization of our mathematical fuzzy logics by sequent—or better: hypersequent—calculi. We refer the interested reader to [97,98].

A second topic is the further formalization of fuzzy set theory and theories involving fuzzy sets. On the one hand side this may mean the development of fuzzy set theory in the form of an axiomatic theory formalized with (the first-order version of) some of the previously mentioned mathematical fuzzy logics.

First approaches have been made by P. Hájek [76,73]. On the other hand side this can be understood as the development of particular mathematical theories in a set theoretic setting, but not in the setting of crisp sets, but of fuzzy sets. Such approaches are given e.g. in [7,11,5,10,6].

And a third topic is the relationship of all the mathematical fuzzy logics to the large class of substructural logics. Also substructural logics have a syntactic, i.e. proof theoretical aspect, as well as a semantical one, cf. [46,95]. On the syntactic side it is rather clear that mathematical fuzzy logics do, in general, not satisfy the structural rule of contraction. On the semantic side, both classes of logics 'meet' one another via lattice ordered residuated semigroups as characterizing algebraic structures. Thus, as explained again e.g. in [99,9], the main systems of mathematical fuzzy logics can be indeed considered as distinguished members of the family of substructural logics.

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