Stochastic independence for probability MV-algebras

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Abstract

We prove that any MV-algebra has a faithful state can be embedded in an *fMV-algebra of integrable functions. As consequence, we prove Hölder's inequality and Hausdorff moment problem for MV-algebras with product and we propose a solution for the stochastic independence of probability MV-algebras.*

Introduction

MV-algebras were defined by Chang [3] and they stand to Łukasiewicz ∞-valued logic as boolean algebras stand to classical logic. The theory of MV-algebras was highlighted by Mundici's categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit (ℓu-groups) [25]. The twofold nature of MV-algebras, generalizations of boolean algebras and unit intervals of ℓu-groups, is also reflected by their probability theory: *the (finite-additive) states* are in one-to-one correspondence with normalized states on ℓu-groups, while

probability MV-algebras are the main ingredient of the extension of Carathéodory boolean algebraic probability theory to many-valued events.

A probability MV-algebra [30] is a pair (A, s), where A is a σ-complete MV-algebra and s is a σ-continuous faithful state. Riečan and Mundici propose in [30] a list of open problems, we recall the fifth one:

"[...] Assuming M and N to be probability MV-algebras, generalize the classical theory of "stochastically independent" σ-subalgebras as defined in Fremlin's treatise [Measure Theory, 325L]."

In [21] the author investigates this problem for MV-algebras endowed with finite-additive states, but no solution is given for probability MV-algebras, as defined in [30]. In the present paper, Theorem 3.1 presents a possible solution and we notice that an analogue result in Fremlin's treatise [13] is [253F].

An important result in our approach is Theorem 2.1 which, combining the results from [21] and [22], prove that any MV-algebra that has a faithful state can be embedded in an fMV-algebra of integrable functions in which the state is represented by the integral. The representation for states is actually the Kroupa-Panti teorem , but we make the context more precise. The representation of the algebraic structure is crucial for our development and is based on Theorem 1.3, which is similar with Kakutani's representation for abstract L-spaces [17]. The *fMV-algebras* are defined in [20] any they are MV-algebras endowed with both an internal product and a scalar product, with scalars from $[0, 1]$. By an extension of Mundici's equivalence, they are categorically equivalent with unital f-algebras [2]. As direct consequences of Theorem 2.1, in Section we prove *Hölder's inequality* and *Hausdorff's moment problem* for *PMV-algebras*, i.e. MV-algebras endowed with an internal product [8], and for fMV-algebras.

Section 3 is focused on the problem of *stochastic independence*. The main idea is the following: given two probability MV-algebras we embed them in corresponding algebras of integrable functions, which allow us to apply the results from [13]. Our final result can be stated as follows:

Given (A, s_A) and (B, s_B) two probability MV-algebras, there exists a probability MV-algebra (T, s_T) and a bilinear function $\beta : A \times B \to T$ such that $s_T(\lambda(a, b)) = s_A(a) \cdot s_B(b)$, for any $a \in A$ and $b \in B$.

By Theorem 3.1, the probability MV-algebra (T, s_T) satisfy an universal property which, however, does not characterize it up to isomorphism.

1 Preliminaries

1.1 Algebraic structures

An *MV-algebra* is an algebraic structure $(A, \oplus, \ {}^*, 0)$, where $(A, \oplus, 0)$ is a commutative monoid, * is an involution and the relation $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$ is satisfied for any $a, b \in A$ [3, 4, 27]. The variety of MV-algebras is generated by $([0,1], \oplus, {}^*, 0)$ where $a \oplus b = min(a + b, 1)$ and $a^* = 1 - a$ for any $a, b \in [0,1]$. The category of MV-algebras is denoted **MV**.

One also defines the constant $1 = 0^*$, the operation $a \odot b = (a^* \oplus b^*)^*$ and the distance function $d(a,b) = (a \odot b^*) \oplus (b \odot a^*)$ for any $a, b \in A$. Setting $a \leq b$ if and only if $a^* \oplus b = 0$, then $(A, \leq, 0, 1)$ is a bounded distributive lattice such that $a \vee b = (a^* \oplus b)^* \oplus b$ and $a \wedge b = (a^* \vee b^*)^*$ for any $a, b \in A$. An MV-algebra A is σ-complete (Dedekind-MacNeille complete) if its lattice reduct is a σ-complete (Dedekind-MacNeille complete) lattice.

If A is an MV-algebra we define a partial operation $+$ as follows: for any $a, b \in A$, $a + b$ is defined if and only if $a \leq b^*$ and, in this case, $a + b = A \oplus b$. This operation is cancellative and any MV-algebra A satisfies the Riesz decomposition property [9, Section 2.9]. Throughout the paper we use the following notation:

$$na = \underbrace{a + \cdots + a}_{n} \text{ and } n_\oplus a = \underbrace{a \oplus \cdots \oplus a}_{n}$$

where $a \in A$ and $n \geq 1$ is a natural number.

If A and B are MV-algebras then a function $\omega : A \to B$ is *linear* if $f(a + b) = f(a) + f(b)$ whenever $a \leq b^*$. Bilinear functions are defined as usual. A bimorphism is a bilinear function that is $\vee$-preserving and $\wedge$-preserving in each component. We refer to [12] for basic results on linear functions.

An *ideal* in A is a lower subset I that contains 0 and it is closed to $\oplus$. A maximal ideal is an ideal that is maximal in the set of all ideals ordered by set-theoretic inclusion. A *semisimple MV-algebra* is an MV-algebra in which the intersection of all maximal ideals is $\{0\}$. Equivalently, an MV-algebra A is semisimple if and only if there exists a compact Hausdorff space X such that A can be embedded in the MV-algebra $C(X) = \{f : X \to [0,1] \mid f \text{ continuous}\}$ with pointwise operations [4, Corollary 3.6.8].

A ℓ-*group* is an abelian group that is also a lattice such that any group translation is isotone. If G is an ℓ-group, an element $u \in G$ is a (*strong order*) *unit* if $u \geq 0$, and for every $x \in G$ there is a natural number $n \geq 1$ such that

$nu \geq |x|$. An ℓ-group is *unital* if it is endowed with a distinguished unit. The category of unital ℓ-groups and their unital homomorphisms is denoted **auG**.

If (G, u) is a unital ℓ-group we denote $[0, u] = \{x \in G | 0 \leq x \leq u\}$ and we define

$$x \oplus y = (x + y) \wedge u \text{ and } \neg x = u - x \text{ for any } x, y \in [0, u].$$

Then $[0, u]_G = ([0, u], \oplus, \neg, 0)$ is an MV-algebra. For any MV-algebra A there exists an ℓu-group (G, u) such that $A \simeq [0, 1]_G$. Moreover, the following property holds: for any $x \geq 0$ in G there exist a natural number $n \geq 1$ and $a_1, \ldots, a_n \in A$ such that $x = a_1 + \cdots + a_n$.

It is possible to define a functor $\Gamma : \mathbf{auG} \to \mathbf{MV}$ by

$$\Gamma(G, u) = [0, u]_G \text{ and } \Gamma(h) = h|_{[0,u]_G}.$$

where (G, u) is an ℓu-group and h is unital homomorphism. In [25] it is proved that Γ establishes a categorical equivalence between the categories **auG** and **MV**. In addition, an MV-algebra A is semisimple if and only if the corresponding ℓu-group (G, u) is archimedean.

Instead of ℓu groups, one may consider ℓ-rings, Riesz spaces(vector lattices) or f-algebras [1, 2] with strong unit and axiomatize the unit interval. The structures obtained in this manner have an MV-algebra reduct endowed with a product operation which can be internal or external.

Product MV-algebras (PMV-algebras for short) have been defined in [8] in the general case and in [24] in a slightly different way for the unital and commutative case. They are MV-algebras endowed with a binary internal product that satisfies the following, for any $x, y, z \in P$:

(PMV1) $c \cdot (a \odot (a \wedge b)^*) = (c \cdot a) \odot (c \cdot (a \wedge b))^*$

(PMV2) $(a \odot (a \wedge b)^*) \cdot c = (a \cdot c) \odot ((a \wedge b) \cdot c)^*$.

(PMV3) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

A PMV-algebra is unital if it has a unit for the product, and a PMVf-algebra [8, Theorem 5.4] is a PMV-algebra that satisfies the f-property:

(f) if $a \wedge b = 0$, then $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$, for any $a, b, c \in P$.

It is straightforward that unital PMV-algebras are PMVf-algebras, and any PMVf-algebra is subdirect product of totally ordered PMV-algebras [8, Proposition 5.5].

Let us denote by **PMV** and **uR** the categories of PMV-algebras and ℓu-rings such that $u \cdot u \leq u$ with suitable morphisms. In [8] the functor Γ was extended to a functor $\Gamma_{(\cdot)} : \mathbf{uR} \to \mathbf{PMV}$ which is also an equivalence.

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A further extension of the notion of MV-algebra has been introduced in [10]. A *Riesz MV-algebra* is a structure $(R, \oplus, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ such that $(R, \oplus, ^*, 0)$ is an MV-algebra and for any $\alpha, \beta \in [0,1]$ and any $a, b \in R$ we have

(RMV1) $\alpha(a \odot b^*) = (\alpha a) \odot (\alpha b)^*$,

(RMV2) $\max(0, \alpha - \beta)a = (\alpha a) \odot (\beta a)^*$,

(RMV3) $\alpha(\beta a) = (\alpha \beta)a$,

(RMV4) $1a = a$.

Any homomorphism of MV-algebras between Riesz MV-algebras preserves the additional unary operations, so it is a homomorphism of Riesz MV-algebras. Riesz MV-algebras are, up to isomorphism, unit intervals of Riesz spaces with strong unit. Let us denote by **RMV** and **uRS** the categories of Riesz MV-algebras and, respectively, Riesz spaces with suitable morphisms. In [10] the functor Γ was extended to a functor $\Gamma_{\mathbb{R}} : \mathbf{uRS} \to \mathbf{RMV}$ which is also an equivalence.

Finally, *fMV-algebras* are introduced in [20] as algebraic structures $(A, \oplus, \ ^*, \cdot, \{\alpha\}_{\alpha \in [0,1]}, 0)$ such that $(A, \oplus, \ ^*, \cdot, 0)$ is a PMVf-algebra, $(A, \oplus, \ ^*, \{\alpha\}_{\alpha \in [0,1]}, 0)$ is a Riesz MV-algebra and the condition $\alpha(a \cdot b) = (\alpha b) \cdot b = a \cdot (\alpha b)$ is satisfied for any $\alpha \in [0,1]$ and $a, b \in A$. If there exist a unit for the product, A will be called unital. The corresponding lattice-ordered structures are the f-algebras with strong unit [2]. If we denot by **fMV** and **fuAlg** the categories of fMV-algebras and fu-algebras with suitable morphisms respectively, we establish a categorical equivalence $\Gamma_f : \mathbf{fuAlg} \to \mathbf{fMV}$ and, in this case, the functor Γ_f extends the previous ones: $\Gamma, \Gamma_{(\cdot)}, \Gamma_{\mathbb{R}}$.

In order to summarize all categorical equivalence, we present the following diagram, in which all horizontal arrows are suitable forgetful functors.

Figure 1.

We finally mention that a PMV-algebra, a Riesz MV-algebra or a fMV-algebra is semisimple if its MV-algebra reduct is semisimple MV-algebra.

1.2 States. The state-completion.

The notion of *state* for an MV-algebra has be introduced in [26], in relation to the notion of "average degree of truth" of a proposition. See also [27, 30] for advanced topics.

Definition 1.1. A state is a linear function $s : A \to [0,1]$ such that $s(1) = 1$.

A state s is *faithful* if $s(a) = 0$ implies $a = 0$ for any $a \in A$. We remark that if there exists a faithful state on A, than A is semisimple.

A state for an ℓu-group (G, u) is a positive normalized additive map $t : G \to \mathbb{R}$. By [27] any state defined on an MV-algebra A can be uniquely extended to a state on the corresponding ℓu-group.

For states in a MV-algebra an equivalent form of the Riesz representation theorem holds, due to Kroupa and Panti [18, 28].

Theorem 1.1. *For any MV-algebra A there is an affine isomorphism $v \mapsto s_v$ of the convex set of regular Borel probability measures on the maximal spectral space $Max(A)$ onto the set of states on A. For every $f \in A$ and $m \in Max(A)$,*

$$s_v(f) = \int_{Max(A)} f^*(m) dv(m)$$

where $f \mapsto f^*$ is the representation for semisimple MV-algebras by continuous functions.

Remark 1.1. The notion of states extends to PMV-algebras, Riesz MV-algebra and fMV-algebras without changes in the the definition. In [10] the authors prove that for Riesz MV-algebras any state is homogeneous, i.e. it preserves the scalar product.

A state is σ-*continuous* if $\lim_n s(a_n) = s(a)$ for any $a_1 \leq \cdots \leq a_n \leq \cdots$ in A such that $\bigvee_n a_n = a$. A pair (A, s) with A a σ-complete MV-algebra and s a faithful σ-continuous state is called *probability MV-algebra* [30].

The metric completion of an MV-algebra with respect to the metric induced by a state was studied in [21]. We remind it in the sequel, since it is important for the present investigation.

The starting point is the remark that, given an MV-algebra A and a state $s : A \to [0,1]$, one can define a pseudo-metric on A by $\rho_s : A \times A \to [0,1]$ defined by $\rho_s(x, y) = s(d(x, y))$ for any $x, y \in A$ [30]. The pseudo-metric ρ_s is a metric iff s is a faithful state.

We say that (A^c, s^c) is *the state-completion* of (A, s) if A^c is the Cauchy completion A w.r.t. the pseudo-metric ρ_s and $s^c([\{a_n\}_n]) = \lim_n s(a_n)$ for any Cauchy sequence $\{a_n\}_n$ in A. We define $\varphi_A : A \to A^c$ by $\varphi_A(a) = [\{a\}]$ for any $a \in A$.

Theorem 1.2. *[21] Let (A, s) be an MV-algebra with a state and (A^c, s^c) be its state-completion then the following hold:*

1) A^c is σ-complete, s^c is a σ-continuous faithful state and $s^c \circ \varphi_A = s$,

2) φ_A is an embedding iff s is faithful,

3) (Universal Property) For any MV-algebra C, for any faithful state m such that C is ρ_m-complete and for any state-preserving homomorphism of MV-algebras $f : A \to C$ there exists a unique state-preserving embedding of MV-algebras $f^c : A^c \to C$ such that $f^c \circ \varphi_A = f$.

Remark 1.2. We remark that any σ-complete MV-algebra is semisimple [26, Proposition 6.6.2], and (A^c, s^c) is a probability MV-algebra.

Proposition 1.1. *If P is a unital and commutative PMV-algebra and s a state, P^c is a PMV-algebra.*

Proof. In order to define the product on P^c, it is enough to prove that the internal product on P is continuous with respect to ρ_s. Following the definition in [21], we define $[\mathbf{x}] \cdot [\mathbf{y}] = [(x_n \cdot y_n)_n]$ whenever $\mathbf{x} = \{x_n\}_n$ and $\mathbf{y} = \{y_n\}_n$. The product is well defined if and only if $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{x} \cdot \mathbf{z} \sim \mathbf{y} \cdot \mathbf{z}$. By definition this holds if and only if $\rho_s(x_n \cdot z_n, y_n \cdot z_n) \to 0$. By property of the unitary product, see [12, Corollary 5.7]

$$\rho_s(x_n \cdot z_n, y_n \cdot z_n) = s(d(x_n \cdot z_n, y_n \cdot z_n)) = s(z_n \cdot d(x_n, y_n)) \leq s(d(x_n, y_n)) =$$
$$\rho_s(x_n, y_n) \to 0 \text{ by hypothesis,}$$

therefore the conclusion follows. $\qquad\square$

Following [22], a *state-complete* Riesz MV-algebra is a structure (A, s) such that A is a Riesz MV-algebra, s is a state on A and (A, ρ_s) is a complete metric space. An L-space [6] is a Banach lattice $(L, \|\cdot\|)$ such that

$$x, y \geq 0 \text{ in } L \text{ implies } \|x + y\| = \|x\| + \|y\|.$$

By [22, Corollary 1] any state-complete Riesz MV-algebra is Dedekind-MacNeille complete. In [22] the author proves the categorical duality between state-complete Riesz MV-algebras and a particular class of measure space. We main point is the following theorem, which is similar with Kakutani representation for abstract L-spaces [17]

Definition 1.2. If (X, Ω, μ) is a measure space then we define

$$L_1(\mu)_u = \{f \in L_1(\mu) \mid \mathbf{0} \leq f \leq \mathbf{1}\}.$$

Remark 1.3. We note that $L_1(\mu)$ is an f-algebra and $\mathbf{1}$ is a weak unit of $L_1(\mu)$. Therefore $L_1(\mu)_u$ is an fMV-algebra.

Theorem 1.3. *[22] For any state complete Riesz MV-algebra (A, s) there exists a measure space (X, Ω, μ) and a isomorphism of Riesz MV-algebras $I_A : A \to L_1(\mu)_u$ such that $s(a) = \int I_A(a)d\mu$ for any $a \in A$. Moreover, (X, Ω, μ) is a probability space such that X is an extremally disconnected compact Hausdorff space, Ω is the Borel σ-algebra of X and μ is a topological finite measure.*

Definition 1.3. [22] A measure space (X, Ω, μ) that satisfies the properties from Theorem 1.3 is called L-space.

2 Embedding in $L^1(\mu)_u$

We prove that any MV-algebra which has a faithful state can be embedded in an fMV-algebra of integrable functions. As a preliminary step, we embed the MV-algebra in its divisible hull.

Remark 2.1.   1. Any MV-algebra can be embedded in a divisible one. For details on divisible MV-algebras see [14]. In the semisimple case, $A^d = \{a \in C(X) \mid a = \frac{a_1}{n} + \ldots + \frac{a_n}{n}, \; a_i \in A, n \in \mathbb{N}\}$, where $A \subseteq C(X)$. Moreover, if P is a unital and commutative PMV-algebra then $P \hookrightarrow P^d$ is an embedding of PMV-algebras.

   2. If $s : A \to [0, 1]$ is a state on A then s can be extended to a state $s^d : A^d \to [0, 1]$ [19, Theorem 6] such that $s^d(\alpha a) = \alpha s^d(a)$ for any $\alpha \in [0, 1] \cap \mathbb{Q}$ and $a \in A$ [10, Lemma 11]. Note that s^d is faithful whenever s is faithful.

Theorem 2.1. *Let A be an MV-algebra and $s : A \to [0, 1]$ a state on A. There exists an L-space (X, Ω, μ) and a homomorphism of MV-algebras $F_A : A \to L_1(\mu)_u$ such that $s(a) = \int F_A(a)d\mu$ for any $a \in A$. If s is faithful then F_A is an embedding.*

Proof. Let $B = A \Big/ Rad(A)$, where $Rad(A)$ is the intersection of all the maximal ideals of A. Denote by $\pi : A \to A \Big/ Rad(A)$ the canonical epimorphism. We

notice that B is semisimple and we can define the divisible hull B^d as in Remark 2.1. Let ι_d be the embedding in the divisible hull. With the notation of Theorem 1.2, we have

$$A \xrightarrow{\ \pi\ } B \xhookrightarrow{\ \iota_d\ } B^d \xhookrightarrow{\ \varphi_{B^d}\ } B^{dc}$$

By [7, Lemma 3.1], B^{dc} is a state-complete Riesz MV-algebra, therefore by Theorem 1.3, $B^{dc} \simeq L_1(\mu)_u$ for a suitable L-space. Finally, $t = \pi \circ s$ is a state on B. By Remark 2.1 t extends to t^d, and by Theorem 1.2 it extends to t^{dc}. The conclusion follows from Theorem 1.3. $\qquad\square$

Remark 2.2. The integral representation of a state from Theorem 2.1 is obviously Kroupa-Panti's result [18, 28]. We mention that, for our development, the representation of the algebraic structures is also crucial. We also mention that we followed the approach from [6], where Riesz integral representation is derived as a consequence of Kakutani's representation for L-spaces.

Proposition 2.1. *If P is a unital and semisimple PMV-algebra (fMV-algebra) then the morphism F_P from Theorem is a morphism of PMV-algebras (fMV-algebras).*

Proof. We first remark that any unital and semisimple PMV-algebra or fMV-algebra is commutative by [20]. If A is a unital and semisimple PMV-algebra, the conclusion follows by Proposition 1.1, Remark 2.1 and [5], since in any archimedean f-rings, the ring structure is generated by the additive group, and therefore any homomorphism of groups for the group reduct is an homomorphism of rings, and the same applies for unital and semisimple PMV-algebras. If A is an unital and semisimple fMV-algebra, the result follows by Remark 1.3, [20, Proposition 3.2] and [10, Corollary 2]. In particular in [20] is proved that any linear homomorphism between unital and semisimple fMV-algebras commutes with the internal product. $\qquad\square$

By means of this representation, we prove two immediate consequences for states using results in functional analysis: Hölder's inequality and the Hausdorff moment problem.

Hölder's inequality for PMV-algebras and fMV-algebras
The first result Hölder's inequality for PMV-algebras and fMV-algebra, in the unital and semisimple case. We recall that any unital and semisimple algebra is commutative.

Theorem 2.2. *Let A be a semisimple PMV^+-algebra ($\mathbb{FR}^+$-algebra) and $s : A \to [0,1]$ a state. If p, $q \in [1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$s(a \cdot b) \le s(a^p)^{\frac{1}{p}} s(b^q)^{\frac{1}{q}} \text{ for any } a, b \in A.$$

Proof. By Proposition 2.1, $F_A : A \to L_1(\mu)_u$ is a morphism of PMV-algebras (fMV-algebras). By Hölder's inequality for $L_1(\mu)$, for any a, $b \in A$ we get

$$\int_X F_A(a \cdot b)d\mu = \int_X (F_A(a) \cdot F_B(b))d\mu \le \left(\int_X F_A(a^p)d\mu\right)^{\frac{1}{p}} \left(\int_X F_A(b^q)d\mu\right)^{\frac{1}{q}},$$

and by Theorem 1.3, $s(a \cdot b) \le s(a^p)^{\frac{1}{p}} s(b^q)^{\frac{1}{q}}$. $\square$

The Hausdorff moment problem for PMV-algebras and fMV-algebras
In statistics and probability a very central subject is the Moment Problem. Given a interval $I \subseteq \mathbb{R}$, the nth-moment of a probability measure μ on I is defined as $\int_I x^n d\mu$. Let $\{m_k\}_{k\ge 0}$ be a sequence of real numbers, the Moment Problems on I consists on finding out the condition on $\{m_k\}_{k\ge 0}$ for which there exists a probability measure μ on I such that m_k is the kth moment of μ.
When $I = [0,1]$ we get the Hausdorff moment problem [15, 16]. We will prove a similar result in the context of MV-algebras.

For any $k \ge 1$ we define $p_k : [0,1] \to [0,1]$ by $p_k(x) = x^k$ for any $x \in [0,1]$. We also set $p_0(x) = 1$ for any $x \in [0,1]$. Note that $p_k \in FR_1$ for any $k \ge 0$.

If $\{m_k | k \ge 0\}$ a sequence of real numbers in $[0,1]$, we define:

$$\Delta^0 m_k = m_k, \quad \Delta^r m_k = \Delta^{r-1} m_{k+1} - \Delta^{r-1} m_k \text{ for any } r, k \ge 0.$$

The sequence $\{m_k\}_k$ satisfies the *Hausdorff moment condition* if $m_0 = 1$ and $(-1)^r \Delta^r m_k \ge 0$ for any r, $k \ge 0$ [11].

Theorem 2.3. *Let C be any unital and semisimple PMV-subalgebra (unital and semisimple fMV-subalgebra) of $C([0,1])$ such that $p_1 \in C$. There exists a state $s : C \to [0,1]$ such that $s(p_k) = m_k$ if and only if the sequence $\{m_k\}$ satisfies the Hausdorff moment condition.*

Proof. Let s be a state such that $s(x^k) = m_k$. Since C is unital, the set of its ideals coincide with the set of ideals of its MV-algebra reduct and by general

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theory of MV-algebras (see for example [4], Chapter 3) and by the integral representation due to Kroupa and Panti [18, 28] we have

$$s(f) = \int_0^1 f d\mu,$$

for any $f \in C$, where μ is a probability measure on $[0,1]$.

By [11] we have

$$(-1)^r \Delta^r m_k = \sum_{h=0}^r \binom{r}{h} (-1)^h m_{k+h},$$

then by the hypothesis,

$$(-1)^r \Delta^r m_k = \sum_{h=0}^r (-1)^h \binom{r}{h} \int_0^1 x^{k+h} d\mu = \int_0^1 \left[x^k \sum_{h=0}^r \binom{r}{h} (-1)^h x^h \right] d\mu =$$

$$= \int_0^1 x^k (1-x)^r d\mu \geq 0,$$

therefore the Hausdorff moment condition is satisfied.

On the other hand, let $\overline{s}$ the functional on the set $\{p_n \mid n = 0, 1, 2, \ldots\}$ such that $\overline{s}(p_k) = m_k$. By [23] $\overline{s}$ has a unique extension $\widetilde{s}$ to a linear prevision (that is a positive and normalized linear functional) from $C([0,1], \mathbb{R})$ to $\mathbb{R}$. In particular, $\widetilde{s}$ is a state between ℓ-groups, then $s : \Gamma(C([0,1], \mathbb{R}), 1) \to [0,1]$ is a state on $C([0,1])$ (see for example [27]). Taking $s|_C$, the restriction of s to C we get the desired result. $\square$

Let A be a unital PMV-algebra (unital fMV-algebra) such that $A \Big/ Rad(A) \subseteq C([0,1])$, and let ϕ_A be the map $\phi_A : A \to C([0,1])$ obtained composing the canonical epimorphism $A \to A \Big/ Rad(A)$ and the embedding of $A \Big/ Rad(A)$ in $C([0,1])$. Moreover, we ask that $p_1 \in \phi_A(A)$.

Corollary 2.1. *Let A be a unital PMV-algebra (unital fMV-algebra) as defined above. If the sequence $\{m_k\}$ satisfies the Hausdorff moment condition, then there exists a state $s : A \to [0,1]$ such that $s(p_k) = m_k$.*

Proof. Theorem 2.3 holds for $A \Big/ Rad(A)$, therefore s will be the composition of the state $t : A/Rad(A) \to [0,1]$ with the canonical epimorphism. $\square$

3 Stochastic independence of probability MV-algebras

The notion of independence for probability MV-algebra is one of the open problems mentioned in [30]. A partial solution has given in [21] for MV-algebras with states, MV-algebras with extremal states and semisimple MV-algebras. In this section we obtain a solution for probability MV-algebras.

Recall that a probability MV-algebra is a pair (A, s) with A a σ-complete MV-algebra and s a faithful σ-continuous state.

Definition 3.1. Let (A, s_A), (B, s_B) and (T, s) probability MV-algebras, and $\beta : A \times B \to T$ a bilinear function. (A, s_A) and (B, s_B) are said to be (T, s, β)-*independent* if $s(\lambda(a, b)) = s_A(a) \cdot s_B(b)$, for any $a \in A$ and $b \in B$.

Given (A, s_A) and (B, s_B) two probability MV-algebras, our problem is to define a probability MV-algebra (T, s) and a bilinear function $\beta : A \times B \to T$ such that (A, s_A) and (B, s_B) are (T, s, β)-independent.

Remark 3.1. We recall some general results from Measure Theory, see [13, 253D, 253G, 253F] for further details. Let (X_A, Ω_A, μ_A) and (X_B, Ω_B, μ_B) be measure spaces. There is a measure space $(X_A \times X_B, \Lambda, \lambda)$ where λ is the c.l.d. product measure on $X_A \times X_B$ and the following properties hold:
(1) $\otimes : L_1(\mu_A) \times L_1(\mu_B) \to L_1(\lambda)$, $(f, g) \mapsto f \otimes g$ is a bounded bilinear operator,
(2) $\int (f \otimes g) d\lambda = \int f d\mu_A \int g d\mu_B$ whenever $f \in L_1(\mu_A)$, $g \in L_1(\mu_B)$,
(3) $f \otimes g \geq 0$ in $L_1(\lambda)$ whenever $f \geq 0$ and $g \geq 0$,
(4) the following universal property is satisfied: for any Banach lattice W (norm complete Riesz space) and bilinear function ϕ there exists a unique linear function ω such that $\omega \circ \otimes = \phi$.

$$
\begin{array}{ccc}
L_1(\mu_A) \times L_1(\mu_B) & \xrightarrow{\otimes} & L_1(\lambda) \\
\phi \downarrow & \swarrow \omega & \\
W & &
\end{array}
$$

Definition 3.2. For any (A, s_A) and (B, s_B) probability MV-algebras and let $(X_A \Omega_A, \mu_A)$, $(X_A \Omega_A, \mu_A)$ be defined by Theorem 1.3 and $(X_A \times X_B, \Lambda, \lambda)$ is the product measure space from Remark 3.1. The *product space* is (T, s_T) where

$$T = L_1(\lambda)_u \text{ and } s_T(f) = \int_{X_A \times X_B} f d\lambda \text{ for any } f \in T.$$

Assume $\beta : A \times B \to L_1(\lambda)_u$ is the bilinear map defined by

$$\beta(a,b) = f_a \otimes f_b, \text{ where}$$
$$A \hookrightarrow A^d \hookrightarrow A^{dc} \simeq L_1(\mu_A)_u, \ a \mapsto F_A(a) = f_a,$$
$$B \hookrightarrow B^d \hookrightarrow B^{dc} \simeq L_1(\mu_B)_u, \ b \mapsto F_B(b) = f_b.$$

Remark 3.2. With the above definition, (T, s_T) is a probability MV-algebra by Theorem 1.3 and Theorem 1.2. Moreover, by Theorem 1.3 and [13, 253D],

$$s_T(\beta(a,b)) = \int (f_a \otimes f_b d\lambda) = \int f_a d\mu_A \int f_b d\mu_B.$$

Definition 3.3. Let (A, s_A), (B, s_B) and (C, s_C) be probability MV-algebras. A linear function $\omega : A \to C$ is *bounded* if there exists a natural number $K \geq 1$ such that

$$s_C(\omega(a)) \leq K_\oplus s_A(a) \text{ for any } a \in A.$$

We say that a bilinear function $\gamma : A \times B \to C$ is *bounded* if there exists a natural number $K \geq 1$ such that

$$s_C(\gamma(a,b)) \leq K_\oplus(s_A(a)s_B(b)) \text{ for any } a \in A \text{ and } b \in B.$$

The bilinear function γ is *continuous* if the following property holds: for any $(x_n)_n \subseteq A$, $x \in A$, $(y_n)_n \subseteq B$, $y \in B$,

$$\rho_{s_A}(x_n, x) \to 0 \text{ and } \rho_{s_B}(y_n, y) \to 0 \text{ imply } \rho_{s_C}(\gamma(x,y), \gamma(x_n, y_n)) \to 0.$$

One can immediately see that any bounded linear function is continuous. In the sequel we prove the same result for bilinear functions.

Lemma 3.1. *If (A, s_A), (B, s_B) and (C, s_C) are probability MV-algebras, $K \geq 1$ is a natural number and the bilinear function $\gamma : A \times B \to C$ is K- bounded then, for any a, $a' \in A$ and b, $b' \in B$*

$$\rho_{s_C}(\gamma(a,b), \gamma^d(a', b')) \leq K_\oplus(\rho_{s_A}(a,a') \oplus \rho_{s_B}(b,b')).$$

Proof. Assume $b \in B$. Since $\gamma(\cdot, b) : A \to C$ is a linear map, we get

$$d(\gamma(a,b), \gamma(a', b)) \leq \gamma(d(a,a'), b) \text{ for any } a, a' \in A.$$

It follows that

$$s_C(d(\gamma(a,b), \gamma(a', b)) \leq s_C(\gamma(d(a,a'), b)) \leq K_\oplus(s_A(d(a,a'))s_B(b)),$$

so $s_C(d(\gamma(a,b), \gamma(a', b)) \leq K_\oplus s_A(d(a,a'))$ for some constant $K \geq 0$.

Similarly we get $s_C(d(\gamma(a,b), \gamma(a, b')) \leq K_\oplus s_B(d(b,b'))$ for any $a \in A$, so

$$s_C(d(\gamma(a,b), \gamma(a', b'))) \leq s_C(d(\gamma(a,b), \gamma(a', b))) \oplus s_C(d(\gamma(a', b), \gamma(a', b')))$$
$$\leq K_\oplus s_A(d(a,a')) \oplus K_\oplus s_B(d(b,b')), \text{ so}$$
$$\rho_{s_C}(\gamma(a,b), \gamma^d(a', b')) \leq K_\oplus(\rho_{s_A}(a,a') \oplus \rho_{s_B}(b,b')).$$

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$\square$

Corollary 3.1. *If (A, s_A), (B, s_B) and (C, s_C) are probability MV-algebras then any bounded bilinear function $\gamma : A \times B \to C$ is continuous.*

Proof. It follows by Lemma 3.1. $\qquad\square$

Proposition 3.1. *If A and B are MV-algebras and $\sigma : A \to B$ is a linear function, then there is a unique linear function $\sigma^d : A^d \to B^d$ that extends σ.*

Proof. Let $a \in A^d$ and $a_1, \ldots, a_n \in A$ such that $a = \frac{a_1}{n} + \cdots + \frac{a_n}{n}$. We set $\sigma^d(a) = \frac{1}{n}(\sigma(a_1) + \cdots + \sigma(a_n))$. Using Riesz decomposition property[9, Section 2.9] in A^d one can prove that σ^d is well-defined. We show that σ^d is linear. Assume that $a + a'$ is defined in A^d. We know that $a = \frac{a_1}{n} + \cdots + \frac{a_n}{n}$ and $a' = \frac{a'_1}{m} + \cdots + \frac{a'_m}{m}$ where $a_1, \ldots, a_n, a'_1, \ldots, a'_n \in A$. It follows that
$a + a' = \frac{a_1}{n} + \cdots + \frac{a_n}{n} + \frac{a'_1}{m} + \cdots + \frac{a'_m}{m} = m\frac{a_1}{nm} + \cdots + m\frac{a_n}{nm} + n\frac{a'_1}{nm} + \cdots + n\frac{a'_m}{nm}$.
We get
$$\sigma^d(a + a') = \frac{1}{nm}(m\sigma(a_1) + \cdots + m\sigma(a_n) + n\sigma(a'_1) + \cdots + n\sigma(a'_m))$$
$$= \frac{1}{n}(\sigma(a_1) + \cdots + \sigma(a_n)) + \frac{1}{m}(\sigma(a'_1) + \cdots + \sigma(a'_m))$$
$$= \sigma(a) + \sigma(a').$$

$\square$

Lemma 3.2. *If (A, s_A), (B, s_B) and (C, s_C) are probability MV-algebras and $\gamma : A \times B \to C$ is a bilinear function, then there exists a unique bilinear $\gamma^d : A^d \times B^d \to C^d$ that extends γ. Moreover, if γ is bounded then γ^d is also bounded.*

Proof. We first recall that any linear function between divisible MV-algebra is linear w.r.t. scalars in $[0, 1] \cap \mathbb{Q}$, has remarked for states in Remark 2.1. If $a \in A^d$ and $b \in B^d$ then there are $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B$ such that $a = \frac{a_1}{n} + \ldots + \frac{a_n}{n}$ and $b = \frac{b_1}{m} + \ldots + \frac{b_m}{m}$. We define $\gamma^d : A^d \times B^d \to C^d$ by
$$\gamma^d(a, b) = \frac{1}{nm} \sum \gamma(a_i, b_j).$$
The fact that γ^d is well-defined and the uniqueness follow by Proposition 3.1. We have that
$s_C^d(\gamma^d(a, b)) = s_C^d(\gamma^d(\frac{a_1}{n} + \ldots + \frac{a_n}{n}, \frac{b_1}{m} + \ldots + \frac{b_m}{m})) = s_C^d(\frac{1}{nm} \sum_{ij} \gamma(a_i, b_j)) =$
$s_C^d(\sum_{ij} \frac{1}{nm} \gamma(a_i, b_j)) = \sum_{ij} \frac{1}{nm} s_C^d(\gamma(a_i, b_j)) \le \sum_{ij} \frac{1}{nm} (K_\oplus(s_A(a_i)s_B(b_j))) =$
$\sum_{i,j} \frac{1}{nm} \min(Ks_A(a_i)s_B(b_j), 1) = \sum_{i,j} \min(Ks_A^d(\frac{a_i}{n})s_B^b(\frac{b_j}{m}), \frac{1}{nm}) \le$
$\min(K(\sum_{i,j} s_A^d(a_i)s_B^d(b_j)), \frac{nm}{nm}) = K_\oplus \left(\sum_{i,j} s_A^d \left(\frac{a_i}{n} \right) s_B^d \left(\frac{b_j}{m} \right) \right) =$
$K \oplus \left(\left(s_A^d \left(\frac{a_1}{n} \right) + \ldots + s_A^d \left(\frac{a_n}{n} \right) \right) \left(s_B^d \left(\frac{b_1}{m} \right) + \ldots + s_B^d \left(\frac{b_m}{m} \right) \right) \right) = K_\oplus(s_A^d(a)s_B^d(b))$

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so γ^d is bounded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 3.2. *Let (A, s_A), (B, s_B) and (C, s_C) be probability MV-algebras, and $\gamma : A \times B \to C$ a bounded bilinear function. Then there exists a unique bounded bilinear function $\gamma^c : A^c \times B^c \to C^c$ that extends γ, defined by*

$$\gamma^c([\{a_n\}_n], [\{b_n\}_n]) = [\{\gamma(a_n, b_n)\}_n]$$

for any Cauchy sequences $\{a_n\}_n$ from A and $\{b_n\}_n$ from B.

Proof. Let $K \geq 1$ be a natural number such that γ is bounded with constant K. If $\{a_n\}_n$, $\{a'_n\}_n$ are Cauchy sequences in A and let $\{b_n\}_n$, $\{b'_n\}_n$ are Cauchy sequences in B, from Lemma 3.1, we infer that $\{\gamma(a_n, b_n)\}_n$ is a Cauchy sequence in C. Moreover, $[\{a_n\}_n] = [\{a'_n\}_n]$ and $[\{b_n\}_n] = [\{b'_n\}_n]$ imply $[\{\gamma(a_n, b_n)\}_n] = [\{\gamma(a'_n, b'_n)\}_n]$. Hence γ^c can be defined by

$$\gamma^c([\{a_n\}_n], [\{b_n\}_n]) = [\{\gamma(a_n, b_n)\}_n]$$

for any Cauchy sequences $\{a_n\}_n$ from A and $\{b_n\}_n$ from B. We get

$$s_C^c(\gamma^c([\{a_n\}_n], [\{b_n\}_n])) = \lim_n s_C(\gamma(a_n, b_n)) \leq K_\oplus(\lim_n s_A(a_n) \lim_n s_B(b_n)),$$
$$s_C^c(\gamma^c([\{a_n\}_n], [\{b_n\}_n])) \leq K_\oplus(s_A^c([\{a_n\}_n]) s_B^c([\{b_n\}_n])).$$

Let $\beta : A^c \times B^c \to C^c$ be another bilinear bounded function that extends γ. Since $\lim_n [a_n] = [\{a_n\}_n]$ in A^c and $\lim_n [b_n] = [\{b_n\}_n]$ in B^c, by Corollary 3.1,

$$\beta([\{a_n\}_n], [\{b_n\}_n]) = \lim_n \beta([a_n], [b_n]) = \lim_n \gamma(a_n, b_n) = [\{\gamma(a_n, b_n)\}_n] =$$
$$\gamma^c([\{a_n\}_n], [\{b_n\}_n]).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.3. If (A, s_A), (B, s_B) and (C, s_C) are probability MV-algebras and $\gamma : A \times B \to C$ is a bounded bilinear function then the following diagram is commutative:

$$
\begin{array}{ccccc}
A \times B & \xrightarrow{\iota_{d_A} \times \iota_{d_B}} & A^d \times B^d & \xrightarrow{\varphi_{A^d} \times \varphi_{B^d}} & A^{dc} \times B^{dc} \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma^d} & & \downarrow{\scriptstyle\gamma^{dc}} \\
C & \xrightarrow{\iota_{dC}} & C^d & \xrightarrow{\varphi_C} & C^{dc}
\end{array}
$$

This is a straightforward consequence of Lemma 3.2 and Proposition 3.2.

We are ready to prove the main result of this section.

Theorem 3.1. *Let (A, s_A), (B, s_B) be probability MV-algebras and assume (T, s_T) and $\beta : A \times B \to T$ are defined as in Definition 3.2. For any probability MV-algebra (C, s_C) and any bounded bilinear function $\gamma : A \times B \to C$ there exists a unique bounded linear function $\omega : T \to C^{dc}$ such that $\omega(\beta(a, b)) = \gamma(a, b)$ for any $a \in A$, $b \in B$, i.e. $\omega(f_a \otimes f_b) = f_{\gamma(a,b)}$ for any $a \in A$, $b \in B$.*

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ \beta\ } & T = L_1(\lambda)_u \\
{\scriptstyle\gamma}\big\downarrow & & \\
C & & \omega \\
{\scriptstyle\varphi_{C^d}}\big\uparrow\big\downarrow & & \\
C^{dc} = L_1(\mu_C)_u & &
\end{array}
$$

Proof. By Proposition 3.2 there exists a unique bounded bilinear function $f\gamma^{dc} : A^{dc} \times B^{dc} \to C^{dc}$ that extends γ. By Theorem 1.3 $A^{dc} \simeq L_1(\mu_A)_u$, $B^{dc} \simeq L_1(\mu_B)_u$ and $C^{dc} \simeq L_1(\mu_C)_u$ for suitable measure spaces.

Note that $\mathbf{1}_A$ and $\mathbf{1}_B$ are weak units in $L_1(\mu_A)$ and $L_1(\mu_B)$. If we set
$$L_A = \{ f \in L_1(\mu_A) \mid |f| \le n\mathbf{1}_A \text{ for some } n \ge 1 \},$$
$$L_B = \{ g \in L_1(\mu_B) \mid |g| \le n\mathbf{1}_B \text{ for some } n \ge 1 \},$$
then $(L_A, \mathbf{1}_A)$ and $(L_B, \mathbf{1}_B)$ are f-algebras with strong unit and they are dense in $L_1(\mu_A)_u$ and $L_1(\mu_B)_u$, respectively. By [12, Proposition 6.5] there exists an extension $\widetilde{\gamma} : L_A \times L_B \to L_1(\mu_C)$ u-bilinear function that extends γ^{dc}. We recall that by construction $\widetilde{\gamma}(\mathbf{1}_A, \mathbf{1}_B) \le \mathbf{1}_C$. One can easily see that the bilinear map $\widetilde{\gamma}$ is bounded.

In order to extend $\widetilde{\gamma}$ to $L_1(\mu_A) \times L_1(\mu_B)$ we shall apply the B.L.T. theorem [29, Theorem I.7] twice. Let $g \in L_B$ be an arbitrary element and apply the B.L.T. theorem for $\widetilde{\gamma}(\cdot, g) : L_A \to L_1(\mu_C)$. Hence there exists a unique bounded linear transformation $\gamma_g : L_1(\mu_A) \to L_1(\mu_C)$ such that $\gamma_g(f) = \widetilde{\gamma}(f, g)$ for any $f \in L_1(\mu_A)$. Now, we fix $f \in L_1(\mu_A)$ and we define $\gamma^f : L_B \to L_1(\mu_C)$ by $\gamma^f(g) = \gamma_g(f)$ for any $g \in L_B$. Applying again the B.L.T. theorem we get a unique bounded linear transformation $\gamma^{f'} : L_1(\mu_B) \to L_1(\mu_C)$ such that $\gamma^{f'}(g) = \beta_g(f)$ for any $g \in L_B$. Finally we define $\gamma' : L_1(\mu_A) \times L_1(\mu_B) \to L_1(\mu_C)$ by $\gamma'(f, g) = \gamma^{f'}(g)$ for any $f \in L_1(\mu_A)$ and $g \in L_1(\mu_B)$. It follows that $\gamma'(f, g) = \widetilde{\gamma}(f, g)$ whenever $f \in L_A$ and $g \in L_B$, so γ' is also bounded.

By [13, 253F, 253G] there exists a unique bounded linear operator Ω : $L_1(\lambda) \to L_1(\mu_C)$ such that $\Omega(f \otimes g) = \gamma'(f,g)$ for any $f \in L_1(\mu_A)$ and $g \in L_1(\mu_B)$. The desired bounded linear function is $\omega = \Omega \mid_T$.

To prove the uniqueness, let $\omega' : T \to L_1(\mu_C)$ be another bounded linear function that closes the above diagram. Using [12, Proposition 4.2] we extend it to the f-algebra generated by T in $L_1(\lambda)$. Applying the B.L.T. theorem we get a bounded bilinear transformation $\Omega' : L_1(\lambda) \to L_1(\mu_C)$ such that $\Omega'(f \otimes g) = \gamma(f,g)$ for any $f \in L_1(\mu_A)_u$ and $g \in L_1(\mu_B)_u$. Following similar arguments as above one gets $\Omega'(f \otimes g) = \gamma'(f,g)$ for any $f \in L_1(\mu_A)$ and $g \in L_1(\mu_B)$, so $\Omega' = \Omega$ and $\omega' = \omega$.

$\square$

Remark 3.4. If (A, s_A) and (B, s_B) are two probability MV-algebras we defined a probability MV-algebra (T, s) and a bilinear function $\beta : A \times B \to T$ such that (A, s_A) and (B, s_B) are (T, s, β)-independent. Thorem 3.1 can be seen as a "universal property" of the product space, but it does not define (T, s_T) up to isomorphism. If (A, s_A) and (B, s_B) are probability MV-algebras and both (T, s_T) and (V, s_V) satisfy the property from Theorem 3.1 then T and V have isomorphic group reducts.

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