# PSEUDO MV-ALGEBRAS AND LEXICOGRAPHIC PRODUCT 

ANATOLIJ DVUREČENSKIJ ${ }^{1,2}$<br>${ }^{1}$ Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia<br>${ }^{2}$ Depart. Algebra Geom., Palacký University<br>17. listopadu 12, CZ-771 46 Olomouc, Czech Republic<br>E-mail: dvurecen@mat.savba.sk


#### Abstract

We study algebraic conditions when a pseudo MV-algebra is an interval in the lexicographic product of an Abelian unital $\ell$-group and an $\ell$ group that is not necessary Abelian. We introduce ( $H, u$ )-perfect pseudo MValgebras and strong $(H, u)$-perfect pseudo MV-algebras, the latter ones will have a representation by a lexicographic product. Fixing a unital $\ell$-group $(H, u)$, the category of strong $(H, u)$-perfect pseudo MV-algebras is categorically equivalent to the category of $\ell$-groups.


## 1. Introduction

MV-algebras were introduced by Chang Cha as the algebraic counterpart of Lukasiewicz infinite-valued calculus and during the last 56 years MV-algebras entered deeply in many areas of mathematics and logics. More than 10 years ago, a non-commutative generalization of MV-algebras has been independently appeared. These new algebras are said to be pseudo MV-algebras in GeIO or a generalized MV-algebras in Rac. For them author Dvu2 generalized a famous Mundici's representation theorem, see e.g. [CDM, Cor 7.1.8], showing that every pseudo MValgebra is always an interval in a unital $\ell$-group not necessarily Abelian. Such algebras have the operation $\oplus$ as a truncated sum and they have two negations. We note that the equality of these two negations does not necessarily imply that a pseudo MV-algebra is an MV-algebra. According to Komori's theorem Kom, CDM, Thm 8.4.4], the variety lattice of MV-algebras is countably, whereas the one of pseudo MV-algebras is uncountable, cf. Jak DvHO. Therefore, the structure of pseudo MV-algebras is much richer than the one of MV-algebras. In DvHo it was shown that the class of pseudo MV-algebras where each maximal ideal is normal is a variety. This variety is also very rich and within this variety many important properties of MV-algebras remain.

In (DDT], perfect pseudo MV-algebras were studied. They are characterized as those that every element of a perfect pseudo MV-algebra is either an infinitesimal or a co-infinitesimal. In DDT we have shown that the category of perfect

[^0]pseudo MV-algebras is equivalent to the variety of $\ell$-groups, and every such an algebra $M$ is in the form of an interval in the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$, i.e. $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$. This generalized the result from DiLe1 for perfect MValgebras. A more general structure, $n$-perfect pseudo MV-algebras were studied in Dvu3. They can be characterized as those pseudo MV-algebras that have $(n+1)$ comparable slices, and their representation is again in the form of an interval in the lexicographic product $\frac{1}{n} \mathbb{Z} \overrightarrow{\times} G$, i.e. every strong $n$-perfect pseudo MV-algebra $M$ is of the form $\Gamma\left(\frac{1}{n} \mathbb{Z} \overrightarrow{\times} G,(1,0)\right)$, where $G$ is any $\ell$-group. In the paper Dvu4, we have studied so-called ( $\mathbb{H}, 1$ )-perfect pseudo MV-algebras, where $(\mathbb{H}, 1)$ is a unital $\ell$-subgroup of the unital $\ell$-group of reals $(\mathbb{R}, 1)$. They can be represented in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$ and such MV-algebras were described in DiLe2].

Recently, lexicographic MV-algebras were studied in DFL. Such algebras are of the form $\Gamma(H \overrightarrow{\times} G,(u, 0))$, where $(H, u)$ is an Abelian unital $\ell$-group and $G$ is an Abelian $\ell$-group. The main aim of the present paper is to generalize such lexicographic MV-algebras also for the case of pseudo MV-algebras. Therefore, we introduce so-called $(H, u)$-perfect and strong $(H, u)$-perfect pseudo MV-algebras, where $(H, u)$ is an Abelian unital $\ell$-group. We show that strong $(H, u)$-perfect pseudo MV-algebras are always of the form $\Gamma(H \overrightarrow{\times} G,(u, 0))$, where $G$ is an $\ell$-group not necessarily Abelian. This category will be always categorically equivalent with the variety of $\ell$-groups. Therefore, we generalize many interesting results that were known only in the realm of MV-algebras, see DiLe2, CiTo, DFL.

The paper is organized as follows. Section 2 gathers necessary properties of pseudo MV-algebras. Section 3 presents a definition of ( $H, u$ )-perfect pseudo MValgebras as those which can be decomposed into a system of comparable slices indexed by the elements of the interval $[0, u]_{H}=\{h \in H: 0 \leq h \leq u]$, where $(H, u)$ is an Abelian unital $\ell$-group. Section 4 defines strong $(H, u)$-perfect pseudo MV-algebras and we show their representation by $\Gamma(H \overrightarrow{\times} G,(u, 0))$. More details on local pseudo MV-algebras with retractive ideals will be done in Section 5. A free product representation of local pseudo MV-algebras will be done in Section 6. In Section 7 we describe pseudo MV-algebras with a so-called lexicographic ideal. A categorical equivalence of the category of strong $(H, u)$-perfect pseudo MV-algebras will be established in Section 8. Finally, in Section 9 we describe weak $(H, u)$-perfect pseudo MV-algebras as those that they can be represented in the form $\Gamma(H \overrightarrow{\times} G,(u, g))$, where $g$ is an arbitrary element (not necessarily $g=0$ ) of an $\ell$-group $G$.

## 2. Pseudo MV-algebras

According to GeIO, a pseudo $M V$-algebras or a GMV-algebra by Rac is an algebra $\left(M ; \oplus,^{-}, \sim, 0,1\right)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$
y \odot x=\left(x^{-} \oplus y^{-}\right)^{\sim}
$$

(A1) $x \oplus(y \oplus z)=(x \oplus y) \oplus z$;
(A2) $x \oplus 0=0 \oplus x=x$;
(A3) $x \oplus 1=1 \oplus x=1$;
(A4) $1^{\sim}=0 ; 1^{-}=0$;
(A5) $\left(x^{-} \oplus y^{-}\right)^{\sim}=\left(x^{\sim} \oplus y^{\sim}\right)^{-}$;
(A6) $x \oplus\left(x^{\sim} \odot y\right)=y \oplus\left(y^{\sim} \odot x\right)=\left(x \odot y^{-}\right) \oplus y=\left(y \odot x^{-}\right) \oplus x{ }^{2}$
$(\mathrm{A} 7) x \odot\left(x^{-} \oplus y\right)=\left(x \oplus y^{\sim}\right) \odot y$;
(A8) $\left(x^{-}\right)^{\sim}=x$.
Any pseudo MV-algebra is a distributive lattice where (A6) and (A7) define the joint $x \vee y$ and the meet $x \wedge y$ of $x, y$, respectively.

We note that a po-group ( $=$ partially ordered group) is a group $(G ;+, 0)$ (written additively) endowed with a partial order $\leq$ such that if $a \leq b, a, b \in G$, then $x+a+y \leq x+b+y$ for all $x, y \in G$. We denote by $\left.G^{+}=\overline{\{ } g \in G: g \geq 0\right\}$ the positive cone of $G$. If, in addition, $G$ is a lattice under $\leq$, we call it an $\ell$-group (= lattice ordered group). An element $u \in G^{+}$is said to be a strong unit (= order unit) if $G=\bigcup_{n}[-n u, n u]$, and the couple ( $G, u$ ) with a fixed strong unit $u$ is said to be a unital po-group or a unital $\ell$-group, respectively. The commutative center of a group $H$ is the set $C(H)=\left\{h \in H: h+h^{\prime}=h^{\prime}+h, \forall h^{\prime} \in H\right\}$. Finally, two unital $\ell$-groups $(G, u)$ and $(H, v)$ is isomorphic if there is an $\ell$-group isomorphism $\phi: G \rightarrow H$ such that $\phi(u)=v$. In a similar way an isomorphism and a homomorphism of unital po-groups are defined. For more information on po-groups and $\ell$-groups and for unexplained notions about them, see [Fuc, Gla].

By $\mathbb{R}$ and $\mathbb{Z}$ we denote the groups of reals and natural numbers, respectively.
Between pseudo MV-algebras and unital $\ell$-groups there is a very close connection: If $u$ is a strong unit of a (not necessarily Abelian) $\ell$-group $G$,

$$
\Gamma(G, u):=[0, u]
$$

and

$$
\begin{aligned}
x \oplus y & :=(x+y) \wedge u, \\
x^{-} & :=u-x, \\
x^{\sim} & :=-x+u, \\
x \odot y & :=(x-u+y) \vee 0,
\end{aligned}
$$

then $\left(\Gamma(G, u) ; \oplus,^{-}, \sim, 0, u\right)$ is a pseudo MV-algebra GeIo.
A pseudo MV-algebra $M$ is an $M V$-algebra if $x \oplus y=y \oplus x$ for all $x, y \in M$. We denote by $\mathcal{P}_{s} \mathcal{M V}$ and $\mathcal{M V}$ the variety of pseudo MV-algebras and MV-algebras, respectively.

The basic representation theorem for pseudo MV-algebras is the following generalization Dvu2] of the Mundici's famous result:

Theorem 2.1. For any pseudo $M V$-algebra $\left(M ; \oplus,,^{\sim}, ~, 0,1\right)$, there exists a unique (up to isomorphism) unital $\ell$-group $(G, u)$ such that $\left(M ; \oplus,^{-}, \sim, 0,1\right)$ is isomorphic to $\left(\Gamma(G, u) ; \oplus,,^{-}, \sim, 0, u\right)$. The functor $\Gamma$ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital $\ell$-groups.

We note that the class of pseudo MV-algebras is a variety whereas the class of unital $\ell$-groups is not a variety because it is not closed under infinite products.

Due to this result, if $M=\Gamma(G, u)$ for some unital $\ell$-group $(G, u)$, then $M$ is linearly ordered iff $G$ is a linearly ordered $\ell$-group, see [Dvu1, Thm 5.3].

Besides a total operation $\oplus$, we can define a partial operation + on any pseudo MV-algebra $M$ in such a way that $x+y$ is defined iff $x \odot y=0$ and then we set

$$
\begin{equation*}
x+y:=x \oplus y \tag{2.1}
\end{equation*}
$$

[^1]In other words, $x+y$ is precisely the group addition $x+y$ if the group sum $x+y$ is defined in $M$.

Let $A, B$ be two subsets of $M$. We define (i) $A \leqslant B$ if $a \leq b$ for all $a \in A$ and all $b \in B$, (ii) $A \oplus B=\{a \oplus b: a \in A, b \in B\}$, and (iii) $A+B=\{a+b:$ if $a+b$ exists in $M$ for $a \in A, b \in B\}$. We say that $A+B$ is defined in $M$ if $a+b$ exists in $M$ for each $a \in A$ and each $b \in B$. (iv) $A^{-}=\left\{a^{-}: a \in A\right\}$ and $A^{\sim}=\left\{a^{\sim}: a \in A\right\}$.

Using Theorem 2.1, we have if $y \leq x$, then $x \odot y^{-}=x-y$ and $y^{\sim} \odot x=-y+x$, where the subtraction - is in fact the group subtraction in the representing unital $\ell$-group.

We recall that if $H$ and $G$ are two po-groups, then the lexicographic product $H \overrightarrow{\times} G$ is the group $H \times G$ which is endowed with the lexicographic order: $(h, g) \leq$ $\left(h_{1}, g_{1}\right)$ iff $h<h_{1}$ or $h=h_{1}$ and $g \leq g_{1}$. The lexicographic product $H \overrightarrow{\times} G$ is an $\ell$-group iff $H$ is linearly ordered group and $G$ is an arbitrary $\ell$-group, Fuc, (d) p. 26]. If $u$ is a strong unit for $H$, then $(u, 0)$ is a strong unit for $H \overrightarrow{\times} G$, and $\Gamma(H \overrightarrow{\times} G,(u, 0))$ is a pseudo MV-algebra.

We say that a pseudo MV-algebra $M$ is symmetric if $x^{-}=x^{\sim}$ for all $x \in M$. The pseudo MV-algebra $\Gamma(G, u)$ is symmetric iff $u \in C(G)$, hence the variety of symmetric pseudo MV-algebras is a proper subvariety of the variety $\mathcal{M V}$. For example, $\Gamma(\mathbb{R} \overrightarrow{\times} G,(1,0))$ is symmetric and it is an MV-algebra iff $G$ is Abelian.

An ideal of a pseudo MV-algebra $M$ is any non-empty subset $I$ of $M$ such that (i) $a \leq b \in I$ implies $a \in I$, and (ii) if $a, b \in I$, then $a \oplus b \in I$. An ideal is said to be (i) maximal if $I \neq M$ and it is not a proper subset of another ideal $J \neq M$; we denote by $\mathcal{M}(M)$ the set of maximal ideals of $M$, (ii) prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, and (iii) normal if $x \oplus I=I \oplus x$ for any $x \in M$; let $\mathcal{N}(M)$ be the set of normal ideals of $M$. A pseudo MV-algebra $M$ is local if there is a unique maximal ideal and, in addition, this ideal also normal.

There is a one-to-one correspondence between normal ideals and congruences for pseudo MV-algebras, GeIO, Thm 3.8]. The quotient pseudo MV-algebra over a normal ideal $I, M / I$, is defined as the set of all elements of the form $x / I:=\{y \in$ $\left.M: x \odot y^{-} \oplus y \odot x^{-} \in I\right\}$, or equivalently, $x / I:=\left\{y \in M: x^{\sim} \odot y \oplus y^{\sim} \odot x \in I\right\}$.

Let $x \in M$ and an integer $n \geq 0$ be given. We define

$$
\begin{gathered}
0 . x:=0, \quad 1 \odot x:=x, \quad(n+1) \cdot x:=(n \cdot x) \oplus x \\
x^{0}:=1, \quad x^{1}:=x, \quad x^{n+1}:=x^{n} \odot x \\
0 x:=0, \quad 1 x:=x, \quad(n+1) x:=(n x)+x
\end{gathered}
$$

if $n x$ and $(n x)+x$ are defined in $M$. An element $x$ is said to be an infinitesimal if $m x$ exists in $M$ for every integer $m \geq 1$. We denote by $\operatorname{Infinit}(M)$ the set of all infinitesimals of $M$.

We define (i) the radical of a pseudo MV-algebra $M, \operatorname{Rad}(M)$, as the set

$$
\operatorname{Rad}(M)=\bigcap\{I: I \in \mathcal{M}(M)\}
$$

and (ii) the normal radical of $M, \operatorname{Rad}_{n}(M)$, via

$$
\operatorname{Rad}_{n}(M)=\bigcap\{I: I \in \mathcal{N}(M) \cap \mathcal{M}(M)\}
$$

whenever $\mathcal{N}(M) \cap \mathcal{M}(M) \neq \emptyset$.
By [DDJ, Prop. 4.1, Thm 4.2], it is possible to show that

$$
\operatorname{Rad}(M) \subseteq \operatorname{Infinit}(M) \subseteq \operatorname{Rad}_{n}(M)
$$

The notion of a state is an analogue of a probability measure for pseudo MValgebras. We say that a mapping $s$ from a pseudo MV-algebra $M$ into the real interval is a state if (i) $s(a+b)=s(a)+s(b)$ whenever $a+b$ is defined in $M$, and (ii) $s(1)=1$. We define the kernel of $s$ as the set $\operatorname{Ker}(s)=\{a \in M: s(a)=0\}$. Then $\operatorname{Ker}(s)$ is a normal ideal of $M$.

If $M$ is an MV-algebra, then at least one state on $M$ is defined. Unlike for MValgebras, there are pseudo MV-algebras that are stateless, Dvu1 (see also a note just before Theorem 8.5 below). We note that $M$ has at least one state iff $M$ has at least one maximal ideal that is also normal. However, every non-trivial linearly ordered pseudo MV-algebra admits a unique state, Dvu1, Thm 5.5].

Let $\mathcal{S}(M)$ be the set of all states on $M$; it is a convex set. A state $s$ is said to be extremal if from $s=\lambda s_{1}+(1-\lambda) s_{2}$, where $s_{1}, s_{2} \in \mathcal{S}(M)$ and $0<\lambda<1$, we conclude $s=s_{1}=s_{2}$. Let $\partial_{e} \mathcal{S}(M)$ denote the set of extremal states. In addition, in view of Dvu1], a state $s$ is extremal iff $\operatorname{Ker}(s)$ is a maximal ideal of $M$ iff $s(a \wedge b)=\min \{s(a), s(b)\}$. Or equivalently, $s$ is a state morphism, i.e., $s$ is a homomorphism from $M$ into the MV-algebra $\Gamma(\mathbb{R}, 1)$. In addition, a normal ideal $I$ is maximal iff $I=\operatorname{Ker}(s)$ for some extremal state $s$.

We say that a net of states $\left\{s_{\alpha}\right\}_{\alpha}$ converges weakly to a state $s$ if $s(a)=$ $\lim _{\alpha} s_{\alpha}(a), a \in M$. Then $\mathcal{S}(M)$ and $\partial_{e} \mathcal{S}(M)$ are compact Hausdorff topological spaces, in particular cases both can be empty, and due to the Krein-Mil'man Theorem [GO Thm 5.17], every state on $M$ is a weak limit of a net of convex combinations of extremal states.

Pseudo MV-algebras can be studied also in the frames of pseudo effect algebra which are a non-commutative generalization of effect algebras introduced by [FoBe].

According to DvVe1, DvVe2, a partial algebraic structure $(E ;+, 0,1)$, where + is a partial binary operation and 0 and 1 are constants, is called a pseudo effect algebra if, for all $a, b, c \in E$, the following hold:
(PE1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case, $(a+b)+c=a+(b+c)$;
(PE2) there are exactly one $d \in E$ and exactly one $e \in E$ such that $a+d=e+a=$ $1 ;$
(PE3) if $a+b$ exists, there are elements $d, e \in E$ such that $a+b=d+a=b+e$;
(PE4) if $a+1$ or $1+a$ exists, then $a=0$.
If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\leq$ is a partial ordering on $E$ such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if and only if $b=a+c=d+a$ for some $c, d \in E$. We write $c=a / b$ and $d=b \backslash a$. Then

$$
(b \backslash a)+a=a+(a / b)=b,
$$

and we write $a^{-}=1 \backslash a$ and $a^{\sim}=a / 1$ for any $a \in E$.
If $(G, u)$ is a unital po-group, then $(\Gamma(G, u) ;+, 0, u)$, where the set $\Gamma(G, u):=$ $\{g \in G: 0 \leq g \leq u\}$ is endowed with the restriction of the group addition + to $\Gamma(G, u)$ and with 0 and $u$ as 0 and 1 , is a pseudo effect algebra. Due to DvVe1, DvVe2, if a pseudo effect algebra satisfies a special type of the Riesz Decomposition Property, $\mathrm{RDP}_{1}$, then every pseudo effect algebra is an interval in some unique (up to isomorphism of unital po-groups) $(G, u)$ satisfying also $\operatorname{RDP}_{1}$ such that $M \cong \Gamma(G, u)$.

We say that a mapping $f$ from one pseudo effect algebra $E$ onto a second one $F$ is a homomorphism if (i) $a, b \in E$ such that $a+b$ is defined in $E$, then $f(a)+f(b)$ is defined in $F$ and $f(a+b)=f(a)+f(b)$, and (ii) $f(1)=1$.

We say that a pseudo effect algebra $E$ satisfies $\mathrm{RDP}_{2}$ property if $a_{1}+a_{2}=b_{1}+b_{2}$ implies that there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that (i) $a_{1}=c_{11}+c_{12}$, $a_{2}=c_{21}+c_{22}, b_{1}=c_{11}+c_{21}$ and $b_{2}=c_{12}+c_{22}$, and (ii) $c_{12} \wedge c_{21}=0$.

In DvVe2, Thm 8.3, 8.4], it was proved that if $(M ; \oplus,-, \sim, 0,1)$ is a pseudo MValgebra, then $(M ;+, 0,1)$, where + is defined by (2.1), is a pseudo effect algebra with $\mathrm{RDP}_{2}$. Conversely, if $(E ;+, 0,1)$ is a pseudo effect algebra with $\mathrm{RDP}_{2}$, then $E$ is a lattice, and by DvVe2, Thm 8.8], $\left(E ; \oplus,^{-}, \sim, 0,1\right)$, where

$$
\begin{equation*}
a \oplus b:=\left(b^{-} \backslash\left(a \wedge b^{-}\right)\right)^{\sim}, \tag{2.2}
\end{equation*}
$$

is a pseudo MV-algebra. In addition, a pseudo effect algebra $E$ has $\mathrm{RDP}_{2}$ iff $E$ is a lattice and $E$ satisfies $\mathrm{RDP}_{1}$, see [DvVe2, Thm 8.8].

Finally, we note that an ideal of a pseudo effect algebra $E$ is a non-empty subset $I$ such that (i) if $x, y \in I$ and $x+y$ is defined in $E$, then $x+y \in I$, and (ii) $x \leq y \in I$ implies $x \in I$. An ideal $I$ is normal if $a+I:=\{a+i: i \in I$ if $a+i$ exists in $E\}=I+a:=\{j+a: j \in I\}$ for any $a \in E$. A maximal ideal is defined in a standard way. If $M$ is a pseudo MV-algebra, then the ideal $I$ of $M$ is also an ideal when $M$ is understood as a pseudo effect algebra; this follows from the fact $x \oplus y=\left(x \wedge y^{-}\right)+y$.

We note that a mapping from a pseudo effect algebra $E$ into a pseudo effect algebra $F$ is a homomorphism if (i) $a+b \in E$ implies $h(a)+h(b) \in F$ and $h(a+b)=$ $h(a)+h(b)$, and (ii) $h(1)=1$. A bijective mapping $h: E \rightarrow F$ is an isomorphism if both $h$ and $h^{-1}$ are homomorphisms of pseudo effect algebras.

## 3. $(H, u)$-Perfect Pseudo MV-algebras

Generalizing ideas from DiLe1, DDT, Dvu3, Dvu4, we introduce the basic notions of our paper.

If $(H, u)$ is a unital $\ell$-group, we set $[0, u]_{H}:=\{h \in H: 0 \leq h \leq u\}$.
Definition 3.1. Let $(H, u)$ be a linearly ordered group and let $u$ belong to the commutative center $C(H)$ of $H$. We say that a pseudo MV-algebra $M$ is $(H, u)$ perfect, if there is a system $\left(M_{t}: t \in[0, u]_{H}\right)$ of nonempty subsets of $M$ such that it is an $(H, u)$-decomposition of $M$, i.e. $M_{s} \cap M_{t}=\emptyset$ for $s<t, s, t \in[0, u]_{H}$ and $\bigcup_{t \in[0, u]_{H}} M_{t}=M$, and
(a) $M_{s} \leqslant M_{t}$ for all $s<t, s, t \in[0, u]_{H}$;
(b) $M_{t}^{-}=M_{u-t}=M_{t}^{\sim}$ for each $t \in[0, u]_{H}$;
(c) if $x \in M_{v}$ and $y \in M_{t}$, then $x \oplus y \in M_{v \oplus t}$, where $v \oplus t=\min \{v+t, u\}$.

We note that (a) can be written equivalently in a stronger way: if $s<t$ and $a \in M_{s}$ and $b \in M_{t}$, then $a<b$. Indeed, by (b) we have $a \leq b$. If $a=b$, then $a \in M_{s} \cap M_{t}=\emptyset$, which is absurd. Hence, $a<b$.

In addition, (i) if $(H, u)=(\mathbb{Z}, 1)$ and $M$ is an MV-algebra, we are speaking on a perfect MV-algebra, DiLe1, (ii) if $(\mathbb{H}, u)=\left(\frac{1}{n} \mathbb{Z}, 1\right)$, a $\left(\frac{1}{n} \mathbb{Z}, 1\right)$-perfect pseudo MValgebra is said to be $n$-perfect, see [Dvu3], (iii) if $\mathbb{H}$ is a subgroup of the group of real numbers $\mathbb{R}$, such that $1 \in \mathbb{H}$, ( $\mathbb{H}, 1)$-perfect pseudo MV-algebras are in Dvu4] called $\mathbb{H}$-perfect pseudo MV-algebras.

For example, let

$$
\begin{equation*}
M=\Gamma(H \overrightarrow{\times} G,(u, 0)) \tag{3.1}
\end{equation*}
$$

where $u \in C(H)$. We set $M_{0}=\left\{(0, g): g \in G^{+}\right\}, M_{u}:=\left\{(u,-g): g \in G^{+}\right\}$and for $t \in[0, u]_{H} \backslash\{0, u\}$, we define $M_{t}:=\{(t, g): g \in G\}$. Then $\left(M_{t}: t \in[0, u]_{H}\right)$ is an ( $H, u$ )-decomposition of $M$ and $M$ is an $(H, u)$-perfect pseudo MV-algebra.

As a matter of interest, if $O$ is the zero group, then $\Gamma(O \overrightarrow{\times} G,(0,0))$ is a oneelement pseudo MV-algebra. The pseudo MV-algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} O,(1,0))$ is a twoelement Boolean algebra. In general, $\Gamma(H \overrightarrow{\times} O,(u, 0)) \cong \Gamma(H, u)$ and it is semisimple (that is, its radical is a singleton) iff $H$ is Archimedean. If $G \neq O \neq H$, $\Gamma(H \overrightarrow{\times} G,(u, 0))$ is not semisimple.

Theorem 3.2. Let $M=\left(M_{t}: t \in[0, u]_{H}\right)$ be an $(H, u)$-perfect pseudo $M V$-algebra.
(i) Let $a \in M_{v}, b \in M_{t}$. If $v+t<u$, then $a+b$ is defined in $M$ and $a+b \in M_{v+t}$; if $a+b$ is defined in $M$, then $v+t \leq u$. If $a+b$ is defined in $M$ and $v+t=u$, then $a+b \in M_{u}$.
(ii) $M_{v}+M_{t}$ is defined in $M$ and $M_{v}+M_{t}=M_{v+t}$ whenever $v+t<u$.
(iii) If $a \in M_{v}$ and $b \in M_{t}$, and $v+t>u$, then $a+b$ is not defined in $M$.
(iv) If $a \in M_{v}$ and $b \in M_{t}$, then $a \vee b \in M_{v \vee t}$ and $a \wedge b \in M_{v \wedge t}$.
(v) $M$ admits a state $s$ such that $M_{0} \subseteq \operatorname{Ker}(s)$.
(vi) $M_{0}$ is a normal ideal of $M$ such that $M_{0}+M_{0}=M_{0}$ and $M_{0} \subseteq \operatorname{Infinit}(M)$.
(vii) The quotient pseudo $M V$-algebra $M / M_{0} \cong \Gamma(H, u)$.
(viii) Let $M=\left(M_{t}^{\prime}: t \in[0, u]_{H}\right)$ be another $(H, u)$-decomposition of $M$ satisfying (a)-(c) of Definition 3.1, then $M_{t}=M_{t}^{\prime}$ for each $t \in[0, u]_{H}$.
(ix) $M_{0}$ is a prime ideal of $M$.

Proof. (i) Assume $a \in M_{v}$ and $b \in M_{t}$. If $v+t<u$, then $b^{-} \in M_{u-t}$, so that $a \leq b^{-}$, and $a+b$ is defined in $M$. Conversely, let $a+b$ be defined, then $a \leq b^{-} \in M_{u-t}$. If $v+t>u$, then $v>u-t$ and $a>b^{-} \geq a$ which is absurd, and this gives $v+t \leq u$. Now let $v+t=u$ and $a+b$ be defined in $M$. By (c) of Definition 3.1, we have $a+b \in M_{u}$.
(ii) By (i), we have $M_{v}+M_{t} \subseteq M_{v+t}$. Suppose $z \in M_{v+t}$. Then, for any $x \in M_{v}$, we have $x \leq z$. Hence, $y=z-x$ is defined in $M$ and $y \in M_{w}$ for some $w \in[0, u]_{H}$. Since $z=y+x \in M_{v+t} \cap M_{v+w}$, we conclude $t=w$ and $M_{v+t} \subseteq M_{v}+M_{t}$.
(iii) It follows from (i).
(iv) Inasmuch as $x \wedge y=\left(x \oplus y^{\sim}\right)-y^{\sim}$, we have by (c) of Definition 3.1, $\left(x \oplus y^{\sim}\right)-y^{\sim} \in M_{s}$, where $s=((v+u-t) \wedge u)-(u-t)=v \wedge t$. Using a de Morgan law and property (d), we have $x \vee y \in M_{v \vee t}$.
(v) Let $s_{0}$ be a unique state on $\Gamma(H, u)$ which exists in view of [Dvu1, Thm 5.5]. Define a mapping $s: M \rightarrow[0,1]$ by $s(x)=s_{0}(t)$ if $x \in M_{t}$. It is clear that $s$ is a well-defined mapping. Take $a, b \in M$ such that $a+b$ is defined in $M$. Then there are unique indices $v$ and $t$ such that $a \in M_{v}$ and $b \in M_{t}$. By (i), $v+t \leq u$ and $a+b \in M_{v+t}$. Therefore, $s(a+b)=s_{0}(v+t)=s_{0}(v)+s_{0}(t)=s(a)+s(b)$. It is evident that $s(1)=1$ and $M_{0} \subseteq \operatorname{Ker}(s)$.
(vi) It is clear that $M_{0}$ is an ideal of $M$. To prove the normality of $M_{0}$, take $x \in M_{v}$ and $y \in M_{0}$. Then $x \oplus y=((x \oplus y)-x)+x \in M_{v}$ which implies by (i)-(ii) $(x \oplus y)-x \in M_{0}$ and $x \oplus M_{0} \subseteq M_{0} \oplus x$. In the same way we proceed for the second implication.

Since $M_{0}+M_{0}=M_{0}$, by (ii) we have $M_{0} \subseteq \operatorname{Infinit}(M)$.
(vii) Since by (vi) $M_{0}$ is a normal ideal, $M / M_{0}$ is a pseudo MV-algebra, too. Using (iv), it is easy to verify that $x \sim_{M_{0}} y$ iff there is an $h \in[0, u]_{H}$ such that $x, y \in M_{h}$. We define a mapping $\phi: M / M_{0} \rightarrow \Gamma(H, u)$ by $\phi(x)=h$ iff $x \in M_{h}$ for some $h \in[0, u]_{H}$. The mapping $\phi$ is an isomorphism from $M / M_{0}$ onto $\Gamma(H, u)$.
(viii) Let $M=\left(M_{t}^{\prime}: t \in[0, u]_{H}\right)$ be another $(H, u)$-decomposition of $M$. We assert $M_{0}=M_{0}^{\prime}$. If not, there are $x \in M_{0} \backslash M_{0}^{\prime}$ and $y \in M_{0}^{\prime} \backslash M_{0}$. By (a), we have $x<y$ as well as $y<x$ which is absurd. Hence, $M_{0}=M_{0}^{\prime}$. By (vi), $M_{0}$ is normal and by (vii), $M_{0} \cong \Gamma(H, u) \cong M / M_{0}^{\prime}$. If $x \sim_{M_{0}} y$, then $x, y \in M_{h}$ for some $h \in[0, u]_{H}$, as well as $x \sim_{M_{0}^{\prime}} y$ implies $x, y \in M_{t^{\prime}}$ and $t=t^{\prime}$ which implies $M_{t}=M_{t}^{\prime}$ for any $t \in[0,1]_{H}$.
(ix) By (vii), $M / M_{0} \cong \Gamma(H, u)$, so that $M / M_{0}$ is a linear pseudo MV-algebra. Applying [Dvu1, Thm 6.1], we conclude that the normal ideal $M_{0}$ is prime.

Example 3.3. We define $M V$-algebras: $M_{1}=\Gamma(\mathbb{Z} \overrightarrow{\times}(\mathbb{Z} \vec{X}),(1,(0,0))), M_{2}=$ $\Gamma((\mathbb{Z} \overrightarrow{\times} \mathbb{Z}) \overrightarrow{\times} \mathbb{Z},((1,0), 0))$, and $M=\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z} \overrightarrow{\times} \mathbb{Z},(1,0,0))$ which are mutually isomorphic. The first one is $(\mathbb{Z}, 1)$-perfect, the second one is $(\mathbb{Z} \overrightarrow{\times},(1,0))$-perfect and of course, the linear unital $\ell$-groups $(\mathbb{Z}, 1)$ and $(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(1,0))$ are not isomorphic while the first one is Archimedean and the second one is not Archimedean. Both pseudo $M V$-algebras define the corresponding natural $(\mathbb{Z}, 1)$-decomposition $\left(M_{t}^{1}\right)_{t}$ and $(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(1,0))$-decompositions $\left(M_{q}^{2}\right)_{q}$ of $M_{1}$ and $M_{2}$, respectively. Then $M_{0}^{1}=$ $\{(0,(0, m)): m \geq 0\} \cup\{(0,(n, m)): n>0, m \in \mathbb{Z}\}=\operatorname{Ker}\left(s_{1}\right)=\operatorname{Infinit}\left(M_{1}\right)$, where $s_{1}$ is a unique state on $M_{1}$; it is two-valued. But $M_{0}^{2}=\{((0,0), m): m \geq$ $0\} \subset \operatorname{Ker}\left(s_{2}\right)=\operatorname{Infinit}\left(M_{2}\right)$, where $s_{2}$ is a unique state on $M_{2}$, it vanishes only on Infinit $\left(M_{2}\right)$; it is two-valued.

In what follows, we show that the normal ideal $M_{0}$ of an $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ of an $(H, u)$-perfect pseudo MV-algebra is maximal iff $(H, u)$ is isomorphic with $(\mathbb{H}, 1)$, where $\mathbb{H}$ is a subgroup of the group of reals $\mathbb{R}$ with natural order, and $1 \in \mathbb{H}$.

Theorem 3.4. Let $\left(M_{t}: t \in[0, u]_{H}\right)$ be an $(H, u)$-decomposition of an $(H, u)$ perfect pseudo $M V$-algebra $M$. The following statements are equivalent:
(i) $M_{0}$ is maximal.
(ii) $(H, u)$ is isomorphic to some $(\mathbb{H}, 1)$, where $\mathbb{H}$ is a subgroup of the group $\mathbb{R}$ and $1 \in \mathbb{H}$.
(iii) $M$ possesses a unique state $s$ and $M_{0}=\operatorname{Ker}(s)$.

Proof. (i) $\Rightarrow$ (ii). By Dvu1, Prop 3.4-3.5], $M_{0}$ is maximal iff $M / M_{0}$ is simple, i.e. it does not contain any non-trivial proper ideal. By (vii) of Theorem 3.2, $\left(M / M_{0}, u / M_{0}\right) \cong \Gamma(H, u)$ which means by Theorem 2.1 that $(H, u)$ is a linear, Archimedean and Abelian unital $\ell$-group, and by Hölder's theorem, Bir, Thm XIII.12] or [Fuc, Thm IV.1.1], it is isomorphic to some $(\mathbb{H}, 1)$, where $\mathbb{H}$ is a subgroup of $\mathbb{R}$ and $1 \in \mathbb{H}$.
(ii) $\Rightarrow$ (iii). If $(H, u) \cong(\mathbb{H}, 1)$, where $\mathbb{H}$ is a subgroup of $\mathbb{R}$ and $1 \in \mathbb{H}$, then $M$ is isomorphic to an ( $\mathbb{H}, 1$ )-perfect pseudo MV-algebra. This kind of pseudo MValgebras was studied in Dvu4, and by [Dvu4, Thm 3.2(iv)], $M$ possesses a unique state $s$. This state has the property $s(M)=\mathbb{H}$ and $\operatorname{Ker}(s)=M_{0}$.
(iii) $\Rightarrow$ (i). If $s$ is a unique state of $M$ and $M_{0}=\operatorname{Ker}(s)$, by [Dvu1], $M_{0}$ is a normal and maximal ideal of $M$.

Remark 3.5. We note that in Corollary 7.7 it will be shown that if an $(H, u)$ perfect pseudo MV-algebra $M$ is of a stronger form, namely a strong ( $H, u$ )-perfect pseudo MV-algebra introduced in the next section, then $M$ has a unique state. In general, the uniqueness of a state for any $(H, u)$-perfect pseudo MV-algebra is unknown.

We note that there is uncountably many non-isomorphic unital $\ell$-subgroups $(\mathbb{H}, 1)$ of the unital group $(\mathbb{R}, 1)$. By [Go, Lem 4.21], every $\mathbb{H}$ is either cyclic, i.e. $\mathbb{H}=\frac{1}{n} \mathbb{Z}$ for some $n \geq 1$ or $\mathbb{H}$ is dense in $\mathbb{R}$.

Therefore, if $\mathbb{H}=\mathbb{H}(\alpha)$ is a subgroup of $\mathbb{R}$ generated by $\alpha \in[0,1]$ and 1 , then $\mathbb{H}=\frac{1}{n} \mathbb{Z}$ for some integer $n \geq 1$ if $\alpha$ is a rational number. Otherwise, $\mathbb{H}(\alpha)$ is countable and dense in $\mathbb{R}$, and $M(\alpha):=\Gamma(\mathbb{H}(\alpha), 1)=\{m+n \alpha: m, n \in \mathbb{Z}, 0 \leq$ $m+n \alpha \leq 1\}$, see [CDM, p. 149]. Therefore, we have uncountably many nonisomorphic ( $\mathbb{H}, 1$ )-perfect pseudo MV-algebras.

## 4. Representation of Strong $(H, u)$-perfect Pseudo MV-algebras

In accordance with Dvu4, we introduce the following notions and generalize results from Dvu4 for strong ( $H, u$ )-perfect pseudo MV-algebras. Our aim is to find an algebraic characterization of pseudo MV-algebras that can be represented in the form of the lexicographic product

$$
\Gamma(H \overrightarrow{\times} G,(u, 0))
$$

where $(H, u)$ is a linearly ordered Abelian unital $\ell$-group and $G$ is an $\ell$-group not necessarily Abelian. In [Dvu4, we have studied a particular case when $(H, u)=$ $(\mathbb{H}, 1)$, where $\mathbb{H}$ is a subgroup of reals.

We say that a pseudo MV-algebra $M$ enjoys unique extraction of roots of 1 if $a, b \in M$ and $n a, n b$ exist in $M$, and $n a=1=n b$, then $a=b$. Every linearly ordered pseudo MV-algebra enjoys due to Theorem 2.1] and [Gla, Lem 2.1.4] unique extraction of roots. In addition, every pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G,(u, 0))$, where $(H, u)$ is a linearly ordered $\ell$-group, enjoys unique extraction of roots of 1 for any $n \geq 1$ and for any $\ell$-group $G$. Indeed, let $k(s, g)=(u, 0)=k(t, h)$ for some $s, t \in[0, u]_{H}, g, h \in G, k \geq 1$. Then $k s=u=k t$ which yields $s=t>0$, and $k g=0=k h$ implies $g=0=h$.

Let $n \geq 1$ be an integer. An element $a$ of a pseudo MV-algebra $M$ is said to be cyclic of order $n$ or simply cyclic if $n a$ exists in $M$ and $n a=1$. If $a$ is a cyclic element of order $n$, then $a^{-}=a^{\sim}$, indeed, $a^{-}=(n-1) a=a^{\sim}$. It is clear that 1 is a cyclic element of order 1.

Let $M=\Gamma(G, u)$ for some unital $\ell$-group $(G, u)$. An element $c \in M$ such that (a) $n c=u$ for some integer $n \geq 1$, and (b) $c \in C(G)$, where $C(G)$ is a commutative center of $G$, is said to be a strong cyclic element of order $n$.

We note that if $\mathbb{H}$ is a subgroup of reals and $t=1 / n$, then $c_{\frac{1}{n}}$ is a strong cyclic element of order $n$.

For example, the pseudo MV-algebra $M:=\Gamma(\mathbb{Q} \overrightarrow{\times} G,(1,0))$, where $\mathbb{Q}$ is the group of rational numbers, for every integer $n \geq 1, M$ has a unique cyclic element of order $n$, namely $a_{n}=\left(\frac{1}{n}, 0\right)$. The pseudo MV-algebra $\Gamma\left(\frac{1}{n} \mathbb{Z},(1,0)\right)$ for a prime number $n \geq 1$, has the only cyclic element of order $n$, namely $\left(\frac{1}{n}, 0\right)$. If $M=\Gamma(G, u)$ and $G$ is a representable $\ell$-group, $G$ enjoys unique extraction of roots of 1 , therefore, $M$ has at most one cyclic element of order $n$. In general, a pseudo MV-algebra $M$ can have two different cyclic elements of the same order. But if $M$ has a strong
cyclic element of order $n$, then it has a unique strong cyclic element of order $n$ and a unique cyclic element of order $n$, DvKo, Lem 5.2].

We say that an $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ of $M$ has the cyclic property if there is a system of elements $\left(c_{t} \in M: t \in[0, u]_{H}\right)$ such that (i) $c_{t} \in M_{t}$ for any $t \in[0, u]_{H}$, (ii) if $v+t \leq 1, v, t \in[0, u]_{H}$, then $c_{v}+c_{t}=c_{v+t}$, and (iii) $c_{1}=1$. Properties: (a) $c_{0}=0$; indeed, by (ii) we have $c_{0}+c_{0}=c_{0}$, so that $c_{0}=0$. (b) If $t=1 / n$, then $c_{\frac{1}{n}}$ is a cyclic element of order $n$.

Let $M=\Gamma(G, u)$, where $(G, u)$ is a unital $\ell$-group. An $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ of $M$ has the strong cyclic property if there is a system of elements $\left(c_{t} \in M: t \in[0, u]_{H}\right)$, called an $(H, u)$-strong cyclic family, such that
(i) $c_{t} \in M_{t} \cap C(G)$ for each $t \in[0, u]_{H}$;
(ii) if $v+t \leq 1, v, t \in[0, u]_{H}$, then $c_{v}+c_{t}=c_{v+t}$;
(iii) $c_{1}=1$.

For example, let $M=\Gamma(H \overrightarrow{\times} G,(u, 0))$, where $(H, u)$ is an Abelian linearly ordered unital $\ell$-group and $G$ is an $\ell$-group (not necessarily Abelian), and $M_{t}=$ $\{(t, g):(t, g) \in M\}$ for $t \in[0, u]_{H}$. If we set $c_{t}=(t, 0), t \in[0, u]_{H}$, then the system $\left(c_{t}: t \in[0, u]_{H}\right)$ satisfies (i)-(iii) of the strong cyclic property, and ( $M_{t}: t \in$ $\left.[0, u]_{H}\right)$ is an $(H, u)$-decomposition of $M$ with the strong cyclic property.

Finally, we say that a pseudo MV-algebra $M$ is strong $(H, u)$-perfect if there is an $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ of $M$ with the strong cyclic property.

A prototypical example of a strong $(H, u)$-perfect pseudo MV-algebra is the following.

Proposition 4.1. Let $G$ be an $\ell$-group and $(H, u)$ an Abelian unital $\ell$-group. Then the pseudo MV-algebra

$$
\begin{equation*}
\mathcal{M}_{H, u}(G):=\Gamma(H \overrightarrow{\times} G,(u, 0)) \tag{4.1}
\end{equation*}
$$

is a strong $(H, u)$-perfect pseudo MV-algebra with a strong cyclic family $((h, 0)$ : $\left.h \in[0, u]_{H}\right)$.

Now we present a representation theorem for strong $(H, u)$-perfect pseudo MValgebras by (4.1). The following theorem uses the basic ideas of the particular situation $(H, u)=(\mathbb{H}, 1)$ which was proved in [Dvu4, Thm 4.3].

Theorem 4.2. Let $M$ be a strong ( $H, u$ )-perfect pseudo $M V$-algebra, where $(H, u)$ is an Abelian unital linearly ordered $\ell$-group. Then there is a unique (up to isomorphism) $\ell$-group $G$ such that $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$.
Proof. Since $M$ is a pseudo MV-algebra, due to Dvu2, Thm 3.9], there is a unique unital (up to isomorphism of unital $\ell$-groups) $\ell$-group $(K, v)$ such that $M \cong \Gamma(K, v)$. Without loss of generality we can assume that $M=\Gamma(K, v)$. Assume $\left(M_{t}: t \in\right.$ $\left.[0, u]_{H}\right)$ is an $(H, u)$-decomposition of $M$ with the strong cyclic property and with an $(H, u)$-strong cyclic family $\left(c_{t} \in M: t \in[0, u]_{H}\right)$.

By (vi) of Theorem 3.2, $M_{0}$ is an associative cancellative semigroup satisfying conditions of Birkhoff's Theorem [Bir, Thm XIV.2.1], [Fuc, Thm II.4], which guarantees that $M_{0}$ is a positive cone of a unique (up to isomorphism) directed po-group $G$. Since $M_{0}$ is a lattice, we have that $G$ is an $\ell$-group.

Take the $(H, u)$-strong perfect pseudo MV-algebra $\mathcal{M}_{H, u}(G)$ defined by (4.1), and define a mapping $\phi: M \rightarrow \mathcal{M}_{H, u}(G)$ by

$$
\begin{equation*}
\phi(x):=\left(t, x-c_{t}\right) \tag{4.2}
\end{equation*}
$$

whenever $x \in M_{t}$ for some $t \in[0, u]_{H}$, where $x-c_{t}$ denotes the difference taken in the group $K$.

Claim 1: $\phi$ is a well-defined mapping.
Indeed, $M_{0}$ is in fact the positive cone of an $\ell$-group $G$ which is a subgroup of $K$. Let $x \in M_{t}$. For the element $x-c_{t} \in K$, we define $\left(x-c_{t}\right)^{+}:=\left(x-c_{t}\right) \vee 0=$ $\left(x \vee c_{t}\right)-c_{t} \in M_{0}$ (when we use (iii) of Theorem 3.2) and similarly $\left(x-c_{t}\right)^{-}:=$ $-\left(\left(x-c_{t}\right) \wedge 0\right)=c_{t}-\left(x \wedge c_{t}\right) \in M_{0}$. This implies that $x-c_{t}=\left(x-c_{t}\right)^{+}-\left(x-c_{t}\right)^{-} \in G$.

Claim 2: The mapping $\phi$ is an injective and surjective homomorphism of pseudo effect algebras.

We have $\phi(0)=(0,0)$ and $\phi(1)=(1,0)$. Let $x \in M_{t}$. Then $x^{-} \in M_{1-t}$, and $\phi\left(x^{-}\right)=\left(1-t, x-c_{1-t}\right)=(1,0)-\left(t, x-c_{t}\right)=\phi(x)^{-}$. In an analogous way, $\phi\left(x^{\sim}\right)=\phi(x)^{\sim}$.

Now let $x, y \in M$ and let $x+y$ be defined in $M$. Then $x \in M_{t_{1}}$ and $y \in M_{t_{2}}$. Since $x \leq y^{-}$, we have $t_{1} \leq 1-t_{2}$ so that $\phi(x) \leq \phi\left(y^{-}\right)=\phi(y)^{-}$which means $\phi(x)+\phi(y)$ is defined in $\mathcal{M}_{H, u}(G)$. Then $\phi(x+y)=\left(t_{1}+t_{2}, x+y-c_{t_{1}+t_{2}}\right)=$ $\left(t_{1}+t_{2}, x+y-\left(c_{t_{1}}+c_{t_{2}}\right)\right)=\left(t_{1}, x-c_{t_{1}}\right)+\left(t_{2}, y-c_{t_{2}}\right)=\phi(x)+\phi(y)$.

Assume $\phi(x) \leq \phi(y)$ for some $x \in M_{t}$ and $y \in M_{v}$. Then $\left(t, x-c_{t}\right) \leq\left(v, y-c_{v}\right)$. If $t=v$, then $x-c_{t} \leq y-c_{t}$ so that $x \leq y$. If $i<j$, then $x \in M_{t}$ and $y \in M_{v}$ so that $x<y$. Therefore, $\phi$ is injective.

To prove that $\phi$ is surjective, assume two cases: (i) Take $g \in G^{+}=M_{0}$. Then $\phi(g)=(0, g)$. In addition $g^{-} \in M_{1}$ so that $\phi\left(g^{-}\right)=\phi(g)^{-}=(0, g)^{-}=(1,0)-$ $(0, g)=(1,-g)$. (ii) Let $g \in G$ and $t$ with $0<t<1$ be given. Then $g=g_{1}-g_{2}$, where $g_{1}, g_{2} \in G^{+}=M_{0}$. Since $c_{t} \in M_{t}, g_{1}+c_{t}$ exists in $M$ and it belongs to $M_{t}$, and $g_{2} \leq g_{1}+c_{t}$ which yields $\left(g_{1}+c_{t}\right)-g_{2}=\left(g_{1}+c_{t}\right) \backslash g_{2} \in M_{t}$. Hence, $g+c_{t}=\left(g_{1}+c_{t}\right) \backslash g_{2} \in M_{t}$ which entails $\phi\left(g+c_{t}\right)=(t, g)$.

Claim 3: If $x \leq y$, then $\phi(y \backslash x)=\phi(y) \backslash \phi(x)$ and $\phi(x / y)=\phi(x) / \phi(y)$.
It follows from the fact that $\phi$ is a homomorphism of pseudo effect algebras.
Claim 4: $\phi(x \wedge y)=\phi(x) \wedge \phi(y)$ and $\phi(x \vee y)=\phi(x) \vee \phi(y)$.
We have, $\phi(x), \phi(y) \geq \phi(x \wedge y)$. If $\phi(x), \phi(y) \geq \phi(w)$ for some $w \in M$, we have $x, y \geq w$ and $x \wedge y \geq w$. In the same way we deal with $\vee$.

Claim 5: $\phi$ is a homomorphism of pseudo MV-algebras.
It is necessary to show that $\phi(x \oplus y)=\phi(x) \oplus \phi(y)$. This follows straightforward from the previous claims and equality (2.2).

Consequently, $M$ is isomorphic to $\mathcal{M}_{H, u}(G)$ as pseudo MV-algebras.
If $M \cong \Gamma\left(H \overrightarrow{\times} G^{\prime},(u, 0)\right)$ for some $G^{\prime}$, then $(H \overrightarrow{\times} G,(u, 0))$ and $\left(H \overrightarrow{\times} G^{\prime},(u, 0)\right)$ are isomorphic unital $\ell$-groups in view of the categorical equivalence, see Dvu2, Thm 6.4] or Theorem [2.1) let $\psi: \Gamma(H \overrightarrow{\times} G,(u, 0)) \rightarrow \Gamma\left(H \overrightarrow{\times} G^{\prime},(u, 0)\right)$ be an isomorphism of the lexicographic products. Hence, by Theorem 3.2 (viii), we see that $\psi\left(\left\{(0, g): g \in G^{+}\right\}\right)=\left\{\left(0, g^{\prime}\right): g^{\prime} \in G^{\prime+}\right\}$ which proves that $G$ and $G^{\prime}$ are isomorphic $\ell$-groups.

We say that a pseudo MV-algebra is lexicographic if there are an Abelian linearly ordered unital $\ell$-group $(H, u)$ and an $\ell$-group $G$ (not necessarily Abelian) such that $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$. In other words, by Theorem4.2 $M$ is lexicographic iff $M$ is
strong $(H, u)$-perfect for some Abelian linear unital $\ell$-group $(H, u)$. We note that in DFL, a lexicographic MV-algebra denotes an MV-algebra having a lexicographic ideal which will be defined below in Section 7. But by Theorem[7.5, we will conclude that both notions are equivalent for symmetric pseudo MV-algebras from $\mathcal{M}$.

It is worthy to note that according to Example 3.3, the pseudo MV-algebra $M$ has two isomorphic lexicographic representations $\Gamma(\mathbb{Z} \overrightarrow{\times}(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}),(1,(0,0)))$ and $\Gamma((\mathbb{Z} \overrightarrow{\times} \mathbb{Z}) \overrightarrow{\times} \mathbb{Z},((1,0), 0))$, but $\left(H_{1}, u_{1}\right):=(\mathbb{Z}, 1)$ and $\left(H_{2}, u_{2}\right):=(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(1,0))$ are not isomorphic, as well as $G_{1}:=\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ and $G_{2}:=\mathbb{Z}$ are not isomorphic $\ell$-groups.

## 5. Local Pseudo MV-algebras with Retractive Radical

In [DiLe2, Cor 2.4], the authors characterized MV-algebras that can be expressed in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$, where $\mathbb{H}$ is a subgroup of $\mathbb{R}$ and $G$ is an Abelian $\ell$ group. In what follows, we extend this characterization for local symmetric pseudo MV-algebras. This result gives another characterization of strong ( $\mathbb{H}, 1$ )-perfect pseudo MV-algebras via lexicographic product.

We denote by $\mathcal{M}$ the set of pseudo MV-algebras $M$ such that either every maximal ideal of $M$ is normal or $M$ is trivial. By [DDT, (6.1)], $\mathcal{M}$ is a variety.

Let $M$ be a symmetric pseudo MV-algebra. For any $x \in M$, we define the order, in symbols $\operatorname{ord}(x)$, as the least integer $n$ such that $n . x=1$ if such $n$ exists, otherwise, $\operatorname{ord}(x)=\infty$. It is clear that the set of all elements with infinite order is an ideal. An element $x$ is finite if $\operatorname{ord}(x)<\infty$ and $\operatorname{ord}\left(x^{-}\right)<\infty$.

Lemma 5.1. Let $M$ be a pseudo $M V$-algebra from $\mathcal{M}$ and $x \in M$. There exists $a$ proper normal ideal of $M$ containing $x$ if and only if $\operatorname{ord}(x)=\infty$.

Proof. Let $x$ be any element of $M$ and let $I(x)$ be the normal ideal of $M$ generated by $x$. Then

$$
\begin{equation*}
I(x)=\{y \in M: y \leq m \cdot x \text { for some } m \in \mathbb{N}\} \tag{5.1}
\end{equation*}
$$

Lemma 5.2. Let $M$ be a symmetric pseudo MV-algebra. If $\operatorname{ord}(x \odot y)<\infty$, then $x \leq y^{-}$.
Proof. By the hypothesis, ord $(x \odot y)=n$ for some integer $n \geq 1$. Hence $\left(y^{-} \oplus x^{-}\right)^{n}=$ 0 . By GeIo, Prop 1.24(ii)], $\left(y^{-} \oplus x^{-}\right) \vee(x \oplus y)=1$ which by GeIo, Lem 1.32] yields $\left(y^{-} \oplus x^{-}\right)^{n} \vee(x \oplus y)^{n}=1$, so that $(x \oplus y)^{n}=1$ and $x \oplus y=1$, consequently, $x \leq y^{-}$.

Lemma 5.3. Let $M \in \mathcal{M}$ be a symmetric pseudo $M V$-algebra. The following statements are equivalent:
(i) $M$ is local.
(ii) For every $x \in M$, ord $(x)<\infty$ or $\operatorname{ord}\left(x^{-}\right)<\infty$.

Proof. Let $M$ be local. There exists a unique maximal ideal $I$ that is normal. Assume that for some $x \in M$, we have $\operatorname{ord}(x)=\infty=\operatorname{ord}\left(x^{-}\right)$. By Lemma 5.1 $x, x^{-} \in I$ which is absurd.

Conversely, let for every $x \in M, \operatorname{ord}(x)<\infty$ or $\operatorname{ord}\left(x^{-}\right)<\infty$. Let $I$ be a maximal ideal of $M$ and assume that $x \notin M$ for some $x \in M$ with $\operatorname{ord}(x)=\infty$. Since $I$ is by the hypothesis normal, by a characterization of normal and maximal ideals, GeIo, Prop 3.5], there is an integer $n \geq 1$ such that $\left(x^{-}\right)^{n} \in I$. By Lemma 5.1, $\operatorname{ord}\left(\left(x^{-}\right)^{n}\right)=\infty$ and $\operatorname{ord}\left(\left(\left(x^{-}\right)^{n}\right)^{-}\right)<\infty$, i.e. $\operatorname{ord}(n . x)<\infty$, which implies
$\operatorname{ord}(x)<\infty$ that is impossible. Hence, every element $x$ with infinite order belongs to $I$, and so $I$ is a unique maximal ideal of $M$, in addition $I$ is normal.
Lemma 5.4. Let $M \in \mathcal{M}$ be a local symmetric pseudo $M V$-algebra and let $I$ be $a$ unique maximal ideal of $M$. For all $x, y \in M$ such that $x / I \neq y / I$, we have $x<y$ or $y<x$.
Proof. By hypothesis, we have that $x \odot y^{-} \notin I$ or $y \odot x^{-} \notin I$. By Lemma 5.3, in the first case we have $\operatorname{ord}\left(x \odot y^{-}\right)<\infty$ which by Lemma 5.2 implies $x \leq y$ and consequently $x<y$. In the second case, we similarly conclude $y<x$.

We introduce the following notion. A normal ideal $I$ of a pseudo MV-algebra $M$ is said to be retractive if the canonical projection $\pi_{I}: M \rightarrow M / I$ is retractive, i.e. there is a homomorphism $\delta_{I}: M / I \rightarrow M$ such that $\pi_{I} \circ \delta_{I}=i d_{M / I}$. If a normal ideal $I$ is retractive, then $\delta_{I}$ is injective and $M / I$ is isomorphic to a subalgebra of M.

For example, if $M=\Gamma(H \overrightarrow{\times} G,(u, 0))$ and $I=\left\{(0, g): g \in G^{+}\right\}$, then $I$ is a normal ideal, see Theorem $3.2($ vi) , and due to $M / I \cong \Gamma(H, u) \cong \Gamma(H \overrightarrow{\times}\{0\},(u, 0)) \subseteq$ $\Gamma(H \overrightarrow{\times} G,(u, 0)), I$ is retractive.

Lemma 5.5. Let I be a normal ideal of a symmetric pseudo MV-algebra. Then the following are equivalent:
(i) $x / I=y / I$.
(ii) $x=(h \oplus y) \odot k^{-}$, where $h, k \in I$.

Proof. (i) $\Rightarrow$ (ii) Assume $x / I=y / I$. Then the elements $k=x^{-} \odot y$ and $h=x \odot y^{-}$ belong to $I$. It is easy to see that $x \oplus k=x \vee y=h \oplus y$. Since $k^{-}=y^{-} \oplus x \geq x$, we have $x=x \wedge k^{-}=(x \oplus k) \odot k^{-}=(h \oplus y) \odot k^{-}$.
(ii) $\Rightarrow$ (i) Then we have $x / I=y / I$.

Let $M$ be a pseudo MV-algebra, and let $\operatorname{Sub}(M)$ be the set of all subalgebras of $M$. Then $\operatorname{Sub}(M)$ is a lattice with respect to set theoretical inclusion with the smallest element $\{0,1\}$ and greatest one $M$. It is easy to see that if $M$ is symmetric and $I$ is an ideal of $M$, then the subalgebra $\langle I\rangle$ of $M$ generated by $I$ is the set $\langle I\rangle=I \cup I^{-}$. We recall a subalgebra $S$ of $M$ is said to be a complement of a subalgebra $A$ of $M$ if $S \cap A=\{0,1\}$ and $S \vee A=M$.

In the following, we characterize retractive ideals of pseudo MV-algebras in an analogous way as it was done for MV-algebras in [CiTo, Thm 1.2].
Theorem 5.6. Let $M$ be a symmetric pseudo $M V$-algebra and I a normal ideal of M. The following statements are equivalent:
(i) $I$ is a retractive ideal.
(ii) $\langle I\rangle$ has a complement.

Proof. Let $I$ be a retractive ideal of $M$ and let $\delta_{I}: M / I \rightarrow M$ be an injective homomorphism such that $\pi_{I} \circ \delta_{I}=i d_{M / I}$. We claim that $\delta_{I}(M / I)$ is a complement of $\langle I\rangle$. Clearly $\delta_{I}(M / I) \cap\langle I\rangle=\{0,1\}$. Let $x \in M$, then $x / I=\delta_{I}(M / I)(x / I) / I$ so that by Lemma 5.5, we have $x=\left(h \oplus \delta_{I}(x / I)\right) \odot k^{-}$for some $h, k \in I$ that implies $x \in \delta_{I}(M / I) \vee\langle I\rangle$.

Conversely, assume that $\langle I\rangle$ has a complement $S \in \operatorname{Sub}(M)$. From $S \cap\langle I\rangle=$ $\{0,1\}$ we conclude that the canonical projection $\pi_{I}$ is injective on $S$. Indeed, if for $x, y \in S$, we have $x / I=y / I$, then also $x / I=(x \vee y) / I=y / I$ which yields
$(x \vee y) \odot x^{-} \in S \cap I=\{0\},(x \vee y) \odot y^{-} \in S \cap I=\{0\}$. Therefore, $x=x \vee y=y$ and this implies that the restriction $\pi_{I 1 S}$ is injective.

From $S \vee\langle I\rangle=M$, we have that for each $x \in M$, there is a term in the language of pseudo MV-algebras, says $p\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)$, such that

$$
x=p^{M}\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right)
$$

for some $x_{1}, \ldots, x_{m} \in S$ and $y_{1}, \ldots, y_{n} \in\langle I\rangle$. Then

$$
x / I=p^{M / I}\left(x_{1} / I, \ldots, x_{m} / I, y_{1} / I, \ldots, y_{n} / I\right)
$$

Since $y_{i} / I \in\{0,1\}$ for each $i=1, \ldots, n$, there is an $n$-tuple $\left(t_{1}, \ldots, t_{n}\right)$ of elements from $\{0,1\}$ such that $x / I=p^{M}\left(x_{1}, \ldots, x_{m}, t_{1}, \ldots, t_{n}\right) / I$. Since

$$
\left(x_{1}, \ldots, x_{m}, t_{1}, \ldots, t_{n}\right) \in S^{m+n}
$$

we have that $p^{M}\left(x_{1}, \ldots, x_{m}, t_{1}, \ldots, t_{n}\right) \in S$. Therefore, the restriction $\pi_{I 1 S}$ is an isomorphism from $S$ onto $M / I$, and setting $\delta_{I}=\left(\pi_{I \uparrow S}\right)^{-1}$, we see that $I$ is retractive.

Theorem 5.7. Let $M$ be a symmetric pseudo $M V$-algebra from $\mathcal{M}$. The following statements are equivalent:
(i) $M$ is local and $\operatorname{Rad}_{n}(M)$ is retractive.
(ii) $M$ is strong $(\mathbb{H}, 1)$-perfect for some subgroup $\mathbb{H}$ of $\mathbb{R}$ with $1 \in \mathbb{H}$.
(iii) There exists a subgroup $\mathbb{H}$ of $\mathbb{R}$ with $1 \in \mathbb{H}$ and an $\ell$-group $G$ such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G,(u, 0))$.

Proof. (i) $\Rightarrow$ (ii) Let $I$ be a unique maximal and normal ideal of $M$ and let $(K, v)$ be a (unique up to isomorphism) unital $\ell$-group given by Theorem 2.1, such that $M \cong \Gamma(K, v)$; without loss of generality we can assume that $M=\Gamma(K, v)$. By [Dvu1, there is an extremal state ( $=$ state morphism) $s_{0}: M \rightarrow[0,1]$ such that $I=\operatorname{Ker}\left(s_{0}\right)$. The range of $s_{0}, s_{0}(M)$, is an MV-algebra which corresponds to a unique subgroup $\mathbb{H}$ of $\mathbb{R}$ such that $s_{0}(M)=\Gamma(\mathbb{H}, 1)$ is a subalgebra of $\Gamma(\mathbb{R}, 1)$.

Since $I=\operatorname{Rad}_{n}(M), I$ is a retractive ideal, and $M / I$ is isomorphic to $\Gamma(\mathbb{H}, 1)$, we have $\Gamma(\mathbb{H}, 1)$ can be injectively embedded into $K$ and $\mathbb{H}$ is isomorphic to a subgroup of $K$.

In addition, let $\langle I\rangle$ be a subalgebra of $M$ generated by $I$. Then $\langle I\rangle=I \cup I^{-}=$ $I \cup I^{-}, I^{-}=I^{\sim}$, and $\langle I\rangle$ is a perfect pseudo MV-algebra. By DDT, Prop 5.2], there is a unique (up to isomorphism) $\ell$-group $G$ such that $\langle I\rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$.

In what follows, we prove that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$.
Define $M_{t}=s^{-1}(\{t\}), t \in[0,1]_{\mathbb{H}}$. We assert that $\left(M_{t}: t \in[0,1]_{\mathbb{H}}\right)$ is an $(\mathbb{H}, 1)$ decomposition of $M$. It is clear that it is a decomposition: Every $M_{t}$ is non-empty, and $M_{t}^{-}=M_{1-t}=M_{t}^{\sim}$ for each $t \in[0,1]_{\mathbb{H}}$. In addition, if $x \in M_{v}$ and $y \in M_{t}$, then $x \oplus y \in M_{v \oplus t}, x \wedge y \in M_{v \wedge t}$ and $x \vee y \in M_{v \vee t}$. By Lemma 5.4, we have $M_{s} \leqslant M_{t}$ for all $s<t, s, t \in[0,1]_{\mathbb{H}}$.

Since $I=\operatorname{Rad}_{n}(M)$ is retractive, there is a unique subalgebra $M^{\prime}$ of $M$ such that $s_{0}\left(M^{\prime}\right)=s_{0}(M)$. For any $t \in[0,1]_{\mathbb{H}}$, there is a unique element $x_{t} \in M^{\prime}$ such that $s_{0}\left(x_{t}\right)=t$. We assert that the system $\left(x_{t}: t \in[0,1]_{\mathbb{H}}\right)$ satisfies the following properties (i) $c_{t} \in M_{t}$ for each $t \in[0,1]_{\mathbb{H}}$, (ii) $c_{v+t}=c_{v}+c_{t}$ whenever $v+t \leq 1$, (iii) $c_{1}=1$. (iv) $c_{t} \in C(K)$. Indeed, since $s_{0}$ is a homomorphism of pseudo MV-algebras, by the categorical equivalence Theorem [2.1, $s_{0}$ can be uniquely extended to a unital $\ell$-group homomorphism $\hat{s}_{0}:(K, v) \rightarrow(\mathbb{H}, 1)$. Now if $x$ is any element of $K$, then $x+c_{t}-x \in M$ because $M$ is symmetric, and hence
$\hat{s}_{0}\left(x+c_{t}-x\right)=\hat{s}_{0}(x)+\hat{s}_{0}\left(c_{t}\right)-\hat{s}_{0}(x)=s_{0}\left(c_{t}\right)=t$ which implies $x+c_{t}-x=c_{t}$ so that $x+c_{t}=c_{t}+x$.

In other words, we have proved that $\left(M_{t}: t \in[0,1]_{\mathbb{H}}\right)$ has the strong cyclic property, and consequently, $M$ is strong ( $\mathbb{H}, 1$ )-perfect. By Theorem 4.2, $M \cong$ $\Gamma\left(\mathbb{H} \overrightarrow{\times} G^{\prime},(1,0)\right)$ for some unique (up to isomorphism) $\ell$-group $G^{\prime}$. Hence $G^{\prime} \cong G$, where $G$ was defined above, which proves (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (i) is evident by the note that is just before Theorem 5.7 .

We note that if $M$ is a local symmetric pseudo MV-algebra with a retractive ideal $\operatorname{Rad}_{n}(M)$, then $M$ is a lexicographic extension of $\operatorname{Ker}_{n}(M)$ in the sense described in HoRa.

Proposition 5.8. Let $\left(M_{\alpha}: \alpha \in A\right)$ be a system of pseudo $M V$-algebras and let $I_{\alpha}$ be a non-trivial normal ideal of $M_{\alpha}, \alpha \in A$. Set $M=\prod_{\alpha} M_{\alpha}$ and $I=\prod_{\alpha} I_{\alpha}$. Then $I$ is a retractive ideal of $M$ if and only if every $I_{\alpha}$ is a retractive ideal of $M_{\alpha}$.
Proof. The set $I=\prod_{\alpha} I_{\alpha}$ is a non-trivial normal ideal of $M$. Then $M / I \cong$ $\prod_{\alpha} M_{\alpha} / I_{\alpha}$ and without loss of generality, we can assume that $M / I=\prod_{\alpha} M_{\alpha} / I_{\alpha}$.

Assume that every $I_{\alpha}$ is retractive. We denote by $\pi_{\alpha}$ the canonical projection of $M_{\alpha}$ onto $M_{\alpha} / I_{\alpha}$ and by $\delta_{\alpha}: M_{\alpha} / I_{\alpha} \rightarrow M_{\alpha}$ its right inversion i.e. $\pi_{\alpha} \circ \delta_{\alpha}=i d_{M_{\alpha} / I_{\alpha}}$. Let $\pi: M \rightarrow M / I$ be the canonical projection. If we set $\delta: M / I \rightarrow M$ by $\delta\left(\left(x_{\alpha} / I_{\alpha}\right)_{\alpha}\right):=\left(\left(\delta_{\alpha}\left(x_{\alpha} / I_{\alpha}\right)\right)_{\alpha}\right)$, then we have $\pi \circ \delta=i d_{M / I}$, so that $I$ is retractive.

Conversely, let $I$ be a retractive ideal of $M$. Let $\pi^{\alpha}: \prod_{\alpha} M_{\alpha}$ be the $\alpha$-th projection of $M$ onto $M_{\alpha}$. We define a mapping $\delta_{\alpha}: M_{\alpha} / I_{\alpha} \rightarrow M_{\alpha}$ by $\delta_{\alpha}=\pi^{\alpha} \circ \delta$ $(\alpha \in A)$. Then $\prod_{\alpha} \pi_{\alpha} \circ \delta_{\alpha}\left(x_{\alpha} / I_{\alpha}\right)=\prod \pi_{\alpha} \circ \pi^{\alpha} \circ \delta\left(x_{\alpha} / I_{\alpha}\right)$ which yields $\pi_{\alpha} \circ \delta_{\alpha}=$ $i d_{M_{\alpha} / I_{\alpha}}$.

Corollary 5.9. Let I be a non-trivial normal ideal of a pseudo MV-algebra M and let $\alpha$ be a cardinal. Then the power $I^{\alpha}$ is a retractive ideal of the power pseudo $M V$-algebra $M^{\alpha}$ if and only if $I$ is a retractive ideal of $M$.

## 6. Free Product and Local Pseudo MV-algebras

In the present section we show that every local pseudo MV-algebra that is a strong ( $\mathbb{H}, 1$ )-perfect pseudo MV-algebra has also a representation via a free product. It will generalize results from DiLe2 known only for local MV-algebras.

Let $\mathcal{V}$ be a class of pseudo MV-algebras and let $\left\{A_{t}\right\}_{t \in T} \subseteq \mathcal{V}$. According to DvHo1, we say that a $\mathcal{V}$-coproduct (or simply a coproduct if $\mathcal{V}$ is known from the context) of this family is a pseudo MV-algebra $A \in \mathcal{V}$, together with a family of homomorphisms

$$
\left\{f_{t}: A_{t} \rightarrow A\right\}_{t \in T}
$$

such that
(i) $\bigcup_{t \in T} f_{t}\left(A_{t}\right)$ generates $A$;
(ii) if $B \in \mathcal{V}$ and $\left\{g_{t}: A_{t} \rightarrow B\right\}_{t \in T}$ is a family of homomorphisms, then there exists a (necessarily) unique homomorphism $h: A \rightarrow B$ such that $g_{t}=f_{t} h$ for all $t \in T$.
Coproducts exist for every variety $\mathcal{V}$ of algebra, and are unique. They are designated by $\bigsqcup_{t \in T}^{\mathcal{V}} A_{t}$ (or $A_{1} \sqcup^{\mathcal{V}} A_{2}$ if $T=\{1,2\}$ ). If each of the homomorphisms $f_{t}$ is an embedding, then the coproduct is called the free product.

By [DvHo1, Thm 2.3], the free product of any set of non-trivial pseudo MValgebras exists in the variety of pseudo MV-algebras.

Now let $M$ be a symmetric local pseudo MV-algebra from $\mathcal{M}$ with a unique maximal and normal ideal $I=\operatorname{Ker}_{n}(M)=\operatorname{Ker}(M)$. Let $\mathbb{H}$ be a subgroup of $\mathbb{R}$ such that $M / I \cong \Gamma(\mathbb{H}, 1)$. Take an $\ell$-group $G$ such that $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0)) \cong\langle I\rangle$. Let $N=\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$. If $I$ is retractive, then by Theorem 5.7, $M \cong N$, and in this section, we describe this situation using free product of $M / I$ and $\langle I\rangle$. We note that this was already established in DiLe2, Thm 3.1] but only for MV-algebras. For our generalization, we introduce a weaker form of our free product of $M / I$ and $\langle I\rangle$ which we will denote $M / I \sqcup_{w}\langle I\rangle$ in the variety of symmetric pseudo MV-algebras from $\mathcal{M}$ and which means that (i) remains and (ii) are changed as follows
$\left(i^{*}\right)$ if $\phi_{1}: M / I \rightarrow M / I \sqcup_{w}\langle I\rangle$ and $\phi_{2}:\langle I\rangle \rightarrow M / I \sqcup_{w}\langle I\rangle$ are injecive homomorphisms, then $\phi_{1}(M / I) \cup \phi_{2}(\langle I\rangle)$ generates $M / I \sqcup_{w}\langle I\rangle$,
$\left(\right.$ ii $\left.^{*}\right)$ if $\kappa_{1}: M / I \rightarrow A$ and $\kappa_{2}:\langle I\rangle \rightarrow A$, where $A$ is a symmetric pseudo MValgebra from $\mathcal{M}$, are such homomorphisms that $\kappa_{1}(a)+\kappa_{2}(b)=\kappa_{2}(b)+$ $\kappa_{1}(a)$, then there is a unique homomorphism $\psi: M / I \sqcup_{w}\langle I\rangle \rightarrow A$ such that $\psi \circ \phi_{1}=\kappa_{1}$ and $\psi \circ \phi_{2}=\kappa_{2}$.
We note that if $M$ is an MV-algebra, then our notion coincides with the original form of the free product of MV-algebras in the class of MV-algebras.

Theorem 6.1. Let $M$ be a symmetric local pseudo $M V$-algebra from $\mathcal{M}, I=$ $\operatorname{Rad}_{n}(I)$ and $N=\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$ for some unital $\ell$-subgroup $(\mathbb{H}, 1)$ of $(\mathbb{R}, 1)$ and some $\ell$-group $G$. The following statements are equivalent:
(i) $M \cong N$.
(ii) The free product $M / I \sqcup_{w}\langle I\rangle$ in the variety of symmetric pseudo $M V$-algebras from $\mathcal{M}$ is isomorphic to $M$.

Proof. (i) $\Rightarrow$ (ii) Let $M=\Gamma(K, v)$. By Theorem 5.7, $I=\operatorname{Ker}_{n}(M)$ is a retractive ideal. Define $\phi_{1}: M / I \rightarrow \Gamma(\mathbb{H} \overrightarrow{\times}\{0\},(1,0)) \subset \Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))=N$ and $\phi_{2}:\langle I\rangle \rightarrow$ $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0)) \subset \Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))=N$ as follows: Let $s_{0}$ be a unique state on $M$ which is guaranteed by Theorem 3.4. We set $M_{t}=s_{0}^{-1}(\{t\})$ for any $t \in[0,1]_{\mathbb{H}}$. Then $\phi_{1}(x / I):=(t, 0)$ whenever $x \in M_{t}$. Since $\langle I\rangle=I \cup I^{-}$, we set $\phi_{2}(x)=(0, x)$ if $x \in I$ and $\phi_{2}(x)=(1, x-1)$ if $x \in I^{-}$. From (4.2) of the proof of Theorem 4.2 we see that $\phi_{1}$ and $\phi_{2}$ are injective homomorphisms of pseudo MV-algebras into $N$. Using again (4.2), we see that $\phi_{1}(M / I) \cup \phi_{2}(\langle I\rangle)$ generates $N$.

Now suppose that there is a symmetric pseudo MV-algebra $A$ from $\mathcal{M}$ and two mutually commuting homomorphisms $\kappa_{1}: M / I \rightarrow A$ and $\kappa_{2}:\langle I\rangle \rightarrow A=\Gamma(W, w)$, i.e. $\kappa_{1}(a)+\kappa_{2}(b)=\kappa_{2}(b)+\kappa_{1}(a)$ for all $a \in M / I$ and $b \in\langle I\rangle$. Then $\kappa_{1}(1 / I)=$ $w=\kappa_{2}(1)$ and $w$ commutes with every $\kappa_{1}(a)$ and $\kappa_{2}(b)$.
Claim 1. Let $a=\kappa_{1} \phi_{1}^{-1}(h, 0)$ with $0<h<1, h \in H$, and $\epsilon=\kappa_{2} \phi_{2}^{-1}(0, g)$ with $g \in G^{+}$. Then $\epsilon<a<\epsilon^{-}$.

Indeed, by the assumption, from the form of the element $a$ we conclude that it is finite and $\epsilon$ and $a$ commute. Then there is an integer $n \geq 1$ such that $n . a=1$. Since $\epsilon \in \operatorname{Rad}(A)$, we have $n . \epsilon=n \epsilon<1=n . a \leq n a$ which yields $0 \leq n(a-\epsilon)$, so that $\epsilon<a$. In a similar way we show $\epsilon<a^{-}$, i.e. $\epsilon<a<\epsilon^{-}$.
Claim 2. Let $\alpha=\kappa_{1} \circ \phi^{-1}: \Gamma(\mathbb{H} \overrightarrow{\times}\{0\},(1,0)) \rightarrow A$ and $\beta=\kappa_{2} \circ \phi_{2}^{-1}: \phi_{2}^{-1}(\langle I\rangle) \rightarrow$ A. Passing to the corresponding representing unital $\ell$-groups, we will denote by
$\hat{\alpha}$ and $\hat{\beta}$ the corresponding extensions of $\alpha$ and $\beta$ to $\ell$-homomorphisms of unital $\ell$-groups into the unital $\ell$-group $\left(G_{A}, w\right)$ such that $\Gamma\left(G_{A}, w\right)=A$. Then $\hat{\alpha}(0, h)+$ $\hat{\beta}(0, g) \geq 0$ for each $h \in \mathbb{H}^{+}$and each $g \in G$.

If $h=0$, the statement is evident. Let $h>0$. Then $a:=\hat{\alpha}(0, h)+\hat{\beta}(0, g)=$ $\hat{\alpha}(h, 0)+\hat{\beta}\left(0, g^{+}\right)+\hat{\beta}\left(0, g^{-}\right)$, where $g^{+}=g \vee 0$ and $g^{-}=g \wedge 0$. Then $a=$ $\hat{\alpha}(h, 0)+\beta\left(0, g^{+}\right)+\beta\left(1, g^{-}\right)-\beta(1,0)$. From Claim 1, we get $\hat{\alpha}(h, 0)+\beta\left(0, g^{+}\right) \geq$ $\beta\left(0,-g^{-}\right)=\beta(1,0)-\beta\left(1, g^{-}\right)$and the claim is proved.

Now we define a mapping $\psi: \mathbb{H} \overrightarrow{\times} G \rightarrow G_{A}$ by

$$
\psi(h, g)=\hat{\alpha}(h, 0)+\hat{\beta}(0, g), \quad(h, g) \in \mathbb{H} \overrightarrow{\times} G
$$

Claim 3. $\psi$ is an $\ell$-group homomorphism of unital $\ell$-groups.
(a) We have $\psi(0,0)=0$ and $\psi(1,0)=w$. Moreover,

$$
\begin{aligned}
\psi\left(h_{1}, g_{1}\right)+\psi\left(h_{2}, g_{2}\right) & =\hat{\alpha}\left(h_{1}, 0\right)+\hat{\beta}\left(0, g_{1}\right)+\hat{\alpha}\left(h_{2}, 0\right)+\hat{\beta}\left(0, g_{2}\right) \\
& =\hat{\alpha}\left(h_{1}, 0\right)+\hat{\alpha}\left(h_{2}, 0\right)+\hat{\beta}\left(0, g_{1}\right)+\hat{\beta}\left(0, g_{2}\right) \\
& =\hat{\alpha}\left(h_{1}+h_{2}, 0\right)+\hat{\beta}\left(0, g_{1}+g_{2}\right) \\
& =\psi\left(h_{1}+h_{2}, g_{1}+g_{2}\right) .
\end{aligned}
$$

(b) According to Claim 2, we see that $\psi(h, g) \geq 0$ whenever $(h, g) \geq(0,0)$.
(c) $\psi$ preserves $\wedge$. For $x:=\left(h_{1}, g_{1}\right) \wedge\left(h_{2}, g_{2}\right)$, we have three cases (i) $x=\left(h_{1}, g_{1}\right)$ if $h_{1}<h_{2}$, (ii) $x=\left(h_{1}, g_{1} \wedge g_{2}\right)$ if $h_{1}=h_{2}$, and (iii) $x=\left(h_{2}, g_{2}\right)$ if $h_{2}<h_{1}$.

In case (i), we have $\psi\left(h_{2}, g_{2}\right)-\psi\left(h_{1}, g_{1}\right)=\psi\left(h_{2}-h_{1}, g_{2}-g_{1}\right) \geq 0$ by Claim 2. Thus $\psi$ preserves $\wedge$. In case (ii), we have

$$
\begin{aligned}
\psi\left(\left(h_{1}, g_{1}\right) \wedge\left(h_{2}, g_{2}\right)\right) & =\psi\left(h_{1}, g_{1} \wedge g_{2}\right)=\hat{\alpha}\left(h_{1}, 0\right)+\hat{\beta}\left(0, g_{1} \wedge g_{2}\right) \\
& =\hat{\alpha}\left(h_{1}, 0\right)+\hat{\beta}\left(0, g_{1}\right) \wedge \hat{\beta}\left(0, g_{2}\right) \\
& =\left(\hat{\alpha}\left(h_{1}, 0\right)+\hat{\beta}\left(0, g_{1}\right)\right) \wedge\left(\hat{\alpha}\left(h_{1}, 0\right)+\hat{\beta}\left(0, g_{2}\right)\right) \\
& =\psi\left(h_{1}, g_{1}\right) \wedge \psi\left(h_{2}, g_{2}\right)
\end{aligned}
$$

Case (iii) follows from (i).
If we restrict $\psi$ to $N$, then we have

$$
\psi(h, g)=\left(\alpha(h, 0) \oplus \beta\left(1, g^{+}\right)\right) \odot \beta\left(1, g^{-}\right), \quad(h, g) \in N
$$

Using that $\psi$ is an $\ell$-group homomorphism, we have that if $g=g_{1}+g_{2}$, where $g_{1} \geq 0$ and $g_{2} \leq 0$, then

$$
\psi(h, g)=\left(\alpha(h, 0) \oplus \beta\left(1, g_{1}\right)\right) \odot \beta\left(1, g_{2}\right) .
$$

Uniqueness of $\psi$. If $\psi^{\prime}$ is another homomorphism from $N$ into $A$ such that $\phi_{i} \circ \psi=\kappa_{i}$ for $i=1,2$, then $\psi^{\prime}(0,0)=\psi(0,0), \psi^{\prime}(0, g)=\psi^{\prime}\left(\phi_{2} \phi_{2}^{-1}(0, g)\right)=$ $\phi_{2} \kappa_{2}(0, g)=\psi(0, g), g \in G^{+} . \psi^{\prime}(h, 0)=\psi^{\prime}\left(\phi_{1} \phi_{1}^{-1}(h, 0)\right)=\phi_{1} \kappa_{1}(h, 0)=\psi(h, 0)$, $h \in[0,1]_{\mathbb{H}}$.

Using all above steps, we have that the free product $M / I \sqcup_{w}\langle I\rangle \cong N$. Since $N \cong M$, we have established (ii).
(ii) $\Rightarrow$ (i) From the proof of the previous implication we have that the free product of $\Gamma(\mathbb{H}, 1)$ and $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$ is isomorphic to $N=\Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$. Since $M / I \cong \Gamma(\mathbb{H}, 1)$ and $\langle I\rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$, we have from (ii) $M \cong N$.

## 7. Pseudo MV-algebras with Lexicographic Ideals

The following notions were introduced in DFL only for MV-algebras, and in this section, we extend them for symmetric pseudo MV-algebras and generalize some results from DFL.

We say that a normal ideal $I$ is (i) commutative if $x / I \oplus y / I=y / I \oplus x / I$ for all $x, y \in M$, (ii) strict if $x / I<y / I$ implies $x<y$.

For example, (i) if $s$ is a state, then $\operatorname{Ker}(s)$ is a commutative ideal, Dvu1, Prop 4.1(ix)], (ii) every maximal ideal that is normal is commutative, Dvu1. If $M$ is a local symmetric pseudo MV-algebra, $\operatorname{Rad}_{n}$ is a strict ideal.

Now we extend for pseudo MV-algebras the notion of a lexicographic ideal introduced in DFL only for MV-algebras. We say that a commutative ideal $I$ of a pseudo MV-algebra $M,\{0\} \neq I \neq M$, is lexicographic if
(i) $I$ is strict,
(ii) $I$ is retractive,
(iii) $I$ is prime.

We note that a lexicographic ideal for MV-algebras was defined in DFL by (i)-(iii) and
(iv) $y \leq x \leq y^{-}$for all $y \in I$ and all $x \in M \backslash\langle I\rangle$, where $\langle I\rangle$ is the subalgebra of $M$ generated by $I$.
But since $I$ is strict, we have $y \in I^{-}$implies $z<y$ for any $z \in I$. Hence, if $z \notin I$, we have $z / I>x / I=0 / I$ for all $x \in I$ which yields $z>x$. Therefore, $\langle I\rangle=I \cup I^{-}$and (iv) holds, and consequently, (iv) from DFL is superfluous, and for the definition of a lexicographic ideal of an MV-algebra we need only (i)-(iii).

Let $\operatorname{LexId}(M)$ be the set of lexicographic ideals of $M$. If we take the MV-algebra $M$ from Example 3.3, we see that $I_{1}=\{(0, m, n): m>0, n \in \mathbb{Z}$ or $m=0, n \geq 0\}$ and $I_{2}=\{(0,0, n): n \geq 0\}$ are two unique lexicographic ideals of $M$ and $I_{1} \subset I_{2}$.

Proposition 7.1. If $I, J \in \operatorname{LexId}(M)$, then $I \subseteq J$ or $J \subseteq I$. In addition, every lexicographic ideal is contained in the radical $\operatorname{Rad}(M)$ of $M$. If one of the lexicographic ideals is a maximal ideal, then $M$ has a unique maximal ideal of $M$.

Proof. Suppose the converse, that is, there are $x \in I \backslash J$ and $y \in J \backslash I$. Then $x / I<y / I$ and $y / J<x / J$ which yields $x<y$ and $y<x$ which is absurd.

Assume that $I$ is a lexicographic ideal of $M$. If $I=\operatorname{Rad}(M)$, the statement is evident. If there is an element $y \in \operatorname{Rad}(M)$ such that $y \notin I$, then by (ii) $x<y$ for any element $x \in I$, so that $I \subseteq \operatorname{Rad}(M)$.

Let $I$ be any lexicographic ideal of $M$. We have two cases. (a) $I$ is a maximal ideal of $M$. We claim $M$ has a unique maximal ideal. Indeed, for any maximal ideal $J$ of $M, J \neq I$, there are $x \in I \backslash J$ and $y \in J \backslash I$ which implies $x<y$ so that $x \in J$ which is a contradiction. Hence, $I$ is a unique maximal ideal of $M$, then $\operatorname{Rad}(M)=I$ and every lexicographic ideal of $M$ is in $\operatorname{Rad}(M)$. (b) $I$ is not a maximal ideal of $M$. Let $J$ be an arbitrary maximal ideal of $M$. There exists $y \in J \backslash I$ which yields $y>x$ for any $x \in I$, so that $x \in J$ and $I \subseteq J$. Hence, again $I \subseteq \operatorname{Rad}(M)$.

Remark 7.2. It is clear that if $\operatorname{LexId}(M) \neq \emptyset$ is finite, then $\operatorname{LexId}(M)$ has the greatest element. If $\operatorname{LexId}(M)$ is infinite, we do not know whether $\operatorname{LexId}(M)$ has the greatest element. And if this element exists, is it a maximal ideal of $M$ ?

We note that in Theorem 7.9(1), we show that if $M$ is symmetric from $\mathcal{M}$ and $\operatorname{LexId}(M) \neq \emptyset$, then $M$ is local.

As an interesting corollary we have the following statement.
Corollary 7.3. If $\operatorname{LexId}(M)$ is non-empty and $s$ is a state on $M$, then $s$ vanishes on each lexicographic ideal of $M$.

Proof. Let $I$ be a lexicographic ideal of $M$. First let $s$ be an extremal state. Then $\operatorname{Ker}(s)$ is by Dvu1, Prop 4.3] a maximal ideal. Hence, by Proposition 7.1. we have $I \subseteq \operatorname{Ker}(M) \subseteq \operatorname{Ker}(s)$, so that each extremal state vanishes on $I$. Therefore, each convex combination of extremal states, and by Krein-Mil'man Theorem, each state on $M$ vanishes on $I$.

A strengthening of the latter corollary for lexicographic pseudo MV-algebras $M$ from $\mathcal{M}$ will be done in Corollary 7.7 showing that then $M$ has a unique state.

Now we present a prototypical examples of a pseudo MV-algebra with lexicographic ideal.

Proposition 7.4. Let $(H, u)$ be an Abelian linear unital $\ell$-group and let $G$ be an $\ell$-group. If we set $I=\left\{(0, g): g \in G^{+}\right\}$, then $I$ is a lexicographic ideal of $M=\Gamma(H \overrightarrow{\times} G,(u, 0))$.

In addition, $M$ is subdirectly irreducible if and only if $G$ is a subdirectly irreducible $\ell$-group.

Proof. It is clear that $I$ is a normal ideal of $M$ as well as it is prime.
We have $x / I=0 / I$ iff $x \in I$. Assume $(0, g) / I<\left(h, g^{\prime}\right) / I$. Then $(h, g) \notin I$ that yields $h>0$ and $(0, g)<\left(h, g^{\prime}\right)$. Hence, if $x / I<y / I$, then $(y-x) / I>0 / I$ and $y-x>0$ and $x<y$.

Since $M / I \cong \Gamma(H \overrightarrow{\times}\{0\},(u, 0)) \subseteq \Gamma(H \overrightarrow{\times} G,(u, 0))$, we see that $I$ is retractive. Finally, let $y \in I$ and $x \in M \backslash\langle I\rangle$. Then $\langle I\rangle=I \cup I^{-}$and $x=\left(h, g^{\prime}\right)$ for some $h$ with $0<h<u$ and $g^{\prime} \in G$. Then $y=(0, g)$ and hence, $y<x<y^{-}$.

The statement on subdirect irreducibility follows from the categorical representation of pseudo MV-algebras, Theorem [2.1]

Theorem 7.5. Let $M$ be a symmetric pseudo $M V$-algebra from $\mathcal{M}$ and let $I$ be a lexicographic ideal of $M$. Then there is an Abelian linear unital $\ell$-group ( $H, u$ ) and an $\ell$-group $G$ such that $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$.

Proof. Similarly as in the proof of Theorem 5.7 we can assume that $M=\Gamma(K, v)$ for some unital $\ell$-group ( $K, v$ ). Since $I$ is lexicographic, then $I$ is normal and prime, so that $M / I$ is a linear, and since $I$ is also commutative, $M / I$ is an MV-algebra. There is an Abelian linear unital $\ell$-group $(H, u)$ such that $M / I \cong \Gamma(H, u)$.

Let $\pi_{I}: M \rightarrow M / I$ be the canonical projection. For any $t \in[0, u]_{H}$, we set $M_{t}:=\pi_{I}^{-1}(\{t)\}$. We assert that $\left(M_{t}: t \in[0, u]_{H}\right)$ is an ( $H, u$ )-decomposition of $M$. Indeed, (a) let $x \in M_{s}$ and $y \in M_{t}$ for $s<t, s, t \in[0, u]_{H}$. Then $\pi_{I}(x)=$ $s<t<\pi(y)$ and $x<y$ because $I$ is strict. (b) Since $\pi_{I}$ is a homomorphism, $M_{t}^{-}=M_{u-t}=M_{t}^{\sim}$ for each $t \in[0, u]_{H}$. (c) Let $x \in M_{s}$ and $y \in M_{t}$, then $\pi_{I}(x \oplus y)=\pi_{I}(x) \oplus \pi_{I}(y)=s \oplus t$.

In addition, $\langle I\rangle=I \cup I^{-}=I \cup I^{-}, I^{-}=I^{\sim}$, and $\langle I\rangle$ is a perfect pseudo MValgebra. By DDT Prop 5.2], there is a unique (up to isomorphism) $\ell$-group $G$ such that $\langle I\rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$.

Now we show that $\left(M_{t}: t \in[0, u]_{H}\right)$ has the strong cyclic property. Being $I$ also retractive, there is a subalgebra $M^{\prime}$ of $M$ such that $M^{\prime} \cong M / I$ and $\pi_{I}\left(M^{\prime}\right)=$ $\pi_{I}(M)$. Then $M^{\prime}$ is in fact an MV-algebra. For any $t \in[0, u]_{H}$, there is a unique $c_{t} \in M_{t}$ such that $\pi_{I}\left(c_{t}\right)=t$. We assert that the system of elements $\left(c_{t}: t \in[0, u]_{H}\right)$ has the following properties: (i) $c_{t} \in M_{t}$, (ii) if $s+t \leq u$, then $c_{s}+c_{t} \in M$ and $c_{s}+c_{t}=c_{s+t}$, (iii) $c_{1}=1$, and (iv) $c_{t} \in C(K)$ for each $t \in[0, u]_{H}$; indeed let $x \in K$. Being $M$ symmetric, the element $x+c_{t}-x \in H$ belongs also to $M$. Due to the categorical equivalence, Theorem 2.1 the homomorphism $\pi_{I}$ can be uniquely extended to a homomorphism $\hat{\pi}_{I}:(K, v) \rightarrow(H, u)$ of unital $\ell$-groups. Hence, $\pi_{I}\left(x+c_{t}-x\right)=\hat{\pi}_{I}\left(x+c_{t}-x\right)=\hat{\pi}_{I}(x)+\hat{\pi}_{I}\left(c_{t}\right)-\hat{\pi}_{I}(x)=\pi_{I}\left(c_{t}\right)=t$ which implies $c_{t}=x+c_{t}-x$ and $x+c_{t}=c_{t}+x$.

Consequently, $M$ is a strong $(H, u)$-perfect pseudo MV-algebra. By Theorem 4.2, there is an $\ell$-group $G^{\prime}$ such that $M \cong \Gamma\left(H \overrightarrow{\times} G^{\prime},(u, 0)\right)$. By uniqueness (up to isomorphism of $\ell$-groups) of $G^{\prime}$ in Theorem4.2, we have $G^{\prime} \cong G$ and consequently $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$.

According to the latter theorem and Proposition [7.4, we see that our notion of a lexicographic pseudo MV-algebra for symmetric pseudo MV-algebras from $\mathcal{M}$ coincides with the notion of one defined for MV-algebras in DFL as those having at least one lexicographic ideal.

In the following result we compare the class of local pseudo MV-algebras with the class of lexicographic pseudo MV-algebras.
Theorem 7.6. (1) The class of lexicographic pseudo $M V$-algebras from $\mathcal{M}$ is strictly included in the class of symmetric local pseudo MV-algebras.
(2) The class of symmetric local pseudo MV-algebras with retractive radical is strictly included in the class of lexicographic pseudo $M V$-algebras from $\mathcal{M}$.
Proof. (1) Let $M$ be a lexicographic pseudo MV-algebra from $\mathcal{M}$. By Theorem 7.5, $M$ is symmetric and it is isomorphic to some $M^{\prime}:=\Gamma(H \overrightarrow{\times} G,(u, 0))$, where $(H, u)$ is an Abelian unital $\ell$-group and $G$ is an $\ell$-group. Then the ideal $I=$ $\left\{(0, g): g \in G^{+}\right\}$is by Proposition 7.4 a retractive ideal of $M^{\prime}$. By Proposition 7.1, we have $I \subseteq \operatorname{Rad}\left(M^{\prime}\right)=\operatorname{Rad}_{n}\left(M^{\prime}\right)$. Since $I$ is prime, so is $\operatorname{Rad}_{n}\left(M^{\prime}\right)$ which yields $M^{\prime} / \operatorname{Rad}_{n}\left(M^{\prime}\right)$ is linearly ordered and semisimple. Hence, $M^{\prime} / \operatorname{Rad}_{n}\left(M^{\prime}\right)$ is a simple MV-algebra. Therefore, by [Dvu1, Prop 3.3-3.5], $\operatorname{Rad}_{n}(M)$ is a maximal ideal which yields that $M^{\prime}$ is local and, consequently $M$ is local.

To show that the class of lexicographic pseudo MV-algebras from $\mathcal{M}$ is strictly included in the class of symmetric local pseudo MV-algebras, we can use an example from the proof of [DFL, Thm 4.7] or the pseudo MV-algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(2,1))$ that has no lexicographic ideal.
(2) By Theorem5.7 we get that the class of symmetric local pseudo MV-algebras with retractive radical is strictly included in the class of lexicographic pseudo MValgebras from $\mathcal{M}$. Using an example from [DFL, Thm 4.7], we conclude that this inclusion is proper.

The latter result entails the following corollary.
Corollary 7.7. Every lexicographic pseudo MV-algebra from $\mathcal{M}$ admits a unique state.
Proof. If $M$ is a lexicographic pseudo MV-algebra from $\mathcal{M}$, by (i) of Theorem 7.6 we see that $M$ is local, that is, it has a unique maximal ideal and this ideal is
normal. Due to a one-to-one relation between extremal states and maximal and normal ideals of $M$, Dvu1, we conclude $M$ admits a unique state.

The following result gives a new look to Theorem 5.7,
Theorem 7.8. Let $M$ be a lexicographic symmetric pseudo $M V$-algebra from $\mathcal{M}$. The following statements are equivalent:
(i) $\operatorname{Rad}_{n}(M)$ is a lexicographic ideal.
(ii) $M$ is strongly $(\mathbb{H}, 1)$-perfect for some unital $\ell$-subgroup $(\mathbb{H}, 1)$ of $(\mathbb{R}, 1)$.

Proof. Let $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$ for some Abelian unital $\ell$-group $(H, u)$ and an $\ell$-group $G$ and let $I$ be a retractive ideal of $M$ such that $M / I \cong \Gamma(H, u)$. By Proposition 7.4 $I \subseteq \operatorname{Rad}_{n}(M)$.
(i) $\Rightarrow$ (ii) If $\operatorname{Rad}_{n}(M)$ is a retractive ideal, then $M / \operatorname{Rad}_{n}(M)$ is a semisimple MV-algebra that is linearly ordered because $\operatorname{Rad}_{n}(M)$ is a prime normal ideal. Again applying by [Dvu1, Prop 3.4-3.5], $M / \operatorname{Rad}_{n}(M) \cong \Gamma(H, u)$ and $\Gamma(H, u)$ is isomorphic to some $(\mathbb{H}, 1)$.
(ii) $\Rightarrow$ (i) Since $M / I \cong \Gamma(\mathbb{H}, 1)$, as a consequence of [Dvu1, Prop 3.4-3.5], we get $I$ is a maximal ideal of $M$. Hence, $I=\operatorname{Rad}_{n}(M)$ and $I$ is a lexicographic ideal of $M$ and $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G,(1,0))$.

We say that a pseudo MV-algebra $M$ from $\mathcal{M}$ is $I$-representable if $I$ is a lexicographic ideal of $M$ and $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$, where $(H, u)$ is an Abelian unital $\ell$-group such that $M / I \cong \Gamma(H, u)$ and $G$ is an $\ell$-group such that $\langle I\rangle \cong$ $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$; the existences of $(H, u)$ and $G$ are guaranteed by Theorem 7.5,
Theorem 7.9. The class of lexicographic pseudo $M V$-algebras from $\mathcal{M}$ is closed under homomorphic images and subalgebras, but it is not closed under direct products.

Moreover, (1) if $N$ is a homomorphic image of $M$, then $N \cong \Gamma\left(H_{1} \overrightarrow{\times} G_{1},\left(u_{1}, 0\right)\right)$, where $\left(H_{1}, u_{1}\right)$ and $G_{1}$ are homomorphic images of $(H, u)$ and $G$, respectively.
(2) If $N$ is a subalgebra of $M$, then $N \cong \Gamma\left(H_{0} \overrightarrow{\times} G_{0},\left(u_{0}, 0\right)\right)$, where $\left(H_{0}, u\right)$ and $G_{0}$ are subalgebras of $(H, u)$ and $G$, respectively.

Proof. Let $I$ be a lexicographic ideal of $M$ such that $M$ is $I$-representable.
(1) Let $f: M \rightarrow N$ be a surjective homomorphism. Then $N$ is symmetric and from $\mathcal{M}$ whilst $\mathcal{M}$ is a variety. If we set $f(I)=\{f(x): x \in I\}$, then $f(I)$ is a normal ideal of $N=f(M)$ that is also commutative, prime and strict. We claim that $f(I)$ is a retractive ideal, too. Let $\pi_{I}: M \rightarrow M / I$ be the canonical projection and let $\delta_{I}: M / I \rightarrow M$ be a homomorphism such that $\pi_{I} \circ \delta_{I}=i d_{M / I}$. Let $M_{0}=\delta_{I}(M / I)$ be a subalgebra of $M$ that is isomorphic to $M / I$. If we define $\hat{f}: M / I \rightarrow N / f(I)$ by $\hat{f}(x / I)=f(x) / f(I)$, then $\hat{f}$ is a well-defined homomorphism such that $\hat{f} \circ \pi_{I}=$ $\pi_{f(I)} \circ f$. Set $N_{0}=f\left(M_{0}\right)$ and let $f_{M_{0}}$ be the restriction of $f$ onto $M_{0}$. We define $\delta_{f(I)}: N / f(I) \rightarrow N$ via $\delta_{f(I)}(f(x) / f(I)):=f_{M_{0}}\left(\delta_{I}(x / I)\right)$; then $\delta_{f(I)}$ is a welldefined homomorphism such that $\delta_{f(I)}(N / f(I))=N_{0}$ and $f_{M_{0}} \circ \delta_{I}=\delta_{f(I)} \circ \hat{f}$. Hence,

$$
\begin{aligned}
\pi_{f(I)} \circ \delta_{f(I)}(f(x) / f(I)) & =\pi_{f(I)} \circ f_{M_{0}} \circ \delta_{I}(x / I) \\
& =\hat{f} \circ \pi_{I} \circ \delta_{I}(x / I)=\hat{f}(x / I) \\
& =f(x) / f(I)
\end{aligned}
$$

that proves $f(I)$ is a retractive ideal of $N$.
Take the unital representation of pseudo MV-algebras given by Theorem 2.1, and let $N \cong \Gamma(K, v)$ and let $f:(H \overrightarrow{\times} G,(u, 0)) \rightarrow(K, v)$ be a surjective homomorphism of unital $\ell$-groups. Let $f_{1}(h)=f(h, 0), h \in H$, and $f_{2}(g)=f(0, g), g \in G$. If we set $H_{1}:=f_{1}(H), u_{1}=f(u, 0)$, and $G_{1}:=f_{2}(G)$. Then $N \cong \Gamma\left(H_{1} \overrightarrow{\times} G_{1},\left(u_{1}, 0\right)\right)$.
(2) Let $N$ be a subalgebra of $M$. Then $N$ is symmetric and belongs to $\mathcal{M}$. We set $J:=N \cap I$. Then $J$ is a normal ideal of $N$ that is also commutative and prime. It is strict, too, because if $x \in N$ and $x \notin J$, then $x \notin I$ and $x>y$ for any $y \in J$ and consequently, for any $y \in J$. Then $N / J$ can be embedded into $M / I$ by a mapping $i_{J}(x / J):=x / I(x \in N)$ and if $i_{0}(x)=x, x \in N$, then $\pi_{I} \circ i_{0}=i_{J} \circ \pi_{J}$. Let $M_{0}:=\delta_{I}(M / I)$ and $N_{0}:=M_{0} \cap N$. Then $\delta_{I}(N / I) \in N_{0}$; indeed, if there is $x \in N_{0}$ such that $\delta_{I}(x / I) \notin N_{0}$, then $\pi_{I} \circ \delta_{I}(x / I)=x / I \notin N_{0} / I$. Define $\delta_{J}: N / J \rightarrow N$ by $\delta_{J}(x / J)=i_{J}^{-1} \circ \delta_{I}(x / I)$. Since $i_{I}^{-1} \circ \pi_{I}(x)=\pi_{J} \circ i_{0}^{-1}(x), x \in N$, then

$$
\begin{aligned}
\pi_{J} \circ \delta_{J}(x / J) & =\pi_{J} \circ i_{0}^{-1} \circ \delta_{I}(x / I) \\
& =i_{I}^{-1} \circ \pi_{I} \circ \delta_{I}(x / I)=i_{I}^{-1}(x / I)=x / J
\end{aligned}
$$

The rest follows the analogous steps as the end of (1).
(3) According to Corollary 7.7, every lexicographic pseudo MV-algebra $M$ admits a unique state. But the pseudo MV-algebra $M \times M$ admits two extremal states, and therefore, $M \times M$ is not lexicographic.

We note that in case (3) of latter Theorem if $I$ is a lexicographic ideal of $M$, then $I \times I$ is by Proposition 5.8 a retractive ideal but not lexicographic.

## 8. Categorical Representation of Strong ( $H, u$ )-perfect Pseudo MV-ALGEBRAS

In this section, we establish the categorical equivalence of the category of strong $(H, u)$-perfect pseudo MV-algebras with the variety of $\ell$-groups. This extends the categorical representation of strong $n$-perfect pseudo MV-algebras from Dvu3 and of $\mathbb{H}$-perfect pseudo MV-algebras from [Dvu4] with the variety of $\ell$-groups. In what follows, we follow the ideas of [Dvu4, Sec 5] and to be self-contained we repeat them mutatis mutandis.

Let $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ be the category of strong $(H, u)$-perfect pseudo MV-algebras whose objects are strong $(H, u)$-perfect pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Now let $\mathcal{G}$ be the category whose objects are $\ell$-groups and morphisms are homomorphisms of $\ell$-groups.

Define a mapping $\mathcal{M}_{H, u}: \mathcal{G} \rightarrow \mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ as follows: for $G \in \mathcal{G}$, let

$$
\mathcal{M}_{H, u}(G):=\Gamma(H \overrightarrow{\times} G,(u, 0))
$$

and if $h: G \rightarrow G_{1}$ is an $\ell$-group homomorphism, then

$$
\mathcal{M}_{H, u}(h)(t, g)=(t, h(g)), \quad(t, g) \in \Gamma(H \overrightarrow{\times} G,(u, 0))
$$

It is easy to see that $\mathcal{M}_{H, u}$ is a functor.
Proposition 8.1. $\mathcal{M}_{H, u}$ is a faithful and full functor from the category $\mathcal{G}$ of $\ell$ groups into the category $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ of strong ( $H, u$ )-perfect pseudo $M V$-algebras.
Proof. Let $h_{1}$ and $h_{2}$ be two morphisms from $G$ into $G^{\prime}$ such that $\mathcal{M}_{H, u}\left(h_{1}\right)=$ $\mathcal{M}_{H, u}\left(h_{2}\right)$. Then $\left(0, h_{1}(g)\right)=\left(0, h_{2}(g)\right)$ for each $g \in G^{+}$, consequently $h_{1}=h_{2}$.

To prove that $\mathcal{M}_{H, u}$ is a full functor, suppose that $f$ is a morphism from a strong $(H, u)$-perfect pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G,(u, 0))$ into some $\Gamma\left(H \overrightarrow{\times} G_{1},(u, 0)\right)$. Then $f(0, g)=\left(0, g^{\prime}\right)$ for a unique $g^{\prime} \in G^{++}$. Define a mapping $h: G^{+} \rightarrow G^{\prime+}$ by $h(g)=g^{\prime}$ iff $f(0, g)=\left(0, g^{\prime}\right)$. Then $h\left(g_{1}+g_{2}\right)=h\left(g_{1}\right)+h\left(g_{2}\right)$ if $g_{1}, g_{2} \in G^{+}$. Assume now that $g \in G$ is arbitrary. Then $g=g^{+}-g_{1}$, where $g^{+}=g \vee 0$ and $g^{-}=-(g \wedge 0)$, and $g=-g^{-}+g^{+}$. If $g=g_{1}-g_{2}$, where $g_{1}, g_{2} \in G^{+}$, then $g^{+}+g_{2}=g^{-}+g_{1}$ and $h\left(g^{+}\right)+h\left(g_{2}\right)=h\left(g^{-}\right)+h\left(g_{1}\right)$ which shows that $h(g)=h\left(g_{1}\right)-h\left(g_{2}\right)$ is a well-defined extension of $h$ from $G^{+}$onto $G$.

Let $0 \leq g_{1} \leq g_{2}$. Then $\left(0, g_{1}\right) \leq\left(0, g_{2}\right)$, which means $h$ is a mapping preserving the partial order.

We have yet to show that $h$ preserves $\wedge$ in $G$, i.e., $h(a \wedge b)=h(a) \wedge h(b)$ whenever $a, b \in G$. Let $a=a^{+}-a^{-}$and $b=b^{+}-b^{-}$, and $a=-a^{-}+a^{+}, b=-b^{-}+b^{+}$. Since, $h\left(\left(a^{+}+b^{-}\right) \wedge\left(a^{-}+b^{+}\right)\right)=h\left(a^{+}+b^{-}\right) \wedge h\left(a^{-}+b^{+}\right)$. Subtracting $h\left(b^{-}\right)$from the right hand and $h\left(a^{-}\right)$from the left hand, we obtain the statement in question.

Finally, we have proved that $h$ is a homomorphism of $\ell$-groups, and $\mathcal{M}_{H, u}(h)=f$ as claimed.

We note that by a universal group for a pseudo MV-algebra $M$ we mean a pair $(G, \gamma)$ consisting of an $\ell$-group $G$ and a $G$-valued measure $\gamma: M \rightarrow G^{+}$(i.e., $\gamma(a+b)=\gamma(a)+\gamma(b)$ whenever $a+b$ is defined in $M)$ such that the following conditions hold: (i) $\gamma(M)$ generates $G$. (ii) If $K$ is a group and $\phi: M \rightarrow K$ is an $K$-valued measure, then there is a group homomorphism $\phi^{*}: G \rightarrow K$ such that $\phi=\phi^{*} \circ \gamma$.

Due to Dvu2, every pseudo MV-algebra admits a universal group, which is unique up to isomorphism, and $\phi^{*}$ is unique. The universal group for $M=\Gamma(G, u)$ is $(G, i d)$ where $i d$ is the embedding of $M$ into $G$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Suppose that $g, h$ be two functors from $\mathcal{B}$ to $\mathcal{A}$ such that $g \circ f=i d_{\mathcal{A}}$ and $f \circ h=i d_{\mathcal{B}}$, then $g$ is a left-adjoint of $f$ and $h$ is a right-adjoint of $f$.

Proposition 8.2. The functor $\mathcal{M}_{H, u}$ from the category $\mathcal{G}$ into $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ has a left-adjoint.

Proof. We show, for a strong ( $H, u$ )-perfect pseudo MV-algebra $M$ with an $(H, u)$ decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ and an $(H, u)$-strong cyclic family $\left(c_{t}: t \in[0, u]_{H}\right)$ of elements of $M$, there is a universal arrow $(G, f)$, i.e., $G$ is an object in $\mathcal{G}$ and $f$ is a homomorphism from the pseudo MV-algebra $M$ into $\mathcal{M}_{H, u}(G)$ such that if $G^{\prime}$ is an object from $\mathcal{G}$ and $f^{\prime}$ is a homomorphism from $M$ into $\mathcal{M}_{H, u}\left(G^{\prime}\right)$, then there exists a unique morphism $f^{*}: G \rightarrow G^{\prime}$ such that $\mathcal{M}_{H, u}\left(f^{*}\right) \circ f=f^{\prime}$.

By Theorem 4.2, there is a unique (up to isomorphism of $\ell$-groups) $\ell$-group $G$ such that $M \cong \Gamma(H \overrightarrow{\times} G,(u, 0))$. By [Dvu2, Thm 5.3], $(H \overrightarrow{\times} G, \gamma)$ is a universal group for $M$, where $\gamma: M \rightarrow \Gamma(H \overrightarrow{\times} G,(u, 0))$ is defined by $\gamma(a)=\left(t, a-c_{t}\right)$, if $a \in M_{t}$.

Define a mapping $\mathcal{P}_{H, u}: \mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u} \rightarrow \mathcal{G}$ via $\mathcal{P}_{H, u}(M):=G$ whenever $(H \overrightarrow{\times} G, f)$ is a universal group for $M$. It is clear that if $f_{0}$ is a morphism from the pseudo MV-algebra $M$ into another one $N$, then $f_{0}$ can be uniquely extended to an $\ell$-group homomorphism $\mathcal{P}_{H, u}\left(f_{0}\right)$ from $G$ into $G_{1}$, where $\left(H \overrightarrow{\times} G_{1}, f_{1}\right)$ is a universal group for the strong $(H, u)$-perfect pseudo MV-algebra $N$.

Proposition 8.3. The mapping $\mathcal{P}_{H, u}$ is a functor from the category $\mathcal{S P} \mathcal{P}_{s} \mathcal{M}_{H, u}$ into the category $\mathcal{G}$ which is a left-adjoint of the functor $\mathcal{M}_{H, u}$.
Proof. It follows from the properties of the universal group.
Now we present the basic result of this section on a categorical equivalence of the category of strong $(H, u)$-perfect pseudo MV-algebras and the category of $\mathcal{G}$.
Theorem 8.4. The functor $\mathcal{M}_{H, u}$ defines a categorical equivalence of the category $\mathcal{G}$ and the category $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ of strong $(H, u)$-perfect pseudo MV-algebras.

In addition, suppose that $h: \mathcal{M}_{H, u}(G) \rightarrow \mathcal{M}_{H, u}\left(G^{\prime}\right)$ is a homomorphism of pseudo MV-algebras, then there is a unique homomorphism $f: G \rightarrow G^{\prime}$ of $\ell$-groups such that $h=\mathcal{M}_{H, u}(f)$, and
(i) if $h$ is surjective, so is $f$;
(ii) if $h$ is injective, so is $f$.

Proof. According to MaL, Thm IV.4.1], it is necessary to show that, for a strong $(H, u)$-perfect pseudo MV-algebra $M$, there is an object $G$ in $\mathcal{G}$ such that $\mathcal{M}_{H, u}(G)$ is isomorphic to $M$. To show that, we take a universal group $(H \overrightarrow{\times} G, f)$. Then $\mathcal{M}_{H, u}(G)$ and $M$ are isomorphic.

An important kind of $\ell$-groups are doubly transitive $\ell$-groups; for more details on them see e.g. Gla. Every such an $\ell$-group generates the variety of $\ell$-groups, Gla, Lem 10.3.1]. The notion of doubly transitive unital $\ell$-group $(G, u)$ was introduced and studied in DvHo, and according to DvHo, Cor 4.9], the pseudo MV-algebra $\Gamma(G, u)$ generates the variety of pseudo MV-algebras.

An example of a doubly transitive permutation $\ell$-group is the system of all automorphisms, $\operatorname{Aut}(\mathbb{R})$, of the real line $\mathbb{R}$, or the next example:

Let $u \in \operatorname{Aut}(\mathbb{R})$ be the translation $t u=t+1, t \in \mathbb{R}$, and

$$
\operatorname{BAut}(\mathbb{R})=\left\{g \in \operatorname{Aut}(\mathbb{R}): \exists n \in \mathbb{N}, u^{-n} \leq g \leq u^{n}\right\}
$$

Then $(\operatorname{BAut}(\mathbb{R}), u)$ is a doubly transitive unital $\ell$-permutation group, and it is a generator of the variety of pseudo MV-algebras $\mathcal{P}_{s} \mathcal{M} \mathcal{V}$. In addition, $\Gamma(\operatorname{BAut}(\mathbb{R}), u)$ is a stateless pseudo MV-algebra.

The proof of the following statement is practically the same as that of Dvu4, Thm 5.6] and therefore, we omit it here.

Theorem 8.5. Let $G$ be a doubly transitive $\ell$-group. Then the variety generated by $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ coincides with the variety generated by $\mathcal{M}_{H, u}(G)$.

## 9. Weak $(H, u)$-perfect Pseudo MV-algebras

In this section, we will study another kind of $(H, u)$-perfect pseudo MV-algebras, called weak $(H, u)$-perfect pseudo MV-algebras. Their prototypical examples are pseudo MV-algebras of the form $\Gamma(H \overrightarrow{\times} G,(1, b))$, where $(H, u)$ is an Abelian unital $\ell$-group, $G$ is an $\ell$-group and $b \in G$. Such pseudo MV-algebras were studied for the case $(H, u)=(\mathbb{H}, 1)$ in Dvu4.

Let $(H, u)$ be an Abelian unital $\ell$-group. We say that a pseudo MV-algebra $M \cong \Gamma(K, v)$ with an $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$ is weak if there is a system $\left(c_{t}: t \in[0, u]_{H}\right)$ of elements of $M$ such that (i) $c_{0}=0$, (ii) $c_{t} \in C(K) \cap M_{t}$, for any $t \in[0, u]_{H}$, and (iii) $c_{v+t}=c_{v}+c_{t}$ whenever $v+t \leq u$.

We notice that in contrast to the strong cyclic property, we do not assume $c_{1}=1$. In addition, a weak $(H, u)$-perfect pseudo MV-algebra $M$ is strong iff $c_{1}=1$.

Example 9.1. Let $(H, u)$ be an Abelian unital $\ell$-group. The pseudo $M V$-algebra $M=\Gamma(H \overrightarrow{\times} G,(u, b))$, where $b \in G, M_{t}=\{(t, g):(t, g) \in M\}, t \in[0, u]_{H}$ form an $(H, u)$-decomposition of $M$, is a weak pseudo $M V$-algebra setting $c_{t}=(t, 0)$, $t \in[0, u]_{H}$.
Proof. We have to verify that $\left(M_{t}: t \in[0, u]_{H}\right)$ is an $(H, u)$-decomposition. To show that it is enough to verify (b) of Definition 3.1, i.e. $M_{t}^{-}=M_{u-t}=M_{t}^{\sim}$ for each $t \in[0, u]_{H}$. Let $(t, g) \in M_{t}$. Then $(t, g)^{-}=(u, b)-(t, g)=(u-t, b-g)$. If we choose $\left(t, g_{0}\right)$, where $g_{0}=b+g-b$, then $\left(t, g_{0}\right)^{\sim}=-\left(t, g_{0}\right)+(u, b)=$ $\left(-t+u,-g_{0}+b\right)=(u-t, b-g)=(t, g)^{-}$which yields $(t, g)^{-} \in M_{t}^{\sim}$, that is $M_{t}^{-} \subseteq M_{t}^{\sim}$. Dually we show $M_{t}^{\sim} \subseteq M_{t}^{-}$. Then $M_{t}^{-}=M_{u-t}=M_{t}^{\sim}$.

Whereas every strong $(H, u)$-perfect pseudo MV-algebra is symmetric, weak ones are not necessarily symmetric.

For example, the pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G,(u, b))$, where $b>0$ and $b \notin C(G)$ and $M_{t}:=\{(t, g) \in \Gamma(H \overrightarrow{\times} G,(u, b))\}$ for each $t \in[0, u]_{H}$, is weak $(H, u)$-perfect but neither strong $(H, u)$-perfect nor symmetric.

We note that $M_{0}$ is a unique maximal and normal ideal of $M$. This ideal is retractive iff $M$ is strongly $(H, u)$-perfect. For example, let $M=\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(2,1))$. Then $M$ is weakly $(\mathbb{Z}, 2)$-perfect that is not strongly $(\mathbb{Z}, 2)$-perfect, and $M_{0}=$ $\{(0, n): n \geq 0\}, M_{1}=\{(1, n): n \in \mathbb{Z}\}, M_{2}=\{(2, n): n \leq 1\}, M / M_{0} \cong \Gamma\left(\frac{1}{2} \mathbb{Z}, 1\right)$ and it has no isomorphic copy in $M$. In addition, $M_{0}$ is not retractive.

We notice that even a pseudo MV-algebra of the form $\Gamma(H \overrightarrow{\times} G,(u, b))$ with $b \neq 0$ can be strongly $(H, u)$-perfect. Indeed, let $M=\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(2,2))$. This MV-algebra is isomorphic with the MV-algebra $M_{1}:=\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z},(2,0))$. In fact, the mapping $\theta: M_{1} \rightarrow M$ defined by $\theta(0, n)=(0, n), \theta(1, n)=(1, n+1)$ and $\theta(2, n)=(2, n+2)$ is an isomorphism in question. In addition, $M_{0}=\{(0, n): n \geq 0\}$ is a retractive ideal and a lexicographic ideal of $M ; M / M_{0}=\Gamma\left(\frac{1}{2} \mathbb{Z}, 1\right)$ and its isomorphic copy in $M$ is the subalgebra $\{(0,0),(1,1),(2,2)\}$.

The next result is a representation of weak $(H, u)$-perfect pseudo MV-algebras by lexicographic product.
Theorem 9.2. Let $M$ be a weak $\mathbb{H}$-perfect pseudo $M V$-algebra which is not strong. Then there is a unique (up to isomorphism) $\ell$-group $G$ with an element $b \in G$, $b \neq 0$, such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G,(1, b))$.

Proof. Assume $M=\Gamma(K, v)$ for some unital $\ell$-group $(H, u)$ is a weak pseudo MValgebra with a $(H, u)$-decomposition $\left(M_{t}: t \in[0, u]_{H}\right)$. Since by (vi) of Theorem 3.2 we have $M_{0}+M_{0}=M_{0}$, in the same way as in the proof of Theorem4.2, there exists an $\ell$-group $G$ such that $G^{+}=M_{0}$ and $G$ is a subgroup of $K$.

Since $M$ is not strong, then $c_{1}<1=: u$. Set $b=1-c_{1} \in M_{0} \backslash\{0\}$, and define a mapping $\phi: M \rightarrow \Gamma(\mathbb{H} \overrightarrow{\times} G,(1, b))$ as follows

$$
\phi(x)=\left(t, x-c_{t}\right)
$$

whenever $x \in M_{t}$; we note that the subtraction $x-c_{t}$ is defined in the $\ell$-group $K$. Using the same way as that in (4.2), we can show that $\phi$ is a well-defined mapping.

We have (1) $\phi(0)=(0,0),(2) \phi(1)=\left(1,1-c_{1}\right)=(1, b),(3) \phi\left(c_{t}\right)=(t, 0),(4)$ $\phi\left(x^{\sim}\right)=\left(1-t,-x+u-c_{1-t}\right)=\left(1-t,-x+b+c_{t}\right), \phi(x)^{\sim}=-\phi(x)+(1, b)=$ $-\left(t, x-c_{t}\right)+(1, b)=\left(1-t,-x+b+c_{t}\right)$, and similarly (5) $\phi\left(x^{-}\right)=\phi(x)^{-}$.

Following ideas of the proof of Theorem 4.2, we can prove that $\phi$ is an injective and surjective homomorphism of pseudo MV-algebras as was claimed.

It is worthy of reminding that Theorem 9.2 is a generalization of Theorem 4.2 because Theorem 4.2 in fact follows from Theorem 9.2 when we have $b=0$. This happens if $c_{1}=1$.

Also in an analogous way as in Dvu4, we establish a categorical equivalence of the category of weak $(H, u)$-perfect pseudo MV-algebras with the category of $\ell$-groups $G$ with a fixed element $b \in G$.

Let $\mathcal{W} \mathcal{P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ be the category of weak $(H, u)$-perfect pseudo MV-algebras whose objects are weak $(H, u)$-perfect pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Similarly, let $\mathcal{L}_{\mathrm{b}}$ be the category whose objects are couples $(G, b)$, where $G$ is an $\ell$-group and $b$ is a fixed element from $G$, and morphisms are $\ell$-homomorphisms of $\ell$-groups preserving fixed elements $b$.

Define a mapping $\mathcal{F}_{H, u}$ from the category $\mathcal{L}_{\mathrm{b}}$ into the category $\mathcal{W} \mathcal{P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ as follows:

Given $(G, b) \in \mathcal{L}_{\mathrm{b}}$, we set

$$
\mathcal{F}_{H, u}(G, b):=\Gamma(H \overrightarrow{\times} G,(u, b)),
$$

and if $h:(G, b) \rightarrow\left(G_{1}, b_{1}\right)$, then

$$
\mathcal{F}_{H, u}(h)(t, g)=(t, h(g)), \quad(t, g) \in \Gamma(H \overrightarrow{\times} G,(u, b)) .
$$

It is easy to see that $\mathcal{F}_{H, u}$ is a functor.
In the same way as the categorical equivalence of strong $(H, u)$-perfect pseudo MV-algebras was proved in the previous section, we can prove the following theorem.

Theorem 9.3. The functor $\mathcal{F}_{H, u}$ defines a categorical equivalence of the category $\mathcal{L}_{\mathrm{b}}$ and the category $\mathcal{W} \mathcal{P P}_{s} \mathcal{M} \mathcal{V}_{H, u}$ of weak $(H, u)$-perfect pseudo $M V$-algebras.

Finally, we present addition open problems.

Problem 9.4. (1) Find an equational basis for the variety generated by the set $\mathcal{S P} \mathcal{P}_{s} \mathcal{M} \mathcal{V}_{H, u}$. For example, if $(H, u)=(\mathbb{Z}, 1)$ the basis is $2 \cdot x^{2}=(2 \cdot x)^{2}$, see DDT, Rem 5.6], and the case $(H, u)=(\mathbb{Z}, n)$ was described in [Dvu3, Cor 5.8].
(2) Find algebraic conditions that entail that a pseudo MV-algebra is of the form $\Gamma(H \overrightarrow{\times} G,(u, 0)$, where $(H, u)$ is a unital $\ell$-group not necessary Abelian.

## 10. Conclusion

In the paper we have established conditions when a pseudo MV-algebra $M$ is an interval in some lexicographic product of an Abelian unital $\ell$-group $(H, u)$ and an $\ell$ group $G$ not necessarily Abelian, i.e. $M=\Gamma(H \overrightarrow{\times} G,(u, 0))$. To show, that we have introduced strong $(H, u)$-perfect pseudo MV-algebras as those pseudo MV-algebras that can be split into comparable slices indexed by the elements from the interval $[0, u]_{H}$. For them we have established a representation theorem, Theorem4.2 and we have shown that the category of strong $(H, u)$-perfect pseudo MV-algebras is categorically equivalent to the variety of $\ell$-groups, Theorem 8.4 .

We have shown that our aim can be solved also introducing so-called lexicographic ideals. We establish their properties and Theorem 7.5 gives also a representation of a pseudo MV-algebra in the form $\Gamma(H \overrightarrow{\times} G,(u, 0))$. We show that every lexicographic pseudo MV-algebra is always local, Theorem 7.6.

Finally, we have studied and represented weak ( $H, u$ )-perfect pseudo MV-algebras as those that they have a form $\Gamma(H \overrightarrow{\times} G,(u, g))$ where $g \in G$ is not necessary the zero element, Theorem 9.2 .

The present study has opened a door into a large class of pseudo MV-algebras and formulated new open questions, and we hope that it stimulate a new research on this topic.

## References

[Bir] G. Birkhoff, "Lattice Theory", Amer. Math. Soc. Coll. Publ., Vol. 25, Providence, Rhode Island, 1967.
[CDM] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, "Algebraic Foundations of Many-valued Reasoning", Kluwer Academic Publ., Dordrecht, 2000.
[CiTo] R. Cignoli, A. Torrens, Retractive MV-algebras, Mathware Soft Comput. 2 (1995), 157165.
[Cha] C.C. Chang, Algebraic analysis of many valued logics, Trans. Amer. Math. Soc. 88 (1958), 467-490.
[DDJ] A. Di Nola, A. Dvurečenskij, J. Jakubík, Good and bad infinitesimals, and states on pseudo MV-algebras, Order 21 (2004), 293-314.
[DDT] A. Di Nola, A. Dvurečenskij, C. Tsinakis, On perfect GMV-algebras, Comm. Algebra 36 (2008), 1221-1249.
[DFL] D. Diaconescu, T. Flaminio, I. Leuştean, Lexicographic MV-algebras and lexicographic states, Fuzzy Sets and System uzzy Sets and Systems 244 (2014), 63-85. DOI 10.1016/j.fss.2014.02.010
[DiLe1] A. Di Nola, A. Lettieri, Perfect MV-algebras are categorical equivalent to abelian $\ell$ groups, Studia Logica 53 (1994), 417-432.
[DiLe2] A. Di Nola, A. Lettieri, Coproduct MV-algebras, nonstandard reals and Riesz spaces, J. Algebra 185 (1996), 605-620.
[Dvu1] A. Dvurečenskij, States on pseudo MV-algebras, Studia Logica 68 (2001), 301-327.
[Dvu2] A. Dvurečenskij, Pseudo MV-algebras are intervals in $\ell$-groups, J. Austral. Math. Soc. 72 (2002), 427-445.
[Dvu3] A. Dvurečenskij, On n-perfect GMV-algebras, J. Algebra 319 (2008), 4921-4946.
[Dvu4] A. Dvurečenskij, $\mathbb{H}$-perfect pseudo MV-algebras and their representations, Math. Slovaca http://arxiv.org/abs/1304.0743
[DvHo] A. Dvurečenskij, W.C. Holland, Top varieties of generalized MV-algebras and unital lattice-ordered groups, Comm. Algebra 35 (2007), 3370-3390.
[DvHo1] A. Dvurečenskij, W.C. Holland, Free products of unital $\ell$-groups and free products of generalized $M V$-algebras, Algebra Universalis 62 (2009), 19-25.
[DvKo] A. Dvurečenskij, M. Kolařík, Lexicographic product vs $\mathbb{Q}$-perfect and $\mathbb{H}$-perfect pseudo effect algebras, Soft Computing 17 (2014), 1041-1053. DOI: 10.1007/s00500-014-1228-6
[DvVe1] A. Dvurečenskij, T. Vetterlein, Pseudoeffect algebras. I. Basic properties, Inter. J. Theor. Phys. 40 (2001), 685-701.
[DvVe2] A. Dvurečenskij, T. Vetterlein, Pseudoeffect algebras. II. Group representation, Inter. J. Theor. Phys. 40 (2001), 703-726.
[FoBe] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. 24 (1994), 1331-1352.
[Fuc] L. Fuchs, "Partially Ordered Algebraic Systems", Pergamon Press, Oxford-New York, 1963.
[GeIo] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, Multiple Val. Logic 6 (2001), 95-135.
[Gla] A.M.W. Glass, "Partially Ordered Groups", World Scientific, Singapore, 1999.
[Go] K.R. Goodearl, "Partially Ordered Abelian Groups with Interpolation", Math. Surveys and Monographs No. 20, Amer. Math. Soc., Providence, Rhode Island, 1986.
[HoRa] D. Hort, J. Rachůnek, Lex ideals of generalized MV-algebras, In: Calude, C. S. (ed.) et al., Combinatorics, computability and logic. Proc. 3rd Inter. Conf., DMTCS '01. Univ. of Auckland, New Zealand and Univ. of Constanta, Romania, 2001. London: Springer. Discrete Mathematics and Theoretical Computer Science. 2001, pp. 125-136.
[Jak] J. Jakubík, On varieties of pseudo MV-algebras, Czechoslovak Math. J. 53 (2003), 10311040.
[Kom] Y. Komori, Super Lukasiewicz propositional logics, Nagoya Math. J. 84 (1981), 119-133.
[MaL] S. Mac Lane, "Categories for the Working Mathematician", Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[Rac] J. Rachůnek, A non-commutative generalization of MV-algebras, Czechoslovak Math. J. 52 (2002), 255-273.


[^0]:    ${ }^{1}$ Keywords: Pseudo MV-algebra, symmetric pseudo MV-algebra, $\ell$-group, strong unit, lexicographic product, ideal, retractive ideal, $(H, u)$-perfect pseudo MV-algebra, lexicographic pseudo MV-algebra, strong ( $H, u$ )-perfect pseudo MV-algebra

    AMS classification: 06D35, 03G12
    This work was supported by the Slovak Research and Development Agency under contract APVV-0178-11, grant VEGA No. 2/0059/12 SAV, and CZ.1.07/2.3.00/20.0051.

[^1]:    ${ }^{2} \odot$ has a higher binding priority than $\oplus$.

