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Representation of BL-algebras with finite independent spectrum

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Abstract

A representation theorem for a class of BL-algebras with finite spectrum is presented. Although the class comprised by our result is not the whole class of BL-algebras with finite spectrum, it applies to some important classes such as finite BL-algebras and BL-chains with finite spectrum among others. Our representation constitutes a generalization of the ordinal sum construction, since we decompose each algebra in terms of totally ordered Wajsberg hoops.

Keywords: BL-algebras; Residuated lattices; Poset product; Wajsberg hoops

1. Introduction

Basic Fuzzy Logic (BL for short) was introduced by Hájek in [10] to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment [0, 1] and the implication by its corresponding adjoint. The equivalent algebraic semantics for BL, in the sense of Blok and Pigozzi, is the class of BL-algebras. Many algebraic properties of BL-algebras correspond to logical properties of BL and, because of the lack of a reasonable proof theory, BL-algebras are the main tool for reasoning inside BL. BL-algebras form a variety (or equational class) of residuated lattices [10]. Subvarieties of the variety of BL-algebras are in correspondence with axiomatic extensions of BL. Important examples of subvarieties of BL-algebras are MV-algebras, Gödel algebras, product algebras and also Boolean algebras. On the other hand, BL-algebras can be characterized as bounded basic hoops ([2]). The theory of hoops has proved to be really useful to work with these algebras (see [2] and [6]).

One of the main tools to understand a class of algebras is to find representation theorems for the algebras in the class. Such theorems allow one to study every algebra in the class in terms of simpler or better known structures.

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That is the case of the famous Stone representation theorem for Boolean algebras and, since that, many others were stated in the field of algebraic logic. In the absence of a representation theorem for a class of algebras, a weaker alternative are embedding theorems. They provide a way to embed every algebra into a product of simpler algebras. For BL-algebras, there is the well-known Hájek's subdirect representation theorem ([10, Lemma 2.3.16]), which is a particular case of Birkhoff's subdirect representation theorem ([7]). It states that each BL-algebra is a subdirect product of totally ordered BL-algebras (BL-chains).

For subclasses of BL-algebras there are two important representation theorems. The first one ([8], see also [11]), is a representation theorem for finite BL-algebras in terms of finite MV-chains. The second one ([2]), is a representation theorem for BL-chains. It states that each BL-chain can be uniquely decomposed as an ordinal sum of totally ordered Wajsberg hoops. This result, together with the subdirect representation of BL-algebras, constitute the most important tools to analyze the variety of BL-algebras. Some of the results that are based on these theorems are amalgamations theorems ([13]), the study of canonical extensions of BL-algebras ([5]), the study of subvarieties of BL-algebras ([2,6]), among others. In [1] finite BL-algebras are represented as algebras of weighted subforests of a labeled forest. Trying to generalize the ordinal sum construction for GBL-algebras, a class of algebras broader than the class of BL-algebras, in [11] and [12] P. Jipsen and F. Montagna introduce the *poset product* construction (called poset sum in [11]). The poset product is also a generalization of the ordinal sum construction for a not necessarily totally ordered index set. Their result turns out to be an embedding theorem: they embed each BL-algebra into a poset indexed family of totally ordered MV-chains and product chains (see [6] or Theorem 4.3, where a detailed proof of this fact is offered).

But so far we lack a representation theorem for the whole class of BL-algebras. Our aim is to find such a theorem, though in this paper we partially succeed. What we present is a representation theorem for the class of BL-algebras whose spectrum (set of prime filters) satisfies two important properties: it is finite and independent (see Definition 3.5). Such a class includes the class of totally ordered BL-algebras with finite spectrum and the class of finite BL-algebras. Unlike the case of poset product in [12], which is stated in terms of MV-chains and product chains, our representation constitutes a genuine generalization of the ordinal sum construction, since we decompose each algebra in terms of totally ordered Wajsberg hoops.

The paper is organized as follows: In Section 2 we present all the background of BL-algebras and hoops necessary to understand the main results of the paper. After that, in Section 3 we introduce the notion of dependent triplets of filters and we analyze BL-algebras with finite and independent spectrum. We also define the necessary spectrum, which is the subset of the spectrum that will be needed for the representation and prove some related theorems, that can be of independent interest. In Section 4 and to make the paper self-contained, we detail the poset product construction and prove that each BL-algebra can be embedded into a poset product of a family of MV-chains and product chains. The reader should notice that this new embedding differs from the one presented in [6, Theorem 3.5.4], which simply relies on the subdirect representation theorem [10, Lemma 2.3.16]. Unlike that embedding, the one in the present paper (Theorem 4.3) takes as an index poset a subposet of the spectrum.

The novelty comes in Section 5, where given a BL-algebra $\mathbf{A}$ of finite and independent spectrum we define BL-functions, which are functions from the poset dual with respect to the necessary spectrum into a union of totally ordered Wajsberg hoops. After defining BL-operations on the functions, we prove that they form a BL-algebra. This BL-algebra is clearly stated in terms of totally ordered Wajsberg hoops. In the last section we prove the main theorem of the paper: each BL-algebra of finite and independent spectrum can be represented as an algebra of functions from a root system into a disjoint union of totally ordered Wajsberg hoops.

2. Preliminaries

A *hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \to, \top\rangle$ of type $\langle 2, 2, 0\rangle$, such that $\langle A, \cdot, \top\rangle$ is a commutative monoid and for all $x, y, z \in A$:

(1) $x \to x = \top$,
(2) $x \cdot (x \to y) = y \cdot (y \to x)$,
(3) $x \to (y \to z) = (x \cdot y) \to z$.

A hoop $\mathbf{A} = \langle A, \cdot, \top\rangle$ is a naturally ordered residuated commutative monoid, where the order is defined by $x \leq y$ iff $x \to y = \top$ and the residuation is

$$x \cdot y \leq z \text{ iff } x \leq y \to z.$$

The partial order on a hoop is a semilattice order, with $x \wedge y = x \cdot (x \to y)$. A *basic hoop* is a hoop that satisfies the equation:

$$(((x \to y) \to z) \cdot ((y \to x) \to z)) \to z = \top.$$

In every basic hoop $\mathbf{A}$ an operation $\vee$ can be defined by

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x),$$

thus $\mathbf{L}(\mathbf{A}) = \langle A, \wedge, \vee, \top \rangle$ is a lattice with greatest element $\top$. In addition, every basic hoop $\mathbf{A}$ satisfies the *prelinear* equation:

$$(x \to y) \vee (y \to x) = \top.$$

A *BL-algebra* is a bounded basic hoop, that is, it is an algebra $\mathbf{A} = \langle A, \cdot, \to, \bot, \top \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ such that $\langle A, \cdot, \to, \top \rangle$ is a basic hoop and $\bot$ is the least element of $\mathbf{L}(\mathbf{A})$.

The varieties of BL-algebras and basic hoops will be denoted by $\mathbb{BL}$ and $\mathbb{BH}$, respectively. It is known that both varieties are congruence distributive and congruence permutable. Some important subvarieties of $\mathbb{BL}$ that we shall need are the one of MV-algebras, that is, BL-algebras satisfying $x \to \bot \to \bot = x$, and the one of product algebras, that is, BL-algebras satisfying $(x \to \bot) \vee ((x \to (x \cdot y)) \to y) = \top$ (see [10]). We presented the definition of BL-algebras as basic hoops because we will be using the hoop structure of each BL-algebra throughout the paper.

Alternatively, an equivalent definition in terms of residuated lattices can be given (see [6] and [9]). Each BL-algebra has a structure of commutative integral and bounded residuated lattice ([10]). The latter definition will also be used in the course of the paper.

We recall the subdirect representation theorem for BL-algebras, which is crucial for our study.

Theorem 2.1 *([10]). Each BL-algebra is a subdirect product of totally ordered BL-algebras (BL-chains).*

Let $\langle I, \leq \rangle$ be a totally ordered set. For each $i \in I$ let $\mathbf{A}_i = \langle A_i, *_i, \to_i, \top \rangle$ be a hoop such that for every $i \neq j$, $A_i \cap A_j = \{\top\}$. Then we can define the ordinal sum as the hoop $\bigoplus_{i \in I} \mathbf{A}_i = \langle \cup_{i \in I} A_i, *, \to, \top \rangle$ where the operations $*, \to$ are given by:

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in A_i; \\ x & \text{if } x \in A_i \setminus \{\top\}, y \in A_j \text{ and } i < j; \\ y & \text{if } y \in A_i \setminus \{\top\}, x \in A_j \text{ and } i < j; \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j \text{ and } i < j; \\ x \to_i y & \text{if } x, y \in A_i; \\ y & \text{if } y \in A_i, x \in A_j \text{ and } i < j. \end{cases}$$

Definition 2.2. A totally ordered hoop is **irreducible** if it cannot be written as the ordinal sum of two non-trivial totally ordered hoops.

A *Wajsberg hoop* is a hoop that satisfies the equation:

$$(x \to y) \to y = (y \to x) \to x.$$

A basic hoop is *cancellative* if it satisfies the equation:

$$x \to (x \cdot y) = y.$$

Notice that cancellative hoops are Wajsberg. A *Wajsberg algebra* is a bounded Wajsberg hoop. It is well-known that Wajsberg algebras are term-equivalent to MV-algebras. The following result may be found in [2].

Theorem 2.3. *For a totally ordered hoop (BL-chain)* $\mathbf{A}$ *the following are equivalent:*

(1) $\mathbf{A}$ *is irreducible;*
(2) $\mathbf{A}$ *is a Wajsberg hoop.*

Example 2.4. We introduce here some Wajsberg hoops that we shall use later to build examples and counterexamples. Let

$$\mathbb{N}^{\partial} = \langle \mathbb{N}, +, \ominus, 0 \rangle ,$$

where $\mathbb{N} = \{0, 1, \ldots\}$ is the set of natural numbers and $x \ominus y = \max\{0, y - x\}$. Then $\mathbb{N}^{\partial}$ is a cancellative hoop which is unbounded.

The disconnected rotation of $\mathbb{N}^{\partial}$ is the structure

$$\mathbf{C} = \langle \{0, 1\} \times \mathbb{N}, *, \to, (0, 0), (1, 0) \rangle ,$$

where

$$(a, b) * (c, d) = \begin{cases} (1, b + d) & \text{if } a = 1 = c\,, \\ (0, b \ominus d) & \text{if } a = 1, c = 0\,, \\ (0, d \ominus b) & \text{if } a = 0, c = 1\,, \\ (0, 0) & \text{if } a = 0 = c\,, \end{cases}$$

$$(a, b) \to (c, d) = \begin{cases} (1, b \ominus d) & \text{if } a = 1 = c\,, \\ (0, b + d) & \text{if } a = 1, c = 0\,, \\ (1, 0) & \text{if } a = 0, c = 1\,, \\ (1, d \ominus b) & \text{if } a = 0 = c\,. \end{cases}$$

The algebra $\mathbf{C}$ is an MV-algebra, called Chang's MV-algebra.

For each integer $n > 1$ we let

$$\mathbf{L}_n = \left\langle \left\{ 0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1 \right\}, \max\{0, x + y - 1\}, \min\{1, 1 - x + y\}, 0, 1 \right\rangle .$$

The structure $\mathbf{L}_n$ is an MV-algebra, called the Łukasiewicz n-valued chain. Observe that if $n = 2$, $\mathbf{L}_2$ is the two elements Boolean chain.

The representation theorem for BL-chains states:

Theorem 2.5 *([2,4]). Each non-trivial BL-chain admits a unique decomposition into an ordinal sum of non-trivial totally ordered Wajsberg hoops.*

While bounded Wajsberg chains coincide with MV-chains, unbounded totally ordered Wajsberg hoops coincide with cancellative totally ordered hoops (see [3]). Another result that we will be using throughout the paper is that for each cancellative totally ordered hoop $\mathbf{W}$, it follows that $\mathbf{L}_2 \oplus \mathbf{W}$ is the product chain that arises by adding an extra bottom element to the Wajsberg hoop $\mathbf{W}$.

2.1. Implicative filters

An *implicative filter* of a BL-algebra (basic hoop) $\mathbf{A}$ is a subset $F \subseteq A$ satisfying the following conditions:

(1) $\top \in F$,
(2) If $x \in F$ and $x \to y \in F$, then $y \in F$.

An implicative filter is called *proper* provided $F \neq A$. Note that every implicative filter F of a BL-algebra $\mathbf{A}$ is the universe of a subhoop of $\mathbf{A}$. Implicative filters characterize congruences in BL-algebras. If F is an implicative filter of a BL-algebra $\mathbf{A}$, then the binary relation $\equiv_F$ on A defined by:

$$x \equiv_F y \quad \text{iff} \quad x \to y \in F \text{ and } y \to x \in F$$

is a congruence of $\mathbf{A}$ (see [10, Lemma 2.3.14]). Moreover, $F = \{x \in A \mid x \equiv_F \top\}$. Conversely, if $\equiv$ is a congruence relation on A, then the set $F = \{x \in A \mid x \equiv \top\}$ is an implicative filter, and $x \equiv y$ iff $x \to y \equiv \top$ and $y \to x \equiv \top$. Therefore, the correspondence

$$F \mapsto \equiv_F$$

is an order isomorphism from the set of implicative filters of $\mathbf{A}$ onto the set of congruences of $\mathbf{A}$, both ordered by inclusion.

Given a BL-algebra $\mathbf{A}$ and a filter F of $\mathbf{A}$, we will denote the quotient algebra $\mathbf{A}/\equiv_F$ by $\mathbf{A}/F$.

We also recall that a *prime filter* of a BL-algebra (basic hoop) $\mathbf{A}$ is a proper filter F of $\mathbf{A}$ such that for all $a, b \in A$, if $a \vee b \in F$, then either $a \in F$, or $b \in F$. It is readily seen that the quotient of a BL-algebra (basic hoop) modulo a prime filter is totally ordered. The set of prime filters of $\mathbf{A}$ is called *the spectrum of* $\mathbf{A}$ and it is denoted by $Spec(\mathbf{A})$. In the present paper we shall only consider BL-algebras $\mathbf{A}$ such that $Spec(\mathbf{A})$ is finite.

Abuse of notation. If $\mathbf{A}$ is a BL-algebra and F is a prime filter of $\mathbf{A}$, following (1) we identify the elements of the quotient $\mathbf{A}/F$ with the elements in the ordinal sum. Therefore we shall sometimes say that for each $b \in W_F$ there is $a \in A$ such that $a/F = b$, meaning that b is the image of a/F under the isomorphism.

Let us fix a BL-algebra $\mathbf{A}$ with finite spectrum. If $F \in Spec(\mathbf{A})$ then the quotient $\mathbf{A}/F$ can be uniquely decomposed into the ordinal sum of non-trivial totally ordered Wajsberg hoops, i.e.,

$$\mathbf{A}/F \cong \bigoplus_{i \in I} \mathbf{W}_i \tag{1}$$

for some totally ordered lower bounded set I. Since for each $j \in I$, if j is not the lower bound of I the hoop $\bigoplus_{i \in I; i \geq j} \mathbf{W}_i$ is a prime filter of $\mathbf{A}$, I has to be finite and with a greatest element $\top$. Under these conditions we define two parameters that will be crucial for what follows:

(1) The *Wajsberg hoop corresponding to* F is the last (*i.e.*, uppermost) nontrivial Wajsberg hoop in the decomposition of $\mathbf{A}/F$ as ordinal sum, and we denote it as $\mathbf{W}_F$.
(2) The *index* of F, denoted I_F, is the cardinality of the finite set I, i.e., the number of non-trivial Wajsberg hoops in the decomposition of $\mathbf{A}/F$ as ordinal sum.

Notice that if $F, G \in Spec(\mathbf{A})$ then $F \subseteq G$ implies $I_G \leq I_F$.

Example 2.6. Recall Chang's MV-algebra $\mathbf{C}$ from Example 2.4. Consider the direct product $(\mathbf{L}_2 \oplus \mathbf{L}_2) \times \mathbf{C}$, where $\mathbf{C}$ is Chang's MV-algebra, and let $\mathbf{A}$ be its subalgebra whose universe is given by

$$\{\langle a, b\rangle \mid a \neq \bot, b > b \to \bot\} \cup \{\langle \bot, b\rangle \mid b < b \to \bot\}.$$

Then $Spec(\mathbf{A}) = \{F_1, F_2, F_3\}$, where $F_1 = \{\langle a, b\rangle \mid a \neq \bot, b > b \to \bot\}$, $F_2 = \{\langle a, \top\rangle \mid a \neq \bot\}$ and $F_3 = \{\langle \top, b\rangle \mid b > b \to \bot\}$. Then $\mathbf{A}/F_1 \cong \mathbf{L}_2$, $\mathbf{A}/F_2 \cong \mathbf{C}$, and $\mathbf{A}/F_3 \cong \mathbf{L}_2 \oplus \mathbf{L}_2$. Whence, $\mathbf{W}_{F_1} \cong \mathbf{L}_2 \cong \mathbf{W}_{F_3}$, $\mathbf{W}_{F_2} \cong \mathbf{C}$, and $I_{F_1} = I_{F_2} = 1 < I_{F_3} = 2$. The lattice structure and the prime spectrum of $\mathbf{A}$ are sketched in Fig. 1(a).

Lemma 2.7. *Following the previous notation, for each $F \in Spec(\mathbf{A})$, $\mathbf{W}_F$ is either the hoop reduct of an MV-chain, or a cancellative totally ordered hoop.*

Finally we list some results that will be needed later:

Lemma 2.8 *([2, Proposition 3.2]). Let $\mathbf{W}_1, \ldots, \mathbf{W}_n$ be hoops and for each i let $\mathbb{H}(\mathbf{W}_i)$ be the set of homomorphic images of $\mathbf{W}_i$. The set of homomorphic images of $\mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_n$ is*

$$\mathbb{H}(\mathbf{W}_1) \cup \{\mathbf{W}_1 \oplus \mathbf{B} : \mathbf{B} \in \mathbb{H}(\mathbf{W}_2)\} \cup \ldots \cup \{\mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_{n-1} \oplus \mathbf{B} : \mathbf{B} \in \mathbb{H}(\mathbf{W}_n)\}.$$

Lemma 2.9. *Let $\mathbf{A}$ be a BL-algebra and F a proper filter of $\mathbf{A}$. The correspondence*

$$G \mapsto G/F$$

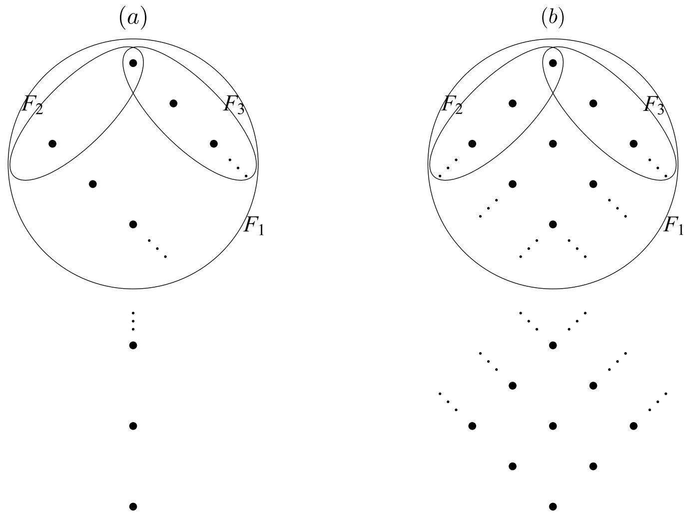

Fig. 1. Hasse diagrams and prime filters of the algebras from Example 2.6 (a) and Example 3.2 (b).

is a bijective order preserving map between the set of proper filters of $\mathbf{A}$ *that include* F *and the set of filters of* $\mathbf{A}/F$*, both sets ordered by inclusion. Moreover,*

$$Spec(\mathbf{A}/F) \cong \{P \in Spec(\mathbf{A}) : F \subseteq P\}.$$

Lemma 2.10. *Let* $\mathbf{A}$ *be a BL-algebra and* F *and* G *proper filters of* $\mathbf{A}$*, such that* $F \subseteq G$*. Then*

$$\mathbf{A}/G \cong \mathbf{A}/F/(G/F).$$

3. BL-algebras with finite independent spectrum

Let $\mathbf{A}$ be a BL-algebra with a finite spectrum $Spec(\mathbf{A})$. A triplet (F_1, F_2, F_3) of elements of $Spec(\mathbf{A})$ is said to be a *V-formation* whenever F_2 and F_3 are properly included in F_1, $F_2 \nsubseteq F_3$ and $F_3 \nsubseteq F_2$.

Definition 3.1. A triplet (F_1, F_2, F_3) of elements of $Spec(\mathbf{A})$ is said to be *dependent* if they form a V-formation and $I_{F_1} = I_{F_2}$ or $I_{F_1} = I_{F_3}$.

Example 3.2. Let $\mathbf{A}$ be the subalgebra of the direct product $(\mathbf{L}_2 \oplus \mathbf{L}_2) \times \mathbf{C}$ introduced in Example 2.6. Then (F_1, F_2, F_3) is a dependent triplet.

Consider now the MV-algebra $\mathbf{C} \times \mathbf{C}$ and let $\mathbf{A}$ be its subalgebra whose universe is given by

$$\{\langle a, b\rangle \mid a > a \to \bot, b > b \to \bot\} \cup \{\langle a, b\rangle \mid a < a \to \bot, b < b \to \bot\}.$$

Then $Spec(\mathbf{A}) = \{F_1, F_2, F_3\}$, where $F_1 = \{\langle a, b\rangle \mid a > a \to \bot, b > b \to \bot\}$, $F_2 = \{\langle a, \top\rangle \mid a > a \to \bot\}$ and $F_3 = \{\langle \top, b\rangle \mid b > b \to \bot\}$. Clearly, (F_1, F_2, F_3) is a V-formation. Moreover, $\mathbf{A}/F_1 \cong \mathbf{L}_2$ and $\mathbf{A}/F_2 \cong \mathbf{C} \cong \mathbf{A}/F_3$. Whence $\mathbf{W}_{F_1} \cong \mathbf{L}_2$, $\mathbf{W}_{F_2} \cong \mathbf{C} \cong \mathbf{W}_{F_3}$ and $I_{F_1} = I_{F_2} = I_{F_3} = 1$. Then (F_1, F_2, F_3) is a dependent triplet. The lattice structure and the prime spectrum of $\mathbf{A}$ are sketched in Fig. 1(b).

Lemma 3.3. *Let* $\mathbf{A}$ *be a BL-algebra with finite spectrum and no dependent triplets. Let* (G, H, F) *be a V-formation such that:*

- $\mathbf{A}/G \cong \mathbf{W}_G$ *is a Wajsberg hoop and*
- $\mathbf{A}/F \cong \mathbf{W}_G \oplus \mathbf{W}_F$.

Then $\mathbf{A}/H \cong \mathbf{W}_G \oplus \cdots \oplus \mathbf{W}_H$.

Proof. Since $I_G = 1$ and the triplet (G, H, F) is not dependent, both I_F and I_H are greater than 1. Thus

$$\mathbf{A}/H \cong \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_n \oplus \mathbf{W}_H$$

for some $n > 0$. Because of Lemma 2.10 we know that $\mathbf{A}/G$ is a homomorphic image of $\mathbf{A}/H$. Following Lemma 2.8, the Wajsberg hoop $\mathbf{W}_G$ is a homomorphic image of $\mathbf{W}_1$.

Let K be the prime filter of $\mathbf{A}/H$ given by $K = \mathbf{W}_2 \oplus \ldots \oplus \mathbf{W}_n \oplus \mathbf{W}_H$. By Lemma 2.9, there is a correspondence between $Spec(\mathbf{A}/H)$ and $\{P \in Spec(\mathbf{A}) : H \subseteq P\}$. Hence, there is H' in $Spec(\mathbf{A})$ with $H \subseteq H'$ such that $\mathbf{A}/H' = \mathbf{A}/H/(K/H) \cong \mathbf{W}_1$.

Assume by contradiction that $\mathbf{W}_G \neq \mathbf{W_1}$. Since $H \subseteq G$ we have that $\mathbf{W}_G$ is a homomorphic image of $\mathbf{W}_1$ and H' is included in G. Observe that F and H' are not comparable, because of the decomposition of $\mathbf{A}/F$. Then we have:

(1) G properly includes F and H',
(2) H' and F are incomparable,
(3) $I_G = I_{H'} < I_F$.

Thus the triplet (G, H', F) of prime filters is dependent. The contradiction implies that $\mathbf{W}_1 \cong \mathbf{W}_G$, thus

$$\mathbf{A}/H \cong \mathbf{W}_G \oplus \cdots \oplus \mathbf{W}_H. \quad \square$$

The following example shows that Lemma 3.3 fails in general if the condition on the absence of dependent triplets is dropped.

Example 3.4. Let $\mathbf{A}$ be the BL-algebra in Example 2.6 and (F_1, F_2, F_3) be its dependent triplet. Then $\mathbf{A}/F_1 \cong \mathbf{L}_2 \cong \mathbf{W}_{F_1}$ and $\mathbf{A}/F_3 \cong \mathbf{L}_2 \oplus \mathbf{L}_2 \cong \mathbf{W}_{F_1} \oplus \mathbf{W}_{F_3}$. Moreover, F_2 and F_3 are properly included in F_1, and form an uncomparable pair with respect to inclusion. Direct inspection shows that $\mathbf{A}/F_2 \cong \mathbf{C}$ is not isomorphic to any ordinal sum of the form $\mathbf{W}_{F_1} \oplus \cdots \oplus \mathbf{W}_{F_2}$.

Definition 3.5. A BL-algebra $\mathbf{A}$ with finite spectrum is said to have *independent spectrum* if it has no triplets of dependent prime filters.

We recall that a hoop is *simple* if its only congruences are the one containing all pairs and the identity. Equivalently, if its only proper homomorphic image is the singleton algebra.

Proposition 3.6. *Let $\mathbf{A}$ be a BL-algebra with finite spectrum. If $\mathbf{W}_F$ is a simple Wajsberg hoop for each $F \in Spec(\mathbf{A})$ then $\mathbf{A}$ has independent spectrum.*

Proof. We prove the contrapositive. Assume (F_1, F_2, F_3) is a dependent triplet of $\mathbf{A}$. Let $n = I_{F_1} = I_{F_2}$. Then $\mathbf{A}/F_2$ uniquely decomposes as an ordinal sum of irreducible Wajsberg hoops as $\mathbf{A}/F_2 \cong \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_{n-1} \oplus \mathbf{W}_n$ with $\mathbf{W}_n \cong \mathbf{W}_{F_2}$. By Lemma 2.10, $\mathbf{A}/F_1$ is a homomorphic image of $\mathbf{A}/F_2$. Then, by Lemma 2.8, for some $1 \leq m \leq n$ we have $\mathbf{A}/F_1 \cong \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_{m-1} \oplus \mathbf{W'}_m$ with $\mathbf{W'}_m \cong \mathbf{W}_{F_1}$, and $\mathbf{W'}_m$ being a homomorphic image of $\mathbf{W}_m$. Since $I_{F_1} = n$ and by uniqueness of the decomposition of $\mathbf{A}/F_1$ as ordinal sum of irreducible Wajsberg hoops, we have $m = n$, whence we conclude $\mathbf{W}_{F_1}$ is a homomorphic image of $\mathbf{W}_{F_2}$, that is, $\mathbf{W}_{F_2}$ is not simple. $\square$

Lemma 3.7. *Let $\mathbf{A}$ be an algebra with finite independent spectrum. Let $F_1 \in Spec(A)$ and F_2 be an implicative filter of $\mathbf{A}$ such that $F_1 \cap F_2 = \{\top\}$. For every $b \in W_{F_1}$ there is an $a \in A$ such that $a/F_1 = b$ and $a/F_2 = \top$.*

Proof. Let G be the filter generated by F_1 and F_2. Following [7, Theorem 7.5, page 52], F_1 and F_2 define a pair of factor congruences of G, thus G is isomorphic as a hoop to the direct product $G/F_1 \times G/F_2$ via the isomorphism given by

$$g \mapsto (g/F_1, g/F_2).$$

Recall that

$$\mathbf{A}/F_1 \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_n$$

with $\mathbf{W}_n = \mathbf{W}_{F_1}$. Then if $G = A$ the result of the theorem follows.

Otherwise, by Lemma 2.10 we know that $\mathbf{A}/G \cong \mathbf{A}/F_1/(G/F_1)$, thus $\mathbf{A}/G \in \mathbb{H}(\mathbf{A}/F_1)$. This implies that G is a prime filter. According to Lemma 2.8 we have

$$\mathbf{A}/G \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}'_k$$

where $\mathbf{W}'_k$ is a homomorphic image of $\mathbf{W}_k$ and $k \leq n$. Since the spectrum of $\mathbf{A}$ is independent, F_1 and F_2 are not comparable and G is generated by these two filters, necessarily k is strictly less than n. Thus we can identify G/F_1 with the hoop

$$G/F_1 \cong \mathbf{B}_k \oplus \mathbf{W}_{k+1} \oplus \ldots \oplus \mathbf{W}_n = \mathbf{W}_{F_1}$$

with $\mathbf{B}_k$ a (possibly trivial) filter of $\mathbf{W}_k$ such that $\mathbf{W}_k/\mathbf{B}_k \cong \mathbf{W}'_k$. Therefore given $b \in W_{F_1}$, there is $g \in G$ such that $(g/F_1, g/F_2) = (b, \top)$. □

Lemma 3.8. *Let* $\mathbf{A}$ *be a BL-algebra with independent spectrum and* H *a filter of* $\mathbf{A}$. *Then* $Spec(\mathbf{A}/H)$ *is independent.*

Proof. The proof of this result strongly relies on Lemma 2.9. Given a BL-algebra $\mathbf{A}$ and a filter H of $\mathbf{A}$ we are going to denote $\tilde{F}$ the filter of $\mathbf{A}/H$ corresponding to $F \in Spec(\mathbf{A})$.

Assume on the contrary that there is a V-formation $(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)$ in $Spec(\mathbf{A}/\mathbf{H})$ such that $I_{\tilde{F}_1} = I_{\tilde{F}_2}$.

Let F_1, F_2 and F_3 be the filters in $Spec(\mathbf{A})$ corresponding to $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ using Lemma 2.9, i.e., $\tilde{F}_1 \cong F_1/H$, $\tilde{F}_2 \cong F_2/H$ and $\tilde{F}_3 \cong F_3/H$. Therefore (F_1, F_2, F_3) is a V-formation.

Using Lemma 2.10 we have that $\mathbf{A}/F_1 \cong \mathbf{A}/H/(F_1/H)$ and $\mathbf{A}/F_2 \cong \mathbf{A}/H/(F_2/H)$. So,

$$\mathbf{A}/F_1 \cong \mathbf{A}/H/(F_1/H) \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_{I_{\tilde{F}_1}}$$
$$\mathbf{A}/F_2 \cong \mathbf{A}/H/(F_2/H) \cong \mathbf{W}'_1 \oplus \ldots \oplus \mathbf{W}'_{I_{\tilde{F}_2}}$$

where $I_{\tilde{F}_1} = I_{\tilde{F}_2}$, thus $\mathbf{A}/F_1$ and $\mathbf{A}/F_2$ have the same number of non-trivial totally ordered Wajsberg hoops in their decomposition, that is $I_{F_1} = I_{F_2}$.

So we have a dependent triplet (F_1, F_2, F_3) in $Spec(\mathbf{A})$, which contradicts the fact that $Spec(\mathbf{A})$ was independent. Therefore $Spec(\mathbf{A}/H)$ must be independent. □

3.1. Necessary spectrum

We shall work only with BL-algebras with finite independent spectrum.

From now on, when we say BL-algebra we mean an algebra of this type.

Definition 3.9. Given a BL-algebra $\mathbf{A}$ we call the *necessary spectrum of* $\mathbf{A}$, denoted by $Spec^*(\mathbf{A})$, the subset of $Spec(\mathbf{A})$ characterized by the following:

$$G \in Spec^*(\mathbf{A}) \text{ if and only if}$$
$$F \subsetneq G \text{ implies } \{x \in A : x/F \in W_F\} \subseteq G\,, \text{ for all } F \in Spec(\mathbf{A})\,.$$

Equivalently, $G \in Spec^*(\mathbf{A})$ iff for no $F \in Spec(\mathbf{A})$ properly contained in G we have $I_F = I_G$ (which in turns would imply that W_F is not simple and W_G is one of its homomorphic images).

Example 3.10. For each $n > 1$ let $\mathbf{L}_n$ denote the n-element Łukasiewicz chain. Let $\mathbf{A}$ be an ordinal sum of the form $\bigoplus_{i=1}^{k} \mathbf{L}_{j_i}$ where $j_i \geq 1$ for all $i = 1, 2, \ldots, k$. Then $Spec^*(\mathbf{A}) = Spec(\mathbf{A})$.

Consider now Chang's MV-algebra $\mathbf{C}$. It is immediate to check that $Spec^*(\mathbf{C}) \neq Spec(\mathbf{C})$, as the prime filter $G = \{b \mid b > b \to \bot\}$ of $\mathbf{C}$ is such that it contains properly the prime filter $F = \{\top\}$, but $\{x \in C \mid x/F \in W_F\} = C \nsubseteq G$. Analogously, the algebra $\mathbf{A}$ of Example 2.6 has a prime filter, namely F_1, which does not belong to its necessary spectrum, as it contains properly F_2 and the set $\{x \in A \mid x/F_2 \in W_{F_2} \cong \mathbf{C}\}$ is clearly not contained in F_1, as it bijects onto the universe of $\mathbf{C}$.

Lemma 3.11. *Let* $\mathbf{A}$ *be a BL-algebra and let* $F \in Spec^*(\mathbf{A})$ *be maximal. Then* $I_F = 1$*, i.e.,* $\mathbf{A}/F$ *is the MV-chain* $\mathbf{W}_F$*.*

Proof. Assume on the contrary that $I_F > 1$ and $\mathbf{A}/F \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_F$, with each $\mathbf{W}_i$ a non-trivial Wajsberg hoop. Then there is a filter G of $\mathbf{A}$ such that G properly includes F and $\mathbf{A}/G \cong \mathbf{W}_1$. Clearly G is a prime filter. From the maximality of F we have that $G \notin Spec^*(\mathbf{A})$. Let $H \in Spec(\mathbf{A})$ be such that $H \subseteq G$. The possibilities are the following:

(1) F and H are comparable. Either if $F \subseteq H$ or $H \subseteq F$, if we assume that H is properly contained in G we have that $\mathbf{A}/H \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_H$. Then $\{x \in A : x/H \in W_H\} \subseteq G$.
(2) F and H are not comparable. In this case, since the spectrum of $\mathbf{A}$ is independent $I_H > 1$. Following Lemma 3.3 we can assert that $\mathbf{A}/H \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_H$. Thus $\{x \in A : x/H \in W_H\} \subseteq G$.

This contradicts the fact that $G \notin Spec^*(\mathbf{A})$, and therefore I_F must be 1. □

Lemma 3.12. *Let* $\mathbf{A}$ *be a BL-algebra and let* $Spec^*(\mathbf{A})$ *be its necessary spectrum. For each* $a, b \in A$ *such that* $a \neq b$*, there is* $G \in Spec^*(\mathbf{A})$ *such that* $a/G \neq b/G$*.*

Proof. From [10, Lemma 2.3.15], if $a \neq b$ are elements of A we know that there is a prime filter G of $\mathbf{A}$ such that $a/G \neq b/G$. If $G \in Spec^*(\mathbf{A})$ we are done. Otherwise there is a prime filter $F \subsetneq G$ such that $\{x \in A : x/F \in W_F\} \nsubseteq G$. Observe that $a/F \neq b/F$ (because otherwise $(a \to b)/F = \top/F$ implies $a \to b \in F \subseteq G$ and $(b \to a)/F = \top/F$ implies $b \to a \in F \subseteq G$ and we get $a/G = b/G$). Thus if $F \in Spec^*(\mathbf{A})$ we are done. If this is not the case the finiteness of the spectrum of $\mathbf{A}$ implies the existence of $H \in Spec^*(\mathbf{A})$ such that H is included in G and $a/H \neq b/H$ as desired. □

Corollary 3.13. $\bigcap_{F \in Spec^*(\mathbf{A})} F = \{\top\}$.

We present two results that will be useful in subsequent chapters:

Lemma 3.14. *Let* $F \in Spec^*(\mathbf{A})$ *be such that* $I_F = n$ *and let* $\mathcal{S} = \{G \in Spec^*(\mathbf{A}) : F \subseteq G\}$*. Then* $\mathcal{S}$ *is a totally ordered set of cardinality* n *and if* $F = F_n \subsetneq F_{n-1} \subsetneq \ldots \subsetneq F_1$ *is an ordered list of the elements of* $\mathcal{S}$ *we have*

$$\mathbf{A}/F \cong \mathbf{W}_{F_1} \oplus \mathbf{W}_{F_2} \oplus \ldots \oplus \mathbf{W}_{F_n}.$$

Proof. First we observe that if $I_F = 1$ then the definition of necessary spectrum implies the maximality of F in $Spec^*(\mathbf{A})$, and Lemma 3.11 does the rest. For the nontrivial case, we set

$$\mathbf{A}/F \cong \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \ldots \oplus \mathbf{W}_n$$

with $\mathbf{W}_n = \mathbf{W}_F$. For each $k < n$ the set $\mathbf{W}_{k+1} \oplus \mathbf{W}_{k+2} \oplus \ldots \oplus \mathbf{W}_n$ is a filter of $\mathbf{A}/F$ that according to Lemma 2.9 corresponds to a filter F_k of $\mathbf{A}$. Then we have a list of prime filters $F = F_n \subsetneq F_{n-1} \subsetneq \ldots \subsetneq F_1$ such that for each k,

$$\mathbf{A}/F_k \cong \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \ldots \oplus \mathbf{W}_k.$$

Since for each k, $\mathbf{W}_k = \mathbf{W}_{F_k}$, it only remains to see that for each $1 \le k \le n$ the filter F_k is in the necessary spectrum and that these are the only filters greater than F in $Spec^*(\mathbf{A})$.

Consider $F \subseteq T \subseteq F_k$ for some $k \le n$. From Lemma 2.8 and Lemma 2.10 the only possibility is that

$$\mathbf{A}/T \cong \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \ldots \oplus \mathbf{W}_k \oplus \ldots \oplus \mathbf{W}_T$$

with $\mathbf{W}_T \in \mathbb{H}(\mathbf{W}_t)$ for some $t > k$. Then $\{x \in A : x/T \in W_T\} \subseteq F_k$ and $F_k \in Spec^*(\mathbf{A})$.

Lastly, assume that H is a prime filter such that $F = F_n \subsetneq H$ and $H \neq F_k$ for all k. From Lemma 2.8 and Lemma 2.10 we know that there is $k \le n$ such that

$$\mathbf{A}/H \cong \mathbf{W}_1 \oplus \ldots \mathbf{W}_{k-1} \oplus \mathbf{W}_H$$

with $\mathbf{W}_H \in \mathbb{H}(\mathbf{W}_k)$ and $\mathbf{W}_H \neq \mathbf{W}_k$. Then $F_k \subseteq H$ and there is a proper prime filter T of $\mathbf{W}_k$ such that $H/F \cong T \oplus \mathbf{W}_{k+1} \oplus \ldots \oplus \mathbf{W}_n$. Let $x \in A$ be such that $x/F \in W_k \setminus T$. Then $x \notin H$ and $H \notin Spec^*(\mathbf{A})$. □

Lemma 3.15. *Let* $\mathbf{A}$ *be a BL-algebra and* H *a filter of* $\mathbf{A}$*. Then*

$$Spec^*(\mathbf{A}/H) \cong \{P/H : P \in Spec^*(\mathbf{A}) \text{ and } H \subseteq P\}.$$

Proof. Let $G \in Spec^*(\mathbf{A})$ be such that $H \subseteq G$. We verify that $\tilde{G} = G/H$ is in $Spec^*(\mathbf{A}/H)$.

Assume that $\tilde{F} \in Spec(\mathbf{A}/H)$ is such that $\tilde{F} \subseteq \tilde{G}$. By Lemma 2.9 we know that $\tilde{F} = F/H$, where F is a prime filter such that $F \subseteq G$.

If $\tilde{x} \in \mathbf{A}/H$ is such that $\tilde{x}/\tilde{F} \in W_{\tilde{F}}$, then $\tilde{x}/\tilde{F} \in (\mathbf{A}/H)/\tilde{F} = (\mathbf{A}/H)/(F/H)$. Since $\mathbf{A}/F \cong (\mathbf{A}/H)/(F/H)$, there is $x \in A$ such that $x/F \in \mathbf{A}/F$ corresponds via the isomorphism to $\tilde{x}/\tilde{F} \in (\mathbf{A}/H)/\tilde{F}$, i.e., $(x/H)/(F/H) = \tilde{x}/\tilde{F}$. But as $\tilde{x}/\tilde{F} \in W_{\tilde{F}}$, x/F must be in $\mathbf{W}_F$, and then $x \in G$ (using that G is necessary in $Spec(\mathbf{A})$).

Therefore $\tilde{x} = x/H \in G/H = \tilde{G}$, and $\tilde{G} \in Spec^*(\mathbf{A}/H)$. □

4. Embedding into the poset product

Definition 4.1. Let $P = \langle P, \leq \rangle$ be a poset and let $\{\mathbf{B}_p \mid p \in P\}$ be a collection of commutative, integral and bounded residuated lattices. Up to isomorphism we can (and we will) assume that all $\mathbf{B}_p$ share the same neutral element $\top$ and the same minimum element $\bot$, only. The *poset product* $\bigotimes_{p\in P} \mathbf{B}_p$ is the algebra $\mathbf{B}$ defined as follows:

(1) The domain of $\mathbf{B}$ is the set of all maps $h \in \{k \in (\bigcup_{p\in P} B_p)^P \mid k(p) \in B_p\} \cong \prod_{p\in P} B_p$ such that for all $p \in P$ if $h(p) \neq \bot$, then for all $q < p$, $h(q) = \top$.
(2) The monoid operation and the lattice operations are defined pointwise.
(3) The residual is as follows:

$$(h \to g)(p) = \begin{cases} h(p) \to_p g(p) & \text{if for all } q < p \quad h(q) \leq_q g(q), \\ \bot & \text{otherwise,} \end{cases}$$

where the subscript p denotes realization of operations and of order in $\mathbf{B}_p$.

The following results, that can be found in [6] are slight modifications of the ones in [12].

Theorem 4.2.

(1) *Suppose that* P *is a forest and that for all* $p \in P$, $\mathbf{B}_p$ *is a BL-chain. Then* $\bigotimes_{p\in P} \mathbf{B}_p$ *is a BL-algebra.*
(2) *Every BL-algebra can be embedded into a poset product of a family of MV-chains and product chains.*

Let $\mathbf{A}$ be a BL-algebra of finite and independent spectrum and let $Spec^*(\mathbf{A})$ be the necessary spectrum of $\mathbf{A}$. Since $Spec^*(\mathbf{A})$ is ordered by inclusion, we consider the poset $P = (Spec^*(\mathbf{A}))^\partial$ the dual of $Spec^*(\mathbf{A})$.

The dual spectrum of any BL-algebra is a forest, that is, the downward closure of any of its elements forms a totally ordered subposet. Since the necessary spectrum $Spec^*(\mathbf{A})$ is a subposet of $Spec(\mathbf{A})$ it is then clear that $(Spec^*(\mathbf{A}))^\partial$ enjoys the property that the downward closure of each one of its elements is a finite subchain. Whence, $P = (Spec^*(\mathbf{A}))^\partial$ is a finite forest.

For each $F \in Spec^*(\mathbf{A})$ we have the corresponding element $p_F \in P$ (given by the identity mapping from $Spec^*(\mathbf{A})$ anti-isomorphically onto P). Analogously, the identity associates with each $p \in P$ a filter $F_p \in Spec^*(\mathbf{A})$. Thus, if $p = p_F$, we denote interchangeably by $\mathbf{W}_F$ or $\mathbf{W}_p$ the Wajsberg hoop corresponding to F, and for each $a \in A$ we write a/p or a/F to indicate the equivalence class of a modulo the congruence determined by p.

For each $p \in P$ we define $\mathbf{W}'_p$ to be the bounded hoop given by

$$\mathbf{W}'_p = \begin{cases} \mathbf{W}_p, & \text{if } \mathbf{W}_p \text{ is lower bounded;} \\ \mathbf{L}_2 \oplus \mathbf{W}_p, & \text{otherwise,} \end{cases}$$

where $\mathbf{L}_2$ is the two-elements Boolean algebra. It is easy to see that in case $\mathbf{W}_p$ is cancellative (unbounded), then $\mathbf{W}'_p$ is the product chain that arises by adding an extra bottom $\bot_p$ to $\mathbf{W}_p$. Under these conditions and notation we have:

Theorem 4.3. $\mathbf{A}$ *can be embedded into the poset product* $\bigotimes_{p\in P}\mathbf{W}'_p$.

Proof. We adapt some of the ideas of the proof of [12, Theorem 3.3]. We define $\psi : \mathbf{A} \to \bigotimes_{p\in P}\mathbf{W}'_p$ by

$$\psi(a)_p = \begin{cases} a/p & \text{if } a/p \in W_p; \\ \bot_p & \text{otherwise,} \end{cases}$$

for each $p \in P$. We first recall that for each $p \in P$, the quotient $\mathbf{A}/F_p$ is a BL-chain. Either $\mathbf{A}/F_p \cong \mathbf{W}_p$ or there are totally ordered nontrivial Wajsberg hoops $\mathbf{W}_1, \ldots \mathbf{W}_n$ such that

$$\mathbf{A}/F_p \cong \mathbf{W}_1 \oplus \ldots \mathbf{W}_n \oplus \mathbf{W}_p.$$

Thus either $a/p \in W_p$ or there is a prime filter $F_q \supset F_p$ (i.e., $q < p$) and $a/q \in W_q$.

We need to check that ψ goes into the poset product and that ψ is an injective morphism.

- $\psi(\mathbf{A}) \subseteq \bigotimes_{p\in P}\mathbf{W}'_p$. Indeed, assume that $\psi(a)_p \neq \bot_p$ and let $q < p$. Then F_p is properly included in F_q. From the definition of necessary spectrum we have that $\{x \in A : x/p \in W_p\} \subseteq F_q$, thus $a \in F_q$ yields $\psi(a)_q = \top$.
- ψ is a morphism of BL-algebras. It is clear that $\psi(\bot)$ is the bottom element of $\bigotimes_{p\in P}\mathbf{W}'_p$ and that $\psi(\top)$ is the top of $\bigotimes_{p\in P}\mathbf{W}'_p$. We leave to the reader the proof of the preservation of the lattice operations and the monoidal operation, which are easy to corroborate, and we embark on the proof that $\psi(a \to b) = \psi(a) \to \psi(b)$, taking into consideration that $\to$ is not defined coordinatewise in the poset product. Thus for each $p \in P$ we shall see that $\psi(a \to b)_p$ equals the p coordinate of $\psi(a) \to \psi(b)$. We split the proof into four cases:
  Case 1: a/p and b/p are in W_p. In this case, using again the definition of necessary spectrum we obtain that for each $q < p$, $a/q = b/q = \top$, thus $\psi(a)_q = \psi(b)_q$. From the definition of the implication in the poset product we get that $(\psi(a) \to \psi(b))_p = \psi(a)_p \to \psi(b)_p = a/p \to b/p$. On the other hand, since $\mathbf{W}_p$ is closed under $\to$ we have that $\psi(a \to b)_p = (a \to b)/p = a/p \to b/p$ as desired.
  Case 2: $a/p \notin W_p$ and $b/p \in W_p$. Therefore $a/p < b/p$ and $\psi(a \to b)_p = \top$. Observe that for each $q < p$, $\psi(b)_q = \top$, once more the definition of the implication in the poset product yields $(\psi(a) \to \psi(b))_p = \psi(a)_p \to \psi(b)_p = \bot_p \to b/p = \top$.
  Case 3: a/p and b/p are not in W_p. This possibility also splits into two cases. If $a/p \leq b/p$ then $(a \to b)/p = \top$ and $\psi(a \to b)_p = \top$. Observe that in this case, if $q < p$ then $a/q \leq b/q$, thus $\psi(a)_q \leq \psi(b)_q$. This implies that $(\psi(a) \to \psi(b))_p = \psi(a)_p \to \psi(b)_p = \bot_p \to \bot_p = \top$. The other case would be $a/p > b/p$. Since a/p and b/p are not in W_p the definition of ordinal sum of Wajsberg hoops implies that $a \to b/p$ is not in $\mathbf{W}_p$. Hence $\psi(a \to b)_p = \bot_p$. Also observe that there is $q < p$ such that $a/q > b/q$. The definition of the implication in the poset product implies that $(\psi(a) \to \psi(b))_p = \bot_p$.
  Case 4: $a/p \in W_p$ and $b/p \notin W_p$. In this case $b/p < a/p$ and clearly $a \to b \notin W_p$. Therefore $\psi(a \to b)_p = \bot_p$. Observe that for all $q < p$, $\psi(a)_q = \top$ and there is $q < p$ such that $\psi(b)_q < \top$. Then $(\psi(a) \to \psi(b))_p = \bot_p$ and we are done.
- We now check that ψ is injective. Because of Lemma 3.12 for each $a, b \in A$ with $a \neq b$ there is a prime filter $G \in Spec^*(\mathbf{A})$ such that $a/G \neq b/G$. Let $I_G = n$ and let $G = G_n \subsetneq \ldots \subsetneq G_1$ be the finite sequence of prime filters in $Spec^*(\mathbf{A})$ given by Lemma 3.14 such that

  $$\mathbf{A}/G \cong \mathbf{W}_{G_1} \oplus \ldots \oplus \mathbf{W}_{G_n}.$$

  Without loss of generality assume that $a/G < b/G$, and let $1 \leq k \leq n$ be such that $a/G \in W_{G_k} \setminus \{\top\}$. If $b/G \in W_{G_k}$ then it is immediate that $a/G_k \neq b/G_k$ and then $\psi(a)_{G_k} = a/G_k \neq b/G_k = \psi(b)_{G_k}$. Otherwise $b/G_k = \top/G_k$ and we also obtain $\psi(a)_{G_k} = a/G_k \neq b/G_k = \psi(b)_{G_k}$. □

Example 4.4. In general, the embedding $\psi : \mathbf{A} \to \bigotimes_{p\in P}\mathbf{W}'_p$ is not onto. As an instance of this fact consider the product chain $\mathbf{A} := \mathbf{L}_2 \oplus \mathbb{N}^\partial$. Now, $Spec(\mathbf{A})$ consists of two distinct prime filters, $F_1 = \mathbb{N}^\partial$, and F_2 being the singleton containing just the top element of $\mathbf{A}$. Moreover, $\mathbf{A}/F_1 \cong \mathbf{L}_2$, $\mathbf{W}_{F_1} \cong \mathbf{L}_2$ and $I_{F_1} = 1$, and $\mathbf{A}/F_2 \cong \mathbf{L}_2 \oplus \mathbb{N}^\partial$, $\mathbf{W}_{F_2} \cong \mathbb{N}^\partial$ and $I_{F_2} = 2$. Then $Spec^*(\mathbf{A}) \cong Spec(\mathbf{A})$. Since $\mathbf{W}_{F_2}$ is not bounded, we have $\mathbf{W}'_{F_2} \cong \mathbf{L}_2 \oplus \mathbb{N}^\partial$, while $\mathbf{W}'_{F_1} \cong \mathbf{W}_{F_1} \cong \mathbf{L}_2 \cong \{\bot, \top\}$. A direct computation shows that the universe of $\mathbf{W}'_{F_1} \otimes \mathbf{W}'_{F_2}$ is $\{(\bot, \bot), (\top, \bot)\} \cup \{(\top, n) \mid n \in \mathbb{N}^\partial\}$.

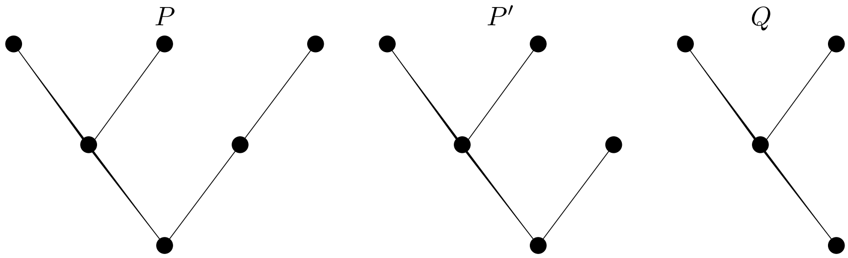

Fig. 2. Given a poset P, P' can be the domain of a BL-function while Q cannot.

Now, for each $n \in \mathbb{N}^\partial$, we have $\psi(n) = (\top, n)$, and $\psi(\bot) = (\bot, \bot)$. It follows that the element $(\top, \bot) \in \mathbf{W}'_{F_1} \otimes \mathbf{W}'_{F_2}$ has no counterimage under ψ.

5. BL-functions

Let $\mathbf{P}$ be a finite forest. We say that $p \in P$ is a *leaf* if p is maximal in P. We say that two elements $p, q \in P$ are *mildly incomparable* and we write $p <> q$ if they are incomparable in the order, i.e., $p \not\leq q$ and $q \not\leq p$ but there is $k \in P$ such that $k \leq p$ and $k \leq q$.

Let $\mathcal{W}$ denote the class of all nontrivial totally ordered Wajsberg hoops. A *BL-pair* is a pair (P, w), with P a finite forest and $w : P \to \mathcal{W}$ a function such that for each minimal element $p \in P$ then $w(p)$ is bounded. We say that a function f is a *BL-function* defined by the BL-pair (P, w) if $f : P' \to \bigcup_{p\in P} w(p)$ is such that $P' \subseteq P$ and it satisfies:

(1) P' is downwards closed and for each minimal element $q \in P$ we have $q \in P'$.
(2) For every $i \in P'$, if $j \in P$ is such that $i <> j$, then there is $k \in P'$ such that $k \leq j$ and $k <> i$.
(3) $f(p) \in w(p)$ for each $p \in P'$.
(4) If $p \in P'$ is not a leaf of P' (there is $q \in P'$ such that $q > p$), then $f(p) = \top$.
(5) If $P' \neq P$, p is a leaf of P' and p is not a leaf of P, then $f(p) \in w(p) \setminus \{\top\}$.

We denote by $R(P, w)$ the set of all BL-functions defined by the BL-pair (P, w) and for any $f \in R(P, w)$ we let P_f be the domain of f.

To understand condition (2) in the definition of the domain of a BL-function, observe that if P is a poset as in Fig. 2, then the poset P' of the figure can be the domain of a BL-function, while Q cannot, because it does not satisfy the condition. Intuitively speaking, if P' is the domain of a BL-function it cannot be the case that $k, p \in P'$, $q \in P \setminus P'$ with $k \leq p$, $k \leq q$ and for all $j \in P$ if $k \leq j < q$ then $j = k$. Then condition (2) is equivalent to requiring that each maximal (that is, not extendable) antichain in P' is a maximal antichain in P, too.

Given $f, g \in R(P, w)$ we define:

(1) $P_{f*g} = P_f \cap P_g$
(2) $P_{f\wedge g} = P_f \cap P_g$
(3) $P_{f\vee g} = P_f \cup P_g$

Observe that all these sets are downwards subsets of P and they contain the minimal elements of P. Therefore we define the structure $\langle R(P, w), *_R, \wedge_R, \vee_R, \bot_R, \top_R\rangle$ to be the algebra of type $\langle 2, 2, 2, 0, 0\rangle$ where the constants $\bot_R$ and $\top_R$ are the BL-functions defined as follows

$$P_{\top_R} = P \text{ and } \top_R(p) = \top \text{ for each } p \in P$$
$$P_{\bot_R} = \{p \in P : p \text{ is minimal in } P\} \text{ and } \bot_R(p) = \bot \text{ for each } p \in P_{\bot_R}$$

and the operations $*_R, \wedge_R, \vee_R$ are given, for all $p \in P_f \cap P_g$, by:

$$(f *_R g)(p) = f(p) * g(p) \quad (f \wedge_R g)(p) = f(p) \wedge g(p)$$

and

$$(f \vee_R g)(p) = \begin{cases} f(p) \vee g(p), & \text{if } p \in P_f \cap P_g; \\ f(p), & \text{if } p \in P_f \setminus P_g; \\ g(p), & \text{if } p \in P_g \setminus P_f. \end{cases}$$

Remarks:

- $\bot_R$ is well defined because if p is minimal in P, $\bot \in w(p)$.
- Since w is a leaf of $P_f \cap P_g$ if and only if w is a leaf of P_f or w is a leaf of P_g or both, it is easy to see that $f *_R g \in R(P, w)$ and $f \wedge_R g \in R(P, w)$.
- It is also easy to check that $f \vee_R g \in R(P, w)$.

The following are consequences of the definitions of the operations:

Lemma 5.1. *For each BL-pair (P, w) the algebra $\langle R(P, w), *_R, \wedge_R, \vee_R, \bot_R, \top_R\rangle$ satisfies:*

(1) $\langle R(P, w), *_R, \top_R\rangle$ *is a commutative monoid.*
(2) $\langle R(P, w), \wedge_R, \vee_R, \bot_R, \top_R\rangle$ *is a bounded lattice.*

Remark 5.2. Let $\preceq$ be the order of $R(P, w)$ given by the lattice structure. Observe that

$$f \preceq g \text{ if and only if } P_f \subseteq P_g \text{ and } f(p) \leq g(p) \text{ for each leaf } p \in P_f.$$

We will dedicate the rest of the section to define the implication of $R(P, w)$. Unfortunately, unlike the cases of $*_R$, $\wedge_R$ and $\vee_R$ the domain of $f \to g$ will not only depend on the domains of f and g.

5.1. Implication of BL-functions

For simplicity we consider BL-pairs (P, w) with P a tree, i.e. there is a minimum element $0 \in P$. Then the definition of the implication naturally extends to a forest, that is, a disjoint union of trees. As usual, for a subset S of a poset P the sets $S\uparrow$ and $S\downarrow$ are defined as

$$S\uparrow = \{y \in P : \exists x \in S \text{ such that } x \leq y\}$$

and

$$S\downarrow = \{y \in P : \exists x \in S \text{ such that } y \leq x\}.$$

Let $f, g \in R(P, w)$. We consider the following subsets of P_g:

(1) $H_g = \{p \in P_g : p \text{ is a leaf of } P_g\} = \{p \in P_g : \forall q \in P_g \text{ if } q \geq p \text{ then } q = p\}$.
(2) $H_g^+ = \{p \in H_g : p \notin P_f \text{ or } f(p) \leq g(p)\}$.
(3) $H_g^- = \{p \in H_g : f(p) > g(p)\} = H_g \setminus H_g^+$.

Then

$$P_{f\to g} = \begin{cases} P & \text{if } f \preceq g; \\ (H_g^+\uparrow \cup H_g^-)\downarrow & \text{otherwise.} \end{cases}$$

Remark 5.3. We list some observations that will be needed later.

(1) If $f \not\preceq g$ then $H_g^- \neq \emptyset$.
(2) Since P is a tree, $H_g^+\uparrow$ and H_g^- are disjoint subsets.
(3) $(H_g^+\uparrow \cup H_g^-)\downarrow = (H_g^+\uparrow)\downarrow \cup (H_g^-)\downarrow$.
(4) $(H_g^+\uparrow)\downarrow \cup (((H_g^-)\downarrow)\uparrow) = P$.

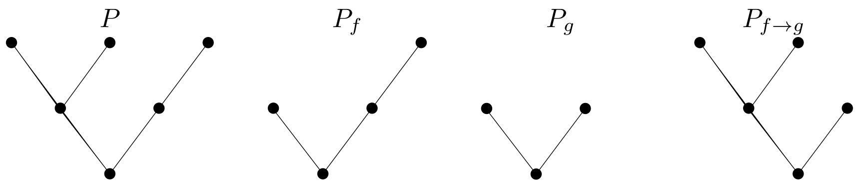

Fig. 3. Example of the domain of the implication of two BL-functions f and g with respect to a tree P.

Example 5.4. In Fig. 3 we present an example of the domain of an implication of two BL-functions. Assume that $\mathbf{W}_p = \mathbf{L}_3 = \{0, \frac{1}{2}, 1\}$ for all $p \in P$. Let f, g have their domains P_f and P_g as in the figure. Assume that both f and g take the value $\frac{1}{2}$ in every leaf of their domain. Then the domain of $f \to g$ is as given in Fig. 3.

Now we define $f \to_R g$ as

$$(f \to_R g)(p) = \begin{cases} f(p) \to g(p) & \text{if } p \in P_f \cap P_g; \\ \top & \text{otherwise.} \end{cases}$$

Remark 5.5. Observe that for every $p \in (H_g^+\uparrow)\downarrow$, we have $(f \to_R g)(p) = \top$. Also if $f \preceq g$ then $H_g^+ = H_g$ therefore $P_{f\to g} = P$ and $f \to_R g = \top_R$.

Lemma 5.6. *Let (P, w) be a BL-pair where P is a tree. The implication $f \to_R g$ of two BL-functions $f, g \in R(P, w)$ is a BL-function defined on the tree $P_{f\to g}$.*

Proof. The statement of the lemma is true in case $f \preceq g$. Otherwise, it is clear that conditions (1), (2) and (3) in the definition of BL-function hold. To check conditions (4) and (5) recall that H_g^- and $H_g^+\uparrow$ are disjoint subsets. Therefore p is a leaf of $P_{f\to g}$ if and only if $p \in H_g^-$ or p is a leaf of $H_g^+\uparrow$, in this last case it is also a leaf of P. Observe also that:

$$\forall p \in H_g^+\uparrow \text{ we have } \; f \to_R g(p) = \top,$$
$$\forall p \in H_g^- \;\text{ we have } f \to_R g(p) \neq \top$$

and any $p \in P_{f\to g} \setminus (H_g^+\uparrow \cup H_g^-)$ was already an element of P_g which was not a leaf. Therefore

$$\forall p \in P_{f\to g} \setminus (H_g^+\uparrow \cup H_g^-) \;\; f \to_R g(p) = \top.$$

After these observations we can conclude that:

(1) if p is a leaf of $P_{f\to g}$ which is not a leaf of P, then $p \in H_g^-$ and $f \to_R g(p) \neq \top$, which is condition 5.
(2) if p is not a leaf of $P_{f\to g}$, then either p is less or equal to a leaf q of H_g^-, in which case $g(p) = \top$ and thus $f \to_R g(p) = \top$ or p is less or equal to some element in $H_g^+\uparrow$ and this also implies that $f \to_R g(p) = \top$. This is condition 4. □

Lemma 5.7. *The operations $(*_R, \to_R)$ form a residuated pair in $R(P, w)$, i.e., for every $f, g, h \in R(P, w)$, we have*

$$h *_R f \preceq g \quad \textit{if and only if} \quad h \preceq f \to_R g$$

Proof. Assume first that $h *_R f \preceq g$. Recall that this means that $P_h \cap P_f \subseteq P_g$ and that for every leaf p of $P_h \cap P_f$ then $(h *_R f)(p) \leq g(p)$. We have to check that $h \preceq f \to_R g$, which amounts to verify that $P_h \subseteq P_{f\to g}$ and for each leaf p of P_h, $h(p) \leq (f \to_R g)(p)$.

- $P_h \subseteq P_{f\to g}$. Let $p \in P_h$. By way of contradiction, assume $p \notin P_{f\to g}$. From the definition of $P_{f\to g}$, since P is a tree, an element is not in $P_{f\to g}$ only if it is greater than a leaf of H_g^-. This means that there is $t \in P_f \cap P_g$ such that $t < p$ and $f(t) > g(t)$. Since $p > t$ we have that $t \in P_h$ and t is not a leaf, therefore $h(t) = \top$. Then $(h *_R f)(t) = h(t) * f(t) = f(t) > g(t)$ contradicting our hypothesis.

- Assume that $p \in P_h$ is a leaf. If $p \in P_f$ then $p \in P_h \cap P_f \subseteq P_g$ and clearly p is a leaf of $P_h \cap P_f$. Thus $h(p) * f(p) = (h *_R f)(p) \leq g(p)$ implies that $(f \to_R g)(p) = f(p) \to g(p) \geq h(p)$. If $p \notin P_f$, since $p \in P_{f \to g}$ we get that $f \to_R g(p) = \top \geq h(p)$.

Now assume that $h \preceq f \to_R g$. Thus $P_h \subseteq P_{f \to g}$ and for each leaf p of P_h, $h(p) \leq (f \to_R g)(p)$. We need to see that $P_h \cap P_f \subseteq P_g$ and that for every leaf p of $P_h \cap P_f$ then $h *_R f(p) \leq g(p)$.

- $P_h \cap P_f \subseteq P_g$. Let $p \in P_h \cap P_f$, and from our hypothesis we also know that $p \in P_{f \to g}$. From the definition of $P_{f \to g}$ one of the following cases occurs. Either $p \leq t$ with t a leaf of P_g in H_g^-, in which case $p \in P_g$, and we are finished, or there is $t \in (H_g^+)\uparrow$, such that $p \leq t$. If $p \notin P_g$ then the definition of $(H_g^+)\uparrow$ implies the existence of a leaf k of P_g such that $k < p$. Observe that k is not a leaf of P, thus $g(k) < \top$. Since $p \in P_h \cap P_f$ we conclude that $k \in P_h \cap P_f$ and it is clear that k is not a leaf of P_h and it is not a leaf of P_f. Therefore $h(k) = \top$, $f(k) = \top$. Then $h(k) = \top > \top \to g(k) = f(k) \to g(k) = (f \to_R g)(k)$, contradicting our hypothesis. The contradiction arises from the assumption that $p \notin P_g$, thus we conclude that $p \in P_g$.
- Assume that $p \in P_h \cap P_f$ is a leaf. We know that $p \in P_g$. Hence $h(p) \leq (f \to_R g)(p) = f(p) \to g(p)$ thus $(h *_R f)(p) = h(p) * f(p) \leq g(p)$ as desired. $\square$

Corollary 5.8. *For each BL-pair (P, w) the algebra*

$$R(P, w) = \langle R(P, w), *_R, \to_R, \wedge_R, \vee_R, \bot_R, \top_R \rangle$$

is a commutative, integral and bounded residuated lattice.

5.2. Embedding $R(P, w)$ into a poset product

As before, let (P, w) be a BL-pair. For each $p \in P$ let $w'(p)$ be the bounded hoop given by

$$w'(p) = \begin{cases} w(p), & \text{if } w(p) \text{ is lower bounded;} \\ \mathbf{L}_2 \oplus w(p), & \text{otherwise,} \end{cases}$$

where $\mathbf{L}_2$ is the two-elements Boolean algebra.

Since for each $p \in P$ the chain $w(p)$ is either the hoop reduct of an MV-chain or a cancellative totally ordered hoop, then for each $p \in P$, $w'(p)$ is either the hoop reduct of an MV-chain or a product chain. Observe that (P, w'), with w' defined as above, is not necessary a BL-pair, since if there is $p \in P$ such that $w(p)$ is unbounded, then $w'(p)$ is not a Wajsberg hoop.

Consider the algebra of BL-functions $R(P, w) = \langle R(P, w), *_R, \to_R, \wedge_R, \vee_R, \bot_R, \top_R \rangle$. Now we will embed $R(P, w)$ into the poset product $\mathbf{B} = \bigotimes_{p \in P} w'(p)$. Because of Theorem 4.2, $\mathbf{B}$ is a BL-algebra. We define $\varphi : R(P, w) \to \mathbf{B}$ as follows:

$$\varphi(f)(p) = \begin{cases} f(p) & \text{if } p \in P_f; \\ \bot & \text{otherwise.} \end{cases}$$

Lemma 5.9. *For every pair of functions $f, g \in R(P, w)$ we have $\varphi(f \to_R g) = \varphi(f) \to \varphi(g)$.*

Proof. If $p \notin P_{f \to g}$, on one side we have $\varphi(f \to_R g)(p) = \bot$. Observe that in this case, $p \notin (H_g^+\uparrow \cup H_g^-)\downarrow$. Since P is a forest, we have that there is $t \in P$ such that $t \in H_g^-$ and $p > t$. From the definition of H_g^- we know that $f(t) > g(t)$, $t \in P_g \cap P_f$ and $\varphi(f)(t) = f(t)$ and $\varphi(g)(t) = g(t)$. The definition of $\to$ in a poset product yields $(\varphi(f) \to \varphi(g))(p) = \bot$.

Now assume that $p \in P_{f \to g}$. From Remark 5.3 (3) we have $p \in (H_g^+\uparrow)\downarrow$ or $p \in H_g^-\downarrow$.

(1) $p \in H_g^-\downarrow$. Then $p \in P_g \cap P_f$ and $\varphi(f \to_R g)(p) = (f \to_R g)(p) = f(p) \to g(p)$. There is a leaf $t \in H_g^-$ such that $f(t) > g(t)$ and $p \leq t$. From the fact that t is a leaf of P_g and the definition of BL-functions, for every $j < t$ we have that $f(j) = \top$, $g(j) = \top$ and $\varphi(f)(j) = \varphi(g)(j) = \top$. Therefore $(\varphi(f) \to \varphi(g))(p) = \varphi(f)(p) \to \varphi(g)(p) = f(p) \to g(p)$.

(2) $p \in (H_g^+\uparrow)\downarrow$. From Remark 5.5 we have $(f \to_R g)(p) = \top$ and then $\varphi(f \to_R g)(p) = \top$. To check that $(\varphi(f) \to \varphi(g))(p)$ also equals $\top$ we verify the following cases:
- $p \in H_g^+$. In this case $\varphi(f)(p) \le \varphi(g)(p)$ (because either $p \notin P_f$ or $f(p) \le g(p)$). For every $j < p$, $g(j) = \varphi(g)(j) = \top$, thus $\varphi(f)(j) \le \varphi(g)(j)$ and we conclude that $(\varphi(f) \to \varphi(g))(p) = \varphi(f)(p) \to \varphi(g)(p) = \top$.
- $p \in H_g^+\uparrow$. In this case $p \ge t$ with $t \in H_g^+$. From the previous item observe that $\varphi(f)(j) \le \varphi(g)(j)$ for every $j < t$ and for every $t \le j \le p$ we have that $\varphi(f)(j) \le \varphi(g)(j)$ (we are using again that either $t \notin P_f$ or $f(t) \le g(t)$ and the definition of BL-function). Since $\{p\}\downarrow$ is totally ordered we conclude that for every $j \le p$, $\varphi(f)(j) \le \varphi(g)(j)$, hence $(\varphi(f) \to \varphi(g))(p) = \varphi(f)(p) \to \varphi(g)(p) = \top$.
- p is less than some $t \in H_g^+$. Then $\varphi(g)(j) = \top$ for each $j \le p$ and we have that $(\varphi(f) \to \varphi(g))(p) = \varphi(f)(p) \to \varphi(g)(p) = \top$. $\square$

Theorem 5.10. *φ is an injective homomorphism from the residuated lattice $R(P, w)$ into the poset product $\mathbf{B} = \bigotimes_{p\in P} w'(p)$.*

Proof. We have already proved that implication is preserved. Trivially $*$ is preserved, because it is coordinatewise. One can easily check that $\perp$ and $\top$ are preserved, therefore it only remains to see that the function φ is injective.

Then let f and g be two BL-functions such that $f \ne g$. If $P_f \ne P_g$, let $p \in P_f \setminus P_g$. Since all minimal elements of P are in both, P_f and P_g, there is $q \in P_g$ such that $q < p$ and q is a leaf of P_g. Because of the definition of BL-function we have $f(q) = \top$ and $g(q) \ne \top$. Then $\varphi(f)(q) \ne \varphi(g)(q)$.

If $P_f = P_g$, since we are assuming that $f \ne g$, there is $p \in P_f$ such that $f(p) \ne g(p)$. Then $\varphi(f)(p) \ne \varphi(g)(p)$ and φ is injective. $\square$

6. Representation theorem

Let $\mathbf{A}$ be a BL-algebra of finite and independent spectrum. We denote by $P^{\mathbf{A}}$ the dual of $Spec^*(\mathbf{A})$, and recall that for every $p, q \in P^{\mathbf{A}}$,

$$p \le q \text{ if and only if } F_q \subseteq F_p$$

for the corresponding filters of $Spec^*(\mathbf{A})$. For each $p \in P^{\mathbf{A}}$ let

$$w^{\mathbf{A}}(p) = \mathbf{W}_{F_p}.$$

Since $P^{\mathbf{A}}$ is a finite forest and from Lemma 3.11 for each minimal $p \in P^{\mathbf{A}}$ the algebra $\mathbf{W}_{F_p}$ is an MV-chain, then $(P^{\mathbf{A}}, w^{\mathbf{A}})$ is a BL-pair. We shall see that $\mathbf{A}$ is isomorphic to $R(P^{\mathbf{A}}, w^{\mathbf{A}})$. First we verify some technical lemmas.

Lemma 6.1. *Assume that $f \in R(P^{\mathbf{A}}, w^{\mathbf{A}})$ with domain P_f. Then there is $a \in A$ such that $a/F_p = f(p)$ for every filter $p \in P_f$.*

Proof. Recall that H_f is the set of leaves of P_f, i.e.,

$$H_f = \{p \in P_f : p \text{ is maximal in } P_f\} = \{p_1, \ldots, p_n\}.$$

Consider the filter $F = \cap_{i=1}^n F_{p_i}$. From Lemma 2.10 we shall use the identification

$$(\mathbf{A}/F)/(F_{p_i}/F) \cong \mathbf{A}/F_{p_i} \cong \mathbf{W_1} \oplus \ldots \oplus \mathbf{W}_{F_{p_i}}.$$

Because of Lemma 2.9 the filters

$$\tilde{F}_i = F_{p_i}/F, \text{ for } i = 1, \ldots, n$$

are minimal in $Spec(\mathbf{A}/F)$. By Lemma 3.8, $Spec(\mathbf{A}/F)$ is independent. We define $\tilde{F}_i{}'$ to be the filter $\cap_{j=1, j\ne i}^n \tilde{F}_i$ of $\mathbf{A}/F$. Then for each $i \in \{1, \ldots, n\}$ the pair $\tilde{F}_i, \tilde{F}_i'$ satisfies $\tilde{F}_i \cap \tilde{F}_i' = \{\top\}$. Since $f(p_i) \in W_{F_{p_i}}$, Lemma 3.7 implies that there is an element $a_i \in \mathbf{A}/F$ such that

$$a_i/\tilde{F}_i = f(p_i) \text{ and } a_i/\tilde{F}_i' = \top.$$

Let

$$\tilde{a} = a_1 \cdot a_2 \cdot \ldots a_n$$

i.e., $\tilde{a} \in \mathbf{A}/F$ is the product of a_i for $i = 1, \ldots, n$. This element satisfies $\tilde{a}/\tilde{F_i} = f(p_i)$ for every $i \in \{1, \ldots, n\}$. Therefore there is an element $a \in A$ such that $a/F = \tilde{a}$ and then $f(p_i) = a/F_{p_i}$ for every $p_i \in H_f$.

Lastly, if $p \in P_f \setminus H_f$, then there is an element $q \in H_f$ such that $q \geq p$, and using Lemma 2.10, $\mathbf{A}/F_p \cong \mathbf{A}/F_q/(F_p/F_q)$. Then $a/F_q = f(q) \in W_q = w(q)$ and $q > p$, and there must be the case that $a/F_p = \top$. $\square$

Theorem 6.2. *Let* $\mathbf{A}$ *be a BL-algebra of finite and independent spectrum. Then* $\mathbf{A} \cong R(P^{\mathbf{A}}, w^{\mathbf{A}})$.

Proof. We will prove that both algebras are the same subalgebra of a poset product. As before, for each $p \in P^{\mathbf{A}}$ we define $w'(p)$ to be the bounded hoop given by

$$w'(p) = \begin{cases} w^{\mathbf{A}}(p), & \text{if } w^{\mathbf{A}}(p) \text{ is lower bounded;} \\ \mathbf{L}_2 \oplus w^{\mathbf{A}}(p), & \text{otherwise.} \end{cases}$$

Again, in case $w^{\mathbf{A}}(p)$ is cancellative (unbounded), then $w'(p)$ is the product chain that arises by adding an extra bottom $\perp_p$ to $w^{\mathbf{A}}(p)$. If $w^{\mathbf{A}}(p)$ is bounded we call $\perp_p$ the bottom element of $w^{\mathbf{A}}(p)$. Let $\mathbf{B}$ be the BL-algebra given by the poset product $\bigotimes_{p \in P^{\mathbf{A}}} w'(p)$. Recall that for each $p \in P^{\mathbf{A}}$, $w^{\mathbf{A}}(p) = \mathbf{W}_{F_p}$ that we simply denote by $\mathbf{W}_p$. Following Theorem 4.3, the function $\psi : \mathbf{A} \to \mathbf{B}$ given by

$$\psi(a)(p) = \begin{cases} a/p & \text{if } a/p \in W_p; \\ \perp_p & \text{otherwise,} \end{cases}$$

is an embedding of $\mathbf{A}$ into the poset product $\mathbf{B}$. On the other hand, by Theorem 5.10, the application $\varphi\colon R(P^{\mathbf{A}}, w^{\mathbf{A}}) \to \mathbf{B}$ defined by

$$\varphi(f)(p) = \begin{cases} f(p) & \text{if } p \in P_f; \\ \perp_p & \text{otherwise,} \end{cases}$$

is an embedding into the same poset product. We shall show that $\psi(\mathbf{A}) = \varphi(R(P^{\mathbf{A}}, w^{\mathbf{A}}))$.

Assume first that $a \in A$. Let $P_a \subseteq P^{\mathbf{A}}$ be defined as

$$P_a = \{p \in P^{\mathbf{A}} : a/p \in W_p\}.$$

Observe that if $p \in P_a$ then for all $q \in P^{\mathbf{A}}$ such that $q < p$ we have that $a \in F_q$. Moreover, P_a is a downwards closed subset of $P^{\mathbf{A}}$.

Consider the function $f_a(p) = \psi(a)(p)$ for each $p \in P_a$. We shall check that f_a is a BL-function.

(1) For each minimal element $p \in P^{\mathbf{A}}$, the filter F_p is maximal. Then $\mathbf{A}/F_p = \mathbf{W}_p$. Thus $a/p \in W_p$. This implies that P_a contains every minimal element of $P^{\mathbf{A}}$. It is easy to verify that P_a is downwards closed.

(2) Assume $p \in P_a$ and $q <> p$. If q is not in the same tree as p, condition (2) in the definition of BL-function follows from the previous item. Otherwise there is $k \leq p$, $k \leq q$ and k is maximum with this property. Lemma 3.14 yields

$$\mathbf{A}/F_p = \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_k \oplus \mathbf{W}_{k+1} \oplus \ldots \oplus \mathbf{W}_p = \mathbf{A}/F_k \oplus \mathbf{W}_{k+1} \oplus \ldots \oplus \mathbf{W}_p$$

and

$$\mathbf{A}/F_q = \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_k \oplus \mathbf{W'}_{k+1} \oplus \ldots \oplus \mathbf{W}_q = \mathbf{A}/F_k \oplus \mathbf{W}'_{k+1} \oplus \ldots \oplus \mathbf{W}_q.$$

Clearly if $a/p \in W_p$ then there is $s > k$ such that $a/s \in W_s$ and $s <> p$.

(3) $f_a(p) = a/p \in W_p$ for each $p \in P_a$.

(4) If $p \in P_a$ is not a leaf of P_a, then there is $q > p$ such that $q \in P_a$. From the definition of P_a, $a \in F_p$. Thus $a/p = \top$, and $\top = \psi(a)(p) = f_a(p)$.

(5) If $P_a \neq P$ and w is a leaf of P_a which is not a leaf of P, then $a \notin F_w$ but for all $p < w$, $a \in F_p$. Then $a/w \neq \top$ and $\top \neq \psi(a)(w) = f_a(w)$.

Therefore f_a is in $R(P^{\mathbf{A}}, w^{\mathbf{A}})$ and $\varphi(f_a) = \psi(a)$.

Now assume that $f \in R(P^{\mathbf{A}}, w^{\mathbf{A}})$ and let P_f be the domain of f. Because of Lemma 6.1 we know that there is $a_f \in A$ such that $a_f/p = a_f/F_p = f(p)$ for each filter $p \in P_f$. Therefore to check that $\varphi(f) = \psi(a_f)$ we need to see that for each $p \in P^{\mathbf{A}} \setminus P_f$, $a_f/p \notin W_p$.

Take $p \in P^{\mathbf{A}} \setminus P_f$. By item (2) in the definition of BL-function, there is a leaf $q \in P_f$ such that $q < p$. Since $F_p \subseteq F_q$ are both in the necessary spectrum of $\mathbf{A}$ from Lemma 3.14 we have

$$\mathbf{A}/F_p \cong \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_q \oplus \ldots \oplus \mathbf{W}_p \cong \mathbf{A}/F_q \oplus \ldots \oplus \mathbf{W}_p.$$

Recall that f is a BL-function, therefore (5) in the definition of BL-function asserts that $f(q) = a_f/q \neq \top$. Thus $a_f/q \in W_q \setminus \{\top\}$, what implies that $a_f/p \notin W_p$, as desired. □

Example 6.3. Theorem 6.2 cannot be generalized dropping the request on the independence of the spectrum. As a matter of fact, consider again the BL-algebra $\mathbf{A}$ of Example 2.6, and its prime spectrum $\{F_1, F_2, F_3\}$. Recall that (F_1, F_2, F_3) is a dependent triplet. As we have seen in Example 3.10, $Spec^*(\mathbf{A}) = \{F_2, F_3\}$. If we denote by (P, w) the BL-pair corresponding to $\mathbf{A}$, then $R(P, w) \cong \mathbf{L}_2 \times \mathbf{C} \ncong \mathbf{A}$, as it can be readily verified.

The reader must by now be convinced that this situation is hardly remediable with the algebraic tools used in the paper. Actually, direct inspection shows that there is no way to reconstruct $\mathbf{A}$ only knowing its spectrum as a poset, and using as building blocks the hoops $\mathbf{W}_{F_1}$, $\mathbf{W}_{F_2}$ and $\mathbf{W}_{F_3}$.

Lastly, note that from the information given above one can construct the BL-algebra formed as the ordinal sum $\mathbf{L}_2 \oplus (\mathbf{L}_2 \times \mathbf{C})$ (with necessary spectrum $\{F_1, F_2, F_3\}$), but again, this algebra is not isomorphic with $\mathbf{A}$.

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